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Perturbative growth of high-multiplicity W, Z and Higgs production processes at high energies

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Abstract: Using the classical recursion relations we compute scattering amplitudes in a spontaneously broken Gauge-Higgs theory into final states involving high multiplicities of massive vector bosons and Higgs bosons. These amplitudes are computed in the kinematic regime where the number of external particles n is $\gg 1$ and their momenta are non-relativistic. Our results generalise the previously known expressions for the amplitudes on the multi-particle thresholds to a more non-trivial kinematic domain. We find that the amplitudes in spontaneously broken gauge theories grow factorially with the numbers of particles produced, and that this factorial growth is only mildly affected by the energy-dependent formfactor computed in the non-relativistic limit. This is reminiscent of the behaviour previously found in massive scalar theories. Cross sections are obtained by integrating the amplitudes squared over the non-relativistic phase-space and found to grow exponentially at energy scales in a few hundred TeV range if we use the non-relativistic high multiplicity limit. This signals a breakdown of perturbation theory and indicates that the weak sector of the Standard Model becomes effectively strongly coupled at these multiplicities. There are interesting implications for the next generation of hadron colliders both for searches of new physics phenomena beyond and within the Standard Model.

Keywords: Higgs Physics, Spontaneous Symmetry Breaking, Scattering Amplitudes, Standard Model

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Contents

1 Introduction

The problem of divergences affecting large orders in perturbation theory is well known [1–5], and is often seen as an academic problem which reflects the asymptotic nature of perturbative series. This problem however is brought to an entirely new level when the perturbation theory breakdown is realised already at the leading order. The physical quantities of interest in this case are associated with the scattering processes involving high multiplicities n of particles produced in the final state in the $n \gg 1$ limit. At sufficiently high energies the production of such high multiplicity final states with n greater than the inverse coupling constant, becomes kinematically allowed and the n-point scattering amplitudes can become large already at leading order — i.e. tree level in a weakly coupled theory.

The motivation of this paper is to study the behaviour of scattering processes involving large numbers massive vector bosons and Higgs bosons produced at high energy collisions. The underlying model is a spontaneously broken gauge theory, and the amplitudes will be computed on and off the multi-particle threshold, thus generalising the previously available results for the on-threshold amplitudes in [6, 7].

In the case of the ϕ^4-type scalar field theories, there is a direct link between the number of contributing Feynman diagrams, which grows as $n!$ at large n, and the and the overall expressions for the scattering amplitudes $\mathcal{A}_n$. Multi-particle amplitudes in scalar theory were studied in detail in the 90s and were found to exhibit factorial growth leading to the ultimate breakdown of the standard weakly-coupled perturbation theory, as reviewed in [8, 9], see also refs. [10–17]. However, this direct connection between the on-shell quantities and the number of Feynman diagrams does not hold in gauge theory because of the cancellations between individual diagrams which is a consequence of gauge invariance and the on-shell conditions. Nevertheless, it was shown in [6, 7] that in the case of spontaneously broken gauge theories, the amplitudes still grow factorially with the numbers of emitted Higgses as well as massive vector bosons.

There are two objectives for our study: one is to demonstrate that the breakdown of perturbation theory previously found in scalar QFTs also applies to the weak sector of the Standard Model — or more precisely in any spontaneously broken gauge theory, and for amplitudes in a generic non-relativistic kinematics. Our second point is to emphasise that the energies and multiplicities involved are not unreachable with future particle experiments. There is an exciting possibility that these processes can be probed at the next generation of hadron colliders. The simple estimates carried out in this paper, however, will assume a conservative non-relativistic limit which is not well-suited for drawing phenomenological conclusions about the lower limit on the energy scale where the weakly coupled perturbation theory would become strong.

Another motivation for studying the high-multiplicity production in the electroweak sector at high energies is their analogy and a potential complementarity with the topologically non-trivial transitions over the sphaleron barrier [18–22] which violate the Baryon plus Lepton $(B + L)$ number in the SM. The common point between these two types of processes is the high multiplicities of the vector bosons and Higgs particles in the final state. In both cases this is the regime where ordinary perturbation theory breaks down.

The paper is organised as follows. In the following section we will describe the method for computing the amplitudes on and off the multi-particle threshold in the double-scaling high-multiplicity low-kinetic-energy limit we introduce. This technique is first applied to a scalar theory with a non-trivial vacuum expectation value of the field. In section 3 this method is applied to our main case of interest — the Gauge-Higgs theory. The phase-space integration is described in section 4, and our conclusions are presented in section 5. Our main results for the amplitudes are in eqs. (3.49)–(3.50), and the cross section in the non-relativistic limit is given in eqs. (4.6), (5.6).

2 Summing tree graphs on and off the multi-particle threshold

Our approach for computing tree-level high-multiplicity amplitudes in the Gauge-Higgs theory is based on solving recursion relations between the n-point amplitudes involving massive vector and Higgs bosons for different values of n. In the high-multiplicity regime, the outgoing particles will have non-relativistic velocities as they are produced not far above their mass thresholds, thus we can simplify the amplitudes recursion relations by assuming the non-relativistic limit.

The main features of the method are best explained following a simple example of a single real scalar field $h(x)$ with non-vanishing VEV $\langle h\rangle = v$

$$\mathcal{L}(h) \,=\, \frac{1}{2}\,\partial^\mu h\,\partial_\mu h \,-\, \frac{\lambda}{4}\left(h^2-v^2\right)^2 . \tag{2.1}$$

This scalar theory with the spontaneously broken $h \to -h$ $\mathcal{Z}_2$ symmetry can be seen as a simplified version of the Higgs sector of the SM in the unitary gauge. In the following section we will apply this approach to our main case of interest — the Gauge-Higgs theory. The goal of the present section is to derive a simple prescription for writing down the relevant recursion relations between the amplitudes in the example of a relatively straightforward scalar QFT case (2.1). Our results for the scalar theory (2.1) are new, while our approach is similar to ref. [16] where the unbroken ϕ^4 theory was considered.

Tree-level amplitudes for production of n bosons from a virtual single-boson state, $\mathcal{A}_n := \mathcal{A}_{1\to n}$, are classical objects (no loops, $\hbar \to 0$), it is well-known that their generating functionals satisfies classical equations of motion with an external source. By differentiating n times with respect to the source and setting it to zero, one obtains the recursion recursion relations for the tree amplitudes, or currents, see e.g. [23, 24]. We introduce the physical VEV-less scalar $\varphi(x) = h(x) - v$, describing bosons of mass $M_h = \sqrt{2\lambda}\,v$. It satisfies the classical equation arising from (2.1),

$$-\,(\partial^\mu\partial_\mu + M_h^2)\,\varphi \,=\, 3\lambda v\,\varphi^2 \,+\, \lambda\,\varphi^3. \tag{2.2}$$

This classical equation in momentum space allows one to read-off directly the structure of the recursion relation for tree-level scattering amplitudes as follows:

$$\begin{aligned}(P_{\rm in}^2 - M_h^2)\,\mathcal{A}_n(p_1\ldots p_n) \,=\, &3\lambda v \sum_{n_1,n_2}^{n} \delta^n_{n_1+n_2} \sum_{\mathcal{P}} \mathcal{A}_{n_1}(p_1^{(1)},\ldots,p_{n_1}^{(1)})\,\mathcal{A}_{n_2}(p_1^{(2)}\ldots p_{n_2}^{(2)})\\ &+\lambda \sum_{n_1,n_2,n_3}^{n} \delta^n_{n_1+n_2+n_3} \sum_{\mathcal{P}} \mathcal{A}_{n_1}(p_1^{(1)}\ldots p_{n_1}^{(1)})\,\mathcal{A}_{n_2}(p_1^{(2)}\ldots p_{n_2}^{(2)})\,\mathcal{A}_{n_3}(p_1^{(3)}\ldots p_{n_2}^{(3)})\,.\end{aligned} \tag{2.3}$$

Here $P_{\rm in}^\mu = \sum_{i=1}^n p_i^\mu$ is the incoming momentum, and the sums over $\mathcal{P}$ involve permutations across the sets of momenta $\{p_i^{(1)}\}$, $\{p_i^{(2)}\}$, and/or $\{p_i^{(1)}\}$, $\{p_i^{(2)}\}$ and $\{p_i^{(3)}\}$ of the the individual amplitudes on the right hand side of eq. (2.3).

