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RAMIFICATION ESTIMATE FOR FONTAINE-LAFFAILLE GALOIS MODULES

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ABSTRACT. Suppose K is unramified over $\mathbb{Q}_p$ and $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$. Let H be a torsion Γ_K-equivariant subquotient of crystalline $\mathbb{Q}_p[\Gamma_K]$-module with HT weights from $[0, p-2]$. We give a new proof of Fontaine's conjecture about the triviality of action of some ramification subgroups $\Gamma_K^{(v)}$ on H. The earlier author's proof from [1] contains a gap and proves this conjecture only for some subgroups of index p in $\Gamma_K^{(v)}$.

INTRODUCTION

Let $W(k)$ be the ring of Witt vectors with coefficients in a perfect field k of characteristic p. Consider the field $K = W(k)[1/p]$, choose its algebraic closure $\bar{K}$ and set $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$. Denote by $\mathbb{C}_p$ the completion of $\bar{K}$ and use the notation $O_{\mathbb{C}_p}$ for its valuation ring.

For $a \in \mathbb{Z}_{\geqslant 0}$, let $\mathrm{M}\Gamma^{cr}_{\mathbb{Q}_p}(a)$ be the category of crystalline $\mathbb{Q}_p[\Gamma_K]$-modules with Hodge-Tate weights from $[0, a]$. Define the full subcategory $\mathrm{M}\Gamma^{cr}_N(a)$ of the category of Γ_K-modules consisting of $H = H_1/H_2$, where H_1, H_2 are Γ_K-invariant lattices in $V \in \mathrm{M}\Gamma^{cr}_{\mathbb{Q}_p}(a)$ and $p^N H_1 \subset H_2 \subset H_1$. J.-M. Fontaine conjectured in [5] that the ramification subgroups $\Gamma_K^{(v)}$ act on $H \in \mathrm{M}\Gamma^{cr}_N(a)$ trivially if $v > N - 1 + a/(p-1)$. The author suggested in [1] a proof of this conjecture under the assumption $0 \leqslant a \leqslant p-2$.

It was pointed recently by Sh. Hattori to the author that the proof in [1] has a gap. More precisely, consider Fontaine's ring $R = \varprojlim_n (O_{\mathbb{C}_p}/p)_n$ where the projective limit is taken with respect to the maps induced by the p-power map in $\mathbb{C}_p$. Let W_N be the functor of Witt vectors of length N. For $r = (o_n \,\mathrm{mod}\, p)_{n\geqslant 0} \in R$ and $m \in \mathbb{Z}$, set $r^{(m)} = \lim_{n\to\infty} o_n^{p^{n+m}} \in O_{\mathbb{C}_p}$ and consider Fontaine's map $\gamma : W_N(R) \longrightarrow O_{\mathbb{C}_p}/p^N$, where $(r_0, \dots, r_{N-1}) \mapsto \sum_{0\leqslant i<N} p^i r_i^{(i)} \,\mathrm{mod}\, p^N$. Consider the projection $(\bar{o}_0, \dots, \bar{o}_N, \dots) \mapsto \bar{o}_N$ from R to $O_{\mathbb{C}_p}/p$ and denote the image of $\mathrm{Ker}\,\gamma$ in $W_N(O_{\mathbb{C}_p}/p)$ by $W_N^1(O_{\mathbb{C}_p}/p)$. This is principal ideal and in order to apply Fontaine's criterion about the triviality of the action of ramification subgroups from [5], we needed an element of $W_N(L)$, where L

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is a finite extension of K with "small" ramification, which generates $W_N^1(O_{\mathbb{C}_p}/p)$. Our "truncation" argument in [1] does not actually work: the resulting element does not belong to $W_N^1(O_{\mathbb{C}_p}/p)$. In the moment the author is inclined to believe that such an element does not exist if $N>1$. Nevertheless, our proof in [1] gives the Fontaine conjecture up to index p: the groups $\Gamma_K^{(v)}$ just should be replaced by the groups $\Gamma_K^{(v)}\cap\Gamma_{K(\zeta_{N+1})}$, where ζ_{N+1} is a primitive p^{N+1}-th root of unity.

The above difficulty appears in many other situations when we try to escape from "R-constructions" (e.g. $W(R)$, A_{cr}, etc) to p-adic constructions inside $\mathbb{C}_p$. In this paper we prove Fontaine's conjecture by applying methods from [2]. These methods were used earlier by the author to study ramification properties in the characteristic p case only. As a matter of fact, this is the first time when we applied them in the mixed characteristic situation.

Note also that if X is a smooth proper scheme over $W(k)$ then our result gives the ramification estimates for the Galois equivariant subquotients of the etale cohomology groups $H^a(X_{\bar K},\mathbb{Q}_p)$. Since the appearance of paper [1] there was a considerable progress in the study of the appropriate ramification estimates in the case of schemes X with semi-stable reduction modulo p [7, 3] but in our case (i.e. the case of schemes with good reduction modulo p) the situation remained unchanged.

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1. Construction of torsion crystalline representations

The ring R is perfect of characteristic p, it is provided with the valution v_R such that $v_R(r):=\lim\limits_{n\to\infty}p^nv_p(o_n)$, where $r=(o_n\bmod p)_{n\geqslant 0}$. With respect to v_R, R is complete and the field $R_0:=\mathrm{Frac}R$ is algebraically closed. Note that R and R_0 are provided with natural Γ_K-action. Denote by σ the Frobenius endomorphism of R and $W(R)$ and by m_R the maximal ideal of R.

