

Entanglement density and gravitational thermodynamicsJyotirmoy Bhattacharya,^{1,*} Veronika E. Hubeny,^{1,†} Mukund Rangamani,^{1,‡} and Tadashi Takayanagi^{2,3,§}¹*Centre for Particle Theory & Department of Mathematical Sciences, Durham University,
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In an attempt to find a quasilocal measure of quantum entanglement, we introduce the concept of entanglement density in relativistic quantum theories. This density is defined in terms of infinitesimal variations of the region whose entanglement we monitor and in certain cases can be mapped to the variations of the generating points of the associated domain of dependence. We argue that strong subadditivity constrains the entanglement density to be positive semidefinite. Examining this density in the holographic context, we map its positivity to a statement of integrated null energy condition in the gravity dual. We further speculate that this may be mapped to a statement analogous to the second law of black hole thermodynamics for the extremal surface.

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I. INTRODUCTION

The holographic AdS/CFT correspondence indicates that the fundamental constituents of spacetime geometry are quanta of a conventional nongravitational field theory. The precise manner in which these nongravitational quanta conspire to construct a smooth semiclassical spacetime, however, still remains obscure. Holography is motivated by black hole thermodynamics, which suggests that the emergence of gravity can be associated with coarse-graining *a la* classical thermodynamics [1]. We then seek to understand what is being coarse-grained, and how.

A crucial hint is provided by the fact that AdS/CFT geometrizes quantum entanglement: entanglement entropy (EE) in the CFT is given by the area of a certain extremal surface in the bulk [2–4]. Indeed, the fascinating idea of spacetime geometry being the encoder of the entanglement structure of the quantum state [5–7] hints at potentially deep insights into the workings of quantum gravity.

As a first step, we would like to decipher the dynamical equations of gravity from these statements. In this regard EE, which motivates the connection to geometry, *a priori* presents a complication: it is nonlocal—even in local quantum field theories (QFTs), it is defined on a causal domain. The corresponding bulk quantity depends on the bulk geometry along a codimension-2 extremal surface. To make contact with local gravitational physics, it would be convenient to work with a more localizable construct in the dual CFT.¹

Inspired by this logic, we propose to study a QFT quantity we call “*entanglement density*.” This effectively measures two-body quantum entanglement between two infinitesimally small regions. To motivate its construction, consider a quantum field theory on a (rigid) background spacetime \mathcal{B} which is foliated by spacelike Cauchy surfaces Σ . We pick a region $\mathcal{A} \subset \Sigma$ and construct the reduced density matrix $\rho_{\mathcal{A}}$. The entanglement entropy $S_{\mathcal{A}} = -\text{Tr}(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}})$ is the von Neumann entropy of this density matrix and is a functional of $\partial\mathcal{A}$. We propose to retain locality by examining EE for infinitesimal variations of $\partial\mathcal{A}$ (and hence \mathcal{A}). Schematically for a configuration ρ_{Σ} on the Cauchy slice, we define the double variation²

$$\hat{n}(\delta_1\mathcal{A}, \delta_2\mathcal{A}) = \delta_1\delta_2 S_{\mathcal{A}}. \quad (1)$$

The construction is pictorially illustrated in Fig. 1.

Let us now simplify \hat{n} by appealing to the fact that $S_{\mathcal{A}}$ is a functional on the entire domain of dependence $D[\mathcal{A}]$. We focus on backgrounds \mathcal{B} and regions \mathcal{A} for which $D[\mathcal{A}]$ is given by the intersection of past and future light cones from two points, \mathcal{C}^{\pm} , respectively. As a consequence we will focus on the variations inherent in (1) which are due to the variations of one of the points, say \mathcal{C}^{-} , keeping the other fixed (or vice versa).³ In this context $\delta_1\delta_2 S_{\mathcal{A}}^{\text{vac}} = 0$ for two-dimensional and three-dimensional CFTs, although (1) pertains in any QFT.

