

# Skyrmions and monopoles — isolated and arrayed

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**Abstract.** There are two basic types of three-dimensional topological soliton, namely skyrmions and monopoles. This article reviews some of the features of such solitons, concentrating on static multi-skyrmions and multi-monopoles with structure group  $SU(2)$ , and focusing in particular on periodic arrays. An isolated multi-skyrmion typically has a regular, for example polyhedral, structure; and there are multi-monopoles which look very similar. However, when it comes to infinite arrays of solitons — chains, walls or crystals — then the skyrmion and monopole systems behave rather differently.

## 1. Introduction: 3-dimensional topological solitons

Topological solitons come in two basic types: skyrmions, which wind around all of space; and monopoles, which wind around spatial infinity. The general background to both may be found in the monograph [1]. Skyrmions were introduced fifty years ago, as a model of nucleons [2]. Magnetic monopoles were introduced even longer (eighty years) ago, by Dirac [3]. Monopoles are predicted in many cosmological models, but as yet there has been no confirmed sighting, and much of the interest in them over the last thirty years has been mathematical: the partial differential equations for static BPS monopoles are completely-integrable, and are associated with some fascinating geometric structure.

Both types of object (skyrmions and monopoles) also occur in condensed-matter contexts. But these applications are more commonly 2-dimensional rather than 3-dimensional, and will not be pursued here.

Topological solitons are stable classical objects, described by fields  $\phi_\alpha(x^j)$ . Here  $x^j = (x^1, x^2, x^3)$  are the standard coordinates on  $\mathbb{R}^3$ ; we shall concentrate here on the static case, so the fields are functions only of these spatial coordinates (but there will be a brief mention of time-dependence later on).

Static solitons correspond to local minima of an energy functional  $E[\phi_\alpha] = \int \mathcal{E}(\phi_\alpha, \partial_j \phi_\alpha) d^3x$ , where the energy density  $\mathcal{E}$  is localized in space. In the best-known cases, one has a high degree of internal symmetry: in particular, the system is invariant under the action of a Lie group  $G$  (for example a gauge group). Soliton stability comes partly from non-trivial topology of the fields  $\phi_\alpha$ ; but topology is scale-free, and so some additional mechanism is needed to fix the soliton size. In our case, this is achieved by the balance of terms in  $\mathcal{E}$ : typically between a term which provides an expanding force and one which provides a contracting force. For each of the two systems — skyrmions and monopoles — we will review isolated solitons in  $\mathbb{R}^3$  with finite total energy, before discussing arrays in  $\mathbb{R}^3$ : singly-periodic chains, doubly-periodic walls, and triply-periodic crystals.

## 2. Skyrmions in $\mathbb{R}^3$

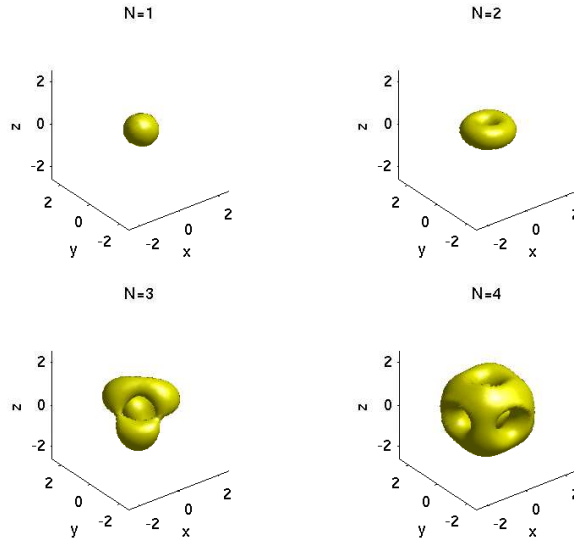
In the Skyrme system, we have a field  $U(x^j)$  which takes values in a Lie group  $G$ ; from now on, take the simplest nontrivial case  $G = \text{SU}(2)$ . The boundary condition is  $U(x^j) \rightarrow \mathbf{1}$  as  $r \rightarrow \infty$ . The energy density is defined to be

$$\mathcal{E} = -\frac{1}{2} \text{tr}(\xi_j \xi_j) - \frac{1}{16} \text{tr}([\xi_j, \xi_k][\xi_j, \xi_k]),$$

where  $\xi_j = U^{-1} \partial_j U$  (note that  $\xi_j$  is Lie-algebra-valued). Topology arises because  $U$  has a winding number  $N \in \mathbb{Z}$ ; in the nuclear-physics application,  $N$  is the baryon number. The energy satisfies a topological lower bound  $E_N = E/(12\pi^2 N) \geq 1$ .