1.1. Let $\mathcal{G}=\mathrm{Spf}\,W(k)[[X]]$ be the Lubin-Tate 1-dimensional formal group over $W(k)$ such that $p\mathrm{id}_{\mathcal{G}}(X)=pX+X^p$. Then $\mathrm{End}_{W(k)}\mathcal{G}=\mathbb{Z}_p$ and for any $l\in\mathbb{Z}_p$, $(l\mathrm{id}_{\mathcal{G}})(X)\equiv lX\bmod X^p$.

Fix $N\in\mathbb{N}$.

For $i\geqslant 0$, choose $o_i\in O_{\mathbb{C}_p}$ such that $o_0=0$, $o_1\neq 0$ and $p\mathrm{id}_{\mathcal{G}}(o_{i+1})=o_i$. Set $\tilde u=(o_{N+i}\bmod p)_{i\geqslant 0}\in R$. Then $\mathcal{K}:=k((\tilde u))$ is a complete discrete valuation closed subfield in R_0. If $\mathcal{K}_{sep}$ is the separable closure of $\mathcal{K}$ in R_0 then $\mathcal{K}_{sep}$ is separably closed and its completion coincides with R_0. The theory of the field-of-norms functor [9] identifies $\Gamma_{\mathcal{K}}$ with a closed subgroup in Γ_K. The quotient $\Gamma_K/\Gamma_{\mathcal{K}}$ acts strictly on $\mathcal{K}$. More precisely, there is a group epimorphism $\kappa:\Gamma_K\longrightarrow\mathrm{Aut}_{W(k)}\mathcal{G}\simeq\mathbb{Z}_p^*$ such that if $g \in \Gamma_K$ then $\kappa(g) \in \mathbb{Z}_p[[X]]$ and $\kappa(g)(X) \equiv \chi(g)X \bmod X^p$ with $\chi(g) \in \mathbb{Z}_p^*$. (Actually, $g \mapsto \chi(g)$ is the cyclotomic character.) With this notation we have $g(\tilde{u}) = \kappa(g)(\tilde{u})$.

Use the p-basis $\{\tilde{u}\}$ for separable extensions $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{K}_{sep}$ to construct the system of lifts $O_N(\mathcal{E})$ of $\mathcal{E}$ modulo p^N. Recall that $O_N(\mathcal{E}) = W_N(\sigma^{N-1}\mathcal{E})[u_N] \subset W_N(\mathcal{E})$ and $O_N(\mathcal{K}) = W_N(k)((u_N))$, where u_N is the Teichmuller representative of $\tilde{u}$ in $W_N(\mathcal{K})$. This construction essentially depends on a choice of p-basis in $\mathcal{K}$. If, say, $\{u'\}$ is another p-basis for $\mathcal{K}$ and $O'_N(\mathcal{E})$ are the appropriate lifts then $O_N(\mathcal{E})$ and $O'_N(\mathcal{E})$ are not very much different one from another: they can be related by the natural embeddings $\sigma^{N-1}O_N(\mathcal{E}) \subset W(\sigma^{N-1}\mathcal{E}) \subset O'_N(\mathcal{E})$. The lifts $O_N(\mathcal{E})$ are provided with the endomorphism σ such that $\sigma u_N = u_N^p$, and $O_N(\mathcal{K}_{sep})$ is provided with continuous $\Gamma_{\mathcal{K}}$-action.

If τ is a continuous automorphism of $\mathcal{E}$ then generally τ can't be lifted to an automorphism of $O_N(\mathcal{E})$ (but it can always be lifted to $W_N(\mathcal{E})$). In many cases it is sufficient to use "the lift" $\hat{\tau} : \sigma^{N-1}O_N(\mathcal{E}) \longrightarrow O_N(\mathcal{E})$ induced by $W_N(\tau) : W_N(\mathcal{E}) \longrightarrow W_N(\mathcal{E})$. In other words, $\hat{\tau}$ is defined only on a part of $O_N(\mathcal{E})$, but $\hat{\tau} \bmod p = \sigma^{N-1} \circ \tau : \sigma^{N-1}\mathcal{E} \longrightarrow \sigma^{N-1}\mathcal{E}$ and, therefore, τ can be uniquely recovered from the "lift" $\hat{\tau}$.

On the other hand, any continuous automorphism τ of $\mathcal{K} = k((\tilde{u}))$ can be lifted to an automorphism $\tau^{(N)}$ of $O_N(\mathcal{K}) = W_N(k)((u_N))$ (use that $u_N \bmod p = \tilde{u}$). Taking into account the existence of a lift τ_{sep} of τ to $\mathcal{K}_{sep}$ we obtain a lift $\tau_{sep}^{(N)}$ of τ to $O_N(\mathcal{K}_{sep}) = W_N(\sigma^{N-1}\mathcal{K}_{sep})[u_N]$.