*jyotirmoy.bhattacharya@durham.ac.uk†veronika.hubeny@durham.ac.uk‡mukund.rangamani@durham.ac.uk§takayana@yukawa.kyoto-u.ac.jp¹For recent progress on directly deriving gravitational dynamics from EE, cf. [8–11].²This construction has some parallels with recent discussions of differential entropy introduced in [12] and explored more thoroughly [13].³A related version of entanglement density was considered earlier in [8,14], without invoking the relativistic causal structure.

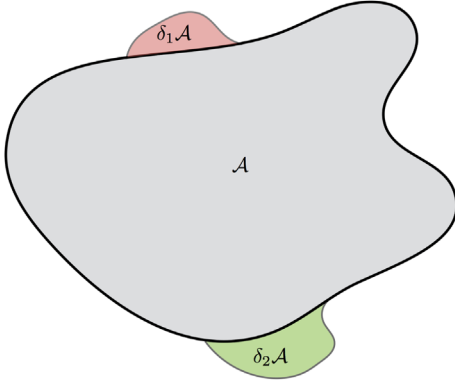


FIG. 1 (color online). Illustration of the generic variations $\delta_1 \mathcal{A}$ and $\delta_2 \mathcal{A}$ which are used to define the entanglement density (1).

We will exploit the fact that \hat{n} is naturally sensitive to a key property of the von Neumann entropy, namely, strong subadditivity (SSA), which states that

$$S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2} \leq S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \quad \forall \mathcal{A}_{1,2}. \quad (2)$$

SSA is a convexity property of entanglement; for regions in (2) being small deformations of a parent region, this has a quadratic structure, which motivates (1). Inspired by a beautiful construction of Casini & Huerta [15,16], we show that entanglement density can be expressed as a second-order differential operator \mathcal{D}_{\pm}^2 acting on EE by differentiating with respect to the coordinates c^{\pm} of \mathcal{C}^{\pm} (specified explicitly for $d=2,3$ in Sec. II). SSA then implies $\mathcal{D}_{c^{\pm}}^2 S(c^+; c^-) \geq 0$.

Exploiting the holographic construction of EE in terms of bulk codimension-2 extremal surfaces $\mathcal{E}_{\mathcal{A}}$, we argue that the variations of interest can be mapped to the motion of the extremal surface along its null normals $N_{(\pm)}^{\mu}$. Using standard differential geometric identities, this in turn can be simplified to a statement about the geometry side of Einstein's equations $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}$, namely,

$$\int_{\mathcal{E}_{\mathcal{A}}} \epsilon E_{\mu\nu} N_{(\pm)}^{\mu} N_{(\pm)}^{\nu} \geq 0, \quad (3)$$

where ϵ is the volume form induced on the extremal surface.⁴ Indeed, as the main result of this paper we will show that the entanglement density is precisely given by (3) for small perturbations in the $\text{AdS}_3/\text{CFT}_2$ setup. We have therefore related SSA (which can be regarded as a physicality condition on EE) to a restriction on the spacetime curvature.⁵

⁴A sufficient condition for this positivity is the null energy condition. The null energy condition has been crucial in the derivations of SSA [17–19].

⁵For other applications of entropic inequalities and related constraints in gravity duals see [20–23].

NB: As this work was nearing completion we received [24], where a similar relation between SSA and bulk energy stress tensor has been discussed. Similar results have been obtained by Arias and Casini (unpublished).

II. SSA IN FIELD THEORY

To set the stage for our analysis let us recall the proof of the c-theorem [25] and F-theorem [26–28] based on SSA, as in [15,16]. We consider subsystems which are defined by the intersection of light cones from two points \mathcal{C}^{\pm} in d -dimensional QFTs. Letting $D[\mathcal{A}] = J^{-}[\mathcal{C}^{+}] \cap J^{+}[\mathcal{C}^{-}]$, we pick \mathcal{A} to be a Cauchy slice for $D[\mathcal{A}]$ at constant time; see, e.g., Figs 2 and 3. Then $S_{\mathcal{A}}$ can be viewed of as a function of the coordinates c^{\pm} of \mathcal{C}^{\pm} ; i.e., $S_{\mathcal{A}} \equiv S(c^+; c^-)$. For $\mathcal{B} = \mathbb{R}^{d-1,1}$ we take $c^{\pm} = (t_{\pm}, \mathbf{x}_{\pm})$. Letting $a = \pm$, we define the entanglement density in $d=2,3$ with respect to varying \mathcal{C}^{\pm} as

$$\hat{n}_a(t_a, \mathbf{x}_a) \equiv \left[\square_a + \frac{2(d-2)}{t_a} \partial_{t_a} \right] S(t_a, \mathbf{x}_a) \geq 0, \quad (4)$$

where the inequality is guaranteed by SSA. We give a quick overview following [16], with some additional generalizations.