The special case of (2.3) where all outgoing n particles are produced on their mass threshold, i.e. with vanishing spatial momenta $\vec{p}_i \equiv 0$, is particularly simple. Here the amplitudes are constants, the kinematics is trivial and one can sum over the permutations with the result,

$$M_h^2(n^2-1)\,\mathcal{A}_n = 3\lambda v \sum_{n_1,n_2}^{n} \delta^n_{n_1+n_2} \frac{n!}{n_1!n_2!}\mathcal{A}_{n_1}\,\mathcal{A}_{n_2} + \lambda \sum_{n_1,n_2,n_3}^{n} \delta^n_{n_1+n_2+n_3} \frac{n!}{n_1!n_2!n_3!}\mathcal{A}_{n_1}\,\mathcal{A}_{n_2}\,\mathcal{A}_{n_3}\,. \tag{2.4}$$

The solution of this momentum-independent recursion relation can be captured and recast in terms of the the amplitudes generating function which solves the original Euler-Lagrange equation (2.2) for the $\vec{x}$-independent field $\varphi(t)$ with the initial condition [14],

$$\varphi(t) \,=\, z(t) + \mathcal{O}(z^2)\,, \quad \text{where} \quad z(t) \,:=\, z_0\, e^{iM_h t} \,=\, z_0\, e^{i\sqrt{2\lambda}\,v\,t}\,. \tag{2.5}$$

The generating function for the on-shell amplitudes is a complex-valued solution of (2.2) which contains only the positive-energy harmonics, $e^{inM_h t}$. For a fixed n each plain wave describes the n-particle final state at rest with n bosons of mass $M_h = \sqrt{2\lambda}\, v$. Hence the generating function is a holomorphic function of the complex argument z,

$$\varphi(z) = \sum_{n=1}^{\infty} d_n\, z^n\,, \quad \text{with} \quad d_1 = 1\,, \tag{2.6}$$

with the individual n-point amplitudes on the multi-particle threshold given by,

$$\mathcal{A}_n \;=\; \left(\frac{\partial}{\partial z}\right)^n \varphi(z)\bigg|_{z=0} \;=\; n!\, d_n\,. \tag{2.7}$$

Substituting the Taylor expansion (2.6) into the classical equation (2.2), one immediately finds the recursion relation between the coefficients d_n,

$$(n^2-1)\, d_n \;=\; 3\,\frac{\lambda v}{M_h^2} \sum_{n_1,n_2}^{n} \delta^n_{n_1+n_2} d_{n_1}\, d_{n_2} + \frac{\lambda}{M_h^2} \sum_{n_1,n_2,n_3}^{n} \delta^n_{n_1+n_2+n_3} d_{n_1}\, d_{n_2}\, d_{n_3}\,, \tag{2.8}$$

which, as expected, is equivalent to eq. (2.4) (note that there are no $n!$ factors when the recursion relations are expressed in terms of the Taylor coefficients d_n, rather than $\mathcal{A}_n$). The generating function $\varphi(z)$ which solves (2.2), is known analytically[1] [14],

$$\varphi \;=\; \frac{z}{1-z/(2v)}\,, \quad \text{hence} \quad d_n \;=\; (2v)^{1-n}\,, \quad \text{and} \quad \mathcal{A}_n \;=\; n!\,(2v)^{1-n} \tag{2.9}$$

The main lesson of this exercise was to show that our tree-level threshold amplitudes $\mathcal{A}_{1\to n}^{\text{threshold}}$ grow factorially with the number of final particles n. However, our main concern are the amplitudes above the threshold, and so we need to recover their dependence on the kinematics of the final state.

Away from the multi-particle threshold, the external particles 3-momenta $\vec{p}_i$ are non-vanishing and in the non-relativistic limit which we will adopt, they are small compared to the particle masses. In this limit we can characterise the process in the COM frame by the non-relativistic kinetic energy E_n^{kin} of the final particles. In general kinematics, the amplitudes are determined by the recursion relation (2.3) which we need to solve. Amplitudes *on* the threshold were already found in (2.9); they correspond to $E_n^{\text{kin}} = 0$. At small values of $E_n^{\text{kin}}/(nM_h)$ the leading correction to these amplitudes turns out to be proportional to E_n^{kin} itself. This fact is simply the consequence of the permutation symmetry acting on the particle momenta $\vec{p}_i$ and of the Galilean invariance of the amplitude [16]. Hence, the scattering amplitude in the non-relativistic approximation takes the form:

$$\mathcal{A}_n(p_1 \ldots p_n) \;=\; \mathcal{A}_n + \mathcal{M}_n\, E_n^{\text{kin}} \;:=\; \mathcal{A}_n + \mathcal{M}_n\, n\, \varepsilon\,, \tag{2.10}$$

where $\mathcal{A}_n$ is the threshold amplitude, $\mathcal{M}_n\, n\, \varepsilon$ denotes the leading order momentum-dependent contribution, and ϵ is the kinetic energy per particle per mass,

$$\varepsilon = \frac{1}{n\, M_h} E_n^{\text{kin}} \;=\; \frac{1}{n}\, \frac{1}{2M_h^2} \sum_{i=1}^{n} \vec{p}_i^{\,2}\,. \tag{2.11}$$

[1] In fact $h(z) = v + \varphi(z)$ in (2.9) corresponds the well-known kink solution in Euclidean time which interpolates between the two vacua at $h = \pm v$.

In the non-relativistic limit we have $\varepsilon \ll 1$.

Working in the CoM frame the incoming momentum is $P^\mu_{\rm in} = (nM_h(1+\epsilon), \vec{0})$ and the left hand side of the recursion relation (2.3) takes the form, $M_h^2\left(n^2(1+\varepsilon)^2 - 1\right)\mathcal{A}_n(p_1\dots p_n)$. Using the relation (2.10) and working at the order-ε^1, the recursion relations (2.3) amount to:

$$\left(n^2-1\right) n\,\varepsilon\,\mathcal{M}_n \,+\, 2n^2\,\varepsilon\,\mathcal{A}_n \,=\, 6\,\frac{\lambda v}{M_h^2}\sum_{n_1,n_2}^{n}\delta^n_{n_1+n_2}\sum_{\mathcal{P}} E^{\rm kin}_{n_1}\,\mathcal{M}_{n_1}\,\mathcal{A}_{n_2} \\ +3\,\frac{\lambda}{M_h^2}\sum_{n_1,n_2,n_3}^{n}\delta^n_{n_1+n_2+n_3}\sum_{\mathcal{P}} E^{\rm kin}_{n_1}\,\mathcal{M}_{n_1}\,\mathcal{A}_{n_2}\,\mathcal{A}_{n_3}\,. \tag{2.12}$$

This is the equation for $\mathcal{M}_n$. On its right hand side we have used the notation $E^{\rm kin}_{n_1}$ (rather than e.g. $n_1\varepsilon$) to denote the total kinetic energy of the outgoing particles of the sub-process $1\to n_1$. This quantity is defined by $E^{\rm kin}_{n_1} := \frac{1}{2M_h}\sum_{i=1}^{n_1}\left(\vec{p}_i - \frac{1}{n_1}\vec{p}_0\right)^2$, where we have taking into account that the initial state of this sub-process is no longer at rest, $\vec{p}_0 := \sum_{i=1}^{n_1}\vec{p}_i \neq \vec{0}$. Hence,

$$E^{\rm kin}_{n_1} = \frac{1}{2M_h}\left[\sum_{i=1}^{n_1}\vec{p}_i^{\,2} - \frac{1}{n_1}\vec{p}_0^{\,2}\right] = \frac{1}{2M_h}\left[\left(1-\frac{1}{n_1}\right)\sum_{i=1}^{n_1}\vec{p}_i^{\,2} - \frac{1}{n_1}\sum_{i=1}^{n_1}\sum_{j\neq i}^{n_1}\vec{p}_i\vec{p}_j\right] \tag{2.13}$$

To simplify this, we note that all n external momenta $\vec{p}_1\dots\vec{p}_n$ will contribute in the sums on the right hand side of the recursion relation (2.12). This allows us to effectively include all n momenta in the double sum above and use the substitution,

$$\sum_{i=1}^{n_1}\sum_{j\neq i}^{n_1}\vec{p}_i\vec{p}_j \Longrightarrow \frac{n_1(n_1-1)}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\vec{p}_i\vec{p}_j = -\frac{n_1(n_1-1)}{n(n-1)}\sum_{i=1}^{n}\vec{p}_i^{\,2}\,. \tag{2.14}$$

This gives the expression for the kinetic energy,

$$E^{\rm kin}_{n_1} \Longrightarrow \frac{n\,(n_1-1)}{n-1}\,\varepsilon\,, \tag{2.15}$$

which we can use on right hand side of (2.12).

We will also use the amplitude's coefficients d_n and f_n defined via:

$$\mathcal{A}_n(p_1\dots p_n) \,=\, n!\,(d_n \,+\, f_n\,\varepsilon)\,, \tag{2.16}$$

rather than the amplitudes $\mathcal{A}_n$ and $\mathcal{M}_n$ in (2.10)). The expression (2.8) is the recursion relation at the order-ε^0, and at the order-ε^1 from eqs. (2.12) and (2.15) we have:

$$\frac{n-1}{n}\left(\left(n^2-1\right)f_n \,+\, 2n^2\,d_n\right) \,=\, 6\,\frac{\lambda v}{M_h^2}\sum_{n_1,n_2}^{n}\delta^n_{n_1+n_2}\,\frac{n_1-1}{n_1}\,f_{n_1}\,d_{n_2} \\ +3\,\frac{\lambda}{M_h^2}\sum_{n_1,n_2,n_3}^{n}\delta^n_{n_1+n_2+n_3}\,\frac{n_1-1}{n_1}\,f_{n_1}\,d_{n_2}\,d_{n_3}\,. \tag{2.17}$$

The factors of $\frac{n-1}{n}$ and $\frac{n_1-1}{n_1}$ on the left and right hand sides of this equation arise from the kinetic energy formula (2.15).

The recursion relation (2.8) for the amplitudes on the multi-particle threshold is solved as before (cf. eq. (2.9)) and determines the full set of the d_n coefficients, $d_n = (2v)^{1-n}$. We can now go ahead and solve the second recursion relation (2.17) to determine the coefficients f_n. Our result is

$$f_n \,=\, -\left(\frac{7}{6}\,n \,+\, \frac{1}{6}\,\frac{n}{n-1}\right) d_n \,, \quad \text{for all } \ n \geq 2\,. \tag{2.18}$$

This result is obtained by solving an ordinary differential equation (as will be explained below) by iterations with *Mathematica*. The resulting amplitude to the order-ε is then given by

$$\mathcal{A}_n(p_1 \ldots p_n) \,=\, n!\,(2v)^{1-n}\left(1 - \frac{7}{6}\,n\,\varepsilon \,-\, \frac{1}{6}\,\frac{n}{n-1}\,\varepsilon \,+\, \ldots\right). \tag{2.19}$$

There are corrections to this expression at higher orders in ε, but it holds to the order ε^1 for any value of n.