For  $N \leq 4$  one gets the following minimal-energy solutions, the energy densities of which are plotted in figure 1:

- $N = 1$ :  $E_N = 1.232$ , sphere;
- $N = 2$ :  $E_N = 1.179$ , torus;
- $N = 3$ :  $E_N = 1.146$ , tetrahedron;
- $N = 4$ :  $E_N = 1.120$ , cube.

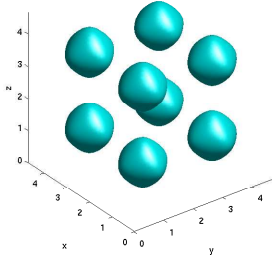


**Figure 1.** Skyrmions for  $1 \leq N \leq 4$ : energy densities.

In general, for  $N \gg 1$ , skyrmions attract and coalesce. There are many local minima, which are only accessible numerically. Typically, the solution resembles a hollow shell ('buckyball'), or a solid crystal chunk. For some shell-like examples with  $N = 32, 37, 47, 67, 97$ , see [4, 5]. In the case  $N = 32$ , the solid crystal chunk has an energy which is slightly higher than that of the shell [5] ( $E_N = 1.072$  for the shell, versus  $E_N = 1.076$  for the crystal); but the belief is that for very large  $N$ , the lowest-energy solution should be a solid crystal chunk.

## 3. Arrays of skyrmions

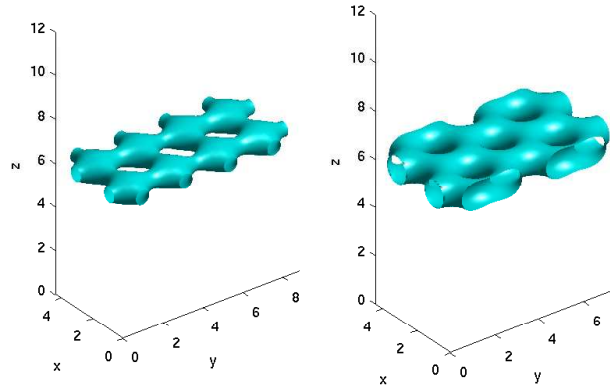
The lowest possible of  $E_N$ , namely  $E_N = 1.038$ , is attained for a triply-periodic configuration, in fact a cubic lattice of half-skyrmions. In what follows, this will be referred to as the Skyrme crystal; its energy density is plotted in figure 2, where one sees eight half-skyrmions in a



**Figure 2.** The energy density of the Skyrme crystal.

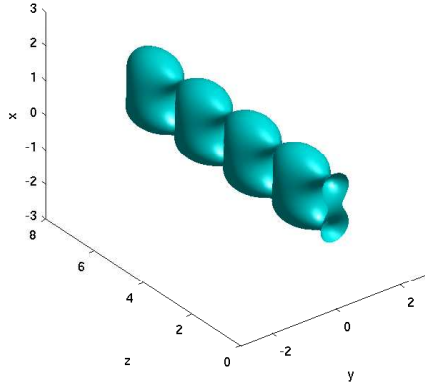
fundamental cell of the crystal. Skyrme crystals (triply-periodic) are in fact closely related [6] to doubly-periodic Skyrme walls, and also to singly-periodic Skyrme chains, each of which will now be described.

