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Geometric analysis aspects of infinite semiplanar graphs with nonnegative curvature

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Abstract. We apply Alexandrov geometry methods to study geometric analysis aspects of infinite semiplanar graphs with nonnegative combinatorial curvature. We obtain the metric classification of these graphs and construct the graphs embedded in the projective plane minus one point. Moreover, we show the volume doubling property and the Poincaré inequality on such graphs. The quadratic volume growth of these graphs implies the parabolicity. Finally, we prove the polynomial growth harmonic function theorem analogous to the case of Riemannian manifolds.

1. Introduction

In this paper, we study systematically (infinite) semiplanar graphs G of nonnegative curvature. This curvature condition can either be formulated purely combinatorically, as in the approach of [24, 31, 48], or as an Alexandrov curvature condition on the polygonal surface $S(G)$ obtained by assigning length one to every edge and filling in faces. The fact that these two curvature conditions – nonnegative combinatorial curvature of G and nonnegative Alexandrov curvature of $S(G)$ – are equivalent will be systematically exploited in the present paper. First of all, we can then classify such graphs. Curiously, as soon as the maximal degree of a face is at least 43, the graph necessarily has a rather special structure. This will simplify our reasoning considerably. Secondly, as Alexandrov geometry is a natural generalization of Riemannian geometry, we can systematically carry over the geometric function theory of nonnegatively curved Riemannian manifolds to the setting of nonnegatively curved semiplanar graphs. Starting with two basic inequalities, the volume doubling property and the Poincaré inequality, which hold for such spaces, we obtain the Harnack inequality for harmonic functions by Moser's iteration scheme. Here, for defining (sub-, super-)harmonic functions, we use the discrete Laplace operator of G. Our main results then say that a nonnegatively curved semiplanar graph is parabolic in

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the sense that it does not support any nontrivial positive superharmonic function (equivalently, Brownian motion is recurrent), and that the dimension of the space of harmonic functions of polynomial growth with exponent at most d is bounded for any d. This is an extension of the solution by Colding–Minicozzi [10] of a conjecture of Yau [55] in Riemannian geometry.

Let us now describe the results in more precise technical terms. The combinatorial curvature for planar graphs was introduced in Stone [48, 49], Gromov [24] and Ishida [31]. In [26], Higuchi conjectured, as a discrete analog of Myers' theorem in Riemannian geometry, that any planar graph with positive curvature everywhere is a finite graph. DeVos and Mohar [19] solved the conjecture by proving the Gauss–Bonnet formula for infinite planar graphs. The combinatorial curvature was studied by many authors [1, 2, 7, 8, 32–34, 43, 50, 51].

In this paper, we are interested in infinite graphs. Let G be an infinite graph embedded in a 2-manifold $S(G)$ such that each face is homeomorphic to a closed disk with finite edges as the boundary. This includes the case of a planar graph, and we call such a $G = (V, E, F)$ with its sets of vertices V, edges E, and faces F, a semiplanar graph. For each vertex $x \in V$, the combinatorial curvature at x is defined as

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)},$$

where d_x is the degree of the vertex x, $\deg(\sigma)$ is the degree of the face σ, and the sum is taken over all faces incident to x (i.e. $x \in \sigma$). The idea of this definition is to measure the difference of 2π and the total angle Σ_x at the vertex x on the polygonal surface $S(G)$ equipped with a metric structure obtained from replacing each face of G with a regular polygon of side lengths one and gluing them along the common edges. That is,

$$2\pi\Phi(x) = 2\pi - \Sigma_x.$$

Let $\chi(S(G))$ denote the Euler characteristic of the surface $S(G)$. The Gauss–Bonnet formula of G in [19] reads as

$$\sum_{x \in G} \Phi(x) \le \chi(S(G)),$$

whenever $\Sigma_{x \in G : \Phi(x) < 0} \Phi(x)$ converges. Furthermore, Chen and Chen [8] proved that if the absolute total curvature $\Sigma_{x \in G} |\Phi(x)|$ is finite, then G has only finitely many vertices with nonvanishing curvature. Then Chen [7] obtained the topological classification of infinite semiplanar graphs with nonnegative curvature: $\mathbb{R}^2$, the cylinder without boundary, and the projective plane minus one point. In addition, at the end of the paper [7], he proposed a question on the construction of semiplanar graphs with nonnegative curvature embedded in the projective plane minus one point.

We note that the definition of the combinatorial curvature is equivalent to the generalized sectional (Gaussian) curvature of the surface $S(G)$. The semiplanar graph G has nonnegative combinatorial curvature if and only if the corresponding regular polygonal surface $S(G)$ is an Alexandrov space with nonnegative sectional curvature, i.e. $\mathrm{Sec}\, S(G) \ge 0$ (or $\mathrm{Sec}(G) \ge 0$ for short).

Here, we are referring to another notion of curvature for such polygonal spaces, or more precisely, of curvature bounds. This paper will derive its insights from comparing these curvature notions. A metric space (X, d) is called an Alexandrov space if it is a geodesic space (i.e. each pair of points in X can be joined by a shortest path called a geodesic) and locally satisfies the Toponogov triangle comparison. For the basic facts of Alexandrov spaces, readers are referred to [3, 4]. In this paper, we shall apply the Alexandrov geometry to study the geometric and analytic properties of semiplanar graphs with nonnegative curvature.

Alexandrov geometry can be seen as a natural generalization of Riemannian geometry, and many fundamental results of Riemannian geometry extend to the more general Alexandrov setting. Firstly, the well-known Cheeger–Gromoll splitting theorem for Riemannian manifolds with nonnegative Ricci curvature was generalized to Alexandrov spaces (see [3, 6, 40, 41, 56]); the result is that if the n-dimensional Alexandrov space (X, d) with nonnegative curvature contains an infinite geodesic γ, i.e. $\gamma : (-\infty, \infty) \to X$, then X isometrically splits as $Y \times \mathbb{R}$, where Y is an $(n-1)$-dimensional Alexandrov space with nonnegative curvature. In the present paper, we shall prove that if the semiplanar graph G with nonnegative curvature has at least two ends (geometric ends at infinity), then $S(G)$ is isometric to the cylinder; this is interesting since we do not use the Gauss–Bonnet formula here. Moreover, we give the metric classification of $S(G)$ for semiplanar graphs G with nonnegative curvature. An orientable $S(G)$ is isometric to a plane, or a cylinder without boundary if it has vanishing curvature everywhere, and isometric to a cap which is homeomorphic but not isometric to the plane if it has at least one vertex with positive curvature. A nonorientable $S(G)$ is isometric to the metric space obtained by gluing in some way the boundary of $[0, a] \times \mathbb{R}$ with vanishing curvature everywhere (see Lemma 3.9). By this lemma, we answer the question of Chen [7].

Secondly, we prove that G inherits some geometric estimates from those of $S(G)$. Let d^G (resp. d) denote the intrinsic metric on the graph G (resp. polygonal surface $S(G)$). It will be proved that these two metrics are bi-Lipschitz equivalent on G, i.e. for any $x, y \in G$,

$$Cd^G(x, y) \le d(x, y) \le d^G(x, y).$$

We denote by $B_R(p) = \{x \in G : d^G(p, x) \le R\}$ the closed geodesic ball in G and by

$$B_R^{S(G)}(p) = \{x \in S(G) : d(p, x) \le R\}$$

the closed geodesic ball in $S(G)$ respectively. The volume of $B_R(p)$ is defined as

$$|B_R(p)| = \sum_{x \in B_R(p)} d_x.$$

The Bishop–Gromov volume comparison holds on the n-dimensional Alexandrov space (X, d) with nonnegative curvature (see [3]). For any $p \in X, 0 < r < R$, we have

$$\frac{\mathcal{H}^n(B_R^X(p))}{\mathcal{H}^n(B_r^X(p))} \le \left(\frac{R}{r}\right)^n, \tag{1.1}$$

$$\mathcal{H}^n(B_{2R}^X(p)) \le 2^n \mathcal{H}^n(B_R^X(p)), \tag{1.2}$$

$$\mathcal{H}^n(B_R^X(p)) \le C(n)R^n, \tag{1.3}$$

where $B_R^X(p)$ is the closed geodesic ball in X and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. We call (1.1) the relative volume comparison and (1.2) the volume doubling property. Note that $S(G)$ is a 2-dimensional Alexandrov space with nonnegative curvature if G is a semiplanar graph with nonnegative combinatorial curvature. Let D_G denote the maximal degree of the faces in G, i.e. $D_G = \max_{\sigma \in F} \deg(\sigma)$ which is finite by [8]. In this paper, for simplicity we also denote $D := D_G$ when it does not make any confusion. The relative volume growth property for the graph G is obtained in the following theorem.

Theorem 1.1. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$*. Then for any $p \in G$, $0 < r < R$, we have*

$$\frac{|B_R(p)|}{|B_r(p)|} \leq C(D)\left(\frac{R}{r}\right)^2, \tag{1.4}$$

$$|B_{2R}(p)| \leq C(D)|B_R(p)|, \tag{1.5}$$

$$|B_R(p)| \leq C(D)R^2 \quad (R \geq 1), \tag{1.6}$$

where $C(D)$ is a constant only depending on D which is the maximal facial degree of G.

Thirdly, we shall show that the Poincaré inequality holds on the semiplanar graph G with nonnegative curvature. The Poincaré inequality has been proved on Alexandrov spaces in [29, 36], and also on graphs (ϵ-nets) embedded into Riemannian manifolds with bounded geometry in [15]. Let u be a local $W^{1,2}$ function on an n-dimensional Alexandrov space (X, d) with Sec $X \geq 0$, then

$$\int_{B_R^X(p)} |u - u_{B_R}|^2 \leq C(n)R^2 \int_{B_R^X(p)} |\nabla u|^2, \tag{1.7}$$

where

$$u_{B_R} = \frac{1}{\mathcal{H}^n(B_R^X(p))} \int_{B_R^X(p)} u.$$

For any function $f : G \to \mathbb{R}$, we extend it to each edge of G by linear interpolation and then to each face nicely with controlled energy (see Lemma 4.6). So we get a local $W^{1,2}$ function on $S(G)$ which satisfies the Poincaré inequality (1.7), and then it implies the Poincaré inequality on the graph G.

Theorem 1.2. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$*. Then there exist two constants $C(D)$ and C such that for any $p \in G$, $R > 0$, $f : B_{CR}(p) \to \mathbb{R}$, we have*

$$\sum_{x \in B_R(p)} (f(x) - f_{B_R})^2 d_x \leq C(D)R^2 \sum_{\substack{x,y \in B_{CR}(p) \\ x \sim y}} (f(x) - f(y))^2, \tag{1.8}$$

where

$$f_{B_R} = \frac{1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x) d_x,$$

and $x \sim y$ means x and y are neighbors.

Finally, we shall study some global properties of harmonic functions on the semiplanar graph G with nonnegative curvature. Let $f : G \to \mathbb{R}$ be a function on the graph G. The Laplace operator L is defined as (see [9, 20, 23])

$$Lf(x) = \frac{1}{d_x} \sum_{y \sim x} (f(y) - f(x)).$$

A function f is called harmonic (subharmonic, superharmonic) if $Lf(x) = 0$ ($\geq 0, \leq 0$), for each $x \in G$.

A manifold or a graph is called parabolic if it does not admit any nontrivial positive superharmonic function. The question when a manifold is parabolic has been studied extensively in the literature; in fact, parabolicity is equivalent to recurrency for Brownnian motion (see [22, 27, 44]). Noticing that the semiplanar graph G with nonnegative curvature has the quadratic volume growth (1.6), we obtain the following theorem in a standard manner (see [27, 52]).

Theorem 1.3. *Any semiplanar graph G with* $\mathrm{Sec}(G) \geq 0$ *is parabolic.*

Since Yau [53] proved the Liouville theorem for positive harmonic functions on complete Riemannian manifolds with nonnegative Ricci curvature, the study of harmonic functions on manifolds has been one of the central fields of geometric analysis. Yau conjectured in [54, 55] that the linear space of polynomial growth harmonic functions with a fixed growth rate on a Riemannian manifold with nonnegative Ricci curvature is of finite dimension. Colding and Minicozzi [10] gave an affirmative answer to the conjecture by the volume doubling property and the Poincaré inequality. An alternative method by the mean value inequality was introduced by Colding and Minicozzi [12] (see also [37]). In this paper, we call this result the polynomial growth harmonic function theorem. Delmotte [16] proved it in the graph setting by assuming the volume doubling property and the Poincaré inequality. Kleiner [35] generalized it to Cayley graphs of groups of polynomial growth, by which he gave a new proof of Gromov's theorem in group theory. The first author [30] generalized it to Alexandrov spaces and gave the optimal dimension estimate analogous to the Riemannian manifold case.

Let G be a semiplanar graph with nonnegative curvature and

$$H^d(G) = \{u : Lu \equiv 0,\ |u(x)| \leq C(d^G(p,x)+1)^d\}$$

which is the space of polynomial growth harmonic functions of growth degree less than or equal to d on G. By the method of Colding and Minicozzi [10, 11, 13], the volume doubling property (1.5) and the Poincaré inequality (1.8) imply that

$$\dim H^d(G) \leq C(D)d^{v(D)} \quad \text{for any } d \geq 1,$$

where $C(D)$ and $v(D)$ depend on D (see [16]). Instead of the volume doubling property (1.5), inspired by [13], we use the relative volume comparison (1.4) to show that

$$\dim H^d(G) \leq C(D)d^2.$$

It seems natural that the dimension estimate of $H^d(G)$ should involve the maximal facial degree D because the relative volume comparison and the Poincaré inequality cannot avoid D, but the estimate is still not satisfactory since $C(D)$ here is only a dimensional constant in the Riemannian case.

Furthermore, we note that a semiplanar graph G with nonnegative curvature and $D_G \geq 43$ has a special structure of linear volume growth like a one-sided cylinder, see Theorem 2.10. Inspired by the work [47], in which Sormani proved that any polynomial growth harmonic function on a Riemannian manifold with one end and nonnegative Ricci curvature of linear volume growth is constant, we obtain the following theorem.

Theorem 1.4. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$ *and* $D_G \geq 43$. *Then for any* $d > 0$,

$$\dim H^d(G) = 1.$$

The final dimension estimate follows from combining the previous two estimates.

Theorem 1.5. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$. *Then for any $d \geq 1$,*

$$\dim H^d(G) \leq Cd^2,$$

where C is an absolute constant.

For convenience, we may change the values of the constants $C, C(D)$ from line to line in the sequel.

2. Preliminaries

A graph is called planar if it can be embedded in the plane without self-intersection of edges. We define a semiplanar graph similarly.

