

# REPRESENTATION ZETA FUNCTIONS OF SOME NILPOTENT GROUPS ASSOCIATED TO PREHOMOGENEOUS VECTOR SPACES

ALEXANDER STASINSKI AND CHRISTOPHER VOLL

ABSTRACT. We compute the representation zeta functions of some finitely generated nilpotent groups associated to unipotent group schemes over rings of integers in number fields. These group schemes are defined by Lie lattices whose presentations are modelled on certain prehomogeneous vector spaces. Our method is based on evaluating  $\mathfrak{p}$ -adic integrals associated to certain rank varieties of matrices of linear forms.

## 1. INTRODUCTION

Let  $G$  be a finitely generated torsion-free nilpotent group (or  $\mathcal{T}$ -group, for short). The representation zeta function of  $G$  is the Dirichlet series

$$\zeta_G(s) := \sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s},$$

where  $\tilde{r}_n(G)$  denotes the number of twist-isoclasses of complex  $n$ -dimensional irreducible representations of  $G$  and  $s$  is a complex variable. If  $K$  is a number field with ring of integers  $\mathcal{O}$  and  $\mathbf{G}$  is a unipotent group scheme over  $\mathcal{O}$ , then  $\mathbf{G}(\mathcal{O})$  is a  $\mathcal{T}$ -group and  $\zeta_{\mathbf{G}(\mathcal{O})}(s)$  has an Euler product indexed by the non-zero prime ideals of  $\mathcal{O}$ :

$$(1.1) \quad \zeta_{\mathbf{G}(\mathcal{O})}(s) = \prod_{\mathfrak{p}} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s),$$

where  $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$  is the zeta function enumerating twist-isoclasses of continuous irreducible representations of the pro- $p$  group  $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$ ; see [11, Proposition 2.2]. General properties of representation zeta functions of the form  $\zeta_{\mathbf{G}(\mathcal{O})}(s)$  were studied in [11]. In particular, it was shown there that for all but finitely many  $\mathfrak{p}$  the factors of the Euler product (1.1) are rational functions in  $q^{-s}$ , where  $q = |\mathcal{O}/\mathfrak{p}|$ .

Almost all of the Euler factors may be expressed in terms of  $\mathfrak{p}$ -adic integrals associated to rank varieties of matrices of linear forms; cf. [11, Corollary 2.11]. In general, computing these integrals is a hard problem. In [11, Theorem B] we computed the representation zeta functions for three infinite, explicitly described families. One motivation for these computations was the idea to construct and study  $\mathcal{T}$ -groups through presentations modelled on prehomogeneous vector spaces (PVSs) and their relative invariants. Indeed, the study of  $\mathfrak{p}$ -adic integrals associated to the relative invariants of PVSs has a comparatively long history; see [4] and [7] for details. These integrals also served as test cases for a number of far-reaching conjectures and explicit formulae for many such integrals are known.

In the current paper as well as in [11, Theorem B], we consider  $\mathcal{T}$ -groups which are groups of rational points of unipotent group schemes defined in terms of certain  $\mathbb{Z}$ -Lie

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Then  $\text{Pf}(\mathcal{R}(\mathbf{Y})) = Y^{m-n}f(\mathbf{Y}_0)$ ; see Lemma 2.1. The matrix  $\mathcal{R}(\mathbf{Y})$  defines the class-2-nilpotent  $\mathbb{Z}$ -Lie lattice

$$\mathcal{Q}_{m,n} = \langle x_r, y_{ij}, y \mid 1 \leq r \leq 2(m+n), 1 \leq i \leq 2m, 1 \leq j \leq 2n, \\ \forall 1 \leq a, b \leq 2(m+n) : [x_a, x_b] = \mathcal{R}(\mathbf{y})_{a,b}, y, y_{ij} \text{ central} \rangle.$$

This Lie lattice in turn defines a unipotent group scheme  $Q_{m,n}$  over  $\mathbb{Z}$  via the Hausdorff series; cf. [11, Section 2.12]. Note that, in this vein, the matrix (1.2) defines the  $\mathbb{Z}$ -Lie lattice  $\mathcal{G}_2$  in [11, Definition 1.2].

Let  $K$  be a number field with ring of integers  $\mathcal{O}$ . Then  $Q_{m,n}(\mathcal{O})$  is a  $\mathcal{T}$ -group of nilpotency class 2. For a non-zero prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$ , we write  $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$  for the completion of  $\mathcal{O}$  at  $\mathfrak{p}$  and  $q = |\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}|$  for the residue cardinality. Let  $\zeta_K(s)$  denote the Dedekind zeta function of  $K$ . Our main results are the following.

**Theorem 1.1.** *For every non-zero prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$ , writing  $t = q^{-s}$ ,*

$$\zeta_{Q_{1,1}(\mathcal{O}_{\mathfrak{p}})}(s) = \frac{(1-t)(1-q^2t)}{(1-q^3t)(1-q^4t)}.$$

**Corollary 1.2.** *The zeta function*

$$\zeta_{Q_{1,1}(\mathcal{O})}(s) = \frac{\zeta_K(s-3)\zeta_K(s-4)}{\zeta_K(s)\zeta_K(s-2)}$$

*has abscissa of convergence 5 and meromorphic continuation to the whole complex plane.*

**Theorem 1.3.** *For every non-zero prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$ , writing  $t = q^{-s}$ ,*

$$\zeta_{Q_{2,1}(\mathcal{O}_{\mathfrak{p}})}(s) = \frac{(1-t)(1+q^3t)(1-q^2t)(q^8t^3 - q^7t^2 + q^6t^2 - q^5t^2 - q^3t + q^2t - qt + 1)}{(1-q^5t)(1-q^5t^2)(1-q^4t)(1-q^8t^2)}.$$

**Corollary 1.4.** *The zeta function  $\zeta_{Q_{2,1}(\mathcal{O})}(s)$  has abscissa of convergence 6 and meromorphic continuation to  $\{s \in \mathbb{C} \mid \text{Re}(s) > 7/2\}$ .*

Our earlier result [11, Theorem B] comprises an explicit computation of the representation zeta functions of infinite families of groups. Its proof is based on a recursive procedure, exploiting the genericity of various matrices of linear forms encoding presentations of the relevant Lie lattices. In the present case, the matrices  $\mathcal{R}(\mathbf{Y})$  do not seem to lend themselves to a similar recursive analysis. This is why we resort to an explicit analysis of  $\mathfrak{p}$ -adic integrals associated to the relevant rank varieties.