If the shell-like solutions of the previous section are visualized, in the language of carbon chemistry, as buckyballs (spherical fullerenes), then a Skyrme wall corresponds to graphene. For a single wall, the field  $U$  is doubly-periodic in  $x$  and  $y$ , and has the asymptotic behaviour  $U \rightarrow \pm 1$  as  $z \rightarrow \pm\infty$ . The corresponding 2-dimensional array of half-skyrmions can be either square or hexagonal, with the latter having an energy-per-charge which is slightly lower than the former:  $E_N = 1.062$  for the hexagonal wall, and  $E_N = 1.068$  for the square wall [1]. The energy densities of these two solutions are plotted in figure 3. Parallel walls attract one another, so there are also doubly-periodic solutions representing  $p$  parallel multi-walls [7]. As one might expect, the  $p$ -square-wall solution tends to the Skyrme crystal as  $p \rightarrow \infty$ .



**Figure 3.** The energy densities of the square wall and the hexagonal wall.

Finally, one can have Skyrme chains: in this case,  $U(x, y, z)$  is periodic in  $z$ , and  $U \rightarrow \pm 1$  as  $x^2 + y^2 \rightarrow \pm\infty$ . The simplest chain may be viewed as a chain of half-skyrmion pairs, and also as a bound vortex-antivortex pair [8], and its energy-per-charge is  $E_N = 1.143$ . More generally, there exist stable chains in which each link consists of  $2p$  half-skyrmions; and (as before) these solutions tend to the Skyrme crystal as  $p \rightarrow \infty$ . The simplest case, a  $p = 1$  chain along the  $z$ -axis, is illustrated in figure 4. This plots the energy density of the solution over two periods — four links of the chain, with each link being antiperiodic.



**Figure 4.** The energy density of the simplest Skyrme chain.

To summarize: in principle, all of nuclear physics is described by quantum chromodynamics (QCD); but direct calculations of nuclear properties in QCD is still rather difficult, and it is worth investigating a simpler approximations such as the Skyrme model. However, fitting to actual nuclear data involves modifications to the basic system, for example, adding quantum corrections and a pion-mass term. The Skyrme system admits arrays of half-skyrmions of various sorts — one may ‘add up’ infinitely many skyrmions without encountering divergences — and such arrays may be relevant to nuclei and exotic nuclear states. In this context, one would need to study the effect of modifications such as those mentioned above. Another interesting question, on which little work has been done, is to see what happens for larger groups  $G > \text{SU}(2)$ .

#### 4. Magnetic monopoles in $\mathbb{R}^3$

The setting here is gauge theory, and more specifically that of a Yang-Mills-Higgs system. The first step is to specify the gauge group  $G$ , a Lie group; in what follows, this will be either  $\text{U}(1)$  (in which case the equations are linear) or  $\text{SU}(2)$  (in which case they are nonlinear). The gauge potential is a 1-form  $A_j(x^k)$  on  $\mathbb{R}^3$ , taking values in the Lie algebra  $\mathfrak{g}$  of  $G$ . The corresponding magnetic field is  $B_j = \varepsilon_{jkl}(\partial_k A_l + A_k A_l)$ , and the energy density of the magnetic field is  $\mathcal{E}_B = |B_j|^2$ .

The case  $\mathfrak{g} = \mathfrak{u}(1)$  corresponds to electromagnetism, and then  $B_j$  is the usual magnetic field vector. Stationary points of the magnetic-field energy are solutions of  $\partial_j B_j = 0$ . The simplest solution incorporating a point source at  $x^j = 0$  is

$$B_j = \frac{1}{2}N r^{-3} x_j, \quad (1)$$

where  $N$ , for topological reasons, has to be an integer. This is the Dirac monopole [3]. Note that it is singular at  $r = 0$ , and it has infinite energy.

The generalization of this to non-Abelian gauge groups allows smooth finite-energy solutions. The simplest version is that of Bogomolny-Prasad-Sommerfield (BPS) monopoles, with gauge algebra  $\mathfrak{g} = \mathfrak{su}(2)$ . In addition to the gauge potential  $A_j(x^k)$ , we now also have a scalar (Higgs) field  $\Phi(x^k)$  taking values in  $\mathfrak{g}$ . The covariant derivative of  $\Phi$  is defined to be  $D_j \Phi = \partial_j \Phi + [A_j, \Phi]$ , and the system satisfies the Bogomolny equations

$$D_j \Phi = B_j. \quad (2)$$

This system of nonlinear partial differential equations is completely-integrable, and consequently has many remarkable properties.