Definition 2.1. A graph $G = (V, E)$ is called semiplanar if it can be embedded into a connected 2-manifold S without self-intersection of edges and each face is homeomorphic to the closed disk with finite edges as the boundary.

The embedding in the definition is called a strong embedding in [7]. Let $G = (V, E, F)$ denote the semiplanar graph with the set of vertices, V, edges, E and faces, F. Edges and faces are regarded as closed subsets of S, and two objects from V, E, F are called incident if one is a proper subset of the other. In this paper, essentially for simplicity, we shall always assume that the surface S has no boundary except in Remark 3.7 and G is a simple graph, i.e. without loops and multi-edges. Throughout this paper, we write $x \in G$ instead $x \in V$ for the vertex x. We denote by d_x the degree of the vertex $x \in G$ and by $\deg(\sigma)$ the degree of the face $\sigma \in F$, i.e. the number of edges incident to σ. Further, we assume that $3 \leq d_x < \infty$ and $3 \leq \deg(\sigma) < \infty$ for each vertex x and face σ, which implies that G is a locally finite graph. For each semiplanar graph $G = (V, E, F)$, there is a unique metric space, denoted by $S(G)$, which is obtained from replacing each face of G by a regular polygon of side length one with the same facial degree and gluing the faces along the common edges in S. The surface $S(G)$ is called the regular polygonal surface of the semiplanar graph G.

For a semiplanar graph G, the combinatorial curvature at each vertex $x \in G$ is defined as

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)},$$

where the sum is taken over all the faces incident to x. This curvature can be read from the corresponding regular polygonal surface $S(G)$ as

$$2\pi\Phi(x) = 2\pi - \Sigma_x,$$

where Σ_x is the total angle of $S(G)$ at x. Positive curvature thus means convexity at the vertex. We shall prove that the semiplanar graph G has nonnegative curvature everywhere if and only if the regular polygonal surface $S(G)$ is an Alexandrov space with nonnegative curvature, which is a generalized sectional (Gaussian) curvature on metric spaces. In this paper, we denote by $\mathrm{Sec}(G) \geq 0$ the semiplanar graph G with nonnegative combinatorial curvature and by $\mathrm{Sec}\, X \geq 0$ the metric space X with nonnegative curvature in the sense of Alexandrov.

We recall some basic facts in metric geometry and Alexandrov geometry. Readers are referred to [3,4].

A curve γ in a metric space (X,d) is a continuous map $\gamma : [a,b] \to X$. The length of a curve γ is defined as

$$L(\gamma) = \sup\left\{\sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i)) : \text{any partition } a = y_0 < y_1 < \cdots < y_N = b\right\}.$$

A curve γ is called rectifiable if $L(\gamma) < \infty$. Given $x, y \in X$, denote by $\Gamma(x,y)$ the set of rectifiable curves joining x and y. A metric space (X,d) is called a length space if

$$d(x,y) = \inf_{\gamma\in\Gamma(x,y)} \{L(\gamma)\} \quad \text{for any } x, y \in X.$$

A curve $\gamma : [a,b] \to X$ is called a geodesic if $d(\gamma(a), \gamma(b)) = L(\gamma)$. It is always true by the definition of the length of a curve that $d(\gamma(a), \gamma(b)) \le L(\gamma)$. A geodesic is a shortest curve (or shortest path) joining the two end-points. A geodesic space is a length space (X,d) satisfying that for any $x, y \in X$, there is a (not necessarily unique) geodesic joining x and y.

Denote by Π_κ, $\kappa \in \mathbb{R}$, the model space which is a 2-dimensional, simply connected space form of constant curvature κ. Typical ones are

$$\Pi_\kappa = \begin{cases} \mathbb{R}^2, & \kappa = 0, \\ S^2, & \kappa = 1, \\ \mathbb{H}^2, & \kappa = -1. \end{cases}$$

In a geodesic space (X,d), we denote by γ_{xy} one of the geodesics joining the points x and y, for $x, y \in X$. Given three points $x, y, z \in X$, denote by $\triangle_{xyz}$ the geodesic triangle with edges $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$. There exists a unique (up to an isometry) geodesic triangle, $\triangle_{\bar{x}\bar{y}\bar{z}}$, in Π_κ ($d(x,y) + d(y,z) + d(z,x) < \frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$) such that

$$d(\bar{x}, \bar{y}) = d(x,y), d(\bar{y}, \bar{z}) = d(y,z) \quad \text{and} \quad d(\bar{z}, \bar{x}) = d(z,x).$$

We call $\triangle_{\bar{x}\bar{y}\bar{z}}$ the comparison triangle in Π_κ.

Definition 2.2. A complete geodesic space (X,d) is called an Alexandrov space with sectional curvature bounded below by κ ($\mathrm{Sec}\, X \ge \kappa$ for short) if for any $p \in X$, there exists a neighborhood U_p of p such that for any $x, y, z \in U_p$ (with $d(x,y) + d(y,z) + d(z,x) < \frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$), any geodesic triangle $\triangle_{xyz}$, and any $w \in \gamma_{yz}$, letting $\bar{w} \in \gamma_{\bar{y}\bar{z}}$ be in the comparison triangle $\triangle_{\bar{x}\bar{y}\bar{z}}$ in Π_κ satisfying $d(\bar{y}, \bar{w}) = d(y,w)$ and $d(\bar{w}, \bar{z}) = d(w,z)$, we have

$$d(x,w) \ge d(\bar{x}, \bar{w}).$$

In other words, an Alexandrov space (X,d) is a geodesic space which locally satisfies the Toponogov triangle comparison theorem for the sectional curvature. It is proved in [4] that the Hausdorff dimension of an Alexandrov space (X,d), $\dim_H(X)$, is an integer or infinity. One-dimensional Alexandrov spaces are: straight line, S^1, ray and closed interval.

Let (X,d) be an Alexandrov space, $B_R^X(p)$ denote the closed geodesic ball centered at $p \in X$ of radius $R > 0$, i.e. $B_R^X(p) = \{x \in X : d(p,x) \le R\}$. The well-known Bishop–Gromov volume comparison theorem holds on Alexandrov spaces [3].

Theorem 2.3. *Let (X, d) be an n-dimensional Alexandrov space with nonnegative curvature, i.e.* $\mathrm{Sec}\, X \geq 0$. *Then for any $p \in X$, $0 < r < R$, it holds that*

$$\frac{\mathcal{H}^n(B_R^X(p))}{\mathcal{H}^n(B_r^X(p))} \leq \left(\frac{R}{r}\right)^n, \tag{2.1}$$

$$\mathcal{H}^n(B_{2R}^X(p)) \leq 2^n \mathcal{H}^n(B_R^X(p)), \tag{2.2}$$

$$\mathcal{H}^n(B_R^X(p)) \leq C(n)R^n, \tag{2.3}$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure.

A curve $\gamma : (-\infty, \infty) \to X$ is called an infinite geodesic if for any $s, t \in (-\infty, \infty)$,

$$d(\gamma(s), \gamma(t)) = L(\gamma|_{[s,t]}),$$

i.e. every restriction of γ to a subinterval is a geodesic (shortest path). For two metric spaces (X, d_X), (Y, d_Y), the metric product of X and Y is a product space $X \times Y$ equipped with the metric $d_{X\times Y}$ which is defined as

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)},$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$. The Cheeger–Gromoll splitting theorem holds on Alexandrov spaces with nonnegative curvature [3, 40, 41, 56].

Theorem 2.4. *Let (X, d) be an n-dimensional Alexandrov space with* $\mathrm{Sec}\, X \geq 0$. *If it contains an infinite geodesic, then X is isometric to a metric product $Y \times \mathbb{R}$, where Y is an $(n-1)$-dimensional Alexandrov space with* $\mathrm{Sec}\, Y \geq 0$.

Let (X, d) be an n-dimensional Alexandrov space with $\mathrm{Sec}\, X \geq \kappa$, $\kappa \in \mathbb{R}$. The tangent space at each point $p \in X$ is well defined, denoted by T_pX, which is the pointed Gromov–Hausdorff limit of the rescaling sequence $(X, \lambda d, p)$ as $\lambda \to \infty$ (see [3]). A point $p \in X$ is called regular (resp. singular) if T_pX is (resp. not) isometric to $\mathbb{R}^n$. Let $S(X)$ denote the set of singular points in X. It is known that $\mathcal{H}^n(S(X)) = 0$. Otsu and Shioya [42] obtained the C^1-differentiable and C^0-Riemannian structure on the regular part of X, $X \setminus S(X)$. A function f defined on a domain $\Omega \subset X$ is called Lipschitz if there is a constant C such that for any $x, y \in \Omega$,

$$|f(x) - f(y)| \leq Cd(x, y).$$

It can be shown that every Lipschitz function is differentiable $\mathcal{H}^n$-almost everywhere and with bounded gradient $|\nabla f|$ (see [5]). Let $\mathrm{Lip}(\Omega)$ denote the set of Lipschitz functions on Ω. For any precompact domain $\Omega \subset X$ and $f \in \mathrm{Lip}(\Omega)$, the $W^{1,2}$ norm of f is defined as

$$\|f\|^2_{W^{1,2}(\Omega)} = \int_\Omega f^2 + \int_\Omega |\nabla f|^2.$$

The $W^{1,2}$ space on Ω, denoted by $W^{1,2}(\Omega)$, is the completion of $\mathrm{Lip}(\Omega)$ with respect to the $W^{1,2}$ norm. We say that a function $f \in W^{1,2}_{\mathrm{loc}}(X)$ if for any precompact domain $\Omega \subset\subset X$, $f|_\Omega \in W^{1,2}(\Omega)$. The Poincaré inequality was proved in [29, 36].

Theorem 2.5. *Let (X,d) be an n-dimensional Alexandrov space with* $\operatorname{Sec} X \geq 0$ *and* $u \in W^{1,2}_{\mathrm{loc}}(X)$*. Then for any* $R > 0$ *and* $p \in X$,

$$\int_{B_R^X(p)} |u - u_{B_R}|^2 \leq C(n) R^2 \int_{B_R^X(p)} |\nabla u|^2, \tag{2.4}$$

where

$$u_{B_R} = \frac{1}{\mathcal{H}^n(B_R^X(p))} \int_{B_R^X(p)} u.$$

Let (X,d) be a geodesic space and $\{B_{R_i}^X(p)\}_{i=1}^\infty$ be an exhaustion of X, i.e.

$$B_{R_i}^X(p) \subset B_{R_{i+1}}^X(p) \quad \text{for any } i \geq 1 \text{ and } \quad X = \bigcup_{i=1}^\infty B_{R_i}^X(p),$$

equivalently $R_i \leq R_{i+1}$ and $R_i \to \infty$ as $i \to \infty$. A connected component E of $X \setminus B_{R_i}^X(p)$ is called connecting to infinity if there is a sequence of points $\{q_j\}_{j=1}^\infty$ in E with $d(p,q_j) \to \infty$ as $j \to \infty$. The number of connected components of $X \setminus B_{R_i}^X(p)$ connecting to infinity, denoted by N_i, is nondecreasing in i. Then the limit $N(X) = \lim_{i\to\infty} N_i$ is well defined and called the number of ends of X. It is easy to show that $N(X)$ does not depend on the choice of the exhaustion of X, $\{B_{R_i}^X(p)\}_{i=1}^\infty$. Given a connected graph $G = (V,E)$, let G_1 denote the 1-dimensional simplicial complex of G, i.e. a metric space obtained from G by assigning each edge the length one. Then G_1 is a geodesic space and $N(G_1)$ is well defined. If G is a semiplanar graph and $S(G)$ is the corresponding regular polygonal surface, then we can also define the number of ends of $S(G)$, $N(S(G))$.

In the sequel, we recall some facts on the combinatorial structure of semiplanar graphs. The Gauss–Bonnet formula for the semiplanar graph was proved in [8, 19].

Theorem 2.6. *Let G be a semiplanar graph, $S(G)$ be the corresponding regular polygonal surface, and $t = N(S(G))$. If G has only finitely many vertices with negative curvature, then there exists a closed 2-manifold M, so that $S(G)$ is homeomorphic to M minus t points, and*

$$\sum_{x \in G} \Phi(x) \leq \chi(S(G)) := \chi(M) - t. \tag{2.5}$$

Moreover, G has at most finitely many vertices with nonvanishing curvature.

By the Gauss–Bonnet formula, Chen [7] gave the topological classification of semiplanar graphs with nonnegative curvature.

Theorem 2.7. *Let G be an infinite semiplanar graph with nonnegative curvature everywhere and $S(G)$ be the regular polygonal surface. Then $S(G)$ is homeomorphic to: $\mathbb{R}^2$, the cylinder without boundary or the projective plane minus one point.*

Let G be a semiplanar graph and $x \in G$. It is straightforward that

$$3 \leq d_x \leq 6 \text{ if } \Phi(x) \geq 0 \quad \text{and} \quad 3 \leq d_x \leq 5 \text{ if } \Phi(x) > 0.$$

A pattern of a vertex x is a vector $(\deg(\sigma_1), \deg(\sigma_2), \ldots, \deg(\sigma_{d_x}))$, where $\{\sigma_i\}_{i=1}^{d_x}$ are the faces incident to x ordered with $\deg(\sigma_1) \leq \deg(\sigma_2) \leq \cdots \leq \deg(\sigma_{d_x})$. The following table is the list of all possible patterns of a vertex x with positive curvature (see [8, 19]).

Patterns		$\Phi(x)$
$(3,3,k)$	$3 \leq k$	$= 1/6 + 1/k$
$(3,4,k)$	$4 \leq k$	$= 1/12 + 1/k$
$(3,5,k)$	$5 \leq k$	$= 1/30 + 1/k$
$(3,6,k)$	$6 \leq k$	$= 1/k$
$(3,7,k)$	$7 \leq k \leq 41$	$\geq 1/1722$
$(3,8,k)$	$8 \leq k \leq 23$	$\geq 1/552$
$(3,9,k)$	$9 \leq k \leq 17$	$\geq 1/306$
$(3,10,k)$	$10 \leq k \leq 14$	$\geq 1/210$
$(3,11,k)$	$11 \leq k \leq 13$	$\geq 1/858$
$(4,4,k)$	$4 \leq k$	$= 1/k$
$(4,5,k)$	$5 \leq k \leq 19$	$\geq 1/380$
$(4,6,k)$	$6 \leq k \leq 11$	$\geq 1/132$
$(4,7,k)$	$7 \leq k \leq 9$	$\geq 1/252$
$(5,5,k)$	$5 \leq k \leq 9$	$\geq 1/90$
$(5,6,k)$	$6 \leq k \leq 7$	$\geq 1/105$
$(3,3,3,k)$	$3 \leq k$	$= 1/k$
$(3,3,4,k)$	$4 \leq k \leq 11$	$\geq 1/132$
$(3,3,5,k)$	$5 \leq k \leq 7$	$\geq 1/105$
$(3,4,4,k)$	$4 \leq k \leq 5$	$\geq 1/30$
$(3,3,3,3,k)$	$3 \leq k \leq 5$	$\geq 1/30$

All possible patterns of a vertex with vanishing curvature are (see [8, 25]):

$$(3,7,42),\ (3,8,24),\ (3,9,18),\ (3,10,15),\ (3,12,12),\ (4,5,20),\ (4,6,12),$$
$$(4,8,8),\ (5,5,10),\ (6,6,6), (3,3,4,12),\ (3,3,6,6),\ (3,4,4,6),\ (4,4,4,4),$$
$$(3,3,3,3,6),\ (3,3,3,4,4),\ (3,3,3,3,3,3).$$

We recall a lemma in [8].