An important observation, first made in [16], is that by exponentiating the order-$n\varepsilon$ contribution, one obtains the expression for the amplitude which solves the original recursion relation (2.3) to all orders in $(n\varepsilon)^m$ in the large-n non-relativistic limit,

$$\mathcal{A}_n(p_1 \ldots p_n) \,=\, n!\,(2v)^{1-n}\,\exp\left[-\frac{7}{6}\,n\,\varepsilon\right]\,, \quad n\to\infty\,, \quad \varepsilon\to 0\,, \quad n\varepsilon = \text{fixed}\,. \tag{2.20}$$

Simple corrections of order ε, with coefficients that are not-enhanced by n are expected, but the expression on right hand side of (2.20) is correct to all orders $n\varepsilon$ in the double scaling large-n limit. This observation follows from the fact that the exponential factor in (2.20) can be absorbed into the z variable so that the expression in (2.6) with the rescaled z on the right hand side,

$$\varphi(z) = \sum_{n=1}^{\infty} d_n\,\left(z\,e^{-\frac{7}{6}\,\varepsilon}\right)^n\,, \tag{2.21}$$

remains a solution to the classical equation

$$-\,(d_t^2 + M_h^2)\,\varphi \,=\, 3\lambda v\,\varphi^2 \,+\, \lambda\,\varphi^3. \tag{2.22}$$

This implies that the individual n-point amplitudes will satisfy the recursion relations (2.4) and by the same token the general-kinematics recursion (2.3). This is because the operator $P_{\rm in}^2 - M_h^2$ on the left hand side of (2.3) becomes $M_h^2\left(n^2(1+\varepsilon)^2-1\right) = M_h^2\left(n^2-1\right)$ in the $n\to\infty$, $\varepsilon\to 0$ double-scaling limit. Then as soon as (2.21) satisfies (2.4), it also satisfies the general-kinematics recursions (2.3) in the double-scaling limit.

Note that the exponentiated expression (2.20) solves (2.22) for any constant factor in the exponent, but having solved the order-ε recursions explicitly we have determined in (2.18) the value of the constant to be $= -7/6$.[2]

[2]In the ϕ^4 theory with no spontaneous symmetry breaking the authors of ref. [16] have shown that the amplitudes scale with energy as $\propto n!\,\exp\left[-\frac{5}{6}\,n\,\varepsilon\right]$. We will re-derive their result in the appendix.

Rescaled variables: to simplify the form of the recursion relations — especially in view of the applications to the coupled Gauge-Higgs equations in the following section — we define new rescaled dimensionless variables:

$$t_{\rm new} \,=\, M_h\, t\,, \quad z_{\rm new} \,=\, \frac{z}{2v} \qquad \text{and} \qquad \phi := \frac{1}{2v}\,\varphi \,=\, \sum_{n=1}^{\infty} d_n^{\rm new}\, z_{\rm new}^n\,, \tag{2.23}$$

so that $d_n^{\rm new} \,=\, (2v)^{n-1} d_n$ which amounts to a particularly simple form of the recursive solution $d_n^{\rm new} \,=\, 1$ or all $n = 1, 2, \ldots \infty$ in eq. (2.9), and is the reason why we introduced factors of 2 in the rescaling (2.23).

In terms of these new variables (and suppressing the superscript 'new') the classical equation (2.22) takes the form

$$-\,(d_t^2 + 1)\phi \,=\, 3\phi^2 + 2\phi^3\,, \tag{2.24}$$

and the amplitudes on the multi-particle threshold (cf. (2.7)) are given by:

$$\mathcal{A}_n \,=\, (2v)^{1-n} \left(\frac{\partial}{\partial z}\right)^n \phi(z)\Big|_{z=0} \,=\, n!\,(2v)^{1-n}\, d_n\,, \tag{2.25}$$

where

$$\phi(z) \,=\, \sum_{n=1}^{\infty} d_n\, z^n\,, \quad \text{and} \quad d_n \equiv 1\,, \quad n = 1, 2, \ldots, \infty\,. \tag{2.26}$$

With the equation (2.24) defining the generating function for amplitudes on the multi-particle threshold, the order-ε correction to the generating function (in our rescaled variables) is determined by the differential equation (cf. (2.17)):

$$\frac{n-1}{n}\left((n^2-1)\, f_n \,+\, 2n^2\, d_n\right) \,=\, 6\,\left(F\phi \,+\, F\phi^2\right)|_{z^n}\,, \tag{2.27}$$

where we defined the new function

$$F(z) \,=\, \sum_{n=2}^{\infty} \frac{n-1}{n}\, f_n\, z^n\,. \tag{2.28}$$

Solving eq. (2.27) by iterations with *Mathematica* gives the f_n coefficients in (2.18), which amounts to the exponentiated form for the amplitude off the multiparticle mass-shell in the non-relativistic limit $\varepsilon \to 0$, with $n\varepsilon$ =fixed,

$$\mathcal{A}_n(p_1 \ldots p_n) \,=\, n!\,(2v)^{1-n}\, \exp\left[-\frac{7}{6}\, n\, \varepsilon\right]\,. \tag{2.29}$$

This is our main result for the tree-level high-multiplicity amplitudes in the scalar theory with SSB in the $\varepsilon \to 0$, with $n\varepsilon$ = fixed limit.

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3 Multiparticle production in the gauge-Higgs theory

We are now ready to consider our main case of interest — the electroweak sector of the Standard Model. In the limit of the vanishing mixing angle $\theta_{\rm W}$, the weak interactions are described by the SU(2) gauge theory spontaneously broken by the vacuum expectation value v of the Higgs doublet,

$$\mathcal{L} \,=\, -\frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu} \,+\, |D_\mu H|^2 \,-\, \lambda\left(|H|^2 - \frac{v^2}{2}\right)^2 . \tag{3.1}$$

We adopt the standard unitary gauge where the Goldstone bosons are gauged away, and the Higgs doublet is described by a single real scalar $h(x)$,

$$H = \frac{1}{\sqrt{2}}\,(0,h)\,, \tag{3.2}$$

The Higgs potential in terms of h takes the same form as in eq. (2.1). The particle content of the model is given by the neutral Higgs state, h, and a triplet of massive vector bosons, $W^\pm$ and Z^0, described by A^a_μ with $a = 1, 2, 3$, which we will collectively refer to as V. The Higgs mass and the mass of the vector boson triplet are given by,

$$M_h \,=\, \sqrt{2\lambda}\, v \,\simeq\, 125.66\,\text{GeV}\,, \qquad M_V \,=\, \frac{gv}{2} \,\simeq\, 80.384\,\text{GeV}\,, \tag{3.3}$$

where we have also shown their numerical values, set by the SM Higgs and W boson masses, which will be uses in our calculations of the amplitudes below.

We want to study the processes where colliding protons first produce an intermediate virtual state, which can be either the Higgs or a gauge boson. This intermediate highly virtual boson then decays into n Higgs bosons and m vector bosons $1 \to n+m$, which is the multi-particle production process we concentrate upon. The multiplicity of the final state $n+m$ is assumed to be large so that most of the energy carried by the virtual state is spent to achieve the multi-particle mass threshold for the $n+m$ final particles. Above the threshold, the momenta of the final state particles are assumed to be non-relativistic. It is convenient to describe the kinematics working in the Lorentz frame where the initial virtual boson is at rest. In this frame,

$$P^\mu_{\rm in} = (P^0_{\rm in}, \vec{0}) \,=\, \sum_{j=1}^{n} p^\mu_j \,+\, \sum_{k=1}^{m} p^\mu_k\,, \tag{3.4}$$

where the first sum on the right hand side is over the n Higgs bosons, and the second sum is over the m vector bosons produced in the final state. We will make an additional simplifying assumption that the total momentum is conserved separately in the Higgs, and in the vector boson sectors, i.e.

$$\sum_{j=1}^{n} \vec{p}_j \,=\, 0\,, \quad \text{and} \quad \sum_{k=1}^{m} \vec{p}_k \,=\, 0\,. \tag{3.5}$$

In the rest frame of $P^\mu_{\rm in} = (P^0_{\rm in}, \vec{0})$ this amounts to a single constraint on the overall kinematics of the multi-particle final state and imposing it should not affect the result of integrating over the $(n+m)$-particle phase space in the $n+m \gg 1$ high-multiplicity limit. With these considerations in mind we can thus express the initial virtual-state momentum in the form,

$$P_{\rm in}^{\mu} = \left(nM_h(1+\varepsilon_h) + mM_V(1+\varepsilon_V)\,,\,\vec{0}\right)\,, \tag{3.6}$$

where ε_h and ε_V denote the average non-relativistic kinetic energies of the Higgs bosons, and of the vector bosons, per particle per mass,

$$\varepsilon_h \,=\, \frac{1}{n}\,\frac{1}{2M_h^2}\sum_{j=1}^{n}\vec{p}_j^{\,2}\,, \qquad \varepsilon_V \,=\, \frac{1}{m}\,\frac{1}{2M_V^2}\sum_{k=1}^{m}\vec{p}_k^{\,2}\,. \tag{3.7}$$

In the non-relativistic limit for the final state we have $0 \le \varepsilon_h \ll 1$ and $0 \le \varepsilon_V \ll 1$.