Set $O_N^0 := O_N(\mathcal{K}_{sep}) \cap W_N(O_{sep})$ and $O_N^+ := O_N(\mathcal{K}_{sep}) \cap W_N(\mathrm{m}_{sep})$, where m_{sep} is the maximal ideal of the valuation ring O_{sep} of $\mathcal{K}_{sep}$. Then $\sigma(O_N^0) \subset O_N^0$, $\sigma(O_N^+) \subset O_N^+$ and $\bigcap_{n\geqslant 0}\sigma^n(O_N^+) = 0$. Note that $O_N^0(\mathcal{K}) := O_N^0 \cap O_N(\mathcal{K}) = W_N(k)[[u_N]]$, $O_N^+(\mathcal{K}) := O_N^+ \cap O_N(\mathcal{K}) = u_N W_N(k)[[u_N]]$ and $O_N(\mathcal{K}) = O_N^0(\mathcal{K})[u_N^{-1}] = W_N(k)((u_N))$.

For $0 \leqslant m \leqslant N$, introduce

$$u_m = (p^{N-m}\mathrm{id}_{\mathcal{G}})(u_N) \in O_N^0(\mathcal{K})$$

Then $u_0 = \sigma u_1 = pu_1 + u_1^p$, $t = u_0/u_1 = p + u_1^{p-1} \in O_N^0(\mathcal{K})$ and $u_0^{p-1} = t^p - pt^{p-1}$. As a matter of fact, u_0, u_1, t depend only on $\tilde{u}$. Indeed, if $u' \in W_N(R)$ and $u' \bmod pW_N(R) = \tilde{u}$ then in $O_N(\mathcal{K})$ we have $u_1 = (p^{N-1}\mathrm{id}_{\mathcal{G}})(u')$.

Lemma 1.1. *Suppose $g \in \Gamma_K$. Then*

a) $g(u_0) \equiv \chi(g)u_0 \bmod u_0^p O_N^0(\mathcal{K})$;

b) $\sigma(g(t)/t) \equiv 1 \bmod u_0^{p-1} O_N^0(\mathcal{K})$.

Proof. $g(u_1) = (p^{N-1}\mathrm{id}_{\mathcal{G}})(g(u_N)) = \kappa(g)(u_1) \equiv \chi(g)u_1 \bmod u_1^p O_N^0(\mathcal{K})$ implies a) because $\sigma(u_1) = u_0$. Then $g(t)/t \equiv 1 \bmod u_1^{p-1} O_N^0(\mathcal{K})$ and applying σ we obtain b). ∎

1.2. Let $\mathcal{MF}$ be the category of $W(k)$-modules M provided with decreasing filtration by $W(k)$-submodules $M=M^0\supset\dots\supset M^{p-1}\supset M^p=0$ and σ-linear morphisms $\varphi_i:M^i\longrightarrow M$ such that for all i, $\varphi_i|_{M^{i+1}}=p\varphi_{i+1}$.

For $0\leqslant a\leqslant p-2$, introduce the filtered module $\mathcal{S}_a$ such that

— $\mathcal{S}_a=O_N^0/u_0^aO_N^+$;

— for $0\leqslant i\leqslant a$, $\mathrm{Fil}^i\mathcal{S}_a=t^i\mathcal{S}_a$;

— $\varphi_i:\mathrm{Fil}^i\mathcal{S}_a\longrightarrow\mathcal{S}_a$ is σ-linear morphism such that $\varphi_i(t^i)=1$.

Clearly, $\mathcal{S}_a\in\mathcal{MF}$ (use that $\sigma t\equiv p\,\mathrm{mod}\,u_0^{p-1}$). In addition, Lemma 1.1 implies also that the action of Γ_K preserves the structure of an object of the category $\mathcal{MF}$ on $\mathcal{S}_a$.

For $0\leqslant a<p$, define the category of filtered Fontaine-Laffaille modules $\mathrm{MF}_N(a)$ as the full subcategory in $\mathcal{MF}$ consisting of modules M of finite length over $W_N(k)$ such that $M^{a+1}=0$ and $\sum\mathrm{Im}\varphi_i=M$. We can assume that M is given together with a functorial splitting of its filtration, i.e. there are submodules N_i in M such that for all i, $M^i=N_i\oplus M^{i+1}$.

Let $M\in\mathrm{MF}_N(a)$ and $\widetilde{U}_a(M)=\mathrm{Hom}_{\mathcal{MF}}(M,\mathcal{S}_a)$. Then the correspondence $M\mapsto\widetilde{U}_a(M)$ determines the functor $\widetilde{U}_a$ from $\mathrm{MF}_N(a)$ to the category of Γ_K-modules.

Proposition 1.2. *If $0\leqslant a\leqslant p-2$ and $H\in\mathrm{M}\Gamma_N^{cr}(a)$ then there is $M\in\mathrm{MF}_N(a)$ such that $\widetilde{U}_a(M)=H$.*

Proof. Recall briefly the main ingredients of the Fontaine-Laffaille theory [6]. The p^N-torsion crystalline ring $A_{cr.N}:=A_{cr}/p^N$ appears as the divided power envelope of $W_N(R)$ with respect to $\mathrm{Ker}\,\gamma$. We need the following construction of a generator of $\mathrm{Ker}\,\gamma$. (Note that we have a natural inclusion of $W(k)$-modules $O_N^0\subset W_N(R)$.)

Lemma 1.3. $\mathrm{Ker}\,\gamma=tW_N(R)$.