Lemma 2.8. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$ *and σ be a face of G with* $\deg(\sigma) \geq 43$. *Then*

$$\sum_{x \in \sigma} \Phi(x) \geq 1.$$

Proof. For completeness, we give the proof of the lemma. Since we have $\mathrm{Sec}(G) \geq 0$ and $\deg(\sigma) \geq 43$, the only possible patterns of the vertices incident to the face σ are

$$(3,3,k),\ (3,4,k),\ (3,5,k),\ (3,6,k),\ (4,4,k),\ (3,3,3,k), \quad \text{where } k = \deg(\sigma).$$

In each case, we have $\Phi(x) \geq \frac{1}{k}$, for $x \in \sigma$. Hence, we get

$$\sum_{x \in \sigma} \Phi(x) \geq 1. \qquad \square$$

Let $G = (V, E, F)$ be a semiplanar graph. We denote by $D_G = \sup\{\deg(\sigma) : \sigma \in F\}$ the maximal degree of faces in G. If G has nonnegative curvature everywhere, then by Theorem 2.6, G has at most finitely many vertices with nonvanishing curvature which implies that $D_G < \infty$.

Lemma 2.9. *Let G denote an infinite semiplanar graph with* $\mathrm{Sec}(G) \geq 0$. *Then either $D_G \leq 42$, or G has a unique face σ with* $\deg(\sigma) \geq 43$ *and has vanishing curvature elsewhere.*

Proof. If G has a face σ with $\deg(\sigma) \geq 43$, then by Lemma 2.8

$$\sum_{x \in \sigma} \Phi(x) \geq 1.$$

Since G is an infinite graph with $\mathrm{Sec}(G) \geq 0$, by the Gauss–Bonnet formula (2.5), we have

$$\sum_{x \in G} \Phi(x) \leq 1,$$

because $\chi(M) \leq 2$ and $t \geq 1$. Hence $\sum_{x \in \sigma} \Phi(x) = 1$ and $\Phi(y) = 0$ for any $y \notin \sigma$. Furthermore, the only possible patterns of the vertices incident to σ are $(3, 6, k)$, $(4, 4, k)$, $(3, 3, 3, k)$, because the other three patterns $(3, 3, k)$, $(3, 4, k)$, $(3, 5, k)$ have curvature strictly larger than $\frac{1}{k}$, where $k = \deg(\sigma)$. $\square$

Let $\mathcal{G}$ denote the set of semiplanar graphs. We define a graph operation on $\mathcal{G}$, $P : \mathcal{G} \to \mathcal{G}$. For any $G \in \mathcal{G}$, we choose a (possibly infinite) subcollection of hexagonal faces of G, add new vertices at the barycenters of the hexagons, and join them to the vertices of the hexagons by new edges. In such a way, we obtain a new semiplanar graph, denoted by $P(G)$, which replaces each hexagon chosen in G by six triangles. We note that $P : \mathcal{G} \to \mathcal{G}$ is a multivalued map depending on which subcollection of hexagons we chosen. The inverse map of P, denoted by $P^{-1} : \mathcal{G} \to \mathcal{G}$, is defined as a semiplanar graph $P^{-1}(G)$ obtained from replacing couples of six triangles incident to a common vertex of pattern $(3, 3, 3, 3, 3, 3)$ in G by a hexagon (we require that the hexagons do not overlap). It is easy to see that $S(P(G))$ and $S(P^{-1}(G))$ are isometric to $S(G)$ which implies that the graph operations P and P^{-1} preserve the curvature condition, i.e. $\mathrm{Sec}\, S(P(G)) \geq 0$ (or $\mathrm{Sec}\, S(P^{-1}(G)) \geq 0$) $\Longleftrightarrow$ $\mathrm{Sec}\, S(G) \geq 0$.

We investigate the combinatorial structure of the semiplanar graph G with nonnegative curvature and large face degree, i.e. $D_G \geq 43$. Lemma 2.9 shows that there is a unique large face σ such that $\deg(\sigma) = D_G = k \geq 43$ and the only patterns of vertices of σ are $(3, 6, k)$, $(4, 4, k)$ and $(3, 3, 3, k)$. Without loss of generality, by the graph operation P, it suffices to assume that the semiplanar graph G has no hexagonal faces. It is easy to show that if one of the vertices of σ is of pattern $(4, 4, k)$ (or $(3, 3, 3, k)$), the other vertices incident to σ are of the same pattern. We denote by L_1 the set of faces attached to the large face σ, which are of the same type (triangle or square) and for which the boundary of $\sigma \cup L_1$ has the same number of edges as the boundary of σ. By Lemma 2.9, G has vanishing curvature except at the vertices incident to σ. Hence, $\sigma \cup L_1$ is in the same situation as σ. To continue the process, we denote by L_2 the set of faces attached to $\sigma \cup L_1$ which are of the same type (triangle or square). In this way, we obtain an infinite sequence of sets of faces, $\sigma, L_1, L_2, \dots, L_m, \dots$, where L_m are the sets of faces of the same type (triangle or square) for $m \geq 1$. The sets L_m and L_n $(m \neq n)$ may be different since they are independent.

Theorem 2.10. *Let G be a semiplanar graph with $\mathrm{Sec}(G) \geq 0$ and $D_G \geq 43$, and let σ be the face of maximal degree. Then either G has no hexagons, constructed from a sequence of sets of faces, $\sigma, L_1, L_2, \dots, L_m, \dots$, where L_m are the sets of faces of the same type (triangle or square), denoted by $S(G) = \sigma \cup \bigcup_{m=1}^{\infty} L_m$, or G has hexagons, i.e. $G = P^{-1}(G')$ where G' has no hexagons.*

3. Metric classification of semiplanar graphs with nonnegative curvature

In this section, we prove that any regular polygonal surface is a complete geodesic space and the combinatorial curvature definition is consistent with the sectional curvature in the sense of Alexandrov. We then obtain the metric classification of semiplanar graphs with nonnegative curvature.

Let G be a semiplanar graph and $S(G)$ be the corresponding regular polygonal surface. Denote by G_1 the 1-dimensional simplicial complex with the metric, denoted by d^{G_1}, by assigning each edge the length one. As a subset of $S(G)$, G_1 has another metric, denoted by d, which is the restriction of the intrinsic metric d of $S(G)$ to G_1. The following lemma says that they are bi-Lipschitz equivalent. We note that $d^G(x, y) = d^{G_1}(x, y)$, for any $x, y \in G$.

Lemma 3.1. *Let G be a semiplanar graph and $S(G)$ be the regular polygonal surface of G. Then there exists a constant C such that for any $x, y \in G_1$,*

$$C d^{G_1}(x, y) \leq d(x, y) \leq d^{G_1}(x, y). \tag{3.1}$$

To prove the lemma, we need the following lemma in Euclidean geometry.

Lemma 3.2. *Let $\triangle_n \subset \mathbb{R}^2$ be a regular n-polygon of side length one ($n \geq 3$). A straight line L intersects the boundary of $\triangle_n$ at two points, A and B. Denote by $|AB| = d$ the length of the segment AB, by l_1, l_2 the length of the two paths P_1, P_2 on the boundary of $\triangle_n$ joining A and B. Then we have*

$$C \min\{l_1, l_2\} \leq d \leq \min\{l_1, l_2\}, \tag{3.2}$$

where the constant C does not depend on n.

Proof. It suffices to prove that $d \geq C \min\{l_1, l_2\}$. Without loss of generality, we may assume $l_1 \leq l_2$. It is easy to prove the lemma for $n = 3$, so we consider $n \geq 4$. If the shorter path P_1 contains no full edges of $\triangle_n$, i.e. A and B are on adjacent edges, then P_1 and AB form a triangle. Denote by a, b the lengths of the two sides in P_1 and by α the angle opposite to AB. Then we have $\alpha = \frac{(n-2)\pi}{n}$ and $l_1 = a + b$. By the cosine rule, we obtain that

$$d \geq a - b\cos\alpha \quad \text{and} \quad d \geq b - a\cos\alpha.$$

Then it follows that

$$2d \geq (a+b)(1-\cos\alpha) \geq (a+b)\left(1-\cos\frac{\pi}{3}\right) = \frac{1}{2}l_1.$$

Hence

$$d \geq \frac{1}{4}l_1. \tag{3.3}$$

If P_1 contains at least one full edge, we consider the following cases.

Case 1. $n \leq 6$. We choose one full edge in P_1 and extend it to a straight line, then project the path P_1 onto the line. It is easy to show that

$$d \geq |\mathrm{Proj}\, P_1| \geq 1,$$

where $\mathrm{Proj}\, P_1$ is the projection of the path P_1. Since $n \leq 6$, we have $l_1 \leq 3$ and

$$d \geq 1 \geq \frac{l_1}{3}. \tag{3.4}$$

Case 2. $n \geq 7$. Denote by l the number of full edges contained in P_1. We draw the circumscribed circle of $\triangle_n$, denoted by C_n, with center O of radius R_n, where $2R_n \sin\frac{\pi}{n} = 1$. Let the straight line L (passing through A and B) intersect the circle C_n at C and D (C is close to A). Denote by d' the length of the segment CD, by θ the angle of $\measuredangle COD$ and by l' the length of the arc $\overset{\frown}{CD}$.

Case 2.1. $l \geq 3$. On one hand, by $l \geq 3$, we have $\theta \geq l\frac{2\pi}{n} \geq 3\frac{2\pi}{n}$. Hence,

$$d' = 2R_n \sin\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\sin\frac{\pi}{n}} \geq \frac{\sin 3\frac{\pi}{n}}{\sin\frac{\pi}{n}} = 3 - 4\sin^2\frac{\pi}{n} \geq 3 - 4\sin^2\frac{\pi}{7} \geq 2.24.$$

On the other hand, by $|AC| \leq 1$ and $|BD| \leq 1$, we obtain that

$$d' - d \leq |AC| + |BD| \leq 2.$$

Then we have

$$\frac{d}{d'} \geq 1 - \frac{2}{d'} \geq 1 - \frac{2}{2.24} = C. \tag{3.5}$$

Since $d' = 2R_n \sin\frac{\theta}{2}$ and $l' = R_n\theta$, we have

$$\frac{d'}{l'} = \frac{2\sin\frac{\theta}{2}}{\theta} \geq \frac{2\cdot\frac{2}{\pi}\cdot\frac{\theta}{2}}{\theta} = \frac{2}{\pi}. \tag{3.6}$$

In addition,

$$l' \geq l \geq l_1 - 2 \geq \frac{l_1}{3}, \tag{3.7}$$

where the last inequality follows from $l_1 \geq l \geq 3$.

Hence, by (3.5), (3.6) and (3.7), we have

$$d \geq Cl_1. \tag{3.8}$$

Case 2.2. $l = 1$. We denote by EF the full edge contained in P_1 (E is close to A) and extend AE and BF to intersect at the point H. It is easy to calculate the angle

$$\measuredangle EHF = \pi - \frac{4\pi}{n} \geq \pi - \frac{4\pi}{7}.$$

By an argument similar to the beginning of the proof, we obtain that

$$\begin{aligned} d = |AB| &\geq \frac{1}{2}(|AH| + |BH|)\left(1 - \cos\left(\pi - \frac{4\pi}{7}\right)\right) \\ &\geq C(|AE| + |EF| + |FB|) = Cl_1, \end{aligned} \tag{3.9}$$

where the last inequality follows from the triangle inequality.

Case 2.3. $l = 2$. We denote by EF and FH the full edges contained in P_1 (E is close to A) and extend AE and BH to intersect at the point K. Easy calculation shows that

$$\measuredangle EKH = \pi - \frac{6\pi}{n} \geq \pi - \frac{6\pi}{7}.$$

By the same argument, we get

$$d = |AB| \geq \frac{1}{2}(|AK| + |BK|)\left(1 - \cos\left(\pi - \frac{6\pi}{7}\right)\right) \geq Cl_1. \tag{3.10}$$

Hence, by (3.3), (3.4), (3.8), (3.9) and (3.10), we obtain that

$$d \geq Cl_1,$$

where C is an absolute constant. Then the lemma follows. □

Now we prove Lemma 3.1.

Proof of Lemma 3.1. For any $x, y \in G_1$, it is obvious that $d(x, y) \leq d^{G_1}(x, y)$. Hence it suffices to show the inequality in the opposite direction. Let $\gamma : [a, b] \to S(G)$ be a geodesic joining x and y. By the local finiteness assumption of the graph G, there exist finitely many faces that cover the geodesic γ. There is a partition of $[a, b]$, $\{y_i\}_{i=0}^N$, where

$$a = y_0 < y_1 < \cdots < y_N = b,$$

such that $\gamma|_{[y_{i-1}, y_i]}$ is a segment on the face σ_i and $\gamma(y_{i-1}), \gamma(y_i)$ are on the boundary of σ_i, for $1 \leq i \leq N$. For each $1 \leq i \leq N$, we choose the shorter path, denoted by l_i, on the boundary of the face σ_i which joins $\gamma(y_{i-1})$ and $\gamma(y_i)$. By Lemma 3.2, we get

$$CL(l_i) \leq d(\gamma(y_{i-1}), \gamma(y_i)) \leq L(l_i),$$

where $L(l_i)$ is the length of l_i. Connecting l_i, we obtain a path l in G_1 joining x and y. Then we have

$$L(l) = \sum_{i=1}^{N} L(l_i) \leq \frac{1}{C}\sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i)) = \frac{1}{C} d(x, y).$$

Hence,

$$d^{G_1}(x, y) \leq L(l) \leq \frac{1}{C} d(x, y).$$

□

Theorem 3.3. *Let $G = (V, E, F)$ be a semiplanar graph and $S(G)$ be the regular polygonal surface. Then $(S(G), d)$ is a complete metric space.*

Proof. We denote by $S(G) = \bigcup_{\sigma \in F} \sigma$ the regular polygonal surface of G, by $\overline{S(G)}$ the completion of $S(G)$ with respect to the metric d. Let $(\sigma)_{\epsilon_0}$ denote the ϵ_0-neighborhood of σ in $\overline{S(G)}$, for $\epsilon_0 > 0$. To prove the theorem, it suffices to show that there exists a constant ϵ_0 such that for any face $\sigma \in F$ we have $(\sigma)_{\epsilon_0} \subset S(G)$.