3.1 Recursion relations for amplitudes on the multi-particle threshold

Following the approach of [6, 7] we will consider the amplitudes for processes with final states which do not contain transverse polarisations of the vector bosons, and concentrate on the production of longitudinal polarisations, A_L^a and Higgses h. The classical equations for spacialy-independent fields readily follow from the Lagrangian (3.1) in the unitary gauge (these are eqs. (3.8)-(3.9) of [6, 7]),

$$-\,d_t^2 h \,=\, \lambda\, h^3 - \lambda v^2\, h + \frac{g^2}{4}(A_L^a)^2 h\,, \tag{3.8}$$

$$-\,d_t^2 A_L^a \,=\, \frac{g^2}{4} h^2 A_L^a\,. \tag{3.9}$$

The generating function of the amplitudes on the multi-h, multi-V_L threshold, is the classical solution of this system of equations given by analytic functions of two variables,

$$z(t) \,=\, z_0\, e^{iM_h t}\,, \quad \text{and} \quad w^a(t) \,=\, w_0^a\, e^{iM_V t}\,, \tag{3.10}$$

with the leading-order terms being,

$$h(t) \,=\, v + z(t) + \ldots\,, \quad \text{and} \quad A_L^a(t) \,=\, w^a(t) + \ldots\,. \tag{3.11}$$

The double Taylor expansion for the two generating functions in terms of the z and w^a variables takes the form:

$$h(z,w^a) = v \,+\, 2v \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d(n,2k)\,\left(\frac{z}{2v}\right)^n \left(\frac{w^a w^a}{(2v)^2}\right)^k\,, \tag{3.12}$$

$$A_L^a(z,w^a) = w^a \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a(n,2k)\,\left(\frac{z}{2v}\right)^n \left(\frac{w^a w^a}{(2v)^2}\right)^k\,, \tag{3.13}$$

where $z(t)$ and $w^a(t)$ are given by eqs. (3.10), and the lowest-order Taylor coefficients are $d(0,0) = 0$ and $a(0,0) = 1$ in agreement with (3.11). The explicit scaling factors of $2v$ are introduced on the right hand side of the above equations to maximally simplify the form of the solutions for the Taylor expansion coefficients. In particular, in this notation we will have $d(n,0) = 1$ for all values of $n \geq 1$, in agreement with the solution of the scalar equation (2.26) in the previous section.

To simplify the form of the classical equations and to emphasise that they depend only on a single numerical parameter κ,

$$\kappa := \frac{g}{2\sqrt{2\lambda}} = \frac{M_V}{M_h}\,, \tag{3.14}$$

we introduce the rescaled dimensionless variables as in (2.23),

$$t_{\rm new} = M_h t\,, \quad z_{\rm new} = \frac{z}{2v} = \frac{z_0}{2v}\,e^{it_{\rm new}}\,, \quad w^a_{\rm new} = \frac{w^a}{2v} = \frac{w^a_0}{2v}\,e^{i\kappa t_{\rm new}}\,, \tag{3.15}$$

and also define the dimensionless fields, ϕ for the VEVless scalar, and A for the vector bosons, via:

$$h = v\,(1+2\phi)\,, \quad A^a_L = w^a \mathrm{A} = 2v\,w^a_{\rm new}\,\mathrm{A}\,. \tag{3.16}$$

Note that the vector boson configuration A on the right hand side of the second equation in (3.16) no longer contains the isospin index $a = 1, 2, 3$ which has been factored out into the w^a prefactor. We use these new dimensionless variables and fields to re-write eqs. (3.8)–(3.9) in the form,

$$-\,(d_t^2+1)\phi = 3\phi^2 + 2\phi^3 + 2\kappa^2\,(1+2\phi)\,(w^a w^a)\,\mathrm{A}^2\,, \tag{3.17}$$

$$-\,(d_t^2+\kappa^2)\,w^a\mathrm{A} = 4\kappa^2(\phi+\phi^2)\,w^a\mathrm{A}\,. \tag{3.18}$$

As expected, this system of equations depends on a single dimensionless parameter κ and we note that in the $\kappa \to 0$ limit the equation (3.17) reproduces the scalar-field equation (2.24) of section 2. The Taylor expansions (3.12)–(3.13) are also simplified in terms of the rescaled variables,

$$\phi = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d(n,2k)\,z^n\,W^k\,, \quad \mathrm{A} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a(n,2k)\,z^n\,W^k\,, \tag{3.19}$$

where we have introduced the squared w variable,

$$W = w^a w^a\,. \tag{3.20}$$

The recursion relations for the coefficients $d(n,2k)$ and $a(n,2k)$ are obtained by substituting the Taylor expansions (3.19) into the classical equations (3.17)–(3.18) and selecting the $z^n\,W^k$ monomials,

$$\left[(n+2k\,\kappa)^2-1\right]\,d(n,2k) = \left[3\phi^2+2\phi^3+2\kappa^2\,(1+2\phi)\,W\mathrm{A}^2\right]\big|_{z^nW^k}\,, \tag{3.21}$$

$$\left[(n+\kappa+2k\,\kappa)^2-\kappa^2\right]\,a(n,2k) = \left[4\kappa^2(\phi+\phi^2)\,\mathrm{A}\right]\big|_{z^nW^k}\,. \tag{3.22}$$

These equations were solved in [6, 7] by iterations using the numerical value of $\kappa = M_W/M_h = 80.384/125.66 \simeq 0.6397$. First we set $k=0$ and solve the Higgs equations (3.21)

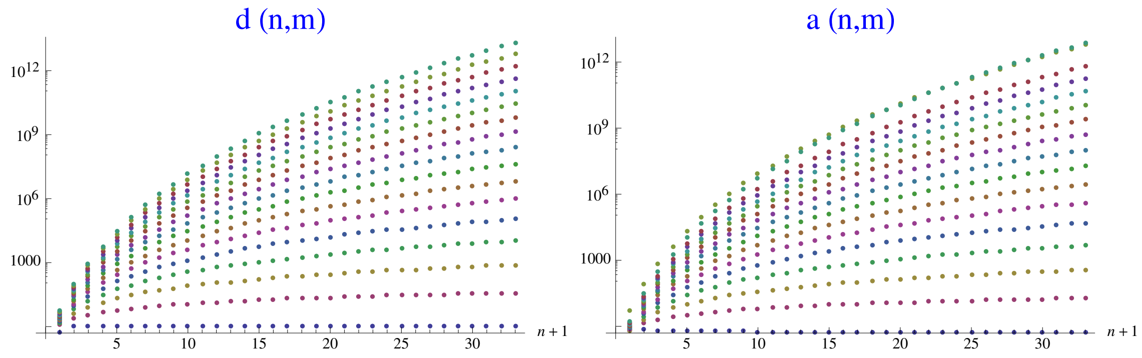

Figure 1. Coefficients $d(n, m)$ and $a(n, m)$ for generating functions of amplitudes (3.23)–(3.24) at threshold, from refs. [6, 7]. The label $n = 0, 1, \ldots, 32$ is shown along the horizontal axis and the sequences of curves correspond to $m = 0, 2, \ldots, 32$ from bottom to top. $\kappa = M_W/M_h \simeq 0.6397$.

for all values of $n \geq 1$ thus determining all coefficients[3] $d(n, 0)$. Then we solve the A-equations (3.22) for the coefficients $a(n, 0)$ for each n. Next we set $k = 1$, and solve equations (3.21) for all n to determine $d(n, 2)$. Following this, the coefficients $a(n, 2)$ are found by solving (3.22) at $k = 1$ for all values of n. This procedure is repeated for all values of k.

This iterative algorithm was implemented in [6, 7] in *Mathematica*. One can solve for $d(n, 2k)$ and $a(n, 2k)$ to any desired values of n and k numerically. Tables 1-4 in refs. [6, 7] list numerical values of the coefficients[4] up to $d(32, 32)$ and $a(32, 32)$. In figure 1 we show the logarithmic plots of all $d(n, m)$ and $a(n, m)$ for $n = 0 \ldots 32$ and $m = 0, 2, \ldots, 32$. These plots can be interpreted as sequences of curves, each curve representing $d(n, m)$ and $a(n, m)$ as functions of n for a fixed value of m. Increasing values of $m = 0, 2, \ldots, 32$ corresponds to moving upwards from lower to higher curves.

The amplitudes on the multi-particle threshold are given by the following expressions in terms of the Taylor expansion coefficients $d(n, 2k)$ and $a(n, 2k)$,

$$\mathcal{A}^{(\mathrm{thr})}_{h^* \to n\times h + m\times Z_L} = (2v)^{1-n-m}\, n!\, m!\, d(n, m)\,, \tag{3.23}$$

and for the longitudinal Z decaying into n Higgses and $m + 1$ vector bosons we have,

$$\mathcal{A}^{(\mathrm{thr})}_{Z_L^* \to n\times h + (m+1)\times Z_L} = \frac{1}{(2v)^{n+m}}\, n!\, (m+1)!\, a(n, m)\,. \tag{3.24}$$

The amplitudes with all varieties of $W_L^\pm$ and Z_L in the final state, one should simply differentiate with respect to w^a with the appropriate values of the isospin index $a = 1, 2, 3$.

At $m = 0$ the coefficients $d(n, 0) = 1$ for all $n \geq 1$ provide a useful reference point. After switching on $m > 0$, the coefficients of the generating functions grow steadily with m, reaching $d(n, m) \sim 10^8$ at $m \geq 16$ and $n \geq 27$; and $d(n, m) \sim 10^{13}$ at $m = 32$ and

[3]The solution is $d(n, 0) = 1$ for all $n \geq 1$ which is in agreement with the pure scalar theory result in (2.26).

