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Symmetric instantons and discrete Hitchin equations

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Self-dual Yang–Mills instantons on $\mathbb{R}^4$ correspond to algebraic ADHM data. The ADHM equations for S^1-symmetric instantons give a one-dimensional integrable lattice system, which may be viewed as an discretization of the Nahm equations. In this note, we see that generalized ADHM data for T^2-symmetric instantons give an integrable two-dimensional lattice system, which may be viewed as a discrete version of the Hitchin equations.

Keywords: discrete integrable systems; Yang–Mills instantons; Hitchin equations.

1. Introduction

The prototype for the idea of this paper is the well-known correspondence between S^1-symmetric instantons (or hyperbolic BPS monopoles) and the discrete Nahm equation. Recall that self-dual Yang–Mills instantons on $\mathbb{R}^4$ correspond to ADHM data [1], which consist of matrices satisfying certain algebraic constraints. If we impose an S^1 symmetry on the instantons, then the corresponding dimensional reduction gives hyperbolic monopoles [2], in other words BPS monopoles on hyperbolic three-space $\mathbb{H}^3$. Such an S^1 action is classified by a positive integer n; and then the monopole mass, or equivalently the asymptotic norm of the monopole Higgs field, is $n/2$. For a given value of n, SU(2) hyperbolic monopoles of charge k are the same as SU(2) instantons of charge nk. With suitable scaling, the $n \to \infty$ limit corresponds to the curvature of the hyperbolic space tending to zero; in other words the hyperbolic monopole tends to a monopole on $\mathbb{R}^3$. Now BPS monopoles on $\mathbb{R}^3$ correspond, via the Nahm transform [3], to solutions of the Nahm equation, which is a set of ordinary differential equations on an interval of the real line. So one might expect the S^1-symmetric ADHM constraints to be a discrete (lattice) version of the Nahm equation, tending to it as $n \to \infty$; and this is exactly what happens [4]. This discrete Nahm equation, which is a special case of the algebraic ADHM constraints, forms an integrable one-dimensional lattice system [5, 6].

The subject of the present paper is to extend this idea to the case where there are two commuting circle symmetries rather than just one. So the starting-point is T^2-symmetric Yang–Mills instantons on $\mathbb{R}^4$, and the corresponding T^2-symmetric ADHM data. Such a T^2-action is characterized by a pair of positive integers n_1 and n_2. One special case which has been known for a long time is where $n_1 = 1$ or $n_2 = 1$: this corresponds to spherically symmetric hyperbolic monopoles of unit charge [7, 8]. The case of general (n_1, n_2) was studied shortly afterwards [9, 10, 11], mainly in the context of instantons invariant under a finite cyclic group $\mathbb{Z}_p$. The relevant expressions for such T^2-symmetric ADHM data are reviewed in Section 2 below. In Section 3, however, we forget about T^2-symmetric instantons as such, and focus on the constraint equations for the ADHM data. It turns out that these may be interpreted as a two-dimensional lattice version of the Hitchin equations [12, 13], a gauge-theory system on $\mathbb{R}^2$ (or more generally on Riemann surfaces). This lattice system is completely integrable, in the sense of being the compatibility condition for a Lax pair of lattice operators, and it tends to the Hitchin system of partial differential equations as $n_1, n_2 \to \infty$.

2. T^2-Symmetric instantons

The structure of the ADHM data for T^2-symmetric SU(2) instantons was described in [9, 10], using the version of the ADHM constraint equations due to Donaldson [14]. This section summarizes the relevant aspects, in a form suitable for our purposes here.

The data for SU(2) instantons of charge N consist [14] of four complex matrices $(\alpha_1, \alpha_2, a, b)$, where the α_i are $N \times N$, a is $2 \times N$, and b is $N \times 2$. They satisfy the equation

$$[\alpha_1, \alpha_2] + ba = 0, \tag{1}$$

and are required to be generic (i.e. to satisfy a maximal-rank condition). The 'gauge freedom' in these data is

$$\alpha_i \mapsto p\alpha_i p^{-1},\ a \mapsto qap^{-1},\ b \mapsto pbq^{-1}, \tag{2}$$

where $p \in \mathrm{GL}(N, \mathbb{C})$ and $q \in \mathrm{SU}(2)$. The $(8N-3)$-dimensional moduli space of instantons is the space of generic solutions of (1), factored out by (2). To convert $(\alpha_1, \alpha_2, a, b)$ into ADHM data, one also needs to solve

$$[\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a = 0, \tag{3}$$

which has an essentially unique solution [14]. Here b^* denotes the complex-conjugate transpose of b.