We remark that the uniformity of the analytic invariants determined in Corollaries 1.2 and 1.4, viz. their independence of  $\mathcal{O}$ , is a general feature of representation zeta functions of  $\mathcal{T}$ -groups obtained from unipotent group schemes; cf. [3] for details.

**1.1. Topological representation zeta functions.** In [10], Rossmann introduced the topological representation zeta function  $\zeta_{\mathbf{G},\text{top}}(s)$  associated to a unipotent group  $\mathbf{G}$  defined over a number field. This is a rational function in a parameter  $s$  which, in a certain precise sense, captures the behaviour of the local representation zeta functions  $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$  associated to  $\mathbf{G}$  in the limit as ‘ $q \rightarrow 1$ ’; cf. [10, Definition 3.5]. Informally, the topological representation zeta function of  $\mathbf{G}$  is the leading term of the expansion in  $q-1$  of the local representation zeta functions  $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ . Given the explicit formulae in Theorems 1.1 and 1.3 it follows easily that

$$\zeta_{Q_{1,1,\text{top}}}(s) = \frac{s(s-2)}{(s-3)(s-4)} \quad \text{and} \quad \zeta_{Q_{2,1,\text{top}}}(s) = \frac{2s(s-2)(s^2-5s+5)}{(s-5)(2s-5)(s-4)^2};$$

cf. [9, Section 4]. Simple computations show that Questions 7.1, 7.2, 7.4, and 7.5 raised in [10, Section 7] have positive answers in the cases under consideration.

**1.2. A potential connection with Coxeter group statistics.** A general theme in [11] is the description of local factors of representation zeta functions of certain  $\mathcal{T}$ -groups in terms of statistics on Coxeter groups of type  $B$  generalizing the classical Coxeter length function. Whenever available, such descriptions afford much more concise and explicit formulae than those given in general by  $\mathfrak{p}$ -adic integrals. They also allow for direct proofs of certain local functional equations; cf. [11, Theorem A]. Indeed, whilst in general these symmetries are consequences of the Weil conjectures for smooth projective algebraic varieties over finite fields, in the presence of Coxeter group theoretic interpretations they are often easily deduced from comparatively elementary symmetry features of Coxeter groups.

We record here an observation that might help explain the “exceptional” factor in the numerator in Theorem 1.3 in such Coxeter group theoretic terms. We write  $B_m$  for the Coxeter group of signed  $m \times m$ -permutation matrices,  $\ell(w)$  for the Coxeter length of an element  $w \in B_m$ , and  $D(w)$  for its (right) descent set; cf. [2, Section 8.1] for details. Set  $Z_0 = q^4t$ ,  $Z_1 = q^8t^2$ . Then, in  $\mathbb{Z}[q, t]$ ,

$$\sum_{w \in B_2} (-q^{-1})^{\ell(w)} \prod_{i \in D(w)} Z_i = q^8t^3 - q^7t^2 + q^6t^2 - q^5t^2 - q^3t + q^2t - qt + 1.$$

By [11, Lemma 4.4], this means that, in  $\mathbb{Q}(q, t)$ ,

$$(1.4) \quad \frac{q^8t^3 - q^7t^2 + q^6t^2 - q^5t^2 - q^3t + q^2t - qt + 1}{(1 - q^4t)(1 - q^8t^2)} = \sum_{I \subseteq \{0,1\}} f_{2,I}(-q^{-1}) \prod_{i \in D(w)} \frac{Z_i}{1 - Z_i},$$

where  $f_{n,I}(X) = \sum_{w \in B_n, D(w) \subseteq I} X^{\ell(w)} \in \mathbb{Z}[X]$  for  $n \in \mathbb{N}$  and  $I \subseteq \{0, 1, \dots, n-1\}$ . Specifically,

$$\begin{aligned} f_{2,\emptyset}(X) &= 1, \\ f_{2,\{0\}}(X) &= f_{2,\{1\}}(X) = 1 + X + X^2 + X^3, \\ f_{2,\{0,1\}}(X) &= 1 + 2X + 2X^2 + 2X^3 + X^4 \quad (\text{the Poincaré polynomial of } B_2). \end{aligned}$$

The contribution to the topological representation zeta function  $\zeta_{Q_{2,1,\text{top}}}(s)$  from the factor (1.4) is  $\frac{2s(s-2)(s^2-5s+5)}{(s-4)^2}$ .

For  $(m, n) = (1, 1)$ , there is a similar interpretation of a factor in Theorem 1.1 in terms of  $B_1(\cong C_2)$ :

$$(1.5) \quad \frac{1 - q^2t}{1 - q^3t} = 1 + (-q^{-1} + 1) \frac{q^3t}{1 - q^3t} = \sum_{I \subseteq \{0\}} f_{1,I}(-q^{-1}) \frac{q^3t}{1 - q^3t}.$$

The contribution to the topological representation zeta function  $\zeta_{Q_{2,1,\text{top}}}(s)$  from the factor (1.5) is  $\frac{s-2}{s-3}$ .

Note that the identities (1.4) and (1.5) are interpretations of our explicit formulae for the respective local zeta functions. Any *a priori* proofs for these formulae seem likely to yield proofs of Theorems 1.1 and 1.3 which are much less arduous than the  $\mathfrak{p}$ -adic computations that make up the bulk of the current paper. Whether analogous identities help to explain the remaining factors and the (presumably involved) formulae for the zeta functions  $\zeta_{Q_{m,n(o)}}(s)$  for further values of  $m$  and  $n$  is therefore an interesting open question.

**1.3. Organisation and notation.** Section 2 comprises a number of elementary preliminary lemmas. In Section 3 we compute  $\zeta_{Q_{m,n}(\mathfrak{o})}(s)$  in the case  $(m, n) = (1, 1)$ . In Section 4 we do the same for  $(m, n) = (2, 1)$ , this case being substantially more involved.