Proof. We have $\gamma(u_N)\equiv o_N\,\mathrm{mod}\,pO_{\mathbb{C}_p}$, therefore, $\gamma(u_0)\equiv 0\,\mathrm{mod}\,p^NO_{\mathbb{C}_p}$ and $t\in\mathrm{Ker}\gamma$. On the other hand, $t\equiv u_1^{p-1}\equiv[r]\,\mathrm{mod}\,pW(R)$, where $r\in R$ is such that $r^{(0)}\equiv o_1^{p-1}\equiv -p\,\mathrm{mod}\,p^{p/(p-1)}O_{\mathbb{C}_p}$. Therefore, $v_p(r^{(0)})=1$ and t generates $\mathrm{Ker}\gamma$, cf. [6]. □

By above Lemma, $A_{cr,N}=W_N(R)[\{\gamma_i(t)\mid i\geqslant 1\}]$, where $\gamma_i(t)$ are the i-th divided powers of t. Then the identity $\gamma_p(t)=t^{p-1}+u_0^{p-1}/p$ implies that $A_{cr,N}=W_N(R)[\{\gamma_i(u_0^{p-1}/p)\mid i\geqslant 1\}]$.

Recall that $A_{cr,N}\in\mathcal{MF}$ with:

— the filtration $\mathrm{Fil}^iA_{cr,N}$, $0\leqslant i<p$, generated as ideal by t^i and all $\gamma_j(u_0^{p-1}/p)$, $j\geqslant 1$;

— the σ-linear morphisms $\varphi_i : \mathrm{Fil}^i A_{cr,N} \longrightarrow A_{cr,N}$ (which come from σ/p^i on A_{cr}) such that $\varphi_i(t^i) = (1+u_0^{p-1}/p)^i$ and $\varphi_i(u_0^{p-1}/p) = p^{p-1-i}(u_0^{p-1}/p)(1+u_0^{p-1}/p)^{p-1}$.

Then the Fontaine-Laffaille functor U_a attaches to $M \in \mathrm{MF}_N(a)$ the Γ_K-module $\mathrm{Hom}_{\mathcal{MF}}(M, A_{cr,N})$. This functor is fully-faithful (we assume that $a \leqslant p-2$) and, therefore, there is $M \in \mathrm{MF}_N(a)$ such that $U_a(M) = H$.

Consider the $W(k)$-module $\mathcal{W}_N^a = W_N(R)/u_0^a W_N(\mathrm{m}_R)$ with the filtration induced by the filtration $W_N^i(R) = t^i W_N(R)$ and σ-linear morphisms φ_i such that $\varphi_i(t^i) = 1$. Prove that we have an identification of Γ_K-modules $H = \mathrm{Hom}_{\mathcal{MF}}(M, \mathcal{W}_N^a)$.

Indeed, let T_a be the maximal element in the family of all ideals I of $A_{cr,N}$ such that φ_a induces a nilpotent endomorphism of I. Then for any $M \in \mathrm{MF}_N(a)$, $U_a(M) = \mathrm{Hom}_{\mathcal{MF}}(M, A_{cr,N}/T_a)$. By straightforward calculations we can see that T_a is generated by the elements of $u_0^a W_N(\mathrm{m}_R)$ and all $\gamma_j(u_0^{p-1}/p)$, $j \geqslant 1$. It remains to note that we have a natural identification $A_{cr,N}/T_a = \mathcal{W}_N^a$ in the category $\mathcal{MF}$.

Consider the natural embedding $O_N^0 \longrightarrow W_N(R)$ and the induced natural map $\iota_a : \mathcal{S}_a \longrightarrow \mathcal{W}_N^a$ in $\mathcal{MF}$. Prove that $\iota_{a*} : \widetilde{U}_a(M) \to H$ is isomorphism of Γ_K-modules.

Choose $W(k)$-submodules N_i in M^i such that $M^i = N_i \oplus M^{i+1}$ and choose vectors $\bar{n}_i$ whose coordinates give a minimal system of generators of N_i. Then the structure of M can be given by the matrix relation $(\varphi_a(\bar{n}_a), \dots, \varphi_0(\bar{n}_0)) = (\bar{n}_a, \dots, \bar{n}_0)C$, where C is an invertible matrix with coefficients in $W(k)$. The elements of H are identified with the residues $(\bar{u}_a, \dots, \bar{u}_0) \bmod u_0^a W_N(\mathrm{m}_{\mathrm{sep}})$ where the vectors $(\bar{u}_a, \dots, \bar{u}_0)$ have coefficients in $W_N(\mathcal{K}_{sep})$ and satisfy the following system of equations (use that φ_a is topologically nilpotent on $u_N^a W_N(\mathrm{m}_{sep})$)

$$\left(\frac{\sigma\bar{u}_a}{\sigma t^a}, \dots, \frac{\sigma\bar{u}_i}{\sigma t^i}, \dots, \sigma(\bar{u}_0)\right) = (\bar{u}_a, \dots, \bar{u}_0)C$$