[4]Note that in the notation of [6, 7] the value of $d(0, 0)$ was 1/2 while in the notation of the present paper $d(0, 0) \equiv 0$. This is the consequence of working with the scalar field ϕ shifted by the VEV rather than with h. The rest of the coefficients in the tables 1-4 of [6, 7] are unchanged.

$n = 31$ and similar growth with m occurs for the $a(n,m)$ coefficients of the gauge field generating function [6, 7]. This numerical growth of the coefficients is on top of $n!$ and $m!$ factors in the expressions for the amplitudes (3.23)–(3.24).

3.2 Amplitudes above the threshold

The operator $d_t^2 + M^2$ on the left hand side of the equations of motion away from the threshold becomes $P_{\rm in}^2 + M^2$. Using the expression for the incoming momentum in (3.6) together with the rescaled dimensionless variables[5] (3.7) this amounts to the substitutions

$$- (d_t^2 + 1)\phi \Longrightarrow \left[(n(1+\varepsilon_h) + 2k\,\kappa\,(1+\varepsilon_V))^2 - 1 \right] \phi(n,2k)\,, \tag{3.25}$$

$$- (d_t^2 + \kappa^2)\mathrm{A} \Longrightarrow \left[(n(1+\varepsilon_h) + (2k+1)\,\kappa\,(1+\varepsilon_V))^2 - \kappa^2 \right] \mathrm{A}(n,2k)\,, \tag{3.26}$$

on the left hand sides of eqs. (3.17) and (3.18) respectively. The quantities $\phi(n,2k)$ and $\mathrm{A}(n,2k)$ appearing on the right hand sides are the Taylor coefficients of the amplitudes including the dependence on the external Higgs and the vector bosons kinematics. Based on the results of section 2, we expect that in the large multiplicity limit $n, 2k \to \infty$ with $n\varepsilon_h$ and $2k\varepsilon_V$ held fixed,

$$\phi(n,2k) \,=\, d(n,2k)\exp\left[-C_h\, n\, \varepsilon_h \,-\, C_V\, 2k\, \varepsilon_V\right]\,, \tag{3.27}$$

$$\mathrm{A}(n,2k) \,=\, a(n,2k)\exp\left[-C_h\, n\, \varepsilon_h \,-\, C_V\, 2k\, \varepsilon_V\right]\,. \tag{3.28}$$

These expressions correspond to a rescaling $z \to z\, e^{-C_h}$ and $W \to W\, e^{-C_V}$ in the Taylor expansions for the generating functions in (3.19). As we already noted in section 2, in the large-multiplicity limit at hand, these expressions with the rescaled variables satisfy the full recursion relations for the amplitudes on and above the threshold. But in order to determine the values of the constants C_h and C_V we should first consider the $\varepsilon_h \to 0$ and $\varepsilon_V \to 0$ limit at finite general values of n and $2k$.

We will first set $2k = 0$ and expand (3.27)–(3.28) to the first order in ε_h as follows,

$$\phi(n,0) \,=\, d(n,0) + \varepsilon_h f(n,0)\,, \quad \mathrm{A}(n,0) \,=\, a(n,0) + \varepsilon_h b(n,0)\,, \tag{3.29}$$

making no a priori assumptions about the form of $f(n,0)$ and $b(n,0)$, and solving for these coefficients to determine the value of C_h and the applicability of (3.27)–(3.28).

Next, we will set $n = 0$ and determine the constant C_V.

3.2.1 n-Higgs production: solving gauge and Higgs equations for 2k=0 for all n

Here we consider the case where no additional gauge bosons were produced in the final state. We thus set $2k = 0$ and substitute for the amplitudes coefficients the leading-order expansion in terms of the Higgs kinetic energy, $d(n,0) + \varepsilon_h f(n,0)$ and $a(n,0) + \varepsilon_h b(n,0)$,

$$\mathcal{A}_{h^* \to n\times h}(\varepsilon_h) \,=\, (2v)^{1-n}\, n!\, \left(d(n,0) + \varepsilon_h f(n,0)\right)\,, \tag{3.30}$$

[5] Note that the number of vector bosons is $m = 2k$ for the amplitude $\mathcal{A}_{h^*\to n\times h+2k\times V_L}$, and $m = 2k+1$ for the amplitude $\mathcal{A}_{V_L^*\to n\times h+(2k+1)\times V_L}$.

$$\mathcal{A}_{Z_L^*\to n\times h+Z_L}(\varepsilon_h) = \frac{1}{(2v)^n}\, n!\, \left(a(n,0)+\varepsilon_h b(n,0)\right)\,. \tag{3.31}$$

Expanding the left hand sides of the classical equations (3.25)–(3.26) up to the order-ε_h^1 we have:

$$\left[n^2(1+\varepsilon_h)^2-1\right](d(n,0)+\varepsilon_h f(n,0)) = (n^2-1)\,d(n,0) \tag{3.32}$$
$$+\varepsilon_h^1\times\left[(n^2-1)f(n,0)+2n^2 d(n,0)\right]\,,$$
$$\left[(n(1+\varepsilon_h)+\kappa)^2-\kappa^2\right](a(n,0)+\varepsilon_h b(n,0)) = (n^2+2n\kappa)\,a(n,0) \tag{3.33}$$
$$+\varepsilon_h^1\times\left[(n^2+2n\kappa)\,b(n,0)+2(n^2+n\kappa)\,a(n,0)\right]$$

The ε_h-independent contributions give rise to the familiar equations for amplitudes on the threshold, (these are eqs. (3.21)–(3.22) restricted to $k=0$) which have provided us with the expressions for $d(n,0)$ and $a(n,0)$.

The order-ε_h^1 contributions result in the equations for the off-threshold corrections to the amplitudes which take the form,

$$\frac{n-1}{n}\left[(n^2-1)f(n,0)+2n^2 d(n,0)\right] = 6\,\left(F\phi\,+\,F\phi^2\right)\big|_{z^n W^0}\,, \tag{3.34}$$
$$\frac{n-1}{n}\left[(n^2+2n\kappa)\,b(n,0)+2(n^2+n\kappa)\,a(n,0)\right] = 4\kappa^2\left[(F+2F\phi)\,\mathrm{A}+(\phi+\phi^2)\,B\right]\big|_{z^n W^0}\,, \tag{3.35}$$

where the function $F(z)$ is the same as in (2.28), and we have similarly defined the new function $B(z)$ appearing in (3.35) via:

$$F(z) = \sum_{n=2}^{\infty}\frac{n-1}{n}\,f(n,0)\,z^n\,,\quad B(z) = \sum_{n=2}^{\infty}\frac{n-1}{n}\,b(n,0)\,z^n\,. \tag{3.36}$$

The equations (3.34)–(3.35) are the recursion relations for the coefficients $f(n,0)$ and $b(n,0)$ which determine the amplitudes' dependence on the kinematics. These equations are obtained following the same prescription as we have used in deriving (2.27) in the scalar theory. In fact, the first equation (3.34) is identical to (2.27) as it does not contain gauge-fields in the W^0 selection rule we have imposed.[6]

Specifically, the left hand sides of the equations (3.34)–(3.35) are given by the order-ε_h^1 contributions to the kinetic terms (3.32)–(3.33) times the overall factor of $\frac{n-1}{n}$ appearing for the same reason as in (2.17). The expressions on the right hand side in (3.34)–(3.35) arise as the leading order ε_h^1 expansion of the expressions on the right hand side of (3.21)–(3.22) with the substitution $\phi\to d(n,0)+\varepsilon_h f(n,0)$ and $\mathrm{A}\to a(n,0)+\varepsilon_h b(n,0)$, and again accompanied by the relevant $\frac{n-1}{n}$ factors, as reflected in the definitions (3.36).

The scalar field equation (3.34) was solved in section 2 with the solution given by (2.18). We can now solve the equation (3.35) for the gauge field recursively with *Mathematica*. We thus determine the coefficients $b(n,0)$ for all values of $n\geq 2$. In the large-n limit we

[6] The the last term on the right hand side of containing the gauge field A vanishes for $W=0$.

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find that solving each of the equations result in the same leading-order behaviour for the coefficients,

$$\frac{f(n,0)}{d(n,0)} \to -\frac{7}{6}\,n\,, \qquad \frac{b(n,0)}{a(n,0)} \to -\frac{7}{6}\,n\,. \tag{3.37}$$

This result (which we also checked does not depend on the value of κ) is not a coincidence. Given the n-Higgs amplitude behaviour which we derived in (2.20), it must be the case that in the double-scaling $n\varepsilon_h$ =fixed large-n limit the amplitudes (3.30)–(3.31) exponentiate

$$\mathcal{A}_{h^*\to n\times h}(\varepsilon_h) \;=\; (2v)^{1-n}\, n!\, d(n,0)\, \exp\left[-\frac{7}{6}\,n\,\varepsilon_h\right]\,, \tag{3.38}$$

$$\mathcal{A}_{Z_L^*\to n\times h+Z_L}(\varepsilon_h) \;=\; \frac{1}{(2v)^n}\, n!\, a(n,0)\, \exp\left[-\frac{7}{6}\,n\,\varepsilon_h\right]\,. \tag{3.39}$$

These expressions for the amplitudes away from the threshold correspond to the rescaling $z \to z\, e^{-\frac{7}{6}}$ in the Taylor expansions for the generating functions in (3.19).

Our next goal is to determine the constant C_V in the exponential factor for the amplitudes (3.27)–(3.28) when the gauge bosons are present in the final state.