We fix some additional notation. Let  $\mathbb{N}$  denote the set of positive integers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Let  $v$  be the valuation on  $\mathfrak{o}$  and for  $x \in \mathfrak{o}$ , let  $|x| = q^{-v(x)}$  denote the  $\mathfrak{p}$ -adic absolute value. For  $d \in \mathbb{N}$  we write  $\mathfrak{p}^d$  for the  $d$ th power of the ideal  $\mathfrak{p}$  and  $\mathfrak{p}^{(d)}$  for the  $d$ -fold Cartesian product of  $\mathfrak{p}$ , considered as a subset of the  $d$ -fold product  $\mathfrak{o}^{(d)} = \mathfrak{o}^d$ . We write  $\mathfrak{o}^\times$  for the group of units of  $\mathfrak{o}$  and  $\varpi$  for a fixed uniformiser of  $\mathfrak{o}$ . We set  $W_d(\mathfrak{o}) = \mathfrak{o}^d \setminus \mathfrak{p}^{(d)}$  and  $A_d$  for a fixed set of representatives of the residue classes under the reduction mod  $\mathfrak{p}$  map  $W_d(\mathfrak{o}) \rightarrow \mathbb{F}_q^d \setminus \{0\}$ . Moreover, let  $A_d^0 = A_d \cup \{0\}$ .

We use boldface letters, such as  $\mathbf{Y}$  and  $\mathbf{y}$ , for vectors of variables. For a finite set  $H$  of polynomial functions corresponding to polynomials in  $\mathfrak{o}[\mathbf{Y}]$ , let  $\|H(\mathbf{y})\| : \mathfrak{o}^d \rightarrow \mathbb{R}$  denote the function defined by  $\mathbf{a} \mapsto \max\{|f(\mathbf{a})| \mid f \in H\}$ . If  $H = \{f_1(\mathbf{Y}), \dots, f_n(\mathbf{Y})\}$  we will often write  $\|f_1(\mathbf{y}), \dots, f_n(\mathbf{y})\|$  for  $\|H(\mathbf{y})\|$ .

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## 2. AUXILIARY LEMMAS

**Lemma 2.1.** *Let  $m \geq n \geq 1$  and  $\mathcal{R}(\mathbf{Y}) \in \text{Mat}_{2(m+n)}(\mathbb{Z}[\mathbf{Y}])$  as in (1.3). Then*

$$\text{Pf}(\mathcal{R}(\mathbf{Y})) = Y^{m-n} \text{Pf}((Y_{ij})^t J_m(Y_{ij})).$$

*Proof.* Consider  $\mathcal{R}(\mathbf{Y})$  as a matrix over the bigger ring  $\mathbb{Z}[\mathbf{Y}, Y^{-1}]$ , so that the block  $Y J_m$  is invertible. It is well known that if  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  is a block matrix over a commutative ring such that  $A$  is invertible, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B).$$

This implies that

$$\begin{aligned} \det(\mathcal{R}(\mathbf{Y})) &= \det(Y J_m) \det(-(Y_{ij})^t Y^{-1} (-J_m)(Y_{ij})) = \\ &= Y^{2m} \det(Y^{-1} (Y_{ij})^t J_m(Y_{ij})) = Y^{2m-2n} \det((Y_{ij})^t J_m(Y_{ij})), \end{aligned}$$

whence the expression for the Pfaffian.  $\square$

**Lemma 2.2.** The following identities hold in the field  $\mathbb{Q}((a, b, c))$ :

- (1)  $\sum_{X \in \mathbb{N}} X a^X = \frac{a}{(1-a)^2}$ ,
- (2)  $\sum_{(X,Y) \in \mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}} = \frac{abc(1-ab)}{(1-abc)(1-a)(1-b)}$ ,
- (3)  $\sum_{(X,Y,Z) \in \mathbb{N}^3} a^X b^{Y+Z} c^{\min\{X,Y+Z\}} = \frac{ab^2c(1-a+ac-2abc+a^2b^2c)}{(1-abc)^2(1-a)(1-b)^2}$ .

*Proof.* For (1), just observe that

$$\sum_{X \in \mathbb{N}} X a^X = a \left( \sum_{X \in \mathbb{N}} X a^{X-1} \right) = a \frac{d}{da} \left( \sum_{X \in \mathbb{N}} a^X \right) = a \frac{d}{da} \left( \frac{a}{1-a} \right) = \frac{a}{(1-a)^2}.$$

For (2), first consider the case  $X \leq Y$  and let  $Y = X + \tilde{Y}$  with  $\tilde{Y} \in \mathbb{N}_0$ . Then

$$\sum_{\substack{(X,Y) \in \mathbb{N}^2 \\ X \leq Y}} a^X b^Y c^{\min\{X,Y\}} = \sum_{(X,\tilde{Y}) \in \mathbb{N} \times \mathbb{N}_0} a^X b^{X+\tilde{Y}} c^X = \frac{abc}{1-abc} \cdot \frac{1}{1-b}.$$

Next, consider the case  $X > Y$  and let  $X = Y + \tilde{X}$  with  $\tilde{X} \in \mathbb{N}$ . Then

$$\sum_{\substack{(X,Y) \in \mathbb{N}^2 \\ X > Y}} a^X b^Y c^{\min\{X,Y\}} = \sum_{(\tilde{X},Y) \in \mathbb{N}^2} a^{Y+\tilde{X}} b^Y c^Y = \frac{abc}{1-abc} \cdot \frac{a}{1-a}.$$

Thus

$$\sum_{(X,Y) \in \mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}} = \frac{abc}{1-abc} \left( \frac{1}{1-b} + \frac{a}{1-a} \right) = \frac{abc(1-ab)}{(1-abc)(1-a)(1-b)}.$$

To prove (3) we make the change of variables  $Y' = Y + Z$ . Then

$$\begin{aligned} (2.1) \quad \sum_{(X,Y,Z) \in \mathbb{N}^3} a^X b^{Y+Z} c^{\min\{X,Y+Z\}} &= \sum_{(X,Y') \in \mathbb{N}^2} (Y' - 1) a^X b^{Y'} c^{\min\{X,Y'\}} \\ &= \sum_{(X,Y') \in \mathbb{N}^2} Y' a^X b^{Y'} c^{\min\{X,Y'\}} - \sum_{(X,Y') \in \mathbb{N}^2} a^X b^{Y'} c^{\min\{X,Y'\}}. \end{aligned}$$