A. QFTs in $d=2$

We start by applying SSA to the configuration in Fig. 2; for space- and time-translation invariant configurations, we can w.l.o.g. fix $\mathcal{C}^{+} = (0,0)$ as a reference and drop subscripts for coordinates of \mathcal{C}^{-} . SSA implies

$$S_{AD} + S_{CB} \geq S_{AB} + S_{CD}. \quad (5)$$

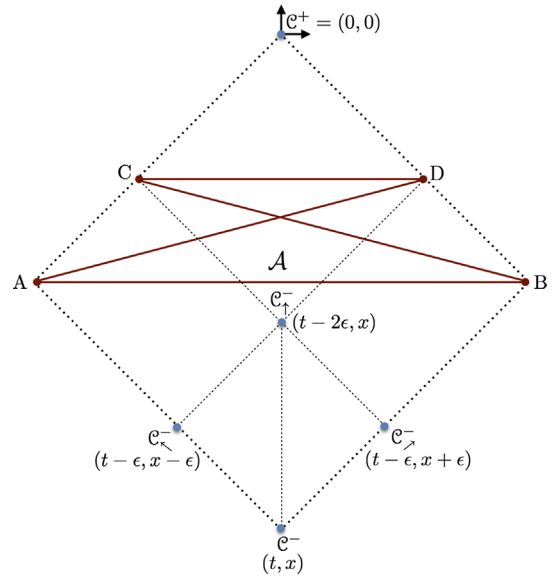


FIG. 2 (color online). Illustration of the setup following [15] in $d=2$. We choose \mathcal{C}^{+} to be the origin and the region \mathcal{A} lies on the time slice with coordinate $\frac{1}{2}t$. We assume $t < 0$ and $\epsilon \leq 0$.

measure from the circular cross sections of $\partial J^-[\mathcal{C}^+]$ onto the vertical segment along the map $\mathcal{C}^- \mapsto \mathcal{C}_\uparrow^-$. Accounting for this as in [16] we find

$$\text{rhs}_{(8)} = \left[1 - \epsilon \partial_t + \frac{\epsilon^2}{4} \left(3\partial_t^2 - \frac{2}{t} \partial_t \right) \right] S(t, \mathbf{x}) + \mathcal{O}(\epsilon^3).$$

Combining the above two expressions we have the inequality resulting from SSA:

$$\hat{n}_- \equiv \left[\square + \frac{2}{t} \partial_t \right] S(t, \mathbf{x}) \geq 0. \quad (9)$$

Repeating the analysis about \mathcal{C}^+ we can show $\hat{n}_+ \geq 0$. This completes the derivation of (4).

Note that in boost invariant states (e.g., vacuum) where $S_{\mathcal{A}}$ is a function of proper length $\ell = \sqrt{t^2 - |\mathbf{x}|^2}$, (4) simply reduces to [15,16]:

$$\ell S''(\ell) - (d-3)S'(\ell) \leq 0. \quad (10)$$

We have however managed to convert this to a local statement for regions \mathcal{A} which are naturally generated by intersecting light cones from two points \mathcal{C}^\pm . Although we have written the expressions (4) and (10) in a manner which suggests an obvious generalization to higher d , there are some subtleties with this interpretation, which we revisit in Sec. IV.