Now the standard action of SO(4) on $\mathbb{R}^4$ induces an action on the space of instantons, and we are interested in instantons which are invariant under the action of the maximal torus $T^2 = S^1 \times S^1$ in SO(4). Such T^2-symmetric instantons are classified by a pair (n_1, n_2) of positive integers, with $n_1 n_2 = N$. For each choice of (n_1, n_2), the solution space is one-dimensional [9, 10]. From the instanton point of view this is because, given the imposed symmetry, the only remaining free parameter is the instanton scale. This in turn corresponds to an overall positive factor on the data $(\alpha_1, \alpha_2, a, b)$. Alternatively, thinking in terms of hyperbolic monopoles, we have a rotationally symmetric hyperbolic monopole with a fixed axis of symmetry, and the free parameter is then the location of the monopole on its axis. The hyperbolic monopole has mass $n_1/2$ and charge n_2. (From the instanton point of view, this is the same solution as a hyperbolic monopole of mass $n_2/2$ and charge n_1: the swap-map $n_1 \leftrightarrow n_2$ has recently been used in the study of symmetric hyperbolic monopoles [15].)

The corresponding T^2-symmetric ADHM-Donaldson data may be written in the following form (cf. [9]). For a positive integer n, define $n \times n$ matrices E_j and E_j^- by

$$E_1 = \mathrm{diag}(1, 0, \ldots, 0), \ldots, E_n = \mathrm{diag}(0, \ldots, 0, 1),$$
$$E_1^- = \mathrm{diag}_{-1}(1, 0, \ldots, 0), \ldots, E_{n-1}^- = \mathrm{diag}_{-1}(0, \ldots, 0, 1).$$

Then set

$$\alpha_1 = \sum_{j=1}^{n_2} \sum_{k=1}^{n_1-1} F_{j,k}\, E_j \otimes E_k^-, \quad \alpha_2 = \sum_{j=1}^{n_2-1} \sum_{k=1}^{n_1} G_{j,k}\, E_j^- \otimes E_k, \tag{4}$$

$$a = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & a_0 \end{bmatrix}, \quad b = \begin{bmatrix} b_0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}^t. \tag{5}$$

Here $A \otimes B$ denotes the Kronecker product of an $n_2 \times n_2$ matrix A and an $n_1 \times n_1$ matrix B. The variables $F_{j,k}$ (for $1 \le j \le n_2$, $1 \le k \le n_1 - 1$), $G_{j,k}$ (for $1 \le j \le n_2 - 1$, $1 \le k \le n_1$), a_0 and b_0 are all positive real numbers. Then the equations (1) and (3) become, respectively,

$$F_{j+1,k}\, G_{j,k} = G_{j,k+1}\, F_{j,k} \text{ for } 1 \le j \le n_2 - 1,\ 1 \le k \le n_1 - 1, \tag{6}$$

$$\begin{aligned} &F_{j,k-1}\, F_{j,k-1}^* - F_{j,k}^*\, F_{j,k} + G_{j-1,k}\, G_{j-1,k}^* - G_{j,k}^*\, G_{j,k} \\ &\quad - a_0^* a_0\, \delta(j - n_2)\delta(k - n_1) + b_0 b_0^*\, \delta(j-1)\delta(k-1) = 0. \end{aligned} \tag{7}$$

In equation (7), the indices have the range $1 \le j \le n_2$, $1 \le k \le n_1$, but undefined terms are to be omitted; for example, if $k = 1$ then the first term is omitted, since there is no variable $F_{j,0}$. The ‘stars’ are unnecessary in (7), since the variables are real numbers; but we retain them because the variables will become complex matrices in the next section.

