Computation of the infimum in the Littlewood Conjecture

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Abstract

The famous Littlewood Conjecture states that for any two real numbers $(\alpha,\beta)\in\mathbb{R}^2$ the value

$$m(\alpha,\beta) := \inf\{\ q\cdot||q\alpha||\cdot||q\beta||\ :\ q\in\mathbb{N}\}$$

is equal to zero. In this paper we provide an algorithm which for given $\epsilon>0$ checks, if the value $m_{LC} := \sup_{\alpha,\beta} m(\alpha,\beta)$ is less than ϵ. In particular with its help we show that $m_{LC} < 1/19$. We also provide a similar algorithm for p-adic counterpart of the Littlewood Conjecture and show that an analogue of m_{LC} in 2-adic case is at most 1/9.

1 introduction

Denote by $||\cdot||$ the distance to the nearest integer. The famous Littlewood Conjecture (LC) states that for any two real numbers α,β,

$$m(\alpha,\beta) := \inf\{\ q\cdot||q\alpha||\cdot||q\beta||\ :\ q\in\mathbb{N}\} = 0. \quad (1)$$

In other words for any $\epsilon>0$ and any $(\alpha,\beta)\in\mathbb{R}^2$ one can find a natural number q such that $q\cdot||q\alpha||\cdot||q\beta||<\epsilon$. Last decades it attracts an attention of many mathematicians mostly after the landmark paper of Einsiedler, Katok and Lindenstrauss [6]. They showed that the set of possible counterexamples to the Littlewood Conjecture is a countable union of sets of box dimension 0. Other related results can be found in [6, 10, 1].

Recently, de Mathan & Teulié in [9] proposed the p-adic variant of LC. It states that for any prime p and any real number α

$$m_p(\alpha) := \inf\{q\cdot|q|_p\cdot||q\alpha||\ :\ q\in\mathbb{N}\} = 0 \quad (2)$$

where $|\cdot|_p$ means the p-adic norm. It is generally believed that the p-adic Littlewood Conjecture (PLC) is easier than classical LC however it is still open as well. At least all the major results achieved for LC have their analogues in PLC language. We refer the reader to [7, 3, 4, 5, 2] for further developments of the problem.

In this paper we consider the following question. Define

$$m_{\mathrm{LC}} := \sup_{\alpha,\beta\in\mathbb{R}} m(\alpha,\beta);\quad m_{\mathrm{PLC}}(p) := \sup_{\alpha\in\mathbb{R}} m_p(\alpha).$$

Given $\epsilon>0$ is the value m_{LC} (correspondingly $m_{\mathrm{PLC}}(p)$) smaller than ϵ? Surely if LC is false then we shall get a negative answer to this question for some small value $\epsilon>0$.

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Classical Hurwitz Theorem [8, Theorem 193] together with the fact that $||\alpha|| \leqslant 1/2$ for all $\alpha \in \mathbb{R}$ provides the positive answer to the question for $\epsilon = \frac{1}{2\sqrt{5}}$. However it seems that there are no any further results for smaller ϵ.

In this paper we provide an algorithm which enables us to answer the question for much smaller values of ϵ. Unfortunately we can not guarantee that it will finish in finite time with either positive or negative answer for an arbitrary small $\epsilon > 0$. However with its help we prove the following

Theorem 1 *For any $(\alpha, \beta) \in \mathbb{R}^2$ there exists a positive integer $q \in \mathbb{N}$ such that*

$$q \cdot ||q\alpha|| \cdot ||q\beta|| < \frac{1}{19}.$$

In other words, $m_{\mathrm{LC}} \leqslant 1/19$.

We also construct an analogous algorithm for PLC (in particular for $p = 2$). With its help we prove

Theorem 2 *For any $\alpha \in \mathbb{R}$ there exists a positive integer $q \in \mathbb{N}$ such that*

$$q \cdot |q|_2 \cdot ||q\alpha|| < \frac{1}{9}.$$

In other words, $m_{\mathrm{PLC}}(2) \leqslant 1/9$.