For any $\sigma \in F$, let $Q = \bigcup\{\tau \in F : \tau \cap \sigma \neq \emptyset\}$. By the local finiteness of G, Q is a union of finitely many faces and the boundary of Q, ∂Q, has finitely many edges. It is easy to see that

$$d^{G_1}(\partial Q, \partial\sigma) = \inf\{d^{G_1}(x, y) : x \in \partial Q,\ y \in \partial\sigma\} \geq 1.$$

By Lemma 3.1, we obtain that for any $x \in \partial Q$, $y \in \partial\sigma$,

$$d(x, y) \geq C = 2\epsilon_0,$$

where we choose $\epsilon_0 = \frac{C}{2}$. Then we have

$$d(\overline{S(G)} \setminus Q, \sigma) = \inf\big\{d(x, y) : x \in \overline{S(G)} \setminus Q,\ y \in \sigma\big\} \geq 2\epsilon_0 > \epsilon_0.$$

Hence, it follows that

$$(\sigma)_{\epsilon_0} \subset Q \subset S(G). \qquad \square$$

Corollary 3.4. *Let G be a semiplanar graph and $S(G)$ be the regular polygonal surface. Then G has nonnegative (resp. nonpositive) curvature everywhere if and only if $S(G)$ is an Alexandrov space with nonnegative (resp. nonpositive) curvature.*

Proof. We prove only the case for nonnegative curvature. The proof for the case of nonpositive curvature is similar.

By Theorem 3.3, $S(G)$ is a complete metric space. It is obvious that $S(G)$ is a geodesic space. Suppose G has nonnegative curvature everywhere. At each point except the vertices, there is a neighborhood which is isometric to the flat disk in $\mathbb{R}^2$. At the vertex $x \in G$, the curvature condition $\Phi(x) \geq 0$ is equivalent to $\Sigma_x \leq 2\pi$. Then there is a neighborhood of x (isometric to a conic surface in $\mathbb{R}^3$) satisfying the Toponogov triangle comparison with respect to the model space $\mathbb{R}^2$. Hence, $S(G)$ is an Alexandrov space with Sec $S(G) \geq 0$. Conversely, if $S(G)$ is an Alexandrov space with Sec $S(G) \geq 0$, then the total angle of each point of $S(G)$ is at most 2π, which implies the nonnegative curvature condition at the vertices. $\square$

In the following, we investigate the metric structure of regular polygonal surfaces by Alexandrov space methods, where we do not use the Gauss–Bonnet formula.

Lemma 3.5. *Let $G = (V, E, F)$ be a semiplanar graph, G_1 be the* 1*-dimensional simplicial complex and $S(G)$ be the regular polygonal surface. Then we have*

$$N(G_1) = N(S(G)).$$

Proof. It is easy to show that $N(S(G)) \leq N(G_1)$, since $G_1 \subset S(G)$. So it suffices to prove that $N(G_1) \leq N(S(G))$.

Let $\{B_{R_i}^{G_1}(p)\}_{i=1}^{\infty}$ be an exhaustion of G_1 such that $G_1 \setminus B_{R_i}^{G_1}(p)$ has N_i different connected components connecting to infinity, denoted by $E_1^i, \ldots, E_{N_i}^i$, and $N(G_1) = \lim_{i\to\infty} N_i$. By the local finiteness of G, $N_i < \infty$. For any $i \geq 1$, let

$$Q_i = \bigcup \{\sigma \in F : \sigma \cap B_{R_i}^{G_1}(p) \neq \emptyset\},$$

i.e. the union of the faces attached to $B_{R_i}^{G_1}(p)$. By the local finiteness of G, Q_i is compact. We shall prove that $S(G) \setminus Q_i$ has at least N_i different connected components connecting to infinity, then we have $N(S(G)) \geq N_i$ for any $i \geq 1$, which implies the lemma.

For fixed $i \geq 1$, let

$$H_j := E_j^i \cap (S(G) \setminus Q_i), \quad j = 1, \ldots, N_i.$$

It is easy to see that $H_j \neq \emptyset$, since E_j^i is connecting to infinity for $1 \leq j \leq N_i$. We shall prove that for any $j \neq k$, H_j and H_k are disconnected in $S(G) \setminus Q_i$. Suppose it is not true; then there exist $x \in H_j$, $y \in H_k$ and a curve $\gamma : [a,b] \to S(G)$ in $S(G) \setminus Q_i$ joining x and y, i.e.

$$\gamma \cap Q_i = \emptyset. \tag{3.11}$$

As in the proof of Lemma 3.1, we can find a curve $\gamma' : [a,b] \to G_1$ in G_1 such that γ' and γ pass through the same faces, i.e. for any $t \in [a,b]$, there is a face τ such that $\gamma(t) \in \tau$ and $\gamma'(t) \in \tau$. Since H_j and H_k are disconnected in $G_1 \setminus B_{R_i}^{G_1}(p)$, we have $\gamma'(t_0) \in B_{R_i}^{G_1}(p)$, for some $t_0 \in [a,b]$. Then there exists a face τ such that $\gamma(t_0) \in \tau$ and $\gamma'(t_0) \in \tau$. Hence $\tau \subset Q_i$ and $\gamma \cap Q_i \neq \emptyset$, which contradicts to (3.11). □

By this lemma, we can apply the Cheeger–Gromoll splitting theorem to the polygonal surface of the semiplanar graph with nonnegative curvature.

Theorem 3.6. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$ *and $S(G)$ be the regular polygonal surface. If $N(G_1) \geq 2$, then $S(G)$ is isometric to a cylinder without boundary.*

Proof. By Lemma 3.5, $N(G_1) \geq 2$ implies that $N(S(G)) \geq 2$. A standard Riemannian geometry argument proves the existence of an infinite geodesic $\gamma : (-\infty, \infty) \to S(G)$. Since $S(G)$ is an Alexandrov space with nonnegative curvature, the Cheeger–Gromoll splitting theorem, Theorem 2.4, shows that $S(G)$ is isometric to $Y \times \mathbb{R}$, where Y is a 1-dimensional Alexandrov space without boundary, i.e. straight line or circle. Because $N(S(G)) \geq 2$, Y must be a circle. Hence, $S(G)$ is isometric to a cylinder without boundary. □

Remark 3.7. Since the Cheeger–Gromoll splitting theorem holds for Alexandrov space with boundary, we may formulate the above theorem in the case of regular polygonal surfaces with boundary (homeomorphic to a manifold with boundary). For the vertex x on the boundary, we define the combinatorial curvature as

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)} = \frac{\pi - \Sigma_x}{2\pi},$$

where Σ_x is the total angle at x. Let G be a semiplanar graph with nonnegative curvature everywhere and $N(G_1) \geq 2$. Then the polygonal surface $S(G)$ is isometric to either the cylinder without boundary or the cylinder with boundary, i.e. $[a,b] \times \mathbb{R}$.

Next we consider the tilings (or tessellations) of the plane (see [25]) and the construction of semiplanar graphs with nonnegative curvature.

Let G be a semiplanar graph with nonnegative curvature and $S(G)$ be the regular polygonal surface of G. If $S(G)$ is isometric to the plane, $\mathbb{R}^2$, then G is just a tiling of the plane by regular polygons called a regular tiling. Then G has vanishing curvature everywhere. There are infinitely many tilings of the plane. A classification is possible only for regular ones. In this paper, we only consider regular tilings. A tiling is called monohedral if all tiles are congruent. The only three monohedral tilings are by triangles, squares or hexagons. There are eleven distinct tilings such that all vertices are of the same pattern:

$$(3^6)(3^4,6)(3^3,4^2)(3^2,4,3,4)(3,4,6,4)(3,6,3,6)(3,12^2)(4^4)(4,6,12)(4,8^2)(6^3).$$

They are called Archimedean tilings and they clearly include the three monohedral tilings.

If $S(G)$ has at least two ends, then by Theorem 3.6 it is isometric to a cylinder without boundary and G has vanishing curvature everywhere. If $S(G)$ is nonorientable, then by Theorem 2.7 and the Gauss–Bonnet formula (2.5) $S(G)$ is homeomorphic to the projective plane minus one point and G has vanishing curvature everywhere.

Conversely, if G has vanishing curvature everywhere, then so does $S(G)$. Hence, $S(G)$ is isometric to $\mathbb{R}^2$, or a cylinder if it is orientable. The surface $S(G)$ is homeomorphic to the projective plane minus one point if it is nonorientable.

In addition, if G has positive curvature somewhere, then so does $S(G)$, which implies that $S(G)$ is not isometric to $\mathbb{R}^2$, but by the Gauss–Bonnet formula (2.5), it is homeomorphic to $\mathbb{R}^2$. We call it a cap.

An isometry of $\mathbb{R}^2$ is a mapping of $\mathbb{R}^2$ onto itself which preserves the Euclidean distance. All isometries of $\mathbb{R}^2$ form a group. It is well known that every isometry of $\mathbb{R}^2$ is of one of four types:

(1) rotation,

(2) translation,

(3) reflection in a given line,

(4) glide reflection, i.e. a reflection in a given line composed with a translation parallel to the same line (see [25]).

For any planar tiling Σ, an isometry is called a symmetry of Σ if it maps every tile of Σ onto a tile of Σ. It is easy to see that all symmetries of Σ form a subgroup of isometries of $\mathbb{R}^2$. We denote by $S(\Sigma)$ the group of symmetries of Σ. For any $\iota \in S(\Sigma)$, we denote by $\langle\iota\rangle$ the subgroup of $S(\Sigma)$ generated by the symmetry ι. The metric quotient of $\mathbb{R}^2$ by $\langle\iota\rangle$, denoted by $\mathbb{R}^2/\langle\iota\rangle$, is a metric space with quotient metric obtained by the group action $\langle\iota\rangle$ (see [3]). The following lemma shows the construction of the tilings of a cylinder.

Lemma 3.8. *There is a correspondence between a planar tiling Σ with a translation symmetry T, (Σ, T) and a tiling of a cylinder.*

Proof. For any planar tiling Σ with a translation symmetry T, the metric quotient $\mathbb{R}^2/\langle T\rangle$ is isometric to a cylinder. The tiling Σ induces a tiling of $\mathbb{R}^2/\langle T\rangle$.

Conversely, given a tiling Σ' of a cylinder W, we lift W to its universal cover $\mathbb{R}^2$ by a map $\pi : \mathbb{R}^2 \to W$. It is easy to see that π is locally isometric, since W is flat. The tiling Σ' can be lifted by π to a tiling Σ of $\mathbb{R}^2$, which has a translation symmetry by construction. □

Next we consider the metric structure of the semiplanar graph with nonnegative curvature such that the corresponding regular polygonal surface is nonorientable, i.e. homeomorphic to the projective plane minus one point.

Lemma 3.9. *There is a correspondence between a planar tiling Σ with a glide reflection symmetry ι, (Σ, ι) and a tiling of the projective plane minus one point with nonnegative curvature.*

Proof. Let Σ be a planar tiling with symmetry of a glide reflection

$$\iota = T_{a,L} \circ F_L = F_L \circ T_{a,L},$$

where $a > 0$, L is a straight line, $T_{a,L}$ is a translation along L through distance a and F_L is a reflection in the line L. The metric quotient $\mathbb{R}^2/\langle\iota\rangle$ is isometric to the metric space obtained from gluing the boundary of $[0,a] \times \mathbb{R}$, which is perpendicular to the line L, by the glide reflection ι. It is easy to see that $\mathbb{R}^2/\langle\iota\rangle$ is homeomorphic to the projective plane minus one point and has vanishing curvature everywhere. Hence the planar tiling Σ and the symmetry ι of Σ induce a tiling of $\mathbb{R}^2/\langle\iota\rangle$.

Conversely, let Σ' be a tiling of $\mathbb{R}P^2 \setminus \{\underline{o}\}$, with nonnegative curvature (actually with vanishing curvature everywhere). We construct a covering map of $\mathbb{R}P^2 \setminus \{\underline{o}\}$ with a $\mathbb{Z}_2$ action,

$$\pi : S^2 \setminus \{S, N\} \to \mathbb{R}P^2 \setminus \{\underline{o}\},$$

where S and N are the south and north pole of S^2. We lift the tiling Σ' to a tiling Σ'' of $S^2 \setminus \{S, N\}$. Since Σ' has vanishing curvature everywhere, so does the lifted tiling Σ''. Note that $S^2 \setminus \{S, N\}$ has two ends. By Theorem 3.6, the regular polygonal surface $S(\Sigma'')$ is isometric to a cylinder, denoted by $(\frac{a}{\pi} S^1) \times \mathbb{R}$. By Lemma 3.8, the tiling of a cylinder Σ'' can be regarded to be a tiling of the cylinder which induces a planar tiling Σ''' and a translation symmetry T_{2a} with T_{2a}-invariant domain $[0, 2a] \times \mathbb{R} \subset \mathbb{R}^2$. Since the $\mathbb{Z}_2$ action of π, the tiling Σ''' has a glide reflection symmetry

$$\iota = F_L \circ T_{a,L}$$

where L is parallel to the direction of the translation T_{2a}. □

By the discussion above, we obtain the metric classification of $S(G)$ for a semiplanar graph G with nonnegative curvature.

Theorem 3.10. *Let G be a semiplanar graph with nonnegative curvature and $S(G)$ be the regular polygonal surface of G. If G has positive curvature somewhere, then $S(G)$ is isometric to a cap which is homeomorphic but not isometric to the plane. If G has vanishing curvature everywhere, then $S(G)$ is isometric to a plane, or a cylinder without boundary if it is orientable, and $S(G)$ is isometric to a metric space obtained from gluing the boundary of $[0,a] \times \mathbb{R}$ by a glide reflection, $\iota = T_{a,L} \circ F_L$, where L is perpendicular to the cylinder, if it is nonorientable.*

At the end of the paper [7], Chen raised a question on the classification of infinite graphs with nonnegative curvature everywhere which can be embedded into the projective plane minus one point. By Lemma 3.9, it suffices to find the planar tiling with a glide reflection symmetry.