The energy density is defined to be  $\mathcal{E} = -\frac{1}{2} \text{tr}[B_j B_j + (D_j \Phi)(D_j \Phi)]$ ; and the boundary condition is  $-\text{tr}(\Phi^2) \rightarrow 1$ ,  $\mathcal{E} \rightarrow 0$  as  $r \rightarrow \infty$ . The asymptotic Higgs field  $\Phi|_{r=\infty}$  is, in effect, a map from  $S^2$  to  $S^2$ , and so it has a winding number  $N \in \mathbb{Z}$ : in other words, configurations are classified topologically by the integer  $N$ . The energy satisfies the topological lower bound  $E \geq 2\pi N$ ; and  $E = 2\pi N$  if and only if the equations (2) hold. For each  $N \in \mathbb{Z}$  there exists a  $4N$ -parameter family of smooth solutions of (2). These  $4N$  moduli correspond to the position in space for each of the  $N$  monopoles (there are no forces between static monopoles, so each of them can have any location), plus a phase for each one.

One consequence of the integrability of (2) is the existence of the Nahm Transform for BPS monopoles. This is a generalized integral transform  $(A_j, \Phi) \longleftrightarrow (\hat{A}_\alpha, \hat{\Phi})$  between Yang-Mills-Higgs systems. In summary, one first solves a linear partial differential equation  $\Delta\psi = 0$  in which the coefficients of the operator  $\Delta$  are constructed from  $(A_j, \Phi)$ ; and then one does an integral involving  $\psi$  to get the transformed field  $(\hat{A}_\alpha, \hat{\Phi})$ . For monopoles on  $\mathbb{R}^3$ , the Nahm transform of an  $N$ -monopole is a set of Nahm data consisting of three  $N \times N$  matrices  $\Phi_j(s)$  on  $(-1, 1)$ , satisfying the ordinary differential equations

$$\Phi'_j = \varepsilon_{jkl} \Phi_k \Phi_l. \quad (3)$$

Since it is generally easier to solve the ODEs (3) than to solve the PDEs (2), this transform is a highly-effective way of understanding and constructing monopole solutions. For example, there are explicit solutions (involving trigonometric and elliptic functions) of (3) corresponding to spherical, toroidal, tetrahedral and cubic monopoles for  $N = 1, 2, 3, 4$  respectively; and there are many similar explicit examples for  $N > 4$ . Each  $4N$ -dimensional moduli space  $\mathcal{M}$  has a natural hyper-Kähler metric (for example, the Atiyah-Hitchin metric in the case  $N = 2$ ), and slow-motion dynamics of monopole systems corresponds to geodesic motion on  $\mathcal{M}$ . More details on all this may be found in [1].

## 5. Arrays of monopoles

In this section, we consider what happens when one tries to form periodic arrays of monopoles. The first issue is that of convergence. Suppose that we have an infinite array of Dirac monopoles of the form (1). The monopole at  $\mathbf{x}_n$  has  $\Phi_n = -1/(2|\mathbf{x} - \mathbf{x}_n|)$ , and  $\sum_n \Phi_n$  diverges for a regular array  $\{\mathbf{x}_n\}$ . So we need to regularize this infinite series, by (in effect) subtracting an infinite constant. For example, if the poles are equally-spaced along the  $z$ -axis chain — say  $\mathbf{x}_n = \mathbf{x} - n \mathbf{e}$ , where  $\mathbf{e} = (0, 0, 2\pi)$  — then the series

$$\Phi = -\frac{1}{2r} - \frac{1}{2} \sum_{n \neq 0} \left[ \frac{1}{|\mathbf{x} - \mathbf{x}_n|} - \frac{1}{2\pi n} \right] \quad (4)$$

converges (except at the poles), and  $\Phi \sim \frac{1}{4\pi} \log(x^2 + y^2)$  as  $x, y \rightarrow \infty$ . So (4) represents a chain of Dirac monopoles. And it motivates the boundary condition that one uses in the non-abelian case [9]: again  $\Phi$  behaves logarithmically as  $x, y \rightarrow \infty$ .