Write  $\sum_{(X,Y') \in \mathbb{N}^2} Y' a^X b^{Y'} c^{\min\{X,Y'\}} = f_1 + f_2$ , where

$$f_1 := \sum_{\substack{(X,Y') \in \mathbb{N}^2 \\ X \leq Y'}} Y' a^X b^{Y'} c^{\min\{X,Y'\}}, \quad f_2 := \sum_{\substack{(X,Y') \in \mathbb{N}^2 \\ X > Y'}} Y' a^X b^{Y'} c^{\min\{X,Y'\}}.$$

Setting  $Y' = X + Y''$ , for  $Y'' \in \mathbb{N}_0$ , we get, by (1),

$$\begin{aligned} f_1 &= \sum_{\substack{X \in \mathbb{N} \\ Y'' \in \mathbb{N}_0}} (X + Y'') a^X b^{Y''+X} c^X = \sum_{\substack{X \in \mathbb{N} \\ Y'' \in \mathbb{N}_0}} X (abc)^X b^{Y''} + \sum_{\substack{X \in \mathbb{N} \\ Y'' \in \mathbb{N}_0}} Y'' (abc)^X b^{Y''} \\ &= \frac{abc}{(1-abc)^2} \cdot \frac{1}{1-b} + \frac{abc}{1-abc} \cdot \frac{b}{(1-b)^2} = \frac{abc(1-ab^2c)}{(1-abc)^2(1-b)^2}. \end{aligned}$$

Moreover, setting  $X = Y' + X'$ , for  $X' \in \mathbb{N}$ , we obtain, using (1), that

$$f_2 = \sum_{(X',Y') \in \mathbb{N}^2} Y' a^{Y'+X'} b^{Y'} c^{Y'} = \sum_{(X',Y') \in \mathbb{N}^2} Y' a^{X'} (abc)^{Y'} = \frac{abc}{(1-abc)^2} \frac{a}{1-a}.$$

We conclude that

$$\sum_{(X,Y') \in \mathbb{N}^2} Y' a^X b^{Y'} c^{\min\{X,Y'\}} = f_1 + f_2 = \frac{abc}{(1-abc)^2} \left( \frac{1-ab^2c}{(1-b)^2} + \frac{a}{1-a} \right).$$

Hence, by (2.1) and (2), we obtain

$$\begin{aligned} &\sum_{(X,Y,Z) \in \mathbb{N}^3} a^X b^{Y+Z} c^{\min\{X,Y+Z\}} \\ &= \frac{abc}{(1-abc)^2} \left( \frac{1-ab^2c}{(1-b)^2} + \frac{a}{1-a} \right) - \frac{abc(1-ab)}{(1-abc)(1-a)(1-b)} \\ &= \frac{ab^2c(1-a+ac-2abc+a^2b^2c)}{(1-abc)^2(1-a)(1-b)^2}. \end{aligned}$$

□

Key tools in our computations are  $\mathfrak{p}$ -adic integrals associated to polynomials or polynomial mappings, known as Igusa's local zeta function; see [4] for a basic introduction. We will use the following simple fact throughout: for  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 0$ ,

$$\int_{\mathfrak{p}} |x|^s d\mu(x) = \frac{(1 - q^{-1})q^{-1-s}}{1 - q^{-1-s}}.$$

Here – and, *mutatis mutandis*, in the sequel – we write  $\mu$  for the additive Haar measure on  $\mathfrak{o}$  normalised such that  $\mu(\mathfrak{o}) = 1$ . This implies that  $\mu(\mathfrak{p}) = q^{-1}$ .

The following identities involving  $\mathfrak{p}$ -adic integrals and rational functions are understood as identities of meromorphic functions. The relevant integrals all converge provided the real parts of their arguments are sufficiently large.

**Lemma 2.3.**

$$\int_{\mathfrak{p}^{(2)}} |x|^s \|x, y\|^t d\mu(x, y) = \frac{q^{-2-s-t}(1 - q^{-2-s})(1 - q^{-1})}{(1 - q^{-2-s-t})(1 - q^{-1-s})}.$$

*Proof.* As, for  $X, Y \in \mathbb{N}$ ,

$$\mu(\{(x, y) \in \mathfrak{p}^{(2)} \mid v(x) = X, v(y) = Y\}) = (1 - q^{-1})^2 q^{-X-Y},$$

we obtain, using Lemma 2.2 (2),

$$\begin{aligned} \int_{\mathfrak{p}^{(2)}} |x|^s \|x, y\|^t d\mu(x, y) &= \sum_{(X, Y) \in \mathbb{N}^2} (1 - q^{-1})^2 q^{-X-Y} q^{-sX-t\min\{X, Y\}} \\ &= (1 - q^{-1})^2 \sum_{(X, Y) \in \mathbb{N}^2} q^{-(1+s)X} q^{-Y} q^{-t\min\{X, Y\}} \\ &= (1 - q^{-1})^2 \frac{q^{-1-s} q^{-1} q^{-t} (1 - q^{-1-s} q^{-1})}{(1 - q^{-1-s} q^{-1} q^{-t})(1 - q^{-1-s})(1 - q^{-1})}, \end{aligned}$$

and the lemma follows.  $\square$

**Lemma 2.4.**

$$\begin{aligned} \int_{\mathfrak{p}^{(3)}} |x|^s \|x, yz\|^t d\mu(x, y, z) &= \\ &= \frac{q^{-3-s-t}(1 - q^{-1})(1 - q^{-1-s} + q^{-1-s-t} - 2q^{-2-s-t} + q^{-4-2s-t})}{(1 - q^{-2-s-t})^2(1 - q^{-1-s})}. \end{aligned}$$

*Proof.* This is very similar to the proof of Lemma 2.3, but uses Lemma 2.2 (3).  $\square$

### 3. THE ZETA FUNCTION OF $Q_{1,1}(\mathfrak{o})$

Let  $m = n = 1$  and recall the notation from Sections 1 and 2. In this case  $\mathcal{R}(\mathbf{Y})$  has Pfaffian

$$f(\mathbf{Y}_0) := \operatorname{Pf}(\mathcal{R}(\mathbf{Y})) = Y_{11}Y_{22} - Y_{12}Y_{21}.$$

For  $0 \leq j \leq 2$  let

$$F_j(\mathbf{Y}) = \{g \mid g = g(\mathbf{Y}) \text{ a principal } 2j \times 2j \text{ minor of } \mathcal{R}(\mathbf{Y})\}.$$