2 Preliminaries

We start with some auxiliary statements. For any positive integer q define

$$\phi_q \ : \ \mathbb{R} \to \mathbb{R}/\mathbb{Z}; \quad \phi_q(\alpha) \mapsto \{q\alpha\}$$

where $\{\cdot\}$ denotes the fractional part of a number. Also define

$$f_q \ : \ \mathbb{R}^2 \to \mathbb{R}; \quad f_q(\alpha, \beta) := q \cdot ||q\alpha|| \cdot ||q\beta||.$$

Notice that the function f_q is periodic in both coordinates with period 1. So it is sufficient to prove Theorems 1 and 2 for $(\alpha, \beta) \in I := [0, 1]^2$. Moreover we have the following fact:

$$\forall (\alpha, \beta) \in I, \quad f_q(\alpha, \beta) = f_q(1 - \alpha, \beta) = f_q(\alpha, 1 - \beta) = f_q(1 - \alpha, 1 - \beta).$$

So we can finally restrict to the case $(\alpha, \beta) \in [0, 1/2]^2$.

Lemma 1 *Let q be a natural number and $R = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$ such that the condition*

$$\phi_q(I) \subset [0, 1/2] \ \ \textit{or} \ \ \phi_q(I) \subset [1/2, 1]. \qquad (*)$$

is satisfied for both intervals $I = [a, b]$ and $I = [c, d]$. Then the condition

$$f_q(a, c) < \epsilon, \ f_q(a, d) < \epsilon, \ f_q(b, c) < \epsilon, \ f_q(b, d) < \epsilon$$

implies that

$$\forall (\alpha, \beta) \in R, \ f_q(\alpha, \beta) < \epsilon.$$

Additionally the condition

$$f_q(a, c) \geqslant \epsilon, \ f_q(a, d) \geqslant \epsilon, \ f_q(b, c) \geqslant \epsilon, \ f_q(b, d) \geqslant \epsilon$$

implies that

$$\forall (\alpha, \beta) \in R, \ f_q(\alpha, \beta) \geqslant \epsilon.$$

PROOF. Since the function $||x||$ is monotonic on intervals $[0, 1/2]$ and $[1/2, 1]$ then for $\alpha \in [a, b]$ the maximal value (and minimal value as well) of $||\phi_q(\alpha)|| = ||q\alpha||$ is achieved at one of the endpoints $\alpha = a$ ar $\alpha = b$. The same condition is true for the maximal (and minimal) value of $||q\beta||$ where $\beta \in [c, d]$. Therefore the maximal value of $f_q(\alpha, \beta)$ restricted to R is achieved at one of the vertices of R. This implies the statement of the lemma.

$\boxtimes$.

Lemma 1 provides an idea of the algorithm. We can choose some large value Q and split $[0, 1/2]^2$ into a large number of small rectangles such that for any such rectangle $R = [a, b] \times [c, d]$ and any natural $1 \leqslant q \leqslant Q$ intervals $[a, b]$ and $[c, d]$ satisfy $(*)$. Then if we check that for some $q \in \{1, \ldots, Q\}$, $f_q(\alpha, \beta) < \epsilon$ for each vertex of R then by Lemma 1 the same inequality is true for all $(\alpha, \beta) \in R$. Finally if we are able to find such a value q for each small rectangle then we prove that $m(\alpha, \beta) < \epsilon$ for all $(\alpha, \beta) \in [0, 1/2]^2$ and therefore for all $(\alpha, \beta) \in \mathbb{R}^2$. The detailed description of the algorithm will be provided in the next section.