The Nahm transform of an  $SU(2)$   $N$ -monopole chain is a solution of the  $U(N)$  Hitchin equations on the cylinder  $\mathbb{R} \times S^1$  [9]. The latter consists of a  $U(N)$  gauge potential  $(\hat{A}_1, \hat{A}_2)$ , plus a Higgs field  $\hat{\Phi}$  taking values in the complexification of  $SU(2)$ , satisfying the Hitchin equations

$$D_{\bar{\zeta}} \hat{\Phi} = 0, \quad \hat{B} = \frac{1}{2} i [\hat{\Phi}, \hat{\Phi}^*]. \quad (5)$$

Here  $\zeta$  is a complex coordinate on  $\mathbb{R} \times S^1$ ,  $\hat{\Phi}^*$  denotes the Hermitian conjugate of  $\hat{\Phi}$ , and  $\hat{B}$  is the magnetic field. In addition,  $(\hat{A}_\alpha, \hat{\Phi})$  are required to satisfy various boundary conditions and

constraints, of which one is

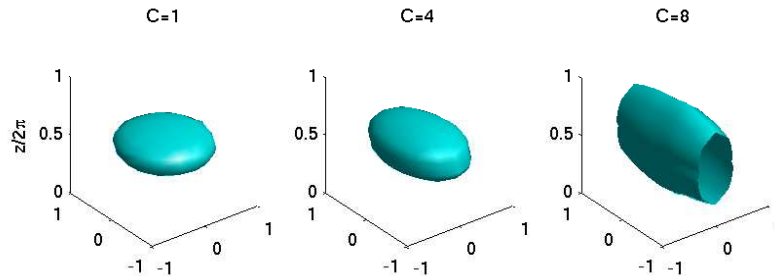
$$\det \widehat{\Phi} = (-1)^{N+1} C \cosh(2\pi\zeta) + K, \quad (6)$$

where  $C$  and  $K$  are complex constants. In fact,  $|C|$  corresponds to the ratio *monopole size/chain period*; so  $|C| \gg 1$  corresponds to a monopole chain in which the individual monopoles spread out and overlap (see below).

The simplest case is  $N = 1$ , corresponding to a chain of 1-monopoles, and in that case (5,6) gives us an explicit solution of the Hitchin equations, namely

$$\widehat{\Phi} = C \cosh(2\pi\zeta) + K, \quad \widehat{A}_\alpha = 0.$$

The (inverse) Nahm transform of this can be implemented numerically, and we get the 1-monopole chain illustrated in figure 5. This plots the quantity  $|\Phi|^2 = -\frac{1}{2} \text{tr}(\Phi^2)$  over one period in the  $z$ -direction, for various values of the parameter  $C$ . If  $C = 1$ , the individual monopoles are localized, but for  $C \gg 1$  they spread out to form a strip in the  $z$ -direction. More details of this may be found in [10].

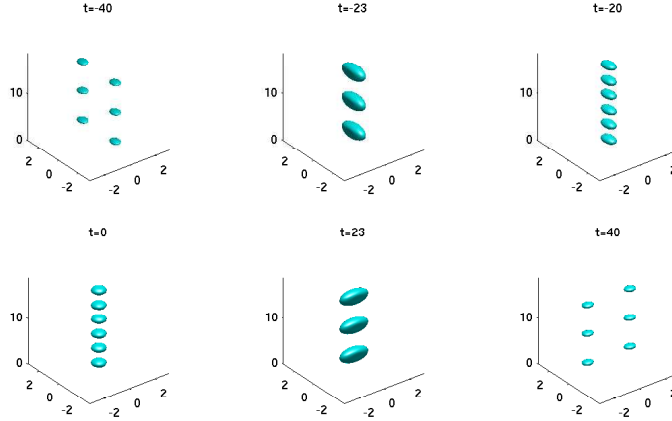


**Figure 5.**  $|\Phi|^2$  over one period, for  $C = 1, 4, 8$ .

For  $N \geq 2$ , things are not quite as explicit, but one can still use the Nahm transform to obtain useful information. For example, a special case of  $N = 2$  describes the slow-motion dynamics of interacting parallel monopole chains, corresponding to geodesics on the  $N = 2$  moduli space [11]. In general, the procedure is to:

- identify a geodesic  $\Gamma$  in the moduli space;
- for each  $t \in \Gamma$ , compute the corresponding solution  $(\widehat{A}_\alpha, \widehat{\Phi})$  of the  $SU(2)$  Hitchin equations;
- Nahm-transform to compute the monopole field  $(A_j, \Phi)$ ;
- display these as a function of  $t$ .