Theorem 3.11. *The monohedral tilings of the projective plane minus one point with nonnegative curvature are of three types: triangle, square, hexagon.*

Proof. By Lemma 3.9, the monohedral tiling of the projective plane minus one point with nonnegative curvature is induced by the monohedral tiling of the plane of triangles, of squares or of hexagons and a glide reflection for the tiling. □

Chen [7] gave two classes of monohedral tilings of the projective plane with nonnegative curvature: PS_n (n is even) and PH_n (n is odd); PS_n is induced by the monohedral tiling of the plane of squares. In fact, PH_n (n is odd) is a proper subset of monohedral tilings of the projective plane minus one point which are induced by the monohedral tiling of the plane by hexagons. We give an example below (see Figure 1, 2) which is induced by the tiling of the plane by hexagons, but is not included in PH_n (n is odd). Let PT, PS, PH denote the tilings of the projective plane minus one point which are induced by the monohedral tiling of the plane of triangles, squares, hexagons and a glide reflection symmetry. They provide the complete classification of monohedral tilings of the projective plane minus one point with nonnegative curvature.

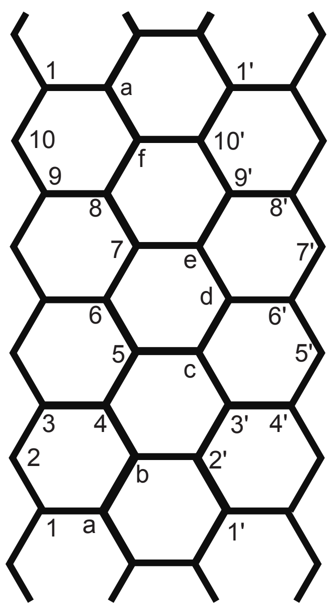

Figure 1. (6,6,6) in $\mathbb{R}^2$.

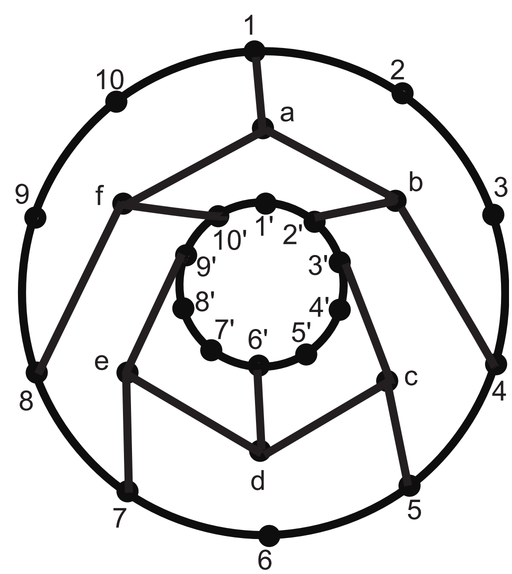

Figure 2. (6,6,6) in $\mathbb{R}P^2 \setminus \{\underline{o}\}$.

In addition, as the Archimedean tilings of the plane, we can classify the tilings of the projective plane minus one point with nonnegative curvature for which each vertex has the same pattern.

Theorem 3.12. *The tilings of the projective plane minus one point with nonnegative curvature such that the pattern of each vertex is the same are induced by the Archimedean tilings of the plane and a gilde reflection symmetry.*

We give two examples of tilings of the projective plane minus one point which are induced by the Archimedean tilings and glide reflection symmetries (see Figure 3, 4, 5, 6). It is easy to see that there are infinitely many tilings of the projective plane minus one point with nonnegative curvature because of the complexity of the tilings of the plane. Another way to see the complexity is that we can apply the graph operation P on the tiling of the projective plane minus one point with hexagonal faces to obtain a new one.

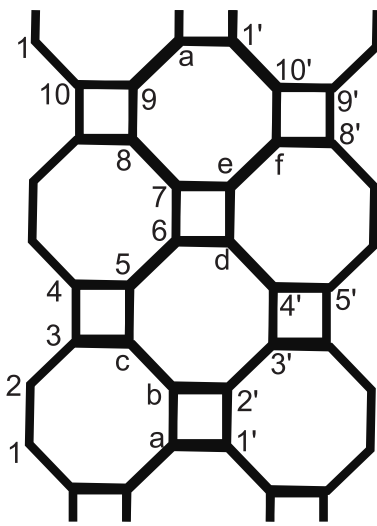

Figure 3. (4,8,8) in $\mathbb{R}^2$.

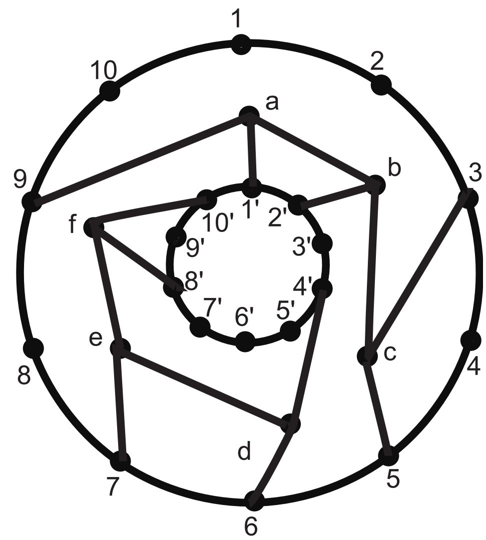

Figure 4. (4,8,8) in $\mathbb{R}P^2 \setminus \{o\}$.

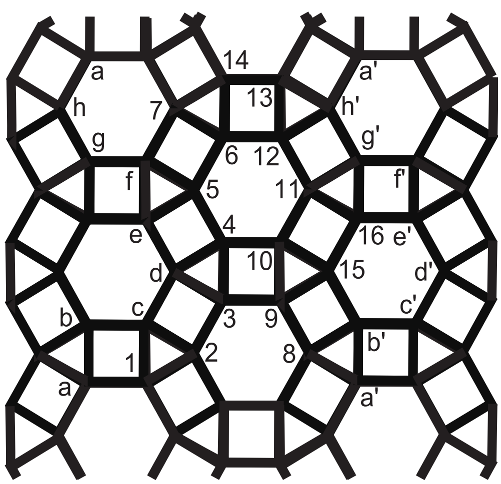

Figure 5. (3,4,6,4) in $\mathbb{R}^2$.

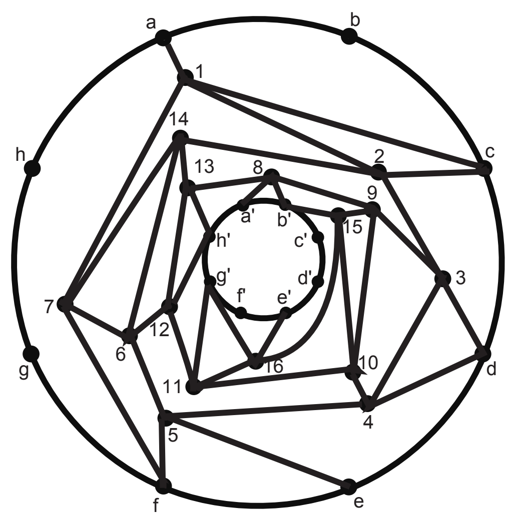

Figure 6. (3,4,6,4) in $\mathbb{R}P^2 \setminus \{o\}$.

4. Volume doubling property and Poincaré inequality

In this section, we shall prove the volume doubling property and the Poincaré inequality for semiplanar graphs with nonnegative curvature.

Let G be a semiplanar graph and $S(G)$ be the regular polygonal surface of G. For any $p \in G$ and $R > 0$, we denote by $B_R(p) = \{x \in G : d^G(p, x) \leq R\}$ the closed geodesic ball in the graph G, and by

$$B_R^{S(G)}(p) = \{x \in S(G) : d(p,x) \le R\}$$

the closed geodesic ball in the polygonal surface $S(G)$. The volume of $B_R(p)$ is defined as

$$|B_R(p)| = \sum_{x \in B_R(p)} d_x,$$

and the volume of $B_R^{S(G)}(p)$ is defined as $|B_R^{S(G)}(p)| = \mathcal{H}^2(B_R^{S(G)}(p))$, where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure. We denote by $\sharp B_R(p)$ the number of vertices in the closed geodesic ball $B_R(p)$. Note that for any semiplanar graph G with nonnegative curvature,

$$3 \le d_x \le 6 \quad \text{for any } x \in G.$$

Hence $|B_R(p)|$ and $\sharp B_R(p)$ are equivalent up to a constant, i.e.

$$3\sharp B_R(p) \le |B_R(p)| \le 6\sharp B_R(p),$$

for any $p \in G$ and $R > 0$.

Theorem 4.1. *Let $G = (V, E, F)$ be a semiplanar graph with* $\mathrm{Sec}(G) \ge 0$. *Then there exists a constant $C(D)$ depending on $D := \max_{\sigma \in F} \deg(\sigma)$ such that for any $p \in G$ and $0 < r < R$, we have*

$$\frac{|B_R(p)|}{|B_r(p)|} \le C(D)\left(\frac{R}{r}\right)^2. \tag{4.1}$$

Proof. We denote $B_R := B_R(p)$ and $B_R^S := B_R^{S(G)}(p)$ for short. By Lemma 3.1, we have $B_{CR}^S \cap G \subset B_R \subset B_R^S \cap G$. For any $\sigma \in F$, $C_1 \le |\sigma| := \mathcal{H}^2(\sigma) \le C_2(D)$. Let

$$H_R := \{\sigma \in F : \sigma \cap B_R \ne \emptyset\}$$

denote the faces attached to B_R. Then

$$|B_R| = \sum_{x \in B_R} d_x \le D \cdot \sharp H_R, \tag{4.2}$$

where $\sharp H_R$ is the number of faces in H_R. For any $\sigma \in F$, since the intrinsic diameter of σ is bounded, i.e. $\operatorname{diam} \sigma := \sup\{d(x,y) : x, y \in \sigma\} \le D$, we have for any face $\sigma \in H_R$

$$\sigma \subset B^S_{R+\operatorname{diam}\sigma} \subset B^S_{R+D}.$$

Hence it follows that

$$C_1 \sharp H_R \le \sum_{\sigma \in H_R} |\sigma| \le |B^S_{R+D}|. \tag{4.3}$$

By the volume comparison of $S(G)$ (2.3), (4.2) and (4.3), we obtain

$$|B_R| \le C(D)|B^S_{R+D}| \le C(D)(R+D)^2. \tag{4.4}$$

For $R \ge D$, we have $|B_R| \le 4C(D)R^2$. For $1 \le R < D$, we have $|B_R| \le C(D)R^2$ by (4.4). Hence, for any $R \ge 1$, the quadratic volume growth property follows

$$|B_R| \le C(D)R^2. \tag{4.5}$$

For any $r > \frac{D}{C}$, where C is the constant in Lemma 3.1, let $r' = Cr - D$. We denote by

$$W_r := \{\sigma \in F : \sigma \cap B^S_{r'} \neq \emptyset\}$$

the faces attached to $B^S_{r'}$, and by

$$\overline{W_r} := \bigcup_{\sigma \in W_r} \sigma$$

the union of faces in W_r. For any vertex $x \in \overline{W_r} \cap G$, there exists a $\sigma \in W_r$ such that $x \in \sigma$ and $d(p,x) \leq r' + \operatorname{diam} \sigma \leq r' + D = Cr$. By Lemma 3.1, we have $d^G(p,x) \leq C^{-1} d(p,x) \leq r$, which implies that $\overline{W_r} \cap G \subset B_r$. It is easy to see that

$$|B^S_{r'}| \leq |\overline{W_r}| = \sum_{\sigma \in W_r} |\sigma| \leq C_2(D) \sharp W_r, \tag{4.6}$$

where $\sharp W_r$ is the number of faces in W_r. Moreover, by $3 \leq \deg(\sigma) \leq D$ for any $\sigma \in F$,

$$3 \sharp W_r \leq \sum_{\sigma \in W_r} \deg(\sigma) \leq \sum_{x \in \overline{W_r} \cap G} d_x \leq 6 \sharp(\overline{W_r} \cap G), \tag{4.7}$$

where $\sharp(\overline{W_r} \cap G)$ is the number of vertices in $\overline{W_r} \cap G$.

Hence by (4.6) and (4.7), we have

$$|B^S_{r'}| \leq C(D) \sharp(\overline{W_r} \cap G) \leq C(D) \sharp B_r \leq C(D)|B_r|. \tag{4.8}$$

By the relative volume comparison (2.1), (4.4) and (4.8), we obtain that for any $r > \frac{D}{C}$,

$$\frac{|B_R|}{|B_r|} \leq C(D) \frac{|B^S_{R+D}|}{|B^S_{r'}|} \leq C(D) \left(\frac{R+D}{Cr - D} \right)^2.$$

Let $r_0(D) := \frac{2D}{C}$. For $r_0(D) \leq r < R < \infty$, $r - \frac{D}{C} \geq \frac{r}{2}$ and $R + D \leq 2R$, so that we have

$$\frac{|B_R|}{|B_r|} \leq C(D) \left(\frac{R}{r} \right)^2. \tag{4.9}$$

For $0 < r < R \leq r_0(D)$, by (4.5), we have

$$\frac{|B_R|}{|B_r|} \leq \frac{|B_{r_0(D)}|}{|B_0|} \leq \frac{1}{3} C(D) r_0^2(D) \leq C(D) \left(\frac{R}{r} \right)^2. \tag{4.10}$$

For $0 < r < r_0(D) < R$, by (4.5), we have

$$\frac{|B_R|}{|B_r|} \leq \frac{C(D) R^2}{|B_0|} \leq C(D) r^2 \left(\frac{R}{r} \right)^2 \leq C(D) r_0^2(D) \left(\frac{R}{r} \right)^2. \tag{4.11}$$

Hence it follows from (4.9), (4.10) and (4.11) that for any $0 < r < R$,

$$\frac{|B_R|}{|B_r|} \leq C(D) \left(\frac{R}{r} \right)^2. \qquad \square$$

From the relative volume comparison, it is easy to obtain the volume doubling property.

Corollary 4.2. *Let G be a semiplanar graph with $\mathrm{Sec}(G) \geq 0$. Then there exists a constant $C(D)$ depending on D such that for any $p \in G$ and $R > 0$, we have*

$$|B_{2R}(p)| \leq C(D)|B_R(p)|. \tag{4.12}$$

In the rest of this section, we shall prove the Poincaré inequality on a semiplanar graph with nonnegative curvature.