III. HOLOGRAPHIC ENTANGLEMENT DENSITY

Having understood the basic constraint on the entanglement density, let us now consider the holographic context, employing the $\text{AdS}_3/\text{CFT}_2$ duality. We focus on linear perturbations around the pure AdS_3 solution, corresponding to small excitations around the vacuum. In the bulk gravity theory, we consider Einstein gravity coupled to arbitrary matter fields, with the energy-momentum tensor $T_{\mu\nu}$ given by the Einstein's equation

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \quad (11)$$

It is now convenient to work directly with the end points of \mathcal{A} , whose null coordinates are (u_L, v_L) and (u_R, v_R) , respectively, in $\mathbb{R}^{1,1}$. In terms of these, the two entanglement densities are given by

$$\hat{n}_+ = -\frac{\partial}{\partial u_R} \frac{\partial}{\partial v_L} \Delta S_{\mathcal{A}}, \quad \hat{n}_- = -\frac{\partial}{\partial u_L} \frac{\partial}{\partial v_R} \Delta S_{\mathcal{A}}. \quad (12)$$

Note that we define the density in terms of $\Delta S_{\mathcal{A}} = S_{\mathcal{A}}^{\rho_\Sigma} - S_{\mathcal{A}}^{\text{vac}}$ which measures the entanglement of the excited state ρ_Σ relative to the vacuum. It is crucial here that \hat{n}_\pm vanishes in the vacuum state, for while SSA holds for any state of the

CFT, it is no longer true that $\Delta S_{\mathcal{A}}$ satisfies SSA.⁷ With this understanding we can replace $S_{\mathcal{A}} \rightarrow \Delta S_{\mathcal{A}}$ and still maintain the sign-definiteness of entanglement densities \hat{n}_\pm defined in (12).

We now evaluate $\Delta S_{\mathcal{A}}$ by analyzing the holographic entanglement entropy in the perturbed geometry around pure AdS_3 described by the (gauge fixed) metric,

$$ds^2 = \frac{dz^2 - du dv}{z^2} + h_{ab}(u, v, z) dx^a dx^b, \quad (13)$$

where h_{ab} captures the perturbation (Latin indices refer to the boundary). For linear order changes of holographic entanglement entropy, we can work with the original geodesic in AdS_3 (parametrized by ξ) which connects the end points of \mathcal{A} ,

$$(u, v, z) = (U + u_\delta \sin \xi, \quad V + v_\delta \sin \xi, \quad \sqrt{|u_\delta v_\delta|} \cos \xi),$$

where $\{U, u_\delta\} = \frac{1}{2}(u_R \pm u_L)$ and $\{V, v_\delta\} = \frac{1}{2}(v_R \pm v_L)$ give the midpoint and separation between the end points of \mathcal{A} .

The first-order perturbation of $\Delta S_{\mathcal{A}}$ is given by

$$\Delta S_{\mathcal{A}} = \frac{1}{8G_N} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\xi \frac{\gamma^{(1)}(\xi)}{\sqrt{\gamma^{(0)}(\xi)}}, \quad (14)$$

where $\gamma^{(0)}$ and $\gamma^{(1)}$ are induced metric ($\gamma_{\xi\xi}$) at leading and first subleading orders, i.e.,

$$\begin{aligned} \gamma^{(0)}(\xi) &= \frac{1}{\cos^2 \xi}, \\ \gamma^{(1)}(\xi) &= \cos^2 \xi (h_{uu} u_\delta^2 + h_{vv} v_\delta^2 + 2h_{uv} u_\delta v_\delta). \end{aligned}$$

After some algebra we arrive at the following simple relations,

$$\begin{aligned} \hat{n}_\pm &= \frac{1}{4G_N |u_\delta v_\delta|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\xi \sqrt{\gamma^{(0)}} (N_{(\pm)}^\mu N_{(\pm)}^\nu E_{\mu\nu}) \\ &= \frac{2\pi}{|u_\delta v_\delta|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\xi \sqrt{\gamma^{(0)}} (N_{(\pm)}^\mu N_{(\pm)}^\nu T_{\mu\nu}) \geq 0, \end{aligned} \quad (15)$$

where $E_{\mu\nu}$ is the lhs of the Einstein's equation (11). The vectors $N_{(\pm)}^\mu$ are the two independent null normals to the extremal surface $\mathcal{E}_{\mathcal{A}}$ in AdS_3 ,

$$N_{(\pm)}^\mu = \left\{ \frac{u_\delta \cos^3 \xi}{(\sin \xi \mp 1)}, \frac{v_\delta \cos^3 \xi}{(\sin \xi \pm 1)}, -\sqrt{|u_\delta v_\delta|} \cos^2 \xi \right\}.$$

Firstly, we note from (15) that the positivity of entanglement density is correlated with null energy condition.