In particular, if $\bar{u} = (\bar{u}_a, \dots, \bar{u}_0)$ then there is an invertible matrix D with coefficients in $O_N(\mathcal{K})$ such that

$$\sigma(\bar{u})D = \bar{u}\,. \tag{1.1}$$

We know that all coordinates of $\sigma^{N-1}\bar{u}$ belong to $\sigma^{N-1}W_N(\mathcal{K}_{sep}) \subset O_N(\mathcal{K}_{sep})$. Then (1.1) implies step-by-step that the vectors $\sigma^{N-2}\bar{u}, \dots, \bar{u}$ have coordinates in $O_N(\mathcal{K}_{sep})$. It remains to note that $O_N^0 = O_N(\mathcal{K}_{sep}) \cap W_N(O_{sep})$ and $O_N^+ = O_N(\mathcal{K}_{sep}) \cap W_N(\mathrm{m}_{sep})$. The proposition is proved. $\square$

2. Reformulation of the Fontaine conjecture

2.1. Review of ramification theory. Let $\mathcal{I}_{\mathcal{K}}$ be the group of all continuous automorphisms of $\mathcal{K}_{sep}$ which keep invariant the residue field of $\mathcal{K}_{sep}$ and preserve the extension of the normalised valuation $v_{\mathcal{K}}$

of $\mathcal{K}$ to $\mathcal{K}_{sep}$. This group has a decreasing filtration by its ramification subgroups $\mathcal{I}_{\mathcal{K}}^{(v)}$ in upper numbering $v\geqslant 0$. Recall basic ingredients of the definition of this filtration following the papers [4, 9, 10].

For any field extension $\mathcal{E}$ of $\mathcal{K}$ in $\mathcal{K}_{sep}$, set $\mathcal{E}_{sep}=\mathcal{K}_{sep}$, in particular, $\mathcal{I}_{\mathcal{E}}=\mathcal{I}_{\mathcal{K}}$. All elements of $\mathcal{I}_{\mathcal{K}}$ preserve the extension $v_{\mathcal{E}}$ of the normalised valuation on $\mathcal{E}$ to $\mathcal{K}_{sep}$.

For $x\geqslant 0$, set $\mathcal{I}_{\mathcal{E},x}=\{\iota\in\mathcal{I}_{\mathcal{E}}\mid v_{\mathcal{E}}(\iota(a)-a)\geqslant 1+x\ \ \forall a\in \mathrm{m}_{\mathcal{E}}\}$, where $\mathrm{m}_{\mathcal{E}}$ is the maximal ideal in $O_{\mathcal{E}}$.

Denote by $\mathcal{I}_{\mathcal{E}/\mathcal{K}}$ the set of all continuous embeddings of $\mathcal{E}$ into $\mathcal{K}_{sep}$ which induce the identity map on $\mathcal{K}$ and the residue field $k_{\mathcal{E}}$ of $\mathcal{E}$. For $x\geqslant 0$, set $\mathcal{I}_{\mathcal{E}/\mathcal{K},x}=\mathcal{I}_{\mathcal{E},x}\bigcap\mathcal{I}_{\mathcal{E}/\mathcal{K}}$.

If $\iota_1,\iota_2\in\mathcal{I}_{\mathcal{E}/\mathcal{K}}$ and $x\geqslant 0$ then ι_1 and ι_2 are *x-equivalent* iff for any $a\in\mathrm{m}_{\mathcal{E}}$, $v_{\mathcal{E}}(\iota_1(a)-\iota_2(a))\geqslant 1+x$. Denote by $(\mathcal{I}_{\mathcal{E}/\mathcal{K}}:\mathcal{I}_{\mathcal{E}/\mathcal{K},x})$ the number of x-equivalent classes in $\mathcal{I}_{\mathcal{E}/\mathcal{K}}$. Then the Herbrand function $\varphi_{\mathcal{E}/\mathcal{K}}$ can be defined for all $x\geqslant 0$, as

$$\varphi_{\mathcal{E}/\mathcal{K}}(x)=\int_0^x(\mathcal{I}_{\mathcal{E}/\mathcal{K}}:\mathcal{I}_{\mathcal{E}/\mathcal{K},x})^{-1}dx\,.$$

This function has the following properties:

- $\varphi_{\mathcal{E}/\mathcal{K}}$ is a piece-wise linear function with finitely many edges;
- if $\mathcal{K}\subset\mathcal{E}\subset\mathcal{H}$ is a tower of finite field extensions in $\mathcal{K}_{sep}$ then for any $x\geqslant 0$, $\varphi_{\mathcal{H}/\mathcal{K}}(x)=\varphi_{\mathcal{E}/\mathcal{K}}(\varphi_{\mathcal{H}/\mathcal{E}}(x))$.

The ramification filtration $\{\mathcal{I}_{\mathcal{K}}^{(v)}\}_{v\geqslant 0}$ appears now as a decreasing sequence of the subgroups $\mathcal{I}_{\mathcal{K}}^{(v)}$ of $\mathcal{I}_{\mathcal{K}}$, where $\mathcal{I}_{\mathcal{K}}^{(v)}$ consists of $\iota\in\mathcal{I}_{\mathcal{K}}$ such that for any finite extension $\mathcal{E}$ of $\mathcal{K}$, $\iota\in\mathcal{I}_{\mathcal{E},v_{\mathcal{E}}}$ with $\varphi_{\mathcal{E}/\mathcal{K}}(v_{\mathcal{E}})=v$.