Theorem 4.3. *Let G be a semiplanar graph with $\mathrm{Sec}(G) \geq 0$. Then there exist two constants $C(D)$ and C such that for any $p \in G$, $R > 0$, $f : B_{CR}(p) \to \mathbb{R}$, we have*

$$\sum_{x \in B_R(p)} (f(x) - f_{B_R})^2 d_x \leq C(D) R^2 \sum_{\substack{x,y \in B_{CR}(p) \\ x \sim y}} (f(x) - f(y))^2, \tag{4.13}$$

where

$$f_{B_R} = \frac{1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x) d_x,$$

and $x \sim y$ means x and y are neighbors.

For any function on a semiplanar graph G, $f : G \to \mathbb{R}$, we shall construct a local $W^{1,2}$ function, denoted by f_2, on $S(G)$ with controlled energy in two steps, and then by the Poincaré inequality (2.4) on $S(G)$, we obtain the Poincaré inequality on the graph G. In step 1, by linear interpolation, we extend f to a piecewise linear function on G_1, $f_1 : G_1 \to \mathbb{R}$. In step 2, we extend f_1 to each face of G. For any regular n-polygon $\triangle_n$ of side length one, there is a bi-Lipschitz map

$$L_n : \triangle_n \to B_{r_n},$$

where B_{r_n} is the closed disk whose boundary is the circumscribed circle of $\triangle_n$ of radius

$$r_n = \frac{1}{2 \sin \frac{\alpha_n}{2}} \quad (\text{for } \alpha_n = \tfrac{2\pi}{n}).$$

Without loss of generality, we may assume that the origin $\underline{o} = (0,0)$ of $\mathbb{R}^2$ is the barycenter of $\triangle_n$, the point $(x, y) = (r_n, 0) \in \mathbb{R}^2$ is a vertex of $\triangle_n$, and $B_{r_n} = B_{r_n}(\underline{o})$. Then in polar coordinates, L_n reads

$$L_n : \triangle_n \ni (r, \theta) \mapsto (\rho, \eta) \in B_{r_n}(\underline{o}),$$

where for $\theta \in [j\alpha_n, (j+1)\alpha_n]$, $j = 0, 1, \ldots, n-1$,

$$\begin{cases} \rho = \dfrac{r \cos\left(\theta - (2j+1)\frac{\alpha_n}{2}\right)}{\cos \frac{\alpha_n}{2}}, \\ \eta = \theta. \end{cases}$$

It maps the boundary of $\triangle_n$ to the boundary of $B_{r_n}(\underline{o})$. Direct calculation shows that L_n is a bi-Lipschitz map, i.e. for any $x, y \in \triangle_n$ we have $C_1|x - y| \leq |L_n x - L_n y| \leq C_2|x - y|$, where C_1 and C_2 do not depend on n. Then for any $\sigma \in F$, we denote $n := \deg(\sigma)$. Let $g : B_{r_n}(\underline{o}) \to \mathbb{R}$ satisfy the following boundary value problem:

$$\begin{cases} \Delta g = 0 \quad \text{in } \mathring{B}_{r_n}(\underline{o}), \\ g|_{\partial B_{r_n}(\underline{o})} = f_1 \circ L_n^{-1}, \end{cases}$$

where $\mathring{B}_{r_n}(\underline{o})$ is the open ball. Then we define $f_2 : S(G) \to \mathbb{R}$ as

$$f_2|_\sigma = g \circ L_n. \tag{4.14}$$

It can be shown that f_2 is a local $W^{1,2}$ function on $S(G)$, since the singular points of $S(G)$ are isolated (see [36]).

We need to control the energy of f_2 by its boundary values. The following lemma is standard. We denote by B_1 the closed unit disk in $\mathbb{R}^2$.

Lemma 4.4. *For any Lipschitz function $h : \partial B_1 \to \mathbb{R}$, let $g : B_1 \to \mathbb{R}$ satisfy the following boundary value problem*

$$\begin{cases} \Delta g = 0 & \text{in } \mathring{B}_1, \\ g|_{\partial B_1} = h. \end{cases}$$

Then we have

$$\int_{B_1} |\nabla g|^2 \le \int_{\partial B_1} h_\theta^2,$$
$$\int_{\partial B_1} h^2 \le C\Big(\int_{B_1} g^2 + \int_{\partial B_1} h_\theta^2\Big),$$

where $h_\theta = \frac{\partial h}{\partial \theta}$.

Proof. Let $\frac{1}{\sqrt{2\pi}}, \frac{\sin n\theta}{\sqrt{\pi}}, \frac{\cos n\theta}{\sqrt{\pi}}$ (for $n = 1, 2, \dots$) be the orthonormal basis of $L^2(\partial B_1)$. Then

$$h : \partial B_1 \to \mathbb{R}$$

can be represented in $L^2(\partial B_1)$ by

$$h(\theta) = a_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \Big(a_n \frac{\cos n\theta}{\sqrt{\pi}} + b_n \frac{\sin n\theta}{\sqrt{\pi}}\Big).$$

So the harmonic function g with boundary value h is

$$g(r,\theta) = a_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \Big(a_n r^n \frac{\cos n\theta}{\sqrt{\pi}} + b_n r^n \frac{\sin n\theta}{\sqrt{\pi}}\Big).$$

Since $\Delta g = 0$, we have $\Delta g^2 = 2|\nabla g|^2$, and then

$$\int_{B_1} |\nabla g|^2 = \frac{1}{2}\int_{B_1} \Delta g^2 = \frac{1}{2}\int_{\partial B_1} \frac{\partial g^2}{\partial r},$$

which follows from integration by parts, so that

$$\int_{B_1} |\nabla g|^2 = \int_{\partial B_1} g g_r = \sum_{n=1}^{\infty} n(a_n^2 + b_n^2).$$

In addition,

$$\int_{\partial B_1} h_\theta^2 = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2).$$

Hence,

$$\int_{B_1} |\nabla g|^2 \le \int_{\partial B_1} h_\theta^2. \tag{4.15}$$

The second part of the theorem follows from an integration by parts and the Hölder inequality:

$$\begin{aligned}\int_{\partial B_1} h^2 &= \int_{\partial B_1} (h^2 x)\cdot x = \int_{B_1} \nabla\cdot(g^2 x)\\ &= 2\int_{B_1} g^2 + 2\int_{B_1} g\nabla g\cdot x\\ &\le \int_{B_1} g^2 + 2\Big(\int_{B_1} g^2\Big)^{\frac{1}{2}}\Big(\int_{B_1} |\nabla g|^2\Big)^{\frac{1}{2}} && \text{(by } |x|\le 1)\\ &\le 3\int_{B_1} g^2 + \int_{B_1} |\nabla g|^2\\ &\le 3\int_{B_1} g^2 + \int_{\partial B_1} h_\theta^2 && \text{(by (4.15)).} \qquad \square\end{aligned}$$

Note that for the semiplanar graph G with nonnegative curvature and any face $\sigma = \triangle_n$ of G, we have

$$3 \le n \le D, \quad \frac{1}{\sqrt{3}} \le r_n = \frac{1}{2\sin\frac{\pi}{n}} \le \frac{1}{2\sin\frac{\pi}{D}} = C(D).$$

Then the scaled version of Lemma 4.4 reads

Lemma 4.5. *For $3 \le n \le D$, and any Lipschitz function $h : \partial B_{r_n} \to \mathbb{R}$, we denote by g the harmonic function satisfying the Dirichlet boundary value problem*

$$\begin{cases} \Delta g = 0 & \text{in } \mathring{B}_{r_n},\\ g|_{\partial B_{r_n}} = h. \end{cases}$$

Then it holds that

$$\begin{aligned}\int_{B_{r_n}} |\nabla g|^2 &\le C(D)\int_{\partial B_{r_n}} h_T^2,\\ \int_{\partial B_{r_n}} h^2 &\le C(D)\Big(\int_{B_{r_n}} g^2 + \int_{\partial B_{r_n}} h_T^2\Big),\end{aligned}$$

where $T = \frac{1}{r_n}\partial_\theta$ is the unit tangent vector on the boundary ∂B_{r_n} and h_T is the directional derivative of h in T.

The following lemma follows from the bi-Lipschitz property of the map $L_n : \triangle_n \to B_{r_n}$.

Lemma 4.6. *Let σ be a face of degree n, i.e. $\sigma = \triangle_n$, in a semiplanar graph G. Let $f_2|_\sigma$ be constructed as* (4.14). *Then we have*

$$\int_{\triangle_n} |\nabla f_2|^2 \le C(D)\int_{\partial\triangle_n} (f_1)_{T_n}^2, \tag{4.16}$$

$$\int_{\partial\triangle_n} f_1^2 \le C(D)\Big(\int_{\triangle_n} f_2^2 + \int_{\partial\triangle_n} (f_1)_{T_n}^2\Big), \tag{4.17}$$

where T_n is the unit tangent vector on the boundary $\partial\triangle_n$ and $(f_1)_{T_n}$ is the directional derivative of f_1 in T_n.

Let $e \subset \triangle_n$ be an edge with two incident vertices, u and v. By linear interpolation, we have

$$\int_e f_1^2 = \int_0^1 (tf(u) + (1-t)f(v))^2 dt = \frac{1}{3}(f(u)^2 + f(u)f(v) + f(v)^2),$$

hence

$$\frac{1}{6}(f(u)^2 + f(v)^2) \le \int_e f_1^2 \le \frac{1}{2}(f(u)^2 + f(v)^2). \tag{4.18}$$

In addition,

$$\int_e (f_1)_{T_n}^2 = (f(u) - f(v))^2. \tag{4.19}$$

Now we can prove the Poincaré inequality.

Proof of Theorem 4.3. Let $B_R^{G_1} := B_R^{G_1}(p)$ denote the closed geodesic ball in G_1. We set the constant

$$c_R = (f_2)_{B^S_{R+1+D}} = \frac{1}{|B^S_{R+1+D}|} \int_{B^S_{R+1+D}} f_2.$$

By the combinatorial relation of vertices and edges and (4.18), we have

$$\begin{aligned} \sum_{x \in B_R} (f(x) - c_R)^2 d_x &\le \sum_{\substack{e = uv \in E \\ e \cap B_R \neq \emptyset}} [(f(u) - c_R)^2 + (f(v) - c_R)^2] \\ &\le 6 \int_{B^{G_1}_{R+1}} (f_1 - c_R)^2. \end{aligned} \tag{4.20}$$

Let

$$W_{R+1} = \{\sigma \in F : \sigma \cap B^{G_1}_{R+1} \neq \emptyset\} \quad \text{and} \quad \overline{W_{R+1}} = \bigcup_{\sigma \in W_{R+1}} \sigma.$$

Since $B^{G_1}_{R+1} \subset \bigcup_{\sigma \in W_{R+1}} \partial\sigma$, we have

$$\begin{aligned} \int_{B^{G_1}_{R+1}} (f_1 - c_R)^2 &\le \sum_{\sigma \in W_{R+1}} \int_{\partial\sigma} (f_1 - c_R)^2 \\ &\le C(D) \sum_{\sigma \in W_{R+1}} \left(\int_\sigma (f_2 - c_R)^2 + \int_{\partial\sigma} (f_1)_T^2 \right), \end{aligned} \tag{4.21}$$

where the last inequality follows from (4.17). For any $y \in \overline{W_{R+1}}$, since $\operatorname{diam} \sigma \le \deg(\sigma) \le D$ for any $\sigma \in F$, we have $d(p, y) \le R + 1 + D$. It implies that $\overline{W_{R+1}} \subset B^S_{R+1+D}$. Hence by (4.21),

$$\begin{aligned} \int_{B^{G_1}_{R+1}} (f_1 - c_R)^2 &\le C(D) \int_{B^S_{R+1+D}} (f_2 - c_R)^2 + C(D) \sum_{\sigma \in W_{R+1}} \int_{\partial\sigma} (f_1)_T^2 \\ &\le C(D)(R+1+D)^2 \int_{B^S_{R+1+D}} |\nabla f_2|^2 \\ &\quad + C(D) \sum_{\sigma \in W_{R+1}} \int_{\partial\sigma} (f_1)_T^2, \end{aligned} \tag{4.22}$$

where we use the Poincaré inequality (2.4).

Let $U_{R+1} := \{\tau \in F : \tau \cap B^S_{R+1+D} \neq \emptyset\}$. Since $B^{G_1}_{R+1} \subset B^S_{R+1} \subset B^S_{R+1+D}$, we have $W_{R+1} \subset U_{R+1}$. By Lemma 3.1, it follows that

$$U_{R+1} \cap G_1 \subset B^{G_1}_{C^{-1}(R+1+2D)}. \tag{4.23}$$

By (4.16), (4.19), (4.22) and (4.23), we obtain that

$$\begin{aligned}
\int_{B^{G_1}_{R+1}} (f_1 - c_R)^2 &\leq C(D)(R+1+D)^2 \sum_{\tau \in U_{R+1}} \int_\tau |\nabla f_2|^2 \\
&\quad + C(D) \sum_{\sigma \in W_{R+1}} \int_{\partial\sigma} (f_1)_T^2 \\
&\leq C(D)(R+1+D)^2 \sum_{\tau \in U_{R+1}} \int_{\partial\tau} (f_1)_T^2 \\
&\quad + C(D) \sum_{\sigma \in W_{R+1}} \int_{\partial\sigma} (f_1)_T^2 \\
&\leq C(D)(R+1+D)^2 \sum_{\tau \in U_{R+1}} \int_{\partial\tau} (f_1)_T^2 \\
&\leq C(D)(R+1+D)^2 \sum_{\substack{x,y \in B_{C^{-1}(R+1+2D)} \\ x \sim y}} (f(x) - f(y))^2.
\end{aligned} \tag{4.24}$$

For $R \geq C^{-1}(1+2D) =: r_0(D)$, we have $C^{-1}(R+1+2D) \leq (C^{-1}+1)R = C_1 R$ and $R+1+2D \leq 2R$. Let

$$f_{B_R} = \frac{1}{|B_R|} \sum_{x \in B_R} f(x) d_x,$$

then by (4.20) and (4.24) we obtain

$$\begin{aligned}
\sum_{x \in B_R} (f(x) - f_{B_R})^2 d_x &\leq \sum_{x \in B_R} (f(x) - c_R)^2 d_x \\
&\leq C(D) R^2 \sum_{\substack{x,y \in B_{C_1 R} \\ x \sim y}} (f(x) - f(y))^2.
\end{aligned} \tag{4.25}$$

For $1 \leq R \leq r_0(D)$, let $G^R = (V^R, E^R)$ be the subgraph induced by B_R. For any $x \in G^R$, we denote by d_{x,G^R} the degree of the vertex x in G^R. The volume of G^R is defined as

$$\operatorname{vol} G^R = \sum_{x \in G^R} d_{x,G^R}$$

and the diameter of G^R is defined as $\operatorname{diam} G^R = \sup_{x,y \in G^R} d^{G^R}(x,y)$. Let $\lambda_1(G^R)$ be the first nonzero eigenvalue of the Laplacian of G^R. Then the Rayleigh principle implies that

$$\lambda_1(G^R) = \inf_{\substack{f : G^R \to \mathbb{R} \\ f \neq \text{const.}}} \frac{\sum_{x,y \in G^R;\, x \sim y} (f(x) - f(y))^2}{\sum_{x \in G^R} (f(x) - f_{G^R})^2 d_{x,G^R}},$$

where

$$f_{G^R} = \frac{1}{\operatorname{vol} G^R} \sum_{x \in G^R} f(x) d_{x,G^R}.$$

We recall a lower bound estimate for $\lambda_1(G^R)$ by the diameter and volume of G^R (see [9]),

$$\lambda_1(G^R) \geq \frac{1}{\operatorname{diam} G^R \cdot \operatorname{vol} G^R}.$$

Since $3 \leq d_x \leq 6$, we have

$$\frac{1}{6} d_x \leq d_{x,G^R} \leq d_x.$$

It is easy to see that $\operatorname{diam} G^R \leq 2R$ and $\operatorname{vol} G^R \leq |B_R| \leq C(D)R^2$ by (4.5). So that we have

$$\lambda_1(G^R) \geq \frac{1}{2R \cdot C(D)R^2} \geq \frac{1}{2r_0(D) \cdot C(D) r_0^2(D)} \geq C(D),$$

which implies that

$$\sum_{x \in G^R} (f(x) - f_{G^R})^2 d_{x,G^R} \leq C(D) \sum_{\substack{x,y \in G^R \\ x \sim y}} (f(x) - f(y))^2,$$

for any $f : G^R \to \mathbb{R}$.