The system (6, 7) consists of $2n_1n_2 - n_1 - n_2 + 1$ equations for $2n_1n_2 - n_1 - n_2 + 2$ variables, and has a one-parameter family of solutions, as mentioned previously. For example, in the case $n_1 = n_2 = 2$ we get five equations for six variables, and the solution is $F_{11} = F_{21} = G_{11} = G_{12} = \lambda$, $a_0 = b_0 = \lambda\sqrt{2}$, with $\lambda > 0$ arbitrary. The case $n_1 = 2$, $n_2 = 3$ is mentioned as an example in [9]. For $n_1 = 2$, $n_2 = 4$ the solution is

$$F_{11} = F_{41} = G_{31} = G_{12} = \lambda\sqrt{2}, \quad F_{21} = F_{31} = \lambda,$$

$$G_{11} = G_{32} = 2\lambda, \quad G_{21} = G_{22} = \lambda\sqrt{3}, \quad a_0 = b_0 = \lambda\sqrt{6}.$$

Several other cases can be solved explicitly, but for general (n_1, n_2) the solution is not known explicitly (and may not be expressible in radicals). A numerical solution for the case $n_1 = n_2 = 50$ is illustrated in the next section.

Finally in this section, let us consider the limit $n_2 = n \to \infty$, with $n_1 = 2$ fixed. One way of interpreting this is as an axially symmetric hyperbolic monopole of charge n on a fixed hyperbolic space, letting $n \to \infty$ to obtain a hyperbolic magnetic disc; such a limit was recently described in detail in [16] for the case $n_1 = 1$. Alternatively, we may think of an axially symmetric two-monopole on a hyperbolic space with curvature $-1/n^2$: then the limit $n \to \infty$ should yield the Nahm data for the axially symmetric two-monopole on $\mathbb{R}^3$. To take this limit, we can follow the same pattern as in [4]. Regard the index j as labelling a one-dimensional lattice with lattice spacing $1/n$, and put $F_{j,1} = f(s)$, $G_{j,1} = n + g(s)$, $G_{j,2} = n + h(s)$. Then, the limit $n \to \infty$ leaves us with the differential equations

$$f' = (h - g)f, \quad g' = -\tfrac{1}{2}f^2 = -h'.$$

The relevant solution of this is

$$h(s) = \frac{\pi}{4}\tan(\pi s/2) = -g(s), \quad f(s) = \frac{\pi}{2}\sec(\pi s/2),$$

which corresponds to the Nahm data for an axially symmetric two-monopole on $\mathbb{R}^3$.

3. Discrete Hitchin equations

In the equations (1) and (3), the vectors a and b play the role of boundary terms, and the interior terms only involve α_i. If we focus on the interior equations by setting $a = b = 0$, then the remaining equations are, in effect, the self-dual Yang–Mills equations reduced to zero dimensions. The idea now is to forget about instantons as such, and simply to regard (6), and (7) with $a_0 = b_0 = 0$, namely

$$F_{j+1,k}\, G_{j,k} = G_{j,k+1}\, F_{j,k}, \quad F_{j,k-1}\, F^*_{j,k-1} + G_{j-1,k}\, G^*_{j-1,k} = F^*_{j,k}\, F_{j,k} + G^*_{j,k}\, G_{j,k} \tag{8}$$

as a two-dimensional lattice system. The objects $F_{j,k}$ and $G_{j,k}$ no longer need to be real numbers: instead, we allow them to be complex $p \times p$ matrices. So the order of the factors in each of the terms of (8) becomes important. The claim is that the resulting system is an integrable lattice version of the U(p) Hitchin equations on $\mathbb{R}^2$.

In what follows, we shall take $n_1 = n_2 = n$ for simplicity, but it is straightforward to relax this condition. The question of what boundary conditions one might want to add to (8) is left open for the moment.

The Hitchin equations [12, 13] may be thought of as the self-dual Yang–Mills equations reduced to $\mathbb{R}^2$, obtained by factoring out two translations in $\mathbb{R}^4$, and the resulting system is as follows. Let (x, y) denote the usual $\mathbb{R}^2$ coordinates, (A_x, A_y) a gauge potential, and (Φ_1, Φ_2) a pair of Higgs fields. Take the gauge group to be U(p), so that $(A_x, A_y, \Phi_1, \Phi_2)$ are antihermitian $p \times p$ matrices. Then the Hitchin equations are

$$F = [\Phi_1, \Phi_2], \quad D_x\Phi_1 = -D_y\Phi_2, \quad D_x\Phi_2 = D_y\Phi_1, \tag{9}$$

where $F = \partial_x A_y - \partial_y A_x + [A_x, A_y]$ is the gauge field, and $D_x\Phi_j = \partial_x\Phi_j + [A_x, \Phi_j]$ (similarly for $D_y\Phi_j$) are the covariant derivatives of Φ_j.