If we replace the lower indices $\mathcal{K}$ to $\mathcal{E}$, the ramification filtration $\{\mathcal{I}_{\mathcal{K}}^{(v)}\}_{v\geqslant 0}$ is not changed as a whole, just only individual subgroups change their upper indices, that is $\mathcal{I}_{\mathcal{K}}^{(v)}=\mathcal{I}_{\mathcal{E}}^{(v_{\mathcal{E}})}$.

Note that the inertia subgroup $\Gamma_{\mathcal{E}}^0$ of $\Gamma_{\mathcal{E}}=\mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{E})$ is a subgroup in $\mathcal{I}_{\mathcal{E}}$ and for any $v\geqslant 0$, the appropriate subgroup $\Gamma_{\mathcal{E}}^{(v)}=\Gamma_{\mathcal{E}}\cap\mathcal{I}_{\mathcal{E}}^{(v)}$ is just the ramification subgroup of $\Gamma_{\mathcal{E}}$ with the upper number v from [8].

2.2. Statement of the main theorem. The main idea of our approach to the Γ_K-modules $\widetilde{U}_a(M)$ is related to the following fact. The filtered module $\mathcal{S}_a$ depends only on the field $\mathcal{K}$ and its uniformizer $\tilde{u}$. Therefore, $\mathcal{S}_a$ can be identified with its analogue $\mathcal{S}'_a$ constructed for any ramified extension $\mathcal{K}'$ of $\mathcal{K}$ together with its uniformizer $\tilde{u}'$. The whole group $\mathcal{I}_{\mathcal{K}}$ does not preserve the structure of $\mathcal{S}_a$ but the ramification subgroups $\mathcal{I}_{\mathcal{K}}^{(v)}$, where $a>a_N^*:=(a+1)p^{N-1}-1$ do preserve this structure because of the following proposition.

Proposition 2.1. *If $v>a_N^*$ and $M\in\mathrm{MF}_N(a)$ then a natural action of $\mathcal{I}_{\mathcal{K}}$ on $W_N(\mathcal{K}_{sep})$ induces the $\mathcal{I}_{\mathcal{K}}^{(v)}$-module structure on $\widetilde{U}_a(M)$.*

Proof. All we need is just the following lemma. □

Lemma 2.2. *If $\tau \in \mathcal{I}_{\mathcal{K}}^{(v)}$ with $v > a_N^*$ then*

a) $\tau(u_0)/u_0 \in O_N^*(\mathcal{K}_{sep})$;

b) *for* $0 \leqslant i \leqslant a$, $\varphi_i(\tau t^i) = 1$.

Proof of Lemma. For $\tau \in \mathcal{I}_{\mathcal{K}}^{(v)}$, we have $\tau(u_N) = u_N + \eta_N + pw$, where $\eta_N \in u_1^{a+1} O_N^+$ and $w \in W_N(\mathcal{K}_{sep})$. For $1 \leqslant i \leqslant N$, this implies

$$\tau(u_i) = u_i + \eta_i + p^{N-i+1} w_i\,,$$

where $\eta_i \in u_1^{a+1} O_N^+$ and $w_i \in W_N(\mathcal{K}_{sep})$. Therefore,

$$\tau(u_1) \equiv u_1 \operatorname{mod} u_1^{a+1} O_N^+\,.$$

This implies part a) because $\tau(u_0) \equiv u_0 \operatorname{mod} u_0^{a+1} O_N^+$ and part b) because $\sigma(\tau t)/\sigma(t) \equiv 1 \operatorname{mod} u_0^a O_N^+$. □

With the relation to the original problem of estimating the upper ramification numbers of the Γ_K-module H notice now that $\mathcal{K} = k((\tilde{u}))$ coincides with $\sigma^{-N}\mathcal{K}_0$, where $\mathcal{K}_0$ is the field-of-norms of the p-cyclotomic extension $\widetilde{K}$ of K. Then for any $v \geqslant 0$, $\Gamma_K^{(v)} = \Gamma_K \cap \mathcal{I}_{\mathcal{K}}^{(v^*)}$, where $\varphi_{\widetilde{K}/K}(v^*) = v$. In particular, $v > N - 1 + a/(p-1)$ if and only if $v^* > a_N^*$.

So, the proof of Fontaine's conjecture is reduced to the proof of the following theorem stated exclusively in terms of the field $\mathcal{K}$ of characteristic p.

Theorem 2.3. *For any $v > a_N^*$, the group $\mathcal{I}_{\mathcal{K}}^{(v)}$ acts trivially on $\widetilde{U}_a(M)$.*

3. Proof of Theorem 2.3

3.1. Auxiliary field $\mathcal{K}'$. Let $N^* \in \mathbb{N}$ and $r^* \in \mathbb{Q}$ be such that for $q := p^{N^*}$, $r^*(q-1) := b^* \in \mathbb{N}$ and $v_p(b^*) = 0$.

Consider the field $\mathcal{K}' = \mathcal{K}(N^*, r^*)$ from [2]. Remind that

— $[\mathcal{K}' : \mathcal{K}] = q$;

— $\mathcal{K}' = k((\tilde{u}'))$, where $\tilde{u} = \tilde{u}'^q E(\tilde{u}'^{b^*})^{-1}$ (here E is the Artin-Hasse exponential);

— the Herbrand function $\varphi_{\mathcal{K}'/\mathcal{K}}$ has only one edge point (r^*, r^*). (In particular, $\varphi_{\mathcal{K}'/\mathcal{K}}(x) < x$ for all $x > r^*$.)