One readily computes that

$$\begin{aligned} F_0(\mathbf{Y}) &= \{1\}, \\ F_1(\mathbf{Y}) &= \{0, Y^2, Y_{11}^2, Y_{12}^2, Y_{21}^2, Y_{22}^2\}, \\ F_2(\mathbf{Y}) &= \{\operatorname{Pf}(\mathcal{R}(\mathbf{Y}))^2\} = \{f(\mathbf{Y}_0)^2\}. \end{aligned}$$

By [11, Corollary 2.11] we can compute  $\zeta_{Q_{1,1}(\mathfrak{o})}(s)$  in terms of the  $\mathfrak{p}$ -adic integral

$$\mathcal{Z}_{\mathfrak{o}}(\rho, \tau) := \int_{\mathfrak{p} \times W_5(\mathfrak{o})} |x|^\tau \prod_{j=1}^2 \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y})x^2\|^\rho}{\|F_{j-1}(\mathbf{y})\|^\rho} d\mu(x, \mathbf{y}),$$

where  $\rho, \tau \in \mathbb{C}$ . More precisely,

$$(3.1) \quad \zeta_{Q_{1,1}(\mathfrak{o})}(s) = 1 + (1 - q^{-1})^{-1} \mathcal{Z}_{\mathfrak{o}}(-s/2, 2s - 6).$$

The integral  $\mathcal{Z}_{\mathfrak{o}}(\rho, \tau)$  is a special case of (2.8) in [11, Section 2.2.3] for the case  $v = 0$  and  $u = 2$ . Since  $\|F_0(\mathbf{y})\| = \|F_1(\mathbf{y})\| = 1$ , we obtain

$$\mathcal{Z}_{\mathfrak{o}}(\rho, \tau) = \int_{\mathfrak{p} \times W_5(\mathfrak{o})} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}).$$

Write  $W_5(\mathfrak{o}) = D_1 \cup D_2$ , where  $D_1 = \mathfrak{o}^\times \times \mathfrak{o}^4$ ,  $D_2 = \mathfrak{p} \times W_4(\mathfrak{o})$ . Thus

$$(3.2) \quad \mathcal{Z}_{\mathfrak{o}}(\rho, \tau) = I_{D_1} + I_{D_2},$$

where

$$\begin{aligned} I_{D_1} &:= \int_{\mathfrak{p} \times \mathfrak{o}^\times \times \mathfrak{o}^4} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}), \\ I_{D_2} &:= \int_{\mathfrak{p}^{(2)} \times W_4(\mathfrak{o})} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}). \end{aligned}$$

In the following we compute each of the integrals  $I_{D_1}$  and  $I_{D_2}$  in turn.

**3.1. Computation of  $I_{D_1}$ .** Clearly

$$I_{D_1} = (1 - q^{-1}) \int_{\mathfrak{p} \times \mathfrak{o}^4} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0).$$

Recall the notation  $A_d^0$  from Section 1.3 and set, for  $z \in A_4^0$ ,

$$(3.3) \quad I_{D_1, z} := (1 - q^{-1}) \int_{\mathfrak{p} \times (z + \mathfrak{p}^{(4)})} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0).$$

Hence

$$(3.4) \quad I_{D_1} = \sum_{z \in A_4^0} I_{D_1, z}.$$

In the sequel we compute each of the integrals  $I_{D_1, z}$ . It will turn out that it suffices to distinguish three cases. Let  $\bar{z}$  denote the image of  $z \bmod \mathfrak{p}$ .

**3.1.1.** Suppose that  $f(z) \not\equiv 0 \bmod \mathfrak{p}$ . Then  $\|f(\mathbf{y}_0), x\|^{2\rho} = 1$  for  $\mathbf{y}_0 \in z + \mathfrak{p}^{(4)}$ , so

$$\begin{aligned} I_{D_1, z} &= (1 - q^{-1}) \int_{\mathfrak{p} \times (z + \mathfrak{p}^{(4)})} |x|^\tau d\mu(x, \mathbf{y}_0) \\ &= (1 - q^{-1}) q^{-4} \int_{\mathfrak{p}} |x|^\tau d\mu(x) \\ &= \frac{(1 - q^{-1})^2 q^{-5 - \tau}}{1 - q^{-1 - \tau}} =: I_{D_1}^1. \end{aligned}$$



3.1.2. Suppose that  $f(z) \equiv 0 \pmod{\mathfrak{p}}$  but  $\bar{z} \neq 0$ . Near any  $\mathfrak{o}$ -point of the hypersurface defined by  $f$  which reduces to a smooth point  $\pmod{\mathfrak{p}}$  (i.e., any point away from the origin) we may replace  $f$  by the first, say, of four coordinate functions  $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4$  in the relevant integral; cf. [1, Section 6.1]. Using Lemma 2.3 we may thus rewrite (3.3) as

$$\begin{aligned} I_{D_1, z} &= (1 - q^{-1}) \int_{\mathfrak{p}^{(5)}} |x|^\tau \|\tilde{y}_1, x\|^{2\rho} d\mu(x, \tilde{\mathbf{y}}) \\ &= (1 - q^{-1}) q^{-3} \int_{\mathfrak{p}^{(2)}} |x|^\tau \|\tilde{y}_1, x\|^{2\rho} d\mu(x, \tilde{y}_1) \\ &= \frac{q^{-5-\tau-2\rho}(1 - q^{-2-\tau})(1 - q^{-1})^2}{(1 - q^{-2-\tau-2\rho})(1 - q^{-1-\tau})} =: I_{D_1}^2. \end{aligned}$$

3.1.3. Suppose finally that  $\bar{z} = 0$ . In this case,

$$I_{D_1, z} = (1 - q^{-1}) \int_{\mathfrak{p} \times \mathfrak{p}^{(4)}} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0).$$

We make the change of variables  $\mathbf{y}_0 = \varpi \mathbf{y}'_0$ , that is,  $y_{ij} = \varpi y'_{ij}$  for  $y'_{ij} \in \mathfrak{o}$ . The norm of the Jacobian of this change of variables is  $q^{-4}$ . As  $f$  is homogeneous of degree 2,

$$I_{D_1, z} = (1 - q^{-1}) \int_{\mathfrak{p} \times \mathfrak{o}^4} |x|^\tau \|\varpi^2 f(\mathbf{y}'_0), x\|^{2\rho} q^{-4} d\mu(x, \mathbf{y}'_0).$$