The lattice equations (8) are a discrete version of (9) in the following sense. Let us take the continuum limit by extending the method of [4] and the previous section. Namely, put $x = j/n$ and $y = k/n$, write

$$G_{j,k} = n - A_x(x, y) + \mathrm{i}\Phi_1(x, y), \quad F_{j,k} = n - A_y(x, y) + \mathrm{i}\Phi_2(x, y), \tag{10}$$

and take the limit $n \to \infty$. The result is the Hitchin system (9).

The lattice system (8) is integrable simply by virtue of being a special case of the ADHM constraints, but one can also see directly that it arises from a lattice Lax pair. This is analogous to the Lax pair for the discrete Nahm equations [5, 6]. Let X be the operator which steps forward in the first index, namely $X : F_{j,k} \mapsto F_{j+1,k}$; and similarly let Y be the operator which steps forward in the second index. Define a pair of operators on lattice p-vectors by

$$\eth_1 = G^*_{j,k}X + \zeta F_{j,k-1}Y^{-1}, \quad \eth_2 = F^*_{j,k}Y - \zeta G_{j-1,k}X^{-1}, \tag{11}$$

where ζ is a complex parameter. Then $[\eth_1, \eth_2] = 0$ for all ζ if and only if the equations (8) hold. In other words, the lattice system (8) is an integrable discretization of the Hitchin equations (9). Note that if we replace $\eth_j$ by the equivalent operators $\eth_1 - n(1+\zeta)$ and $\eth_2 - n(1-\zeta)$, and take the limit $n \to \infty$ using the expressions (10), then we get the usual Lax pair $\{(D_x + \mathrm{i}\Phi_1) - \zeta(D_y - \mathrm{i}\Phi_2), (D_y + \mathrm{i}\Phi_2) + \zeta(D_x - \mathrm{i}\Phi_1)\}$ for the Hitchin equations (9).

Since the Hitchin system has a U(p) gauge invariance, one might expect the discrete version to have a local gauge invariance on the lattice, and this is indeed the case. Indeed, the equations (8) are invariant under

$$G_{j,k} \mapsto G'_{j,k} = \Lambda_{j+1,k} G_{j,k} \Lambda_{j,k}^{-1}, \quad F_{j,k} \mapsto F'_{j,k} = \Lambda_{j,k+1} F_{j,k} \Lambda_{j,k}^{-1}, \tag{12}$$

where Λ is a U(p)-valued function on the lattice.

As an example, consider the simplest case $p = 1$, so the gauge group is U(1). The continuum system is then linear, and in fact reduces to the Cauchy–Riemann equation $(\partial_x + \mathrm{i}\partial_y)(\Phi_1 - \mathrm{i}\Phi_2) = 0$. But the discrete system remains nonlinear. One may choose a gauge such that $F_{j,k}$ and $G_{j,k}$ are real-valued and positive, and the equations (8) then become

$$\Delta_x^+ \log(F) = \Delta_y^+ \log(G), \ \ \Delta_y^- F^2 + \Delta_x^- G^2 = 0, \tag{13}$$

where Δ_x^+ denotes forward difference in the first index, Δ_x^- backward difference in the first index, and similarly with y referring to the second index. So from this point of view, the system (13) is an integrable nonlinear discretization of the Cauchy–Riemann equations. One particular solution corresponds to instanton data: for this one imposes boundary conditions as in (7). Numerical solution of the equations when $n_1 = n_2 = n$ indicates that this solution satisfies $G_{j,k} = F_{k,j}$ and $b_0 = a_0$; and such a numerically obtained solution is plotted in Fig. 1 for the case $n = 50$.