For $\mathcal{K}'$ and its above uniformiser $\tilde{u}'$ proceed as earlier to construct the lifts $O'_N(\mathcal{K}')$ and $O'_N(\mathcal{K}_{sep})$ obtained with respect to the p-basis $\tilde{u}'$. Introduce similarly the modules O'^0_N, O'^+_N, the elements $u'_0, t' \in O_N(\mathcal{K}')$ and the filtered module $\mathcal{S}'_a$.

3.2. Compare the old and the new lifts using their canonical embeddings into $W_N(\mathcal{K}_{sep})$. Note that u_N is not generally an element of $O'_N(\mathcal{K}')$ because the Teichmuller representative $u_N=[\tilde u]$ can't be written as a power series in $u'_N=[\tilde u']$ if $N>1$. However, we can easily see that for $1\leqslant i<N$, $u_{N-i}\in O'_N(\mathcal{K}')\operatorname{mod} p^{i+1}W_N(\mathcal{K}')$. In particular, $u_1,u_0,t\in O'_N(\mathcal{K}')$.

Proposition 3.1. *If $\xi\in\widetilde U_a(M)$ then for any $m\in M$, $\xi(m)\in O'_N(\mathcal{K}_{sep})$.*

Proof. Proceed as we proceeded at the end of Section 1. Then the vectors $(\xi(\bar n_a),\dots,\xi(\bar n_0))$ appear in the form $\bar\xi\operatorname{mod} u_0^aO_N^+$, where $\bar\xi$ is a vector with coefficients in $O_N(\mathcal{K}_{sep})$ such that

$$\sigma(\bar\xi)D=\bar\xi\,, \tag{3.1}$$

and the matrix D has coefficients in $O'_N(\mathcal{K}')$ (use that $t\in O'_N(\mathcal{K}')$). We know that all coordinates of $\sigma^{N-1}\bar\xi$ belong to $\sigma^{N-1}O_N(\mathcal{K}_{sep})\subset O'_N(\mathcal{K}_{sep})$. Then (3.1) implies step-by-step that the vectors $\sigma^{N-2}\bar\xi,\dots,\bar\xi$ have coordinates in $O'_N(\mathcal{K}_{sep})$. □

3.3. Now suppose $v^*\geqslant a_N^*$, $\mathcal{I}_{\mathcal{K}}^{(v)}$ acts trivially on $\widetilde U_a(M)$ for all $v>v^*$ and v^* is the minimal with this property. The existence of v^* follows from the left-continuity of the ramification filtration with respect to the upper numbering.

If $v^*=a_N^*$ then our theorem is proved.

Suppose that $v^*>a_N^*$. Choose the parameters r^* and N^* from Subsection 3.1 such that $a_N^*q/(q-1)<r^*<v^*$.

For any $\alpha\in O'_N(\mathcal{K}_{sep})$, set $\alpha^{(q)}=\sigma^{N^*}\alpha$.

Lemma 3.2. $u_1/u_1'^{(q)}\equiv 1\operatorname{mod} u_1'^{(q)a}O_N'^{+}(\mathcal{K}')$.

Proof. Consider $b^*=r^*(q-1)\in\mathbb{N}$ from Subsection 3.1. Then $b^*+q>q(a_N^*+1)=q(a+1)p^{N-1}$ and

$$u_N\equiv u_N'^{(q)}\operatorname{mod}\left(u_1'^{(q)\,a+1}O_N^+(\mathcal{K}')+pO_N(\mathcal{K}')\right)$$

This implies $u_1\equiv u_1'^{(q)}\operatorname{mod} u_1'^{(q)\,a+1}O_N'^{+}(\mathcal{K}')$ and the lemma is proved. □

Corollary 3.3. a) *$u_0/u_0'^{(q)}$ is invertible in $O_N'^{0}(\mathcal{K}')$;*

b) $\sigma(t/t'^{(q)})\equiv 1\operatorname{mod} u_0'^{(q)\,a}O_N'^{+}(\mathcal{K}')$.

3.4. $\mathcal{I}_{\mathcal{K}'}^{(v^*)}$-action. Introduce the filtered module $\mathcal{S}_a'^{(q)}$ as follows.

— $\mathcal{S}_a'^{(q)}=O_N'^{0}/u_0'^{(q)\,a}O_N'^{+}$;

— for $0\leqslant i\leqslant a$, $\mathrm{Fil}^i\mathcal{S}_a'^{(q)}=t'^{(q)\,i}\mathcal{S}_a'^{(q)}$;

— $\varphi_i'^{(q)}:\mathrm{Fil}^i\mathcal{S}_a'^{(q)}\longrightarrow\mathcal{S}_a'^{(q)}$ is σ-linear such that $\varphi_i'^{(q)}(t'^{(q)\,i})=1$.

Suppose $M'\in \mathrm{MF}_N(a)$ is given similarly to M by the relation

$$(\varphi_a(\bar n_a),\dots,\varphi_0(\bar n_0))=(\bar n_a,\dots,\bar n_0)\sigma^{-N^*}C$$

Then we can use σ^{N^*} to identify the modules $\widetilde{U}'_a(M'):=\mathrm{Hom}_{\mathcal{MF}}(M',\mathcal{S}'_a)$ and $\widetilde{U}_a'^{(q)}(M):=\mathrm{Hom}_{\mathcal{MF}}(M,\mathcal{S}_a'^{(q)})$. This identification is compatible with the action of the subgroups $\mathcal{I}_{\mathcal{K}'}^{(v)}$, where $v>a_N^*$.