Write  $I_{D_1, z} = I_1 + I_2$ , where

$$\begin{aligned} I_1 &:= (1 - q^{-1}) q^{-4} \int_{\mathfrak{p}^2 \times \mathfrak{o}^4} |x|^\tau \|\varpi^2 f(\mathbf{y}'_0), x\|^{2\rho} d\mu(x, \mathbf{y}'_0), \\ I_2 &:= (1 - q^{-1}) q^{-4} \int_{\mathfrak{p} \setminus \mathfrak{p}^2 \times \mathfrak{o}^4} |x|^\tau \|\varpi^2 f(\mathbf{y}'_0), x\|^{2\rho} d\mu(x, \mathbf{y}'_0). \end{aligned}$$

Consider  $I_1$  and make the change of variables  $x = \varpi^2 x'$  for  $x' \in \mathfrak{o}$ . The norm of the Jacobian of this change of variables is  $q^{-2}$ . This yields

$$\begin{aligned} (3.5) \quad I_1 / (1 - q^{-1}) &= q^{-4} \int_{\mathfrak{o} \times \mathfrak{o}^4} |\varpi^2 x'|^\tau \|\varpi^2 f(\mathbf{y}'_0), \varpi^2 x'\|^{2\rho} q^{-2} d\mu(x', \mathbf{y}'_0) \\ &= q^{-6-2\tau-4\rho} \int_{\mathfrak{o} \times \mathfrak{o}^4} |x'|^\tau \|f(\mathbf{y}'_0), x'\|^{2\rho} d\mu(x', \mathbf{y}'_0) \\ &= q^{-6-2\tau-4\rho} \left( \int_{\mathfrak{o} \times \mathfrak{o}^4} d\mu(x', \mathbf{y}'_0) + \frac{1}{1 - q^{-1}} I_{D_1} \right) \\ &= q^{-6-2\tau-4\rho} \left( 1 - q^{-1} + \frac{1}{1 - q^{-1}} I_{D_1} \right). \end{aligned}$$

Next, consider  $I_2$ . When  $x \in \mathfrak{p} \setminus \mathfrak{p}^2$ , then  $\|\varpi^2 f(\mathbf{y}'_0), x\|^{2\rho} = |x|^{2\rho} = q^{-2\rho}$ , so

$$I_2 = (1 - q^{-1}) q^{-4} \int_{\mathfrak{p} \setminus \mathfrak{p}^2 \times \mathfrak{o}^4} |x|^{\tau+2\rho} d\mu(x, \mathbf{y}'_0) = (1 - q^{-1})^2 q^{-5-\tau-2\rho}.$$

Thus

$$\begin{aligned} I_{D_1, z} &= I_1 + I_2 \\ &= (1 - q^{-1}) q^{-6-2\tau-4\rho} \left( 1 - q^{-1} + \frac{1}{1 - q^{-1}} I_{D_1} \right) + (1 - q^{-1})^2 q^{-5-\tau-2\rho} =: I_{D_1}^3. \end{aligned}$$

3.1.4. *Conclusion.* Taking the above three cases together and noting that

$$|\{z \in A_4^0 \mid f(z) \neq 0 \pmod{\mathfrak{p}}\}| = |\mathrm{GL}_2(\mathbb{F}_q)| = q(q-1)^2(q+1),$$

(3.4) yields

$$\begin{aligned} I_{D_1} &= |\mathrm{GL}_2(\mathbb{F}_q)| I_{D_1}^1 + (q^4 - |\mathrm{GL}_2(\mathbb{F}_q)| - 1) I_{D_1}^2 + I_{D_1}^3 \\ &= |\mathrm{GL}_2(\mathbb{F}_q)| \frac{(1 - q^{-1})^2 q^{-5-\tau}}{1 - q^{-1-\tau}} \\ &\quad + (q^4 - |\mathrm{GL}_2(\mathbb{F}_q)| - 1) \frac{q^{-5-\tau-2\rho} (1 - q^{-2-\tau}) (1 - q^{-1})^2}{(1 - q^{-2-\tau-2\rho}) (1 - q^{-1-\tau})} \\ &\quad + (1 - q^{-1}) q^{-6-2\tau-4\rho} \left( 1 - q^{-1} + \frac{1}{1 - q^{-1}} I_{D_1} \right) + (1 - q^{-1})^2 q^{-5-\tau-2\rho}. \end{aligned}$$

Solving for  $I_{D_1}$ , we obtain

$$\begin{aligned} I_{D_1} &= \frac{(1 - q^{-1})^2 q^{-5-\tau}}{1 - q^{-6-2\tau-4\rho}} \\ &\quad \left( \frac{q(q-1)^2(q+1)}{1 - q^{-1-\tau}} + \frac{q^{-2\rho}(q^3 + q^2 - q - 1)(1 - q^{-2-\tau})}{(1 - q^{-2-\tau-2\rho})(1 - q^{-1-\tau})} + q^{-1-\tau-4\rho} + q^{-2\rho} \right). \end{aligned}$$

3.2. **Computation of  $I_{D_2}$ .** Recall that  $F_2(\mathbf{Y}) = \{f(\mathbf{Y}_0)^2\}$ . Hence

$$\begin{aligned} (3.6) \quad I_{D_2} &= \int_{\mathfrak{p} \times \mathfrak{p} \times W_4(\mathfrak{o})} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, y, \mathbf{y}_0) \\ &= q^{-1} \int_{\mathfrak{p} \times \mathfrak{o}^4} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0) \\ &\quad - q^{-1} \int_{\mathfrak{p} \times \mathfrak{p}^{(4)}} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0) \\ &= \frac{q^{-1}}{1 - q^{-1}} I_{D_1} - q^{-1} \int_{\mathfrak{p} \times \mathfrak{p}^{(4)}} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0). \end{aligned}$$

To compute the integral

$$I := \int_{\mathfrak{p} \times \mathfrak{p}^{(4)}} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0)$$

we write it as  $I = J_1 + J_2$ , where

$$\begin{aligned} J_1 &:= \int_{\mathfrak{p} \setminus \mathfrak{p}^2 \times \mathfrak{p}^{(4)}} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0), \\ J_2 &:= \int_{\mathfrak{p}^2 \times \mathfrak{p}^{(4)}} |x|^\tau \|f(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0). \end{aligned}$$