3.2.2 2k-vector production: solving gauge and Higgs equations for n=0 for all k

We now consider the case where only the vector bosons are produced in the final state, thus we keep $2k$ general and set $n = 0$. The equations (3.27)–(3.28) are expanded to the first order in ε_V for $\varepsilon_h = 0$. We have,

$$\phi(0,2k) \;=\; d(0,2k) + \varepsilon_V\, f(0,2k)\,, \quad \mathrm{A}(0,2k) \;=\; a(0,2k) + \varepsilon_V\, b(0,2k)\,, \tag{3.40}$$

once again, making no a priori assumptions about the form of $f(0,2k)$ and $b(0,2k)$, and solving for these coefficients to determine the value of C_V in (3.27)–(3.28).

Repeating the same steps as in the previous sub-section we can write down the recursion relations at the order-ε_V^0 (cf. (3.21)= (3.22)):

$$\left[(2k\,\kappa)^2 - 1\right]\, d(0,2k) \;=\; \left[3\phi^2 + 2\phi^3 + 2\kappa^2\,(1+2\phi)\,W\mathrm{A}^2\right]\big|_{z^0W^k}\,, \tag{3.41}$$

$$\left[(1+2k)^2 - 1\right]\, a(0,2k) \;=\; 4\left[(\phi+\phi^2)\,\mathrm{A}\right]\big|_{z^0W^k}\,, \tag{3.42}$$

and at the order-ε_V^1 the Higgs-field equation is,

$$\begin{aligned}\frac{2k-1}{2k}&\left[((2k\kappa)^2-1)f(0,2k) + 8(k\kappa)^2 d(0,2k)\right] \;=\\ &=\left[6F\phi(1+\phi) + 4\kappa^2 FW\mathrm{A}^2 + 4\kappa^2\,(1+2\phi)\,W\mathrm{A}B\right]\big|_{z^0W^k}\,,\end{aligned} \tag{3.43}$$

and the gauge-field equation is,

$$\frac{2k-1}{2k}\left[4k(k+1)\,b(0,2k) + 2(k+1)^2\,a(0,2k)\right] \;=\; 4\left[(F+2\phi F)\,\mathrm{A} + (\phi+\phi^2)\,B\right]\big|_{z^0W^k} \tag{3.44}$$

The functions $F(W)$ and $B(W)$ are defined here via:

$$F(W) \;=\; \sum_{k=2}^{\infty}\frac{2k-1}{2k}\, f(0,2k)\,W^k\,, \quad B(W) \;=\; \sum_{k=2}^{\infty}\frac{2k-1}{2k}\, b(0,2k)\,W^k\,. \tag{3.45}$$

We have solved numerically the order-ε_V^0 equations (3.41)–(3.42) by iterations with *Mathematica* to determine the coefficients $d(0,2k)$ and $a(0,2k)$. Using these we solved the order-ε_V^1 equations (3.43)–(3.44) for the coefficients $f(0,2k)$ and $b(0,2k)$ for different numerical values of the κ parameter.

In the large-multiplicity limit $2k \to \infty$ our numerical results for the ratios of the coefficients,

$$\frac{f(0,2k)}{d(0,2k)} \to -C_V(\kappa)\,2k\,, \qquad \frac{b(0,2k)}{a(0,2k)} \to -C_V(\kappa)\,2k\,, \tag{3.46}$$

confirm that both ratios: for the scalar-field coefficients, and for the gauge-field coefficients approach the same numerical constant C_V, which itself depends on the value of the mass-parameter κ. For the physical value $\kappa = M_W/M_h = 80.384/125.66 \simeq 0.64$, we get

$$C_V \simeq 1.7\,, \quad \text{for} \quad \kappa = 0.64\,. \tag{3.47}$$

More generally, defining

$$-\frac{1}{2}\left(\frac{f(0,2k)}{d(0,2k)} - \frac{f(0,2k-2)}{d(0,2k-2)}\right) \to C_V^{\text{scal.}}(\kappa)\,, \quad -\frac{1}{2}\left(\frac{b(0,2k)}{a(0,2k)} - \frac{b(0,2k-2)}{a(0,2k-2)}\right) \to C_V^{\text{vect.}}(\kappa)\,,$$

we get with $2k = 54$,

$$\begin{array}{lll}
C_V^{\text{scal.}} \simeq 3.342\,, & C_V^{\text{vect.}} \simeq 3.336\,, & \text{for} \quad \kappa = 0.55 \\
C_V^{\text{scal.}} \simeq 1.702\,, & C_V^{\text{vect.}} \simeq 1.696\,, & \text{for} \quad \kappa = 0.64 \\
C_V^{\text{scal.}} \simeq 0.996\,, & C_V^{\text{vect.}} \simeq 0.996\,, & \text{for} \quad \kappa = 1. \\
C_V^{\text{scal.}} \simeq 0.829\,, & C_V^{\text{vect.}} \simeq 0.829\,, & \text{for} \quad \kappa = 2. \\
C_V^{\text{scal.}} \simeq 0.805\,, & C_V^{\text{vect.}} \simeq 0.805\,, & \text{for} \quad \kappa = 3. \\
C_V^{\text{scal.}} \simeq 0.794\,, & C_V^{\text{vect.}} \simeq 0.794\,, & \text{for} \quad \kappa = 5. \\
C_V^{\text{scal.}} \simeq 0.789\,, & C_V^{\text{vect.}} \simeq 0.789\,, & \text{for} \quad \kappa = 10. \\
C_V^{\text{scal.}} \simeq 0.787\,, & C_V^{\text{vect.}} \simeq 0.787\,, & \text{for} \quad \kappa = 100.
\end{array} \tag{3.48}$$

The convergence of the series is improved at higher values of κ. A very likely conclusion is that at an unphysical value of $\kappa = 1$ which corresponds to $M_h = M_V$ the value of the vector constant is $C_V = 1$.

Our final result for the tree-level multi-vector-boson multi-Higgs production amplitudes in the high-multiplicity double-scaling limit for $\kappa = M_W/M_h = 80.384/125.66 \simeq 0.64$ is:

$$\mathcal{A}_{h^* \to n\times h + m\times Z_L} = (2v)^{1-n-m}\, n!\, m!\, d(n,m)\, \exp\left[-\frac{7}{6}\, n\, \varepsilon_h - 1.7\, m\, \varepsilon_V\right]\,, \tag{3.49}$$

$$\mathcal{A}_{Z_L^* \to n\times h + (m+1)\times Z_L} = \frac{1}{(2v)^{n+m}}\, n!\, (m+1)!\, a(n,m)\, \exp\left[-\frac{7}{6}\, n\, \varepsilon_h - 1.7\, m\, \varepsilon_V\right] \tag{3.50}$$

The expressions (3.49)–(3.50) constitute our main results, as far as the scattering amplitudes at high multiplicities are concerned. They incorporate the dependence on the momenta of final particles (computed in the $n\,\varepsilon_h + m\,\varepsilon_V =$ fixed regime) and hence can be integrated over the phase-space. The lesson we draw from the amplitude expressions above is that even away from the multi particle thresholds they mainain the factorial dependence on the multiplicities and the further enhancement by the growing coefficients $d(n,m)$ and $a(n,m)$, found in the threshold amplitudes. The dependence on the kinematic variables of the final space provides only a mild suppression of the result on the threshold — at least in the regime where the derivation of (3.49)–(3.50) is valid. For example, for $n = 30$ and $\varepsilon_h = 0.1$ so that $n\varepsilon_h = 3$ is an order-one constant, we have $e^{-\frac{7}{6}n\varepsilon_h} \simeq 0.03$ and similarly for $m = 30$ and $\varepsilon_V = 0.1$ we have $e^{-1.7\, m\,\varepsilon_V} \simeq 0.006$ as overall multiplicative factors in the amplitudes.

In the next section we will integrate these amplitudes over the phase space in order to estimate the rates for these processes.

4 Integrating over the phase space

The scattering cross sections for multi-particle production rates arise from integrating the squared amplitudes (3.49)–(3.50) over the Lorentz-invariant phase space,

$$\sigma_{n,m} = \int d\Phi_{n,m}\, \frac{1}{n!\, m!}\, \left|\mathcal{A}_{h^*\to n\times h+m\times Z_L}\right|^2 , \tag{4.1}$$

where $1/n!$ and $1/m!$ are the Bose statistics factors accounting for the n identical Higgses and m identical longitudinal vector boson states, and we have dropped the overall flux factor on the r.h.s. of (4.1). The next step is to integrate over phase space. The n-particle Lorentz-invariant phase space volume element has the familiar form,

$$\int d\Phi_n = (2\pi)^4 \delta^{(4)}(P_{\rm in} - \sum_{j=1}^{n} p_j) \prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3\, 2p_j^0} , \tag{4.2}$$

but in order to use in (4.1) our results for the amplitudes (3.49) within their region of validity — i.e. the high-multiplicity non-relativistic limit — the phase space integrations have to be performed in the same non-relativistic approximation.

We note that it should not come as a surprise that the large-n small-ε limit will amount to a very small phase-space volume. Indeed, a very rough estimate for the phase-space volume in this approximation will be $\Phi_n \propto M^{3n} \times \varepsilon^{3n/2}$. In dimensionless units, it arises from the product of n three-dimensional spherical volumes obtained by integrating over each of the final particle momenta $|p|_i \lesssim M\sqrt{2\varepsilon}$. It is then not surprising that the resulting volume of the non-relativistic n-particle phase-space reduces the cross section by the factor $\propto \varepsilon^{3n/2}$ which is $\ll 1$ in the limit $\varepsilon \to 0$ and $n \gg 1$. We will confirm this estimate with a more precise computation below, but it is important to stress from the outset that the suppression of the resulting cross sections at moderate energies is entirely caused by the non-relativistic approximation used in computing the phase-space volume, and is not driven by the form of the amplitudes squared. In order to compute the rate in the more realistic settings, one should integrate over a larger portion of the phase-space. In the present paper we will not pursue this route as this would require knowing the amplitudes beyond the non-relativistic limit.