⁷It is easy to verify this statement explicitly say by considering ρ_Σ to be the thermal state.

While we have established the above result explicitly only for linear deviations away from the vacuum, the fact that \hat{n}_\pm vanishes in vacuum, and its positive semidefiniteness from SSA for any excited state, makes it natural for us to conjecture that the relation

$$\hat{n}_\pm = \frac{1}{8G_N} \int_{\mathcal{E}_A} d\xi \sqrt{\gamma_{\xi\xi}} (N_{(\pm)}^\mu N_{(\pm)}^\nu E_{\mu\nu}) \geq 0 \quad (16)$$

holds for any asymptotically AdS_3 backgrounds, with \mathcal{E}_A being the extremal surface (spacelike geodesic parametrized ξ) which holographically encodes S_A . We leave a more complete exploration of this relation for the future.

It is interesting to note that for normalizable states of pure gravity in AdS_3 , the entanglement density always vanishes. This is consistent with our earlier observation that entanglement density is vanishing for any state obtained by conformal transformations of ground states in two-dimensional CFTs. Indeed, solutions in the pure AdS_3 gravity can be obtained by bulk diffeomorphisms corresponding to boundary conformal transformations [30].

IV. DISCUSSION

In this paper we have introduced a new quantity, the entanglement density \hat{n} for relativistic field theories, and argued that it provides a useful encoding of certain aspects of gravitational dynamics via holography. We have directly argued for its positivity using the SSA property of EE in two-dimensional and three-dimensional field theories. More generally, we see from our explicit analysis that the positivity of \hat{n} and the gravitational null energy condition go hand in hand. At the same time, we anticipate (16) to be of fundamental importance, since it geometrically encodes the SSA and captures second-order variations of holographic entanglement entropy.

While our holographic analysis was carried out for linearized fluctuations around AdS_3 , we anticipate that (16) holds at the nonlinear level. In fact, it is tempting to conjecture a more general statement valid in any dimension: SSA implies that the entanglement density $\hat{n} \geq 0$ for any state of a QFT with $\hat{n}^{\text{vac}} = 0$. Furthermore, translating the description of \hat{n} into holography one finds that (3) holds for any deformation away from pure AdS in arbitrary spacetime dimensions. To wit,

$$\begin{aligned} \text{SSA} &\Rightarrow \hat{n}_\pm \geq 0, \quad \hat{n}_\pm^{\text{vac}} = 0, \\ &\Rightarrow \int_{\mathcal{E}_A} \epsilon N_{(\pm)}^\mu N_{(\pm)}^\nu E_{\mu\nu} \geq 0. \end{aligned} \quad (17)$$

One could try to follow the logic of Sec. II B to arrive at the conclusions above, by considering variations of the past tip of $D[\mathcal{A}]$ (cf. Fig. 3 with each point replaced by S^{d-3}). However, this attempt runs afoul of subleading divergences in the entanglement entropy from the rhs of (8) as explained in [16]. It is nevertheless interesting to contemplate whether the entanglement density can be used to provide further insight into c - and F -theorems and generalizations thereof.

Nevertheless, we may draw the following analogy based on the conjecture above: the statement of SSA is reminiscent of the second law of thermodynamics since it asserts convexity of entanglement (but under region variation as opposed to time variation). We are arguing that this guarantees positivity of the entanglement density. Via holography, generic deformations about the CFT vacuum (equilibrium) then increase the “cosmological Einstein tensor” $E_{\mu\nu}$ when suitably averaged over the extremal surface. In essence, this quantity codifies a version of the gravitational second law for entanglement density. Indeed, in the “long-wavelength” (hydrodynamic) regime, one may capture the thermal entropy production via the entanglement density by taking \mathcal{A} to be suitably large.

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