We shall need some basic facts about Farey fractions. For $Q \in \mathbb{N}$ the Farey fractions of order Q are defined as

$$\mathcal{F}_Q := \left\{ \frac{a}{q} \in \mathbb{Q} \ : \ 1 \leqslant q \leqslant Q, \ \ 0 \leqslant a \leqslant q, \ \ (a, q) = 1 \right\}.$$

where (a, q) denotes the greatest common divisor of a and q. Usually all the elements in $\mathcal{F}_Q$ are sorted in ascending order. Let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be two consecutive Farey fractions from $\mathcal{F}_Q$. Then they satisfy the following properties:

- **(A)** $\dfrac{p_2}{q_2} - \dfrac{p_1}{q_1} = \dfrac{1}{q_1 q_2}$;
- **(B)** The rational number p/q with the smallest possible denominator such that

$$\frac{p_1}{q_1} < \frac{p}{q} < \frac{p_2}{q_2} \qquad \text{is} \qquad \frac{p}{q} = \frac{p_1 + p_2}{q_1 + q_2}.$$

  Moreover rational numbers $\dfrac{p_1}{q_1}, \dfrac{p_1 + p_2}{q_1 + q_2}, \dfrac{p_2}{q_2}$ are consecutive Farey fractions in $\mathcal{F}_{q_1 + q_2}$.

For these and other properties of Farey fractions we refer reader to [8, Chapter III].

Lemma 2 *Let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be two consecutive Farey fractions. Then the interval $[p_1/q_1, p_2/q_2]$ satisfies $(*)$ for all $q < \frac{q_1 + q_2}{2}$.*

PROOF. The condition $(*)$ is equivalent to the following: interval $(q \cdot \frac{p_1}{q_1}, q \cdot \frac{p_2}{q_2})$ does not contain any integer point and any rational point with the denominator 2. However latter is true since there is no rational number of the form p/q and $p/2q$ between p_1/q_1 and p_2/q_2 with $q < \frac{1}{2}(q_1 + q_2)$.

$\boxtimes$

3 Description of the algorithm

Fix $\epsilon > 0$. In this section we provide an algorithm which checks whether $m_{\mathrm{LC}} < \epsilon$. On each iteration of the algorithm we shall have a list **RList** of potential rectangles R for which we still don't find the value q such that $f_q(\alpha, \beta) < \epsilon$ for all $(\alpha, \beta) \in R$. We will prove by induction that the endpoints of both intervals which form the sides of any rectangle $R \in$ **RList** are indeed consecutive Farey fractions of some orders q_1 and q_2 correspondingly.

Step 1. Initialization. We start with the list of just one rectangle **RList** $:= \{[0, 1/2]^2\}$.

Note that $0/1$ and $1/2$ are consecutive Farey fractions of order 2. This forms the base of the induction.

Step 2. While the list **RList** is nonempty we do the iterations.

Step 3. For each rectangle $R \in$ **RList** we do the procedure

Step 4. Check the rectangle. Let

$$R = \left[\frac{p_{1x}}{q_{1x}}, \frac{p_{2x}}{q_{2x}}\right] \times \left[\frac{p_{1y}}{q_{1y}}, \frac{p_{2y}}{q_{2y}}\right].$$

By inductional assumption $\frac{p_{1x}}{q_{1x}}$ and $\frac{p_{2x}}{q_{2x}}$ are consecutive Farey fractions in $\mathcal{F}_{\max\{q_{1x},q_{2x}\}}$ and $\frac{p_{1y}}{q_{1y}}$ and $\frac{p_{2y}}{q_{2y}}$ are consecutive Farey fractions in $\mathcal{F}_{\max\{q_{1y},q_{2y}\}}$. For each q between 1 and $\min\{q_{1x}, q_{1y}, q_{2x}, q_{2y}\}$ we calculate the values

$$f_q\left(\frac{p_{1x}}{q_{1x}}, \frac{p_{1y}}{q_{1y}}\right), f_q\left(\frac{p_{2x}}{q_{2x}}, \frac{p_{1y}}{q_{1y}}\right), f_q\left(\frac{p_{1x}}{q_{1x}}, \frac{p_{2y}}{q_{2y}}\right) \text{ and } f_q\left(\frac{p_{2x}}{q_{2x}}, \frac{p_{2y}}{q_{2y}}\right).$$

Step 5. If for some q we find that all these 4 values are less than ϵ then by Lemmata 1 and 2 the same will be true for all $(\alpha, \beta) \in R$. If this happens the rectangle R fails the check.