The phase-space integration in the large-n non-relativistic limit with $n\varepsilon_h$ fixed is easily carried out by integrating over the $d^{3n}p$ volume of the $3n$-dimensional of radius

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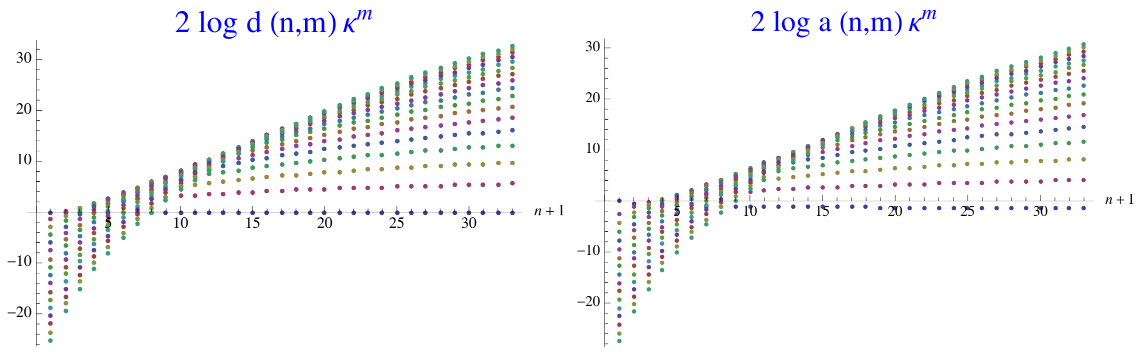

Figure 2. Amplitude coefficients of figure 1 in the form $2\log(\kappa^m d(n,m))$ and $2\log(\kappa^m a(n,m))$ appearing in eqs. (4.6)–(5.6). The label $n = 0, 1, \ldots, 32$ is shown along the horizontal axis and the sequence of curves corresponds to $m = 0, 2, \ldots, 32$ with m increasing from bottom to top (on the right of each plot).

$|p| = M_h\sqrt{2n\varepsilon_h}$. The resulting non-relativistic phase space volume in the large-n limit is (see e.g. [17]),

$$\Phi_n \simeq \frac{1}{\sqrt{n}}\left(\frac{M_h^2}{2}\right)^n \exp\left[\frac{3n}{2}\left(\log\frac{\varepsilon_h}{3\pi}+1\right) + \frac{n\varepsilon_h}{4} + \mathcal{O}(n\varepsilon_h^2)\right]. \tag{4.3}$$

Combining this with the n-Higgs amplitude squared (with $m = 0$ vector bosons), we get,

$$\begin{aligned}\frac{1}{n!}\,\Phi_n\,|\mathcal{A}_n|^2 &\simeq \Phi_n\,(2v)^{-2n}\,n!\,d(n,0)^2\,\exp\left[-\frac{7}{3}\,n\,\varepsilon_h\right]\\ &\simeq \frac{1}{\sqrt{n}}\exp\left[2\log d(n,0) + n\left(\log\frac{\lambda n}{4}-1\right) + \frac{3n}{2}\left(\log\frac{\varepsilon_h}{3\pi}+1\right) - \frac{25}{12}\,n\,\varepsilon_h\right]\end{aligned} \tag{4.4}$$

Repeating the same steps for vector boson emissions we now can write down the rate for the high multiplicity n-Higgs + m-vector boson production corresponding to the amplitude (3.49),

$$\begin{aligned}\sigma_{n,m} \sim \exp\Bigg[&2\log d(n,m) + n\left(\log\frac{\lambda n}{4}-1\right) + m\left(\log\left(\frac{g^2 m}{32}\right)-1\right)\\ &+\frac{3n}{2}\left(\log\frac{\varepsilon_h}{3\pi}+1\right) + \frac{3m}{2}\left(\log\frac{\varepsilon_V}{3\pi}+1\right) - \frac{25}{12}n\varepsilon_h - 3.15\,m\varepsilon_V + \mathcal{O}(n\varepsilon_h^2 + m\varepsilon_V^2)\Bigg]\end{aligned} \tag{4.5}$$

The cross section arising from the amplitude (3.50) takes the same form as (4.5) but with the $2\log a(n,m)$ factor on the right hand side. The numerical coefficients $d(n,m)$ and $a(n,m)$ were derived in [6, 7] by solving recursion relations for the amplitudes on the multi-particle mass threshold; they are plotted in figure 1.

At $m = 0$ all d-coefficients are equal to one, hence the first term on the right hand side vanishes in this case, $2\log d(n, m = 0) = 0$. At higher values of m, however the coefficients $d(n,m)$ and $a(n,m)$ start growing. To somewhat tame the numerical growth of the Taylor coefficients we can rescale them with a factor of κ^m and this can be nicely combined with the observation that $m\log(\frac{g^2 m}{32}) = m\log(\kappa^2) + m\log(\frac{\lambda m}{4})$ which facilitates a re-write of (4.5) in the form:

$$\sigma_{n,m} \sim \exp\Big[2\log(\kappa^m d(n,m)) + n\log\frac{\lambda n}{4} + m\log\frac{\lambda m}{4} \tag{4.6}$$
$$+\frac{n}{2}\left(3\log\frac{\varepsilon_h}{3\pi}+1\right) + \frac{m}{2}\left(3\log\frac{\varepsilon_V}{3\pi}+1\right) - \frac{25}{12}n\,\varepsilon_h - 3.15\,m\,\varepsilon_V + \mathcal{O}(n\varepsilon_h^2+m\varepsilon_V^2)\Big]$$

The amplitudesTaylor coefficients in the form $2\log(\kappa^m d(n,m))$ and $2\log(\kappa^m a(n,m))$ appearing on the right hand side are shown in figure 2.

5 Conclusions

The expressions in eq. (4.5) or equivalently in eq. (4.6) characterise the cross section $\sigma_{n,m}$ for the multi-particle n-Higgs m-vector boson production obtained in the high-multiplicity non-relativistic limit. They were derived in the Gauge-Higgs theory and based on computing all tree-level scattering amplitudes with n-Higgs m-longitudinal-vector boson final states, derived on- and off- the multi-particle mass threshold in eqs. (3.49)–(3.50). These are our main results.

As we have already noted, the imposition of the non-relativistic limit dramatically reduces the otherwise available phase-space to a tiny volume $\propto \exp\left[-\frac{3n}{2}\log\frac{3\pi/e}{\varepsilon_h} - \frac{3m}{2}\log\frac{3\pi/e}{\varepsilon_V}\right]$. At moderately high energies this suppression factor will dominate the cross section, as we will illustrate below. However, one should keep in mind that this effect is is simply an artifact of the approximation used for computing the phase-space.

Before we conclude, it will be useful to sketch some simple estimates for the energy scales involved. First, we would like to estimate the value of $\log(\sigma_{n,m})$ for a "minimal interesting value" of final particle multiplicities, $n = m = 30$ which is roughly $1/\alpha_W$ (they of course also correspond to the highest multiplicities where we have calculated the values of the Taylor coefficients $d(n,m)$ and $d(n,m)$). Following our expression in eq. (4.6) we have,

$$n = m = 30 \Longrightarrow 2\log(\kappa^m d(n,m)) \simeq 30\,, \tag{5.1}$$
$$\sqrt{\hat{s}} \simeq 6.8\,\text{TeV} \qquad n\log\frac{\lambda n}{4} = m\log\frac{\lambda m}{4} \simeq -0.02 \simeq 0\,, \tag{5.2}$$

$$\varepsilon_h = \varepsilon_V = 0.1 \Longrightarrow \frac{n}{2}\left(3\log\frac{\varepsilon_h}{3\pi}+1\right) = \frac{m}{2}\left(3\log\frac{\varepsilon_V}{3\pi}+1\right) \simeq -190\,, \tag{5.3}$$
$$-\frac{25}{12}n\varepsilon_h - 3.15\,m\varepsilon_V \simeq -15\,, \tag{5.4}$$

where we have been careful in (5.3) to select an appropriately small value 0.1 of the kinetic energy per particle to be consistent with the non-relativistic limit. This amounts to

$$n = m \simeq 30 \Longrightarrow \qquad \log(\sigma) \simeq 30 - 190 - 190 - 15 = -365\,, \tag{5.5}$$

which amounts to a negligibly small rate $\sigma_{30,30} \simeq 0.3\times10^{-160}$. Clearly, to have a higher rate, we need to increase the number of particles in the final state.

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Still, at even higher multiplicities perturbation theory will break down and perturbative unitarity will be violated by exponentially growing rates even within the current non-relativistic phase-space limit. To see this, let us re-arrange the expression in eq. (4.6) as follows:

$$\sigma_{n,m} \sim \exp\left[2\log(\kappa^m d(n,m)) + m\log\left(\frac{\lambda\, m\varepsilon_V}{12\pi}\sqrt{\frac{\varepsilon_V}{3\pi}}\right) + m\,(0.5 - 3.15\,\varepsilon_V)\right.$$
$$\left. + n\log\left(\frac{\lambda\, n\varepsilon_h}{12\pi}\sqrt{\frac{\varepsilon_h}{3\pi}}\right) + n\left(\frac{1}{2} - \frac{25}{12}\,\varepsilon_h\right) + \mathcal{O}(n\varepsilon_h^2 + m\varepsilon_V^2)\right]. \quad (5.6)$$

This result holds in the double scaling limit, $n \to \infty$, $m \to \infty$, $\varepsilon_h \to 0$, $\varepsilon_V \to 0$ with $n\varepsilon_h$ and $m\varepsilon_V$ held fixed.