The simplest example is obtained by fixing the centre-of-mass of the 2-monopole system, and imposing a  $D_2$  symmetry corresponding to invariance under rotations by  $\pi$  about each of the  $x$ -,  $y$ - and  $z$ -axes: this selects a particular geodesic in the moduli space. The Hitchin equations then reduce to an elliptic sinh-Gordon equation on the cylinder, which can be solved numerically. Finally, the Nahm transform can be implemented numerically, and  $|\Phi|^2 = -\text{tr} \Phi^2$  plotted for a sequence of  $t$ -values, to visualize the dynamics. This was the process used to produce figure 6, which shows snapshots of the scattering of two monopole chains (each plotted over three  $z$ -periods). One sees the right-angle scattering which is typical of topological-soliton interactions.



**Figure 6.** Scattering of two monopole chains.

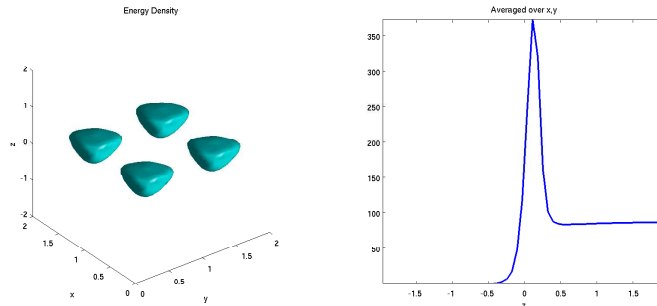
Our final topic is that of doubly-periodic monopole solutions, or monopole walls. (It is worth pointing out that smooth triply-periodic monopoles, or monopole crystals, cannot exist — the divergences become uncontrollable. So double periodicity is as far as one can go.) So we impose periodicity in the  $x$ - and  $y$ -directions. There are now two topological charges  $N_+$  and  $N_-$ , corresponding to the winding number of the Higgs field  $\Phi$  as  $z \rightarrow \pm\infty$  respectively. The appropriate boundary condition in  $z$  turns out to be

$$\Phi \sim 2\pi N_{\pm} z \text{ as } z \rightarrow \pm\infty,$$

with the energy density behaving like  $\mathcal{E} \rightarrow 8\pi^2(N_{\pm})^2$ . The interpretation is that one has a wall of monopoles separating two domains, in each of which the energy density is constant. In this case, the Nahm transform maps walls to walls, with (in general) different gauge group  $G$  and topological charges  $N_{\pm}$ . For example, the simplest non-trivial case of a Nahm correspondence is

$$\{G = \text{U}(1), \quad N_{\pm} = \pm 1\} \leftrightarrow \{G = \text{SU}(2), \quad N_- = 0, \quad N_+ = 1\}.$$

The energy density  $\mathcal{E}$  of the  $\text{SU}(2)$  monopole wall in this case is illustrated in figure 7: the



**Figure 7.**  $\{G = \text{SU}(2), N_- = 0, N_+ = 1\}$ : an example solution

left-hand plot is a 3-dimensional plot of  $\mathcal{E}$  over two periods in  $x$  and  $y$ , so one sees four of the

monopoles in the wall; while the right-hand plot shows  $\mathcal{E}$  integrated over  $x$  and  $y$ , as a function of  $z$ , revealing how the wall separates two domains, one with zero energy and one with constant positive energy density.

Monopole arrays have many other nice mathematical features which follow from the complete-integrability of the equations (2). For example, associated with any solution is a set of “spectral data” which are usually much simpler than the field itself, are invariant under the Nahm transform, and give one a lot of information about the solution. For example, in the monopole-wall case, the holonomy of  $D_x + i\Phi$  over one period in the  $x$ -direction is holomorphic in  $z - iy$  (as a consequence of (2)), and it has a very simple explicit form. All this structure ensures that BPS monopoles will remain of great mathematical interest, even if their physical importance remains unclear.

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