Step 6. Split the rectangle. If R passes the checks for all values q, we split it into two smaller rectangles and add both of them to the new list **NRList**. Splitting procedure is performed as follows:
If $q_{1x} + q_{2x} \leqslant q_{1y} + q_{2y}$ then R splits into the rectangles

$$\left[\frac{p_{1x}}{q_{1x}}, \frac{p_{1x}+p_{2x}}{q_{1x}+q_{2x}}\right] \times \left[\frac{p_{1y}}{q_{1y}}, \frac{p_{2y}}{q_{2y}}\right] \text{ and } \left[\frac{p_{1x}+p_{2x}}{q_{1x}+q_{2x}}, \frac{p_{2x}}{q_{2x}}\right] \times \left[\frac{p_{1y}}{q_{1y}}, \frac{p_{2y}}{q_{2y}}\right].$$

Otherwise it splits into the rectangles

$$\left[\frac{p_{1x}}{q_{1x}}, \frac{p_{2x}}{q_{2x}}\right] \times \left[\frac{p_{1y}}{q_{1y}}, \frac{p_{1y}+p_{2y}}{q_{1y}+q_{2y}}\right] \text{ and } \left[\frac{p_{1x}}{q_{1x}}, \frac{p_{2x}}{q_{2x}}\right] \times \left[\frac{p_{1y}+p_{2y}}{q_{1y}+q_{2y}}, \frac{p_{2y}}{q_{2y}}\right].$$

Note that by property **(B)** of Farey fractions the endpoints of both intervals which form the sides of new rectangles are still consecutive Farey fractions.

Step 7. End of the procedure.

Step 8. RList := **NRList**; **NRList** is emptied.

On step 6 we showed that for each $R \in$ **NRList** the endpoints of both intervals which form the sides of R are consecutive Farey fractions. This completes the step of the induction.

Step 9. End of the iteration.

To make the algorithm clearer let's manually run it for relatively big value ϵ, for example $\epsilon = 0.23$.

- (Step 1) We start the first iteration with **RList** consisting of one rectangle: **RList** $= \{[0/1, 1/2] \times [0/1, 1/2]\}$.

- (Steps 4 and 5) We perform the check for $R = [0/1, 1/2]^2$ and $q = 1$. The rectangle passes it because $f_1(1/2, 1/2) = 1/4 > \epsilon$.
- (Step 6) So we split R into two parts

  $$R_1 := [0/1, 1/3] \times [0/1, 1/2], \quad R_2 := [1/3, 1/2] \times [0/1, 1/2]$$

  and add both of them to the list **NRList**.
- (Steps 7 – 9) There are no more rectangles in **RList**, so we end the first iteration with the new set **RList** $= \{R_1, R_2\}$.
- (Steps 4,5 for R_1). In the second iteration the rectangle R_1 fails the check. Indeed,

  $$f_1(0, 0) = f_1(0, 1/2) = f_1(1/3, 0) = 0 < \epsilon; \quad f_1(1/3, 1/2) = 1/6 < \epsilon.$$
- (Steps 4 – 6) On the other hand, R_2 passes the check since we still have $f_1(1/2, 1/2) = 1/4 \geqslant \epsilon$. So we split R_2 into two parts

  $$R_3 := [1/3, 1/2] \times [0/1, 1/3], \quad R_2 := [1/3, 1/2] \times [1/3, 1/2].$$
- (Steps 7 – 9) We end this iteration with the set **RList** $= \{R_3, R_4\}$.
- (Iteration 3) Rectangle R_3 fails the check for $q = 1$:

  $$f_1(1/3, 0) = f_1(1/2, 0) = 0 < \epsilon; \quad f_1(1/3, 1/3) = 1/9 < \epsilon; \quad f_1(1/2, 1/3) = 1/6 < \epsilon.$$

  But R_4 also fails the check for $q = 2$:

  $$f_2(1/3, 1/3) = 2/9 < \epsilon, f_2(1/2, 1/3) = f_2(1/3, 1/2) = f_2(1/2, 1/2) = 0 < \epsilon.$$

  Therefore we end the third iteration with the empty set **RList**! This ends the algorithm with the conclusion that $m_{\mathrm{LC}} < \epsilon = 0.23$.