Note that the fields $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic (as any two fields of formal power series with the same residue field). Choose an isomorphism $\kappa:\mathcal{K}\longrightarrow\mathcal{K}'$ such that $\kappa(\tilde u)=\tilde u'$ and $\kappa|_k=\sigma^{-N^*}$. We can extend κ to an isomorphism of separable closures of $\mathcal{K}$ and $\mathcal{K}'$. This allows us to identify the groups $\mathcal{I}_{\mathcal{K}}$ and $\mathcal{I}_{\mathcal{K}'}$ and this identification is compatible with the appropriate ramification filtrations. Even more, we obtain an identification of $\widetilde{U}_a(M)$ with $\widetilde{U}'_a(M')$ and this identification respects the action of $\mathcal{I}_{\mathcal{K}}^{(v)}$ on $\widetilde{U}_a(M)$ and the action of $\mathcal{I}_{\mathcal{K}'}^{(v)}$ on $\widetilde{U}'_a(M')$ for any $v>a_N^*$. Therefore, v^* is the maximal number such that $\mathcal{I}_{\mathcal{K}'}^{(v^*)}$ acts non-trivially on $\widetilde{U}'_a(M')$ and

- *v^* is the maximal such that $\mathcal{I}_{\mathcal{K}'}^{(v^*)}$ acts non-trivially on $\widetilde{U}_a'^{(q)}(M)$.*

3.5. $\mathcal{I}_{\mathcal{K}}^{(v^*)}$-action. Introduce the filtered module $\mathcal{S}_a^\star$ as follows:

— $\mathcal{S}_a^\star=O_N^0\cap O'_N(\mathcal{K}_{sep})/u_0^aO_N^+\cap O'_N(\mathcal{K}_{sep})$;

— $\mathrm{Fil}^i\mathcal{S}_a^\star=t^i\mathcal{S}_a\cap\mathcal{S}_a^\star$;

— $\varphi_i^\star=\varphi_i|_{\mathrm{Fil}^i\mathcal{S}_a^\star}:\mathrm{Fil}^i\mathcal{S}_a^\star\longrightarrow\mathcal{S}_a^\star$.

The results from Subsection 3.2 allow us to identify $\widetilde{U}_a(M)$ with $U_a^\star(M)=\mathrm{Hom}_{\mathcal{MF}}(M,\mathcal{S}_a^\star)$. By the results from Subsection 3.3, there is a natural embedding of filtered modules $\mathcal{S}_a^\star\longrightarrow\mathcal{S}_a'^{(q)}$ and, therefore, we can identify $\widetilde{U}_a(M)$ with $\widetilde{U}_a'^{(q)}(M')$. This identification is compatible with the action of ramification subgroups $\mathcal{I}_{\mathcal{K}}^{(v)}$ for all $v>a_N^*$. So,

- *v^* is the maximal such that $\mathcal{I}_{\mathcal{K}}^{(v^*)}$ acts non-trivially on $\widetilde{U}_a'^{(q)}(M)$.*

3.6. The end of proof of Theorem. It remains to notice that $\mathcal{I}_{\mathcal{K}'}^{(v^*)}=\mathcal{I}_{\mathcal{K}}^{(v_0^*)}$, where $v_0^*=\varphi_{\mathcal{K}'/\mathcal{K}}(v^*)<v^*$.

The contradiction.

References

[1] V. Abrashkin, *Ramification in etale cohomology*, Invent. math. (1990) **101**, 631–640

[2] V.Abrashkin, *Ramification filtration of the Galois group of a local field. III*, Izvestiya RAN: Ser. Mat., **62**, no.5 (1998), 3-48; English transl. Izvestiya: Mathematics **62**, no.5, 857–900

[3] X. Caruso, T. Liu, *Some bounds for ramification of p-torsion semi-stable representations*, J. Algebra **325** (2011), 70-96

[4] P.Deligne *Les corps locaux de caractéristique p, limites de corps locaux de caractéristique* 0, Representations of reductive groups over a local field, Travaux en cours, Hermann, Paris, 1973, 119-157

[5] J.-M. Fontaine, *Il n'y a pas de variété abelienne sur* $\mathbb{Z}$, Invent. math. (1990) **101**, 631–640

[6] J.-M. Fontaine, G. Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup., 4 Ser. **15** (1982), 547–608

[7] Sh. Hattori, *On a ramification bound of torsion semi-stable representations over a local field*, J. Number Theory **129** (2009), no. 10, 2474-2503

[8] J.-P.Serre, *Local Fields* Berlin, New York: Springer-Verlag, 1980

[9] J.-P. Wintenberger, *Le corps des normes de certaines extensions infinies des corps locaux; application* Ann. Sci. Ec. Norm. Super., IV. Ser, **16** (1983), 59–89

[10] J.-P.Wintenberger, *Extensions de Lie et groupes d'automorphismes des corps locaux de caractéristique p.* (French) C. R. Acad. Sci. Paris Sér. **A-B** **288** (1979), no. 9, A477-A479

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