Clearly

$$J_1 = \int_{\mathfrak{p} \setminus \mathfrak{p}^2 \times \mathfrak{p}^{(4)}} |x|^{\tau+2\rho} d\mu(x, \mathbf{y}_0) = q^{-4} \int_{\mathfrak{p} \setminus \mathfrak{p}^2} |x|^{\tau+2\rho} d\mu(x) = q^{-5-\tau-2\rho} (1 - q^{-1}).$$

Next, consider  $J_2$  and make the change of variables  $\mathbf{y}_0 = \varpi \mathbf{y}'_0$  for  $\mathbf{y}_0 \in \mathfrak{o}^4$ . The norm of the Jacobian of this change of variables is  $q^{-4}$ , whence, using (3.5),

$$J_2 = \int_{\mathfrak{p}^{(2)} \times \mathfrak{o}^4} |x|^\tau \|\varpi^2 f(\mathbf{y}'_0), x\|^{2\rho} q^{-4} d\mu(x', \mathbf{y}'_0) = \frac{I_1}{1 - q^{-1}} = q^{-6-2\tau-4\rho} \left( (1 - q^{-1}) + \frac{I_{D_1}}{1 - q^{-1}} \right).$$

Hence, by (3.6), we get

$$\begin{aligned} I_{D_2} &= \frac{q^{-1}}{1 - q^{-1}} I_{D_1} - q^{-1} (J_1 + J_2) \\ &= \frac{q^{-1}}{1 - q^{-1}} I_{D_1} - q^{-1} \left( q^{-5-\tau-2\rho} (1 - q^{-1}) + q^{-6-2\tau-4\rho} \left( (1 - q^{-1}) + \frac{I_{D_1}}{1 - q^{-1}} \right) \right) \\ &= \frac{q^{-1} (1 - q^{-6-2\tau-4\rho}) I_{D_1}}{1 - q^{-1}} - q^{-6-\tau-2\rho} (1 - q^{-1}) (1 + q^{-1-\tau-2\rho}). \end{aligned}$$

**3.3. Conclusion.** With the computations of the two previous sections, equation (3.2) implies that, for  $a = q^{-1}$ ,  $b = q^{-\tau}$ ,  $c = q^{-\rho}$ ,

$$\begin{aligned} \mathcal{Z}_{\mathfrak{o}}(\rho, \tau) &= I_{D_1} + I_{D_2} \\ &= \frac{ab(a-1)^2}{(1 - a^3bc^2)(1 - a^2bc^2)(1 - ab)} \\ &\quad \left( a^9b^2c^4 + a^8b^2c^4 + a^7b^2c^4 - a^7bc^4 + a^6b^2c^4 - 2a^6bc^4 + a^5b^2c^4 - a^5bc^4 - a^4bc^4 \right. \\ &\quad \left. - a^5bc^2 - a^4bc^2 + a^4c^2 - 2a^3bc^2 + a^3c^2 - a^2bc^2 + 2a^2c^2 + ac^2 - a^2 + 1 \right) \end{aligned}$$

With (3.1) and a direct computation, this completes the proof of Theorem 1.1.

*Remark 3.1.* As in [11, Section 6], it is interesting to compare  $\zeta_{Q_{1,1}}(s)$  to the Igusa integral of the relative invariant of the corresponding PVS. By [6, p. 165], with  $t = q^{-s}$ ,

$$Z_{1,1}(s) := \int_{\text{Mat}_2(\mathfrak{o})} |f(\mathbf{y}_0)|^s d\mu(\mathbf{y}_0) = \frac{(1 - q^{-2})(1 - q^{-1})}{(1 - q^{-2}t)(1 - q^{-1}t)}.$$

We note that the real parts of the poles of  $\zeta_{Q_{1,1}(\mathfrak{o})}(s)$  are given by an additive translation of the real parts of the poles of  $Z_{1,1}(s)$ , just as for the groups  $F_{n,\delta}(\mathfrak{o})$ ,  $G_n(\mathfrak{o})$ ,  $H_n(\mathfrak{o})$  considered in [11].

#### 4. THE ZETA FUNCTION OF $Q_{2,1}(\mathfrak{o})$

Let  $m = 2$ ,  $n = 1$ , and recall the notation from Sections 1 and 2. In this case the Pfaffian of  $\mathcal{R}(\mathbf{Y})$  is  $\text{Pf}(\mathcal{R}(\mathbf{Y})) = Yh(\mathbf{Y}_0)$ , where

$$h(\mathbf{Y}_0) = Y_{11}Y_{22} - Y_{12}Y_{21} + Y_{31}Y_{42} - Y_{32}Y_{41}.$$

For  $0 \leq j \leq 3$  let

$$F_j(\mathbf{Y}) = \{g \mid g = g(\mathbf{Y}) \text{ a principal } 2j \times 2j \text{ minor of } \mathcal{R}(\mathbf{Y})\}.$$

Given integers  $r, s$  such that  $1 \leq r < s \leq 4$ , we write  $m_{rs} = m_{rs}(\mathbf{Y}_0)$  for the  $2 \times 2$ -minor of  $(\mathbf{Y}_0) \in \text{Mat}_{4 \times 2}(\mathbb{Z}[\mathbf{Y}_0])$  indexed by the  $r$ th and  $s$ th rows. Note that  $h = m_{12} + m_{34}$ .

One readily computes that

$$\begin{aligned} F_0(\mathbf{Y}) &= \{1\}, \\ F_1(\mathbf{Y}) &= \{0, Y^2, Y_{ij}^2 \mid i \in \{1, 2, 3, 4\}, j \in \{1, 2\}\}, \\ F_2(\mathbf{Y}) &= \{Y^4, Y^2 Y_{ij}^2, m_{rs}(\mathbf{Y}_0)^2 \mid i \in \{1, 2, 3, 4\}, j \in \{1, 2\}, 1 \leq r < s \leq 4\}, \\ F_3(\mathbf{Y}) &= \{\text{Pf}(\mathcal{R}(\mathbf{Y}))^2\} = \{Y^2 h(\mathbf{Y}_0)^2\}. \end{aligned}$$