Hence we obtain that

$$\begin{aligned}
\sum_{x \in B_R} (f(x) - f_{B_R})^2 d_x &\leq 6 \sum_{x \in G^R} (f(x) - f_{G^R})^2 d_{x,G^R} \\
&\leq C(D) \sum_{\substack{x,y \in G^R \\ x \sim y}} (f(x) - f(y))^2 \\
&\leq C(D) R^2 \sum_{\substack{x,y \in B_R \\ x \sim y}} (f(x) - f(y))^2.
\end{aligned} \tag{4.26}$$

For $0 < R < 1$, the Poincaré inequality (4.13) is trivial. The theorem is proved by (4.25) and (4.26). □

5. Analysis on semiplanar graphs with nonnegative curvature

In this section, we shall study the analytic consequences of the volume doubling property and the Poincaré inequality.

In Riemannian manifolds, it is well known that the volume doubling property and the Poincaré inequality are sufficient for the Nash–Moser iteration which implies the Harnack inequality for positive harmonic functions (see [21, 45]).

Let G be a graph. For a function $f : G \to \mathbb{R}$, the Laplace operator L is defined as

$$Lf(x) = \frac{1}{d_x} \sum_{y \sim x} (f(y) - f(x)).$$

The gradient of f is defined as

$$|\nabla f|^2(x) = \sum_{y \sim x} (f(y) - f(x))^2.$$

Given a subset $\Omega \subset G$, a function f is called harmonic (subharmonic, superharmonic) on Ω if $Lf(x) = 0$ ($\geq 0, \leq 0$) for any $x \in \Omega$. We denote by

$$H^d(G) = \{f : Lf \equiv 0, |f(x)| \leq C(d^G(p,x)+1)^d\}$$

the space of polynomial growth harmonic functions of growth degree less than or equal to d on G.

It was proved by Delmotte [17] and Holopainen–Soardi [28] independently that the Harnack inequality for positive harmonic functions holds on graphs satisfying the volume doubling property and the Poincaré inequality. Applying their results to our case, we obtain the following theorem.

Theorem 5.1 ([17,28]). *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$*. Then there exist constants* $C_1 > 1$, $C_2(D) < \infty$ *such that for any* $R > 0$, $p \in G$ *and any positive harmonic function u on* $B_{C_1 R}(p)$*, we have*

$$\max_{B_R(p)} u \leq C_2(D) \min_{B_R(p)} u. \tag{5.1}$$

Remark 5.2. In [18], Delmotte obtained the parabolic Harnack inequality and the Gaussian estimate for the heat kernel which is stronger than the elliptic one of the preceding theorem.

In the Nash–Moser iteration, the mean value inequality for nonnegative subharmonic functions is obtained (see [14]). Since the square of a harmonic function is subharmonic, we obtain

Lemma 5.3. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$*. Then there exist two constants* C_1 *and* $C_2(D)$ *such that for any* $R > 0$, $p \in G$, *any harmonic function u on* $B_{C_1 R}(p)$, *we have*

$$u^2(p) \leq \frac{C_2(D)}{|B_{C_1 R}(p)|} \sum_{x \in B_{C_1 R}(p)} u^2(x) d_x. \tag{5.2}$$

The Liouville theorem for positive harmonic functions follows from the Harnack inequality (see [46]).

Theorem 5.4. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$*. Then any positive harmonic function on G must be constant.*

Proof. Since G is a semiplanar graph with $\mathrm{Sec}(G) \geq 0$, it follows that $D_G < \infty$. Let u be a positive harmonic function on G. By the Harnack inequality (5.1), we obtain

$$\max_{B_R}(u - \inf_G u) \leq C_2(D_G) \min_{B_R}(u - \inf_G u), \tag{5.3}$$

for any $R > 0$. The right hand side of (5.3) tends to 0 if $R \to \infty$. Hence,

$$u \equiv \inf_G u = \text{const.}$$

□

A manifold or a graph is called parabolic if it does not admit any nontrivial positive superharmonic function. The parabolicity of a manifold has extensively been studied in the literature (see [22, 27, 44]). In the graph setting, it is equivalent to the fact that the simple random walk on the graph is recurrent, see e.g. [52]. We already prove the quadratic volume growth, (4.5) in Theorem 4.1, of the semiplanar graph with nonnegative curvature. Note that [52, Lemma 3.12] yields the parabolicity of such graphs.

Theorem 5.5. *Any semiplanar graph G with* $\mathrm{Sec}(G) \geq 0$ *is parabolic.*

In the last part of the section, we investigate the polynomial growth harmonic function theorem on graphs. For Riemannian manifolds, the polynomial growth harmonic function theorem was proved by Colding and Minicozzi in [10]. By assuming the volume doubling property (4.12) and the Poincaré inequality (4.13) on the graph, Delmotte [16] proved the polynomial growth harmonic function theorem with the dimension estimate in our case

$$\dim H^d(G) \leq C(D) d^{v(D)},$$

where $C(D)$ and $v(D)$ depend on the maximal facial degree D of the semiplanar graph G with nonnegative curvature. We improve Delmotte's dimension estimate of $H^d(G)$ by using the relative volume comparison (4.1) instead of the volume doubling property (4.12).

Theorem 5.6. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$. *Then*

$$\dim H^d(G) \leq C(D) d^2, \tag{5.4}$$

for any $d \geq 1$.

We will use the argument by the mean value inequality (see [12, 37]). From now on, we fix some vertex $p \in G$, and denote $B_R = B_R(p)$ for short. We need the following lemmas.

Lemma 5.7. *For any finite dimensional subspace $K \subset H^d(G)$, there exists a constant $R_0(K)$ depending on K such that for any $R \geq R_0(K)$, $u, v \in K$,*

$$A_R(u, v) := \sum_{x \in B_R} u(x) v(x) d_x$$

is an inner product on K.

Proof. The lemma is easily proved by a contradiction argument (see [30]). □

Lemma 5.8. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$ *and K be a k-dimensional subspace of $H^d(G)$. Given $\beta > 1, \delta > 0$, for any $R_1 \geq R_0(K)$ there exists an $R > R_1$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of K with respect to the inner product $A_{\beta R}$, then*

$$\sum_{i=1}^k A_R(u_i, u_i) \geq k \beta^{-(2d+2+\delta)}.$$

Proof. The proof is same as [16, 30, 38]. □

Lemma 5.9. *Let G be a semiplanar graph with $\mathrm{Sec}(G) \geq 0$ and K be a k-dimensional subspace of $H^d(G)$. Then there exists a constant $C(D)$ such that for any basis of K, $\{u_i\}_{i=1}^k$, $R > 0, 0 < \epsilon < \frac{1}{2}$, we have*

$$\sum_{i=1}^{k} A_R(u_i, u_i) \leq C(D)\epsilon^{-2} \sup_{u \in \langle A,U \rangle} \sum_{y \in B_{(1+\epsilon)R}} u^2(y) d_y,$$

where $\langle A, U\rangle := \{w = \sum_{i=1}^k a_i u_i : \sum_{i=1}^k a_i^2 = 1\}$.

Proof. For any $x \in B_R$, we set

$$K_x = \{u \in K : u(x) = 0\}.$$

It is easy to see that $\dim K/K_x \leq 1$. Hence there exists an orthonormal linear transformation $\phi : K \to K$, which maps $\{u_i\}_{i=1}^k$ to $\{v_i\}_{i=1}^k$ such that $v_i \in K_x$, for $i \geq 2$. The mean value inequality (5.2) implies that

$$\begin{aligned}
\sum_{i=1}^{k} u_i^2(x) &= \sum_{i=1}^{k} v_i^2(x) = v_1^2(x) \\
&\leq C(D)|B_{(1+\epsilon)R-r(x)}(x)|^{-1} \sum_{y \in B_{(1+\epsilon)R-r(x)}(x)} v_1^2(y) d_y \\
&\leq C(D)|B_{(1+\epsilon)R-r(x)}(x)|^{-1} \sup_{u \in \langle A,U \rangle} \sum_{y \in B_{(1+\epsilon)R}} u^2(y) d_y,
\end{aligned}$$

where $r(x) = d^G(p, x)$.

By the relative volume comparison (4.1), we have

$$\begin{aligned}
|B_{(1+\epsilon)R-r(x)}| &\geq C(D)\left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^2 |B_{2R}(x)| \\
&\geq C(D)\left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^2 |B_R(p)| \geq C(D)\epsilon^2 |B_R|.
\end{aligned}$$

Hence, by $3 \leq d_x \leq 6$ for any $x \in G$,

$$\sum_{i=1}^{k} \sum_{x \in B_R} u_i^2(x) d_x \leq 6 \sum_{i=1}^{k} \sum_{x \in B_R} u_i^2(x) \leq C(D)\epsilon^{-2} \sup_{u \in \langle A,U \rangle} \sum_{y \in B_{(1+\epsilon)R}} u^2(y) d_y. \qquad \square$$

Proof of Theorem 5.6. For any k-dimensional subspace $K \subset H^d(G)$, we set $\beta = 1+\epsilon$. By Lemma 5.8, there exists an $R > R_0(K)$ such that for any orthonormal basis $\{u_i\}_{i=1}^k$ of K with respect to $A_{(1+\epsilon)R}$, we have

$$\sum_{i=1}^{k} A_R(u_i, u_i) \geq k(1+\epsilon)^{-(2d+2+\delta)}.$$

Lemma 5.9 implies that

$$\sum_{i=1}^{k} A_R(u_i, u_i) \leq C(D)\epsilon^{-2}.$$

Setting $\epsilon = \frac{1}{2d}$, and letting $\delta \to 0$, we obtain

$$k \le C(D)\left(\frac{1}{2d}\right)^{-2}\left(1+\frac{1}{2d}\right)^{2d+2+\delta} \le C(D)d^2. \qquad \square$$

The dimension estimate in (5.4) is not satisfactory since in Riemannian geometry the constant $C(D)$ depends only on the dimension of the manifold rather than the maximal facial degree of G. Note that Theorem 2.10 shows that the semiplanar graph G with $\mathrm{Sec}(G) \ge 0$ and $D_G \ge 43$ has a special structure, i.e. the one-side cylinder structure of linear volume growth. In Riemannian geometry, Sormani [47] used Yau's gradient estimate and the nice behavior of the Busemann function on a one-end Riemannian manifold with nonnegative Ricci curvature of linear volume growth to show that it does not admit any nontrivial polynomial growth harmonic function. Inspired by the work [47] and the special structure of semiplanar graphs with nonnegative curvature and large face degree, we shall prove the following theorem.

Theorem 5.10. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \ge 0$ *and $D_G \ge 43$. Then for any $d > 0$,*

$$\dim H^d(G) = 1.$$

To prove the theorem, we need a weak version of the gradient estimate [39]. We recall the Cacciappoli inequality for harmonic functions on the graph G.

Theorem 5.11 ([39]). *Let G be a graph and $d_m = \sup_{x\in G} d_x$. For any harmonic function u on B_{6r}, $r \ge 1$, we have*

$$\sum_{x\in B_r} |\nabla u|^2(x) \le \frac{C(d_m)}{r^2} \sum_{y\in B_{6r}} u^2(y)d_y.$$

Moreover for any $x \in B_r$,

$$|\nabla u|^2(x) \le \frac{C(d_m)}{r^2} \sum_{y\in B_{6r}} u^2(y)d_y. \tag{5.5}$$

Corollary 5.12. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \ge 0$ *and $D_G \ge 43$. For any harmonic function u on G, we have*

$$|\nabla u|(x) \le \frac{C(D)}{\sqrt{r}} \mathrm{osc}_{B_{6r}(x)} u, \tag{5.6}$$

where $\mathrm{osc}_{B_{6r}(x)} u = \max_{B_{6r}(x)} u - \min_{B_{6r}(x)} u$.

Proof. By Theorem 2.10, the regular polygonal surface $S(G)$ has linear volume growth. As same as in the proof of (4.5) in Theorem 4.1, we obtain that for any $x \in G$ and $r \ge 1$,

$$|B_r(x)| \le C(D)r. \tag{5.7}$$

By (5.5) in Theorem 5.11 and $d_m \le 6$, we have

$$|\nabla u|^2(x) \le \frac{C}{r^2} \sum_{y\in B_{6r}(x)} u^2(y)d_y \le \frac{C}{r^2}|B_{6r}(x)| \max_{B_{6r}(x)} |u|^2. \tag{5.8}$$

We replace u by $u - \min_{B_{6r}(x)} u$ in (5.8), noting that (5.7), to obtain that

$$|\nabla u|(x) \leq \frac{C(D)}{\sqrt{r}} \mathrm{osc}_{B_{6r}(x)} u. \qquad \square$$

Remark 5.13. We call (5.5) the weak version of the gradient estimate since its scaling is not as usual, but it suffices for our application.