As yet, not much is known about the possible existence of explicit solutions—even in the simplest U(1) case—except for small values of n. The simplest one which is defined on the infinite lattice seems to be

$$F_{j,k} = \alpha \exp(\gamma j), \ \ G_{j,k} = \beta \exp(\gamma k),$$

where α, β and γ are real constants. Whether one can find more general families of explicit solutions is an open question.

4. Concluding remarks

We have seen that the algebraic constraints on ADHM data for T^2-symmetric instantons can be generalized to give a set of lattice-gauge equations on a two-dimensional square lattice; and this lattice system may be viewed as an integrable discrete version of the Hitchin equations.

Since S^1-symmetric instantons may be interpreted as hyperbolic monopoles, we may view T^2-symmetric instantons in that context as axially symmetric hyperbolic monopoles. But this treats the two circle actions differently, and a more even-handed interpretation of T^2-symmetric instantons is as a variant of the two-dimensional Hitchin system. Let us write the $\mathbb{R}^4$ coordinates x_μ as $x_0 + \mathrm{i}x_3 = r_1 \exp(\mathrm{i}\theta_1)$ and $x_1 + \mathrm{i}x_2 = r_2 \exp(\mathrm{i}\theta_2)$, and reduce by invariance in the θ_j-directions. The gauge-potential components A_{θ_j} then become Higgs fields Φ_j. If we write $A_j = A_{r_j}$, $\partial_j = \partial_{r_j}$ and $F = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$, then the

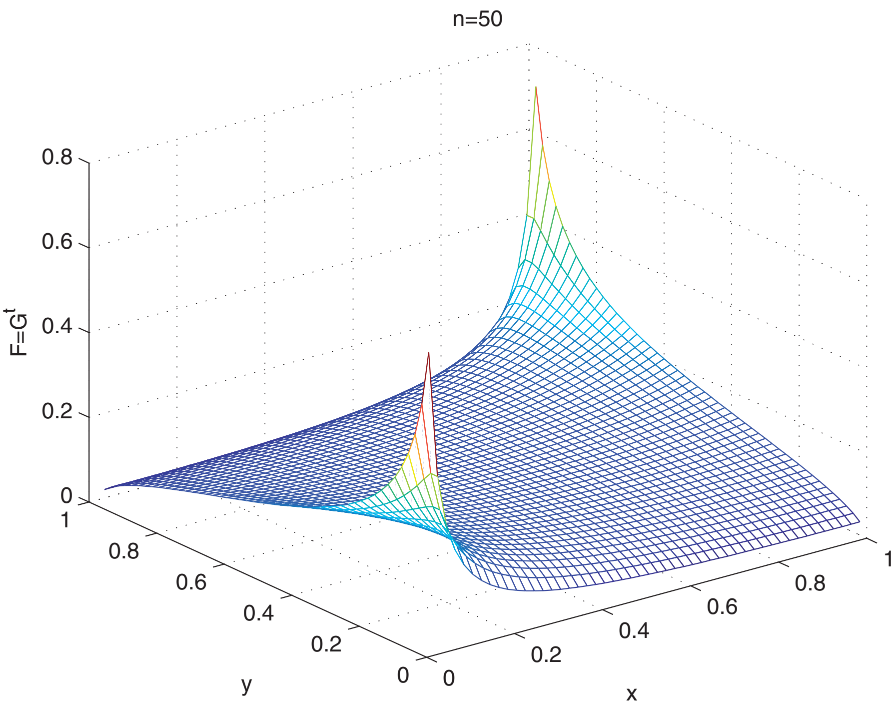

FIG. 1. Instanton data for $n_1 = n_2 = 50$.

reduced self-dual Yang–Mills equations become

$$F = (r_1 r_2)^{-1}[\Phi_1, \Phi_2], \quad D_1\Phi_1 = -(r_1/r_2)D_2\Phi_2. \quad D_1\Phi_2 = (r_2/r_1)D_2\Phi_1. \tag{14}$$

This is therefore an integrable variant of the standard Hitchin system (9). Note that (14) is invariant under $r_j \mapsto \kappa r_j$, with Φ_j having conformal weight zero. Solutions of (14) which actually correspond to instantons satisfy the boundary conditions $|\Phi_j| \to \frac{1}{2}n_j$ as $r_j \to 0$ for $j = 1, 2$, in terms of the two integers n_j. For each pair (n_1, n_2), the equation (14) has a one-parameter of 'instantonic' solutions, corresponding to the solutions of (6, 7). One open question is whether less restrictive boundary conditions would allow more interesting moduli spaces of solutions of (14), no longer corresponding to instantons.