4 Some modifications

Firstly note that Step 4 of the algorithm can be made more efficient. If for some $R \in$ **RList** and $q \in \mathbb{N}$ we prove that $\forall(\alpha, \beta) \in R$ the value $f_q(\alpha, \beta) > \epsilon$ then in future iterations we do not need to check that q for all rectangles sitting inside R. The condition mentioned above is easy to check. By Lemma 1 it is sufficient to check that all four values f_q in Step 4 are bigger than ϵ.

So additionally in Step 4 we can store in **RList** the information about values of q which do not need to be checked in the next iterations. In our implementation of the algorithm we just store the minimal value n which should be checked in the next iteration. Then on Step 6 we send this information together with both rectangles that are sent to **NRList**. This improvement dramatically decreases the number of operations in Step 4 (in practise it increases the speed of algorithm approximately two times).

Secondly one can be interested in finding the minimal value $Q(\epsilon) \in \mathbb{N}$ such that

$$\forall(\alpha, \beta) \in \mathbb{R}^2, \quad \inf\{\, q \cdot ||q\alpha|| \cdot ||q\beta|| \;:\; 1 \leqslant q \leqslant Q(\epsilon)\} < \epsilon.$$

Unfortunately our algorithm can not give a precise answer to this questions. However it can provide us with lower and upper bounds for $Q(\epsilon)$. At least it may give us some understanding about the rate of growth of $Q(\epsilon)$ as $\epsilon \to 0$.

Indeed if for a given $R \in$ **RList** and $q_{max}(R) \in \mathbb{N}$ all the values f_q in Step 5 are less than ϵ then for all $(\alpha, \beta) \in R$,

$$\inf\{\, q \cdot ||q\alpha|| \cdot ||q\beta|| \;:\; 1 \leqslant q \leqslant q_{max}(R)\} < \epsilon.$$

Therefore the final value $Q(\epsilon)$ is not bigger than q_{max}, which is defined as the maximum of $q_{max}(R)$ over all rectangles R considered in the algorithm.

Analogously given $R \in$ **RList** if for all $1 \leqslant q \leqslant q_{min}(R)$ all four inequalities in Step 5 are false then for all $(\alpha, \beta) \in R$,

$$\inf \{ \ q \cdot ||q\alpha|| \cdot ||q\beta|| \ : \ 1 \leqslant q \leqslant q_{min}(R)\} \geqslant \epsilon.$$

So the final value $Q(\epsilon)$ is not smaller than q_{min}, which is defined as the maximum of $q_{min}(R)$ over all rectangles R considered in the algorithm. Finally notice that q_{min} and q_{max} can be easily calculated within the cycle (Steps 4 and 5).

The last observation is that the algorithm can be easily parallelized. if instead of $[0, 1/2]^2$ we start with any other rectangle R such that the endpoints of both of its sides form the consecutive Farey fractions of some order then the algorithm will finally prove that $m(\alpha, \beta) < \epsilon$ for all $(\alpha, \beta) \in R$.

So we can proceed as follows. Firstly start the algorithm with the rectangle $[0, 1/2]^2$. However after some number of iterations we stop. Then we divide the list **RList** into some number of parts (equal to the number of parallel processes) working with each part in parallel. Each process will perform the same algorithm but will be initialized by its own part of **RList**.