Proof of Theorem 5.10. Let G be a semiplanar graph with $\mathrm{Sec}(G) \geq 0$ and $D_G \geq 43$. Let σ be the largest face with $\deg(\sigma) = D_G = D \geq 43$. By Theorem 2.10, either G has no hexagons, which looks like $\sigma, L_1, L_2, \ldots, L_m, \ldots$ where each L_m has the same type of faces (triangle or square), i.e. $G = \sigma \cup \bigcup_{m=1}^{\infty} L_m$, or G has hexagons, $G = P^{-1}(\sigma \cup \bigcup_{m=1}^{\infty} L_m)$, where P^{-1} is the graph operation defined in Section 2. Denote by $A = \sigma \cap G$ the set of vertices incident to σ, by $d^G(x, A) = \min_{y \in A} d^G(x, y)$ the distance function of A in G. Let $B_r(A) = \{x \in G : d^G(x, A) \leq r\}$ and $\partial B_r(A) = \{x \in G : d^G(x, A) = r\}$.

We claim that by the construction of G, for any $x, y \in \partial B_r(A)$, there is a path joining x and y in $B_r(A)$ with length less than or equal to $3D$. For the case that G has no hexagons, we know that $\sharp \partial B_r(A) = D$ and $\partial B_r(A)$ is a closed path (a cycle). This is trivial. For the case that G has hexagons, by induction on $r \in \mathrm{N}$, we have $\sharp \partial B_r(A) \leq D$ and the distance to A, i.e. $d(x, A)$ is invariant under the operation P^{-1}. Since for each two vertices in $\partial B_r(A)$ sharing one hexagon, there is a path on the boundary of the hexagon with length ≤ 3 which lies in $B_r(A)$. Note that $\sharp \partial B_r(A) \leq D$, the claim follows.

In addition, for any $q \in A$, we have

$$\partial B_r(A) \subset B_{r+D}(q). \tag{5.9}$$

Let $u \in H^d(G)$ and $M(r) = \mathrm{osc}_{\partial B_r(A)} u = \max_{\partial B_r(A)} u - \min_{\partial B_r(A)} u$. By the maximum principle which is a direct consequence of the definition of the harmonic function, we have $\max_{\partial B_r(A)} u = \max_{B_r(A)} u$ and $\min_{\partial B_r(A)} u = \min_{B_r(A)} u$, so that $M(r)$ is nondecreasing in r. To prove the theorem, it suffices to show that $M(r) = 0$ for any large r. Let $y_r, x_r \in \partial B_r(A)$ satisfy $u(y_r) = \max_{\partial B_r(A)} u$ and $u(x_r) = \min_{\partial B_r(A)} u$. Then there exists a path in $B_r(A)$ such that

$$y_r = z_0 \sim z_1 \sim \cdots \sim z_l = x_r,$$

where $z_i \in B_r(A)$ for $0 \leq i \leq l$ and $l \leq 3D$. Hence

$$\begin{aligned} M(r) = u(y_r) - u(x_r) &\leq \sum_{i=0}^{l-1} |\nabla u|(z_i) \\ &\leq C \sum_{i=0}^{l-1} \frac{C(D)}{\sqrt{r}} \mathrm{osc}_{B_{6r}(z_i)} u \\ &\leq C(D) \frac{\mathrm{osc}_{B_{7r+D}(q)} u}{\sqrt{r}} \cdot (3D) \\ &\leq C(D) \frac{\mathrm{osc}_{B_{7r+2D}(A)} u}{\sqrt{r}} \\ &\leq C(D) \frac{M(9r)}{\sqrt{r}} \quad \text{for } r \geq D, \end{aligned} \tag{5.10}$$

where we use (5.6) in Corollary 5.12, $B_{6r}(z_i) \subset B_{7r+D}(q) \subset B_{7r+2D}(A)$.

Let $r \geq R_0(D,\delta) := \left(\frac{C(D)}{\delta}\right)^2$, for $\delta < 1$ which will be chosen later. Then we have

$$\frac{C(D)}{\sqrt{r}} \leq \delta < 1.$$

By (5.10), for any $r \geq R_0(D,\delta)$, we obtain that for $k \geq 1$,

$$M(r) \leq \delta M(9r) \leq \delta^k M(9^k r).$$

Since $u \in H^d(G)$, by (5.9),

$$M(r) \leq 2 \max_{B_{r+D}(q)} |u| \leq 2C(r+D+1)^d.$$

Hence for large k with $9^k r \geq D+1$,

$$M(r) \leq \delta^k M(9^k r) \leq 2C\delta^k (9^k r + D + 1)^d \leq C 2^{d+1} \delta^k (9^k r)^d = C(d)\left(\frac{1}{2}\right)^k r^d$$

if we choose $\delta = \frac{1}{2 \cdot 9^d}$. Then for any $r \geq R_0(D,\delta) = (C(D)2 \cdot 9^d)^2$, we have

$$M(r) \leq C(d)\left(\frac{1}{2}\right)^k r^d.$$

By $k \to \infty$, we obtain $M(r) = 0$, which proves the theorem. □

Combining Theorem 5.6 and Theorem 5.10 with Lemma 2.9, we obtain a dimension estimate that does not depend on the maximal facial degree D_G.

Theorem 5.14. *Let G be a semiplanar graph with* $\mathrm{Sec}(G) \geq 0$. *Then for any $d \geq 1$,*

$$\dim H^d(G) \leq C d^2,$$

where C is an absolute constant.

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References

[1] *O. Baues* and *N. Peyerimhoff*, Curvature and geometry of tessellating plane graphs, Discrete Comput. Geom. **25** (2001), 141–159.

[2] *O. Baues* and *N. Peyerimhoff*, Geodesics in non-positively curved plane tessellations, Adv. Geom. **6** (2006), no. 2, 243–263.

[3] *D. Burago*, *Y. Burago* and *S. Ivanov*, A course in metric geometry, Grad. Stud. Math. **33**, American Mathematical Society, Providence 2001.

[4] *Y. Burago*, *M. Gromov* and *G. Perelman*, A. D. Aleksandrov spaces with curvatures bounded below, Russian Math. Surveys **47** (1992), 1–58.

[5] *J. Cheeger*, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. **9** (1999), no. 3, 428–517.

[6] *J. Cheeger* and *D. Gromoll*, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geom. **6** (1971/72), 119–128.

[7] *B. Chen*, The Gauss–Bonnet formula of polytopal manifolds and the characterization of embedded graphs with nonnegative curvature, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1601–1611.
[8] *B. Chen* and *G. Chen*, Gauss–Bonnet formula, finiteness condition, and characterizations of graphs embedded in surfaces, Graphs Combin. **24** (2008), no. 3, 159–183.
[9] *F. R. K. Chung*, Spectral graph theory, CBMS Reg. Conf. Ser. Math. **92**, American Mathematical Society, Providence 1997.
[10] *T. H. Colding* and *W. P. Minicozzi II*, Harmonic functions on manifolds, Ann. of Math. (2) **146** (1997), no. 3, 725–747.
[11] *T. H. Colding* and *W. P. Minicozzi II*, Harmonic functions with polynomial growth, J. Differential Geom. **46** (1997), no. 1, 1–77.
[12] *T. H. Colding* and *W. P. Minicozzi II*, Liouville theorems for harmonic sections and applications, Comm. Pure Appl. Math. **51** (1998), no. 2, 113–138.
[13] *T. H. Colding* and *W. P. Minicozzi II*, Weyl type bounds for harmonic functions, Invent. Math. **131** (1998), no. 2, 257–298.
[14] *T. Coulhon* and *A. Grigoryan*, Random walks on graphs with regular volume growth, Geom. Funct. Anal. **8** (1998), no. 4, 656–701.
[15] *T. Coulhon* and *L. Saloff-Coste*, Variétés riemanniennes isométriques á l'infini, RevṀat. Iberoam. **11** (1995), no. 3, 687–726.
[16] *T. Delmotte*, Harnack inequalities on graphs, Sémin. Théor. Spectr. Géom. **16** (1997/1998), 217–228.
[17] *T. Delmotte*, Inégalité de Harnack elliptique sur les graphes, Colloq. Math. **72** (1997), no. 1, 19–37.
[18] *T. Delmotte*, Parabolic Harnack inequality and estimates of Markov chains on graphs, Rev. Mat. Iberoam. **15** (1999), no. 1, 181–232.
[19] *M. DeVos* and *B. Mohar*, An analogue of the Descartes–Euler formula for infinite graphs and Higuchi's conjecture, Trans. Amer. Math. Soc. **359** (2007), no. 7, 3287–3300 (electronic).
[20] *J. Dodziuk* and *L. Karp*, Spectral and function theory for combinatorial Laplacians, in: Geometry of random motion (Ithaca 1987), Contemp. Math. **73**, American Mathematical Society, Providence (1988), 25–40.
[21] *A. Grigor'yan*, The heat equation on noncompact Riemannian manifolds (Russian), Mat. Sb. **182** (1991), no. 1, 55–87; translation in Math. USSR-Sb. **72** (1992), no. 1, 47–77.
[22] *A. Grigor'yan*, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 2, 135–249.
[23] *A. Grigor'yan*, Analysis on graphs, Lecture notes, University of Bielefeld, 2009.
[24] *M. Gromov*, Hyperbolic groups, in: Essays in group theory, Math. Sci. Res. Inst. Publ. **8**, Springer-Verlag, New York (1987), 75–263.
[25] *B. Grünbaum* and *G. C. Shephard*, Tilings and patterns, W. H. Freeman and Company, New York 1987.
[26] *Y. Higuchi*, Combinatorial curvature for planar graphs, J. Graph Theory **38** (2001), no. 4, 220–229.
[27] *I. Holopainen* and *P. Koskela*, Volume growth and parabolicity, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3425–3435 (electronic).
[28] *I. Holopainen* and *P. M. Soardi*, A strong Liouville theorem for p-harmonic functions on graphs, Ann. Acad. Sci. Fenn. Math. **22** (1997), no. 1, 205–226.
[29] *B. Hua*, Generalized Liouville theorem in nonnegatively curved Alexandrov spaces, Chin. Ann. Math. Ser. B **30** (2009), no. 2, 111–128.
[30] *B. Hua*, Harmonic functions of polynomial growth on singular spaces with nonnegative Ricci curvature, Proc. Amer. Math. Soc. **139** (2011), 2191–2205.
[31] *M. Ishida*, Pseudo-curvature of a graph, Lecture at "Workshop on topological graph theory", Yokohama National University, 1990.
[32] *M. Keller*, The essential spectrum of the Laplacian on rapidly branching tessellations, Math. Ann. **346** (2010), no. 1, 51–66.
[33] *M. Keller*, Curvature, geometry and spectral properties of planar graphs, Discrete Comput. Geom. **46** (2011), no. 3, 500–525.
[34] *M. Keller* and *N. Peyerimhoff*, Cheeger constants, growth and spectrum of locally tessellating planar graphs, Math. Z. **268** (2011), no. 3–4, 871–886.
[35] *B. Kleiner*, A new proof of Gromov's theorem on groups of polynomial growth, J. Amer. Math. Soc. **23** (2010), no. 3, 815–829.
[36] *K. Kuwae*, *Y. Machigashira* and *T. Shioya*, Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces, Math. Z. **238** (2001), no. 2, 269–316.
[37] *P. Li*, Harmonic sections of polynomial growth, Math. Res. Lett. **4** (1997), no. 1, 35–44.
[38] *P. Li*, Harmonic functions and applications to complete manifolds, Lecture notes, University of California, Irvine 2004, `http://math.uci.edu/~pli/lectures_harmonic.pdf`.

[39] *Y. Lin* and *L. Xi*, Lipschitz property of harmonic function on graphs, J. Math. Anal. Appl. **366** (2010), no. 2, 673–678.
[40] *A. D. Milka*, Metric structure of a certain class of spaces that contain straight lines (Russian), Ukrain. Geometr. Sb. Vyp. **4** (1967), 43–48.
[41] *A. Mitsuishi*, A splitting theorem for infinite dimensional Alexandrov spaces with nonnegative curvature and its applications, Geom. Dedicata **144** (2010), 101–114.
[42] *Y. Otsu* and *T. Shioya*, The Riemannian structure of Alexandrov spaces, J. Differential Geom. **39** (1994), no. 3, 629–658.
[43] *T. Réti*, *E. Bitay* and *Z. Kosztolányi*, On the polyhedral graphs with positive combinatorial curvature, Acta Polytech. Hungar. **2** (2005) no. 2, 19–37.
[44] *M. Rigoli*, *M. Salvatori* and *M. Vignati*, Subharmonic functions on graphs, Israel J. Math. **99** (1997), 1–27.
[45] *L. Saloff-Coste*, Uniformly elliptic operators on Riemannian manifolds, J. Differential Geom. **36** (1992), no. 2, 417–450.
[46] *L. Saloff-Coste*, Aspects of Sobolev-type inequalities, London Math. Soc. Lecture Note Ser. **289**, Cambridge University Press, Cambridge 2002.
[47] *C. Sormani*, Harmonic functions on manifolds with nonnegative Ricci curvature and linear volume growth, Pacific J. Math. **192** (2000), no. 1, 183–189.
[48] *D. Stone*, A combinatorial analogue of a theorem of Myers, Illinois J. Math. **20** (1976), no. 1, 12–21.
[49] *D. Stone*, Correction to my paper: "A combinatorial analogue of a theorem of Myers" [Illinois J. Math. **20** (1976), no. 1, 12–21], Illinois J. Math. **20** (1976), no. 3, 551–554.
[50] *L. Sun* and *X. Yu*, Positively curved cubic plane graphs are finite, J. Graph Theory **47** (2004), 241–274.
[51] *W. Woess*, A note on tilings and strong isoperimetric inequality, Math. Proc. Cambridge Philos. Soc. **124** (1998), 385–393.
[52] *W. Woess*, Random walks on infinite graphs and groups, Cambridge Tracts in Math. **138**, Cambridge University Press, Cambridge 2000.
[53] *S. T. Yau*, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. **28** (1975), 201–228.
[54] *S. T. Yau*, Nonlinear analysis in geometry, Enseign. Math. (2) **33** (1987), 109–158.
[55] *S. T. Yau*, Differential geometry: Partial differential equations on manifolds, Proc. Sympos. Pure Math. **54**, part 1, American Mathematical Society, Providence 1993.
[56] *H. C. Zhang* and *X. P. Zhu*, Ricci curvature on Alexandrov spaces and rigidity theorems, Comm. Anal. Geom. **18** (2010), no. 3, 503–553.

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