We now consider a somewhat extreme case with the number of produced vector bosons is very large, $m \simeq 7500$ and we keep the number of Higgs bosons small, for simplicity. Then only the terms on the first line of eq. (5.6) matter. If we also assume $\varepsilon_V = 0.5$ we would get,

$$m = 7500\,,\ \varepsilon_V = 0.5 \Longrightarrow m\log\left(\frac{\lambda\, m\varepsilon_V}{12\pi}\sqrt{\frac{\varepsilon_V}{3\pi}}\right) + m\,(0.5 - 3.15\,\varepsilon_V) \simeq 0\,, \quad (5.7)$$

$$\sqrt{\hat{s}} \simeq 845\,\text{TeV} \qquad \sigma_{n,m} \sim \exp\left[2\log(\kappa^m d(n,m))\right], \quad (5.8)$$

and if we increase the number of vector bosons by 100 more at a cost of extra 10 TeV at these energies, we would get

$$m = 7600\,,\ \varepsilon_V = 0.5 \Longrightarrow m\log\left(\frac{\lambda\, m\varepsilon_V}{12\pi}\sqrt{\frac{\varepsilon_V}{3\pi}}\right) + m\,(0.5 - 3.15\,\varepsilon_V) \simeq 102\,, \quad (5.9)$$

$$\sqrt{\hat{s}} \simeq 855\,\text{TeV} \qquad \sigma_{n,m} \sim \exp\left[2\log(\kappa^m d(n,m))\right] \times e^{102} \ggg 1\,, \quad (5.10)$$

This behaviour is obviously in violation of perturbative unitarity even if we do not worry about the additional factor of $2\log(\kappa^m d(n,m))$ which is likely to continue growing beyond the value of 30 in (5.2) at these multiplicities. This regimes is also beyond the validity region of eq. (5.6) since the unknown corrections of the order $m\varepsilon_V^2$ are large.

Similarly, in the case of mostly Higgs production, i.e. at low m and $n \simeq 4000$ we find,

$$n = 4000\,,\ \varepsilon_h = 0.65 \Longrightarrow n\log\left(\frac{\lambda\, n\varepsilon_h}{12\pi}\sqrt{\frac{\varepsilon_h}{3\pi}}\right) + n\,(1/2 - 25/12\,\varepsilon_h) \simeq 23\,, \quad (5.11)$$

$$\sqrt{\hat{s}} \simeq 830\,\text{TeV} \qquad \sigma_{n,m} \sim \exp\left[2\log(\kappa^m d(n,m))\right] \times e^{23} \ggg 1\,, \quad (5.12)$$

The main conclusion we want to draw from the computations presented in this paper is that perturbation theory does break down in the weak sector of the Standard Model. This breakdown occurs already at leading order (i.e. tree-level) in the perturbative expansion.[7] To accurately determine *the lower bound* on the energy scale where this breakdown does

[7]The higher-loop corrections are expected to make this worse by introducing corrections of the order $\lambda n^2/(4\pi)$ and $\alpha_W m^2/(4\pi)$ which are $\gg 1$ at the sufficiently high multiplicities.

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occur would require going beyond the double-scaling high-multiplicity non-relativistic limit we have assumed and used throughout (and would also require including higher-order corrections as well as computing even higher multiplicity amplitudes). Our rough estimate is that the perturbative meltdown energy range is not far from a few hundred TeV even after including the effect of the highly suppressed non-relativistic phase-space volume.

To establish whether these very high multiplicity processes become observable and even dominant at future circular hadron colliders, to determine what is the precise energy scale where this happens and what is the average number of bosons produced, ultimately requires developing of a non-perturbative (possibly semi-classical) technique in the electroweak sector of the Standard Model. In particular, it was argued in the 90-s (see [8, 9, 15–17, 25–27] and references therein) that multi-particle production cross-sections in a scalar ϕ^4-type field theory take the form:

$$\sigma_n \sim \exp\left[-\frac{1}{\lambda}F(\lambda n, \varepsilon)\right] , \tag{5.13}$$

in the double-scaling $\lambda \to 0$, $n \to \infty$ limit with λn and ε held fixed. This exponential behaviour of the n-particle cross-section was obtained from extrapolating results derived in perturbation theory. The original perturbative computations are valid in the small-λn limit where $F(\lambda n, \varepsilon) > 0$ and the cross-section is exponentially suppressed. This is extrapolated to the regime of interest where λn is no longer small and is treated as a finite fixed quantity. In this regime the function F is unknown, but the exponential behaviour of the cross-section has a suggestive semi-classical form. The function $F(z, \varepsilon)$ appearing in the exponent is sometimes referred to as the "holy grail" function; in the simplest case scenario of a one scale scalar theory[8] it depends on two arguments, $z = \lambda n$ and the scaled kinetic energy ε. In order to determine whether or not $F(z, \varepsilon)$ can approach zero at some finite value of z for the high-multiplicity cross-section to become observable and even dominant, one would require a non-perturbative technique accessible and applicable at finite values of z. This non-perturbative technique is still lacking.

The calculation presented in this paper is purely perturbative, in fact tree-level, and it does not attempt to uncover the relevant higher-order and non-perturbative dependence on λn. We instead discuss the dependence of the cross-section on the scaled kinetic energy ε (still at the leading tree-level order in perturbation theory) and the leading-order dependence on λn. These considerations give an indication of interesting and non-trivial behaviour of cross-sections at large multiplicities and they also confirm the breakdown of ordinary perturbation theory — now in the case of the Standard Model - type Gauge-Higgs theory, i.e. beyond the pure scalar field theory example studied before. By extrapolating the leading-order perturbative treatment, it is clear from the results presented in this paper that the energy scale does exist where cross sections not only become large, but also grow and break perturbative unitarity. The discussion of scale where perturbative cross-sections become large is is addressed in more detail in a companion publication [28]. At the same time, the fully non-perturbative behaviour of the theory remains unknown. This is an open problem and we plan to return it in future work.

[8]With a single coupling λ and a single mass scale m.

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A Unbroken ϕ^4 theory

For completeness and to provide another illustration for using the formalism outlined in section 2, we will re-derive here the results of ref. [16] for multi-particle amplitudes in the ϕ^4 theory with no spontaneous symmetry breaking,

$$\mathcal{L} \,=\, \frac{1}{2}\,(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{1}{4}\lambda\phi^4\,. \tag{A.1}$$

The corresponding classical equation for the theory (A.1) is

$$-\,(\partial^\mu\partial_\mu + M^2)\,\phi \,=\, \lambda\,\phi^3\,, \tag{A.2}$$

and we are after the n-point amplitude in the non-relativistic limit $\varepsilon \ll 1$ in the form (2.16),

$$\mathcal{A}_n(p_1\ldots p_n) \,=\, n!\,(d_n \,+\, f_n\,\varepsilon)\,. \tag{A.3}$$

At the order ε^0 the classical equation (A.2) gives the recursion relation for d_n (cf. eq. (2.8)):

$$(n^2-1)\,d_n \,=\, \frac{\lambda}{M^2}\sum_{n_1,n_2,n_3}^{n}\delta^n_{n_1+n_2+n_3} d_{n_1}\,d_{n_2}\,d_{n_3}\,, \qquad d_1 = 1\,, \tag{A.4}$$

with the solution, $d_n = (\lambda/(8M^2))^{(n-1)/2}$ for $n = 3, 5, 7, \ldots$. At the order-ε^1 we can write down the recursion relation for the f_n coefficients following the same routine as we did in writing (2.17),

$$\frac{n-1}{n}\,\left((n^2-1)\,f_n \,+\, 2n^2\,d_n\right) \,=\, 3\,\frac{\lambda}{M^2}\sum_{n_1,n_2,n_3}^{n}\delta^n_{n_1+n_2+n_3}\,\frac{n_1-1}{n_1}\,f_{n_1}\,d_{n_2}\,d_{n_3}\,. \tag{A.5}$$

The factor of 3 on the right hand side of (A.5) accounts for the combinatorial factor which occurs in the order-ε expansion of $(d_{n_1}+\varepsilon f_{n_1})(d_{n_2}+\varepsilon f_{n_2})(d_{n_3}+\varepsilon f_{n_3})$ to obtain $3\,\varepsilon\,f_{n_1}\,d_{n_2}\,d_{n_3}$. The factors of $\frac{n-1}{n}$ and $\frac{n_1-1}{n_1}$ on both sides of (A.5) arise as before from the form of kinetic energy in (2.15).

The recursion relation (A.5) is solved by introducing the function $F(z)$, exactly as in (2.27), and solving the ordinary differential equation

$$\frac{n-1}{n}\,\left((n^2-1)\,f_n \,+\, 2n^2\,d_n\right) \,=\, 3\,\frac{\lambda}{M^2}\,F(z)\phi(z)^2|_{z^n}\,. \tag{A.6}$$

It is straightforward to solve this equation with *Mathematica* and we have checked that the solution is the same as found in [16], which reads $f_n \,=\, -\,\left(\frac{5}{6}\,n \,-\, \frac{1}{6}\,\frac{n}{n-1}\right)\,d_n$. The resulting amplitude in the double-scaling limit $n\to\infty$, $\varepsilon\to 0$, $n\varepsilon$ fixed, is then given by

$$\mathcal{A}_n^{\rm noSSB} \,=\, n!\,\left(\frac{\lambda}{8M^2}\right)^{\frac{n-1}{2}}\,\exp\left[-\frac{5}{6}\,n\,\varepsilon\right]\,, \tag{A.7}$$

in agreement with ref. [16].

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