5 Numerical results

The algorithm was implemented on C++ with use of NTL library. It was launched on Intel Core Q6600 CPU with parameters $\epsilon = 1/n$ where $n \in \mathbb{N}$, $8 \leqslant n \leqslant 20$. The results are presented in the table 1 below. We can see from the table that as the function of $1/\epsilon$, $Q(\epsilon)$ grows exponentially. This result heuristically supports the following conjecture introduced in [2]:

Conjecture A *For each* $\lambda \geqslant 0$ *define the set*

$$\mathbf{Mad}^{\lambda} := \{(\alpha, \beta) \in \mathbb{R}^2 \ : \ \inf\{f_q(\alpha, \beta) \cdot (\log q)^{\lambda} \ : \ q \in \mathbb{N}\} > 0\}.$$

Then $\mathbf{Mad}^1 \neq \emptyset$ *and for any* $\lambda < 1$ *the set* $\mathbf{Mad}^{\lambda}$ *is empty.*

Table 1. Results for LC.

ϵ	Q_{min}	Q_{max}	Number of iterations	Time spent
1/8	2	7	9	0
1/9	5	7	12	0
1/10	7	18	19	0
1/11	22	49	20	0
1/12	75	107	27	0.016 s
1/13	285	285	32	0.046 s
1/14	695	1268	39	0.312 s
1/15	5551	5551	50	4.38 s
1/16	12398	12398	58	38.5 s
1/17	29863	29863	63	4 m 40 s
1/18	377503	377503	77	2 h 2 m
1/19	1272121	1526726	79+23	29 h 11 m+7 h 3 m*
1/20	⩾3162586	⩾3533493	$\geqslant 31 + 63$	⩾2 m+110 h*

* The computation was made in two stages. The first stage was proceeded on one core of CPU. Then **RList** was splitted into four equal parts and the computation continued on four cores.

6 Algorithm for 2-adic PLC

The algorithm for PLC is analogous to that for original LC. It is even simpler to implement since here we work with intervals instead of rectangles. We will present the algorithm here without too much discussion.

Step 1. Initialization. Start with the list of one interval **Ilist** $:= \{[0, 1/2]\}$.

Step 2. While the list **IList** is nonempty do the iterations.

Step 3. For each interval $I \in$ **IList** do the procedure

Step 4. Check the interval. Let

$$I = \left[\frac{p_1}{q_1}, \frac{p_2}{q_2}\right].$$

For each q between 1 and $\min\{q_1, q_2\}$ calculate values

$$g_q\left(\frac{p_1}{q_1}\right) \quad \text{and} \quad g_q\left(\frac{p_2}{q_2}\right)$$

where $g_q(\alpha) := q \cdot |q|_2 \cdot ||q\alpha||$.

Step 5. If for some q we find that both these values are less than ϵ then by Lemma 2 and analogue of Lemma 1 the same will be true for all $\alpha \in I$. If this happens the interval I fails the check.

Step 6. Split the interval. If I passes the checks for all values q, we split it into two smaller intervals

$$\left[\frac{p_1}{q_1}, \frac{p_1+p_2}{q_1+q_2}\right] \quad \text{and} \quad \left[\frac{p_1+p_2}{q_1+q_2}, \frac{p_2}{q_2}\right].$$

Then add both of them to the new list **NIList**.

Step 7. End of the procedure.

Step 8. IList := **NIList**; **NIList** is emptied.

Step 9. End of the iteration.

This algorithm was also implemented on C++ with help of NTL library. For Intel Core Q6600 CPU it shows the following results:

Table 2. Results for 2-adic LC.

ϵ	Q_{min}	Q_{max}	Number of iterations	Time spent
1/4	8	8	6	0
1/5	16	16	11	0
1/6	128	128	18	0
1/7	1024	1024	23	0
1/8	10240	10240	34	0.234 s
1/9	65536	65536	39	3.02 s
1/10	$\geqslant$446431232	$\geqslant$446431232	>54	>60 h

For $\epsilon = 1/10$ we see a big "jump" of the value $Q(\epsilon)$. It raises some doubts whether the 2-adic analogue of Conjecture A is actually true.

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