Clearly many other questions remain open as well. For example, it is likely that one could develop a more comprehensive treatment of the complete-integrability of the discrete system, in particular through understanding its spectral data, along the lines of what was done for the discrete Nahm equations [6].

Another question is whether it is possible to impose boundary conditions on the lattice system (8) which allow nice moduli spaces of solutions, and/or a generalized Nahm transform. In the special case corresponding to T^2-symmetric instantons, this Nahm transform is the ADHM transform; but this case is rather special, with each solution space being just one-dimensional. The Hitchin equations (9) admit a rich collection of moduli spaces of solutions, and whether any of this extends to the discrete version remains open.

A final remark is that our starting-point above was the ADHM construction for S^1- and T^2-symmetric instantons with gauge group SU(2). Recently the S^1-symmetric case has been generalized to describe the discrete Nahm equations corresponding to SU(N) hyperbolic monopoles [17], and it might be interesting to see what happens for T^2-symmetric instantons for larger gauge group.

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References

1. Atiyah, M. F., Drinfeld, V. G., Hitchin, N. J. & Manin, Y. I. (1978) Construction of instantons. *Phys. Lett. A*, **65**, 185–187.

2. Atiyah, M. F. (1987) Magnetic monopoles in hyperbolic space. *Proceedings of Bombay Colloquium on Vector Bundles*. Oxford: Tata Institute and Oxford University Press, pp. 1–34.

3. Nahm, W. (1983) All self-dual multimonopoles for arbitrary gauge groups. *Structural Elements in Particle Physics and Statistical Mechanics* (J. Honerkamp et al. eds). NATO ASI Series B82. New York: Plenum Press, pp. 301–310.

4. Braam, P. J. & Austin, D. M. (1990) Boundary values of hyperbolic monopoles. *Nonlinearity*, **3**, 809–823.

5. Ward, R. S. (1999) Two integrable systems related to hyperbolic monopoles. *Asian. J. Math.*, **3**, 325–332.

6. Murray, M. K. & Singer, M. A. (2000) On the complete integrability of the discrete Nahm equations. *Commun. Math. Phys.*, **210**, 497–519.

7. Nash, C. (1986) Geometry of hyperbolic monopoles. *J. Math. Phys.*, **27**, 2160–2164.

8. Chakrabarti, A. (1986) Construction of hyperbolic monopoles. *J. Math. Phys.*, **27**, 340–348.

9. Furuta, M. (1990) Z_a-invariant SU(2) instantons over the four sphere. *Geometry of Low-Dimensional Manifolds 1* (S. K. Donaldson & C. B. Thomas eds.). Cambridge: Cambridge University Press, pp. 161–174.

10. Furuta, M. & Hashimoto, Y. (1990) Invariant instantons on S^4. *J. Fac. Sci. Univ. Tokyo*, **37**, 585–600.

11. Austin, D. M. (1990) SO(3)-instantons on $L(p,q) \times \mathbf{R}$. *J. Diff. Geom.*, **32**, 383–413.

12. Lohe, M. A. (1977) Two- and three-dimensional instantons. *Phys. Lett. B.*, **70**, 325–328.

13. Hitchin, N. J. (1987) The self-duality equations on a Riemann surface. *Proc. Lond. Math. Soc.*, **55**, 59–126.

14. Donaldson, S. K. (1984) Instantons and geometric invariant theory. *Commun. Math. Phys.*, **93**, 453–460.

15. Cockburn, A. (2014) Symmetric hyperbolic monopoles. *J. Phys. A.*, **47**, 395401.

16. Bolognesi, S., Harland, D. & Sutcliffe, P. (2015) Magnetic bags in hyperbolic space. *Phys. Rev. D.*, **92**, 025052.

17. Chan, J. Y. C. Discrete Nahm equations for SU(N) hyperbolic monopoles. arXiv:1506.08736