By [11, Corollary 2.11] we can compute  $\zeta_{Q_{2,1}(\mathfrak{o})}(s)$  in terms of the  $\mathfrak{p}$ -adic integral

$$\mathcal{Z}_{\mathfrak{o}}(\rho, \tau) := \int_{\mathfrak{p} \times W_9(\mathfrak{o})} |x|^\tau \prod_{j=1}^3 \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y})x^2\|^\rho}{\|F_{j-1}(\mathbf{y})\|^\rho} d\mu(x, \mathbf{y}),$$

where  $\rho, \tau \in \mathbb{C}$ . More precisely,

$$(4.1) \quad \zeta_{Q_{2,1}(\mathfrak{o})}(s) = 1 + (1 - q^{-1})^{-1} \mathcal{Z}_{\mathfrak{o}}(-s/2, 3s - 10).$$

The integral  $\mathcal{Z}_{\mathfrak{o}}(\rho, \tau)$  is a special case of (2.8) in [11, Section 2.2.3] for the case  $v = 0$  and  $u = 3$ . Since  $\|F_0(\mathbf{y})\| = \|F_1(\mathbf{y})\| = 1$ , we obtain

$$\mathcal{Z}_{\mathfrak{o}}(\rho, \tau) = \int_{\mathfrak{p} \times W_9(\mathfrak{o})} |x|^\tau \|F_2(\mathbf{y}), x^2\|^\rho \frac{\|F_3(\mathbf{y}) \cup F_2(\mathbf{y})x^2\|^\rho}{\|F_2(\mathbf{y})\|^\rho} d\mu(x, \mathbf{y}).$$

Write  $W_9(\mathfrak{o}) = D_1 \cup D_2$ , where  $D_1 = \mathfrak{o}^\times \times \mathfrak{o}^8$   $D_2 = \mathfrak{p} \times W_8(\mathfrak{o})$ . Thus

$$(4.2) \quad \mathcal{Z}_{\mathfrak{o}}(\rho, \tau) = I_{D_1} + I_{D_2},$$

where

$$\begin{aligned} I_{D_1} &:= \int_{\mathfrak{p} \times \mathfrak{o}^\times \times \mathfrak{o}^8} |x|^\tau \|F_2(\mathbf{y}), x^2\|^\rho \frac{\|F_3(\mathbf{y}) \cup F_2(\mathbf{y})x^2\|^\rho}{\|F_2(\mathbf{y})\|^\rho} d\mu(x, \mathbf{y}), \\ I_{D_2} &:= \int_{\mathfrak{p} \times \mathfrak{o}^\times \times W_8(\mathfrak{o})} |x|^\tau \|F_2(\mathbf{y}), x^2\|^\rho \frac{\|F_3(\mathbf{y}) \cup F_2(\mathbf{y})x^2\|^\rho}{\|F_2(\mathbf{y})\|^\rho} d\mu(x, \mathbf{y}). \end{aligned}$$

In the following we compute each of the integrals  $I_{D_1}$  and  $I_{D_2}$  in turn. We will need to refer to the following determinantal varieties (schemes over  $\mathbb{Z}$ ):

$$\begin{aligned} V_2 &= \text{Spec } \mathbb{Z}[\mathbf{Y}_0]/(m_{rs}(\mathbf{Y}_0), 1 \leq r < s \leq 4), \\ V_3 &= \text{Spec } \mathbb{Z}[\mathbf{Y}_0]/(h(\mathbf{Y}_0)). \end{aligned}$$

Clearly  $V_2$  is a closed subscheme of  $V_3$  and each fibre of  $V_2$  can be interpreted as the space of  $4 \times 2$ -matrices of rank at most 1 over a field. Similarly, each fibre of  $V_3$  can be seen as the space of  $4 \times 2$ -matrices of rank at most 1 over a field. It is well known that  $V_2$  has codimension 3.

**4.1. Computation of  $I_{D_1}$ .** Note that  $\|F_2(\mathbf{y})\| = 1$  and  $\|F_3(\mathbf{y})\| = \|h(\mathbf{y}_0)^2\|$  as  $y \in \mathfrak{o}^\times$ . Thus

$$\begin{aligned} I_{D_1} &= \int_{\mathfrak{p} \times \mathfrak{o}^\times \times \mathfrak{o}^8} |x|^\tau \|F_3(\mathbf{y}), x^2\|^\rho d\mu(x, \mathbf{y}) \\ &= \int_{\mathfrak{p} \times \mathfrak{o}^\times \times \mathfrak{o}^8} |x|^\tau \|h(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}) \\ &= (1 - q^{-1}) \int_{\mathfrak{p} \times \mathfrak{o}^8} |x|^\tau \|h(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0). \end{aligned}$$

We set, for  $z \in A_8^0$ ,

$$(4.3) \quad I_{D_1, z} := (1 - q^{-1}) \int_{\mathfrak{p} \times (z + \mathfrak{p}^{(8)})} |x|^\tau \|h(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0).$$

Hence

$$(4.4) \quad I_{D_1} = \sum_{z \in A_8^0} I_{D_1, z}.$$

In the sequel we compute each of the integrals  $I_{D_1, z}$ . It will turn out that it suffices to distinguish three cases.

4.1.1. Suppose that  $\bar{z} \notin V_3(\mathbb{F}_q)$ . In this case,  $\|h(\mathbf{y}_0), x\|^{2\rho} = 1$  for  $\mathbf{y}_0 \in z + \mathfrak{p}^{(8)}$ , so

$$\begin{aligned} I_{D_1, z} &= (1 - q^{-1}) \int_{\mathfrak{p} \times (z + \mathfrak{p}^{(8)})} |x|^\tau d\mu(x, \mathbf{y}_0) \\ &= (1 - q^{-1}) q^{-8} \int_{\mathfrak{p}} |x|^\tau d\mu(x) = \frac{(1 - q^{-1})^2 q^{-9-\tau}}{1 - q^{-1-\tau}} =: I_{D_1}^1. \end{aligned}$$

4.1.2. Suppose that  $\bar{z} \in V_3(\mathbb{F}_q) \setminus \{0\}$ . Near any  $\mathfrak{o}$ -point of  $V_3$  which reduces to a smooth point mod  $\mathfrak{p}$  (i.e, a point away from the origin), we may replace  $h$  by the first, say, of eight coordinate functions  $\tilde{y}_{ij}$ , where  $i \in \{1, \dots, 4\}$ ,  $j \in \{1, 2\}$ , in the integral (4.3). It may thus be rewritten, using Lemma 2.3, as

$$\begin{aligned} I_{D_1, z} &= (1 - q^{-1}) \int_{\mathfrak{p}^{(9)}} |x|^\tau \|\tilde{y}_{11}, x\|^{2\rho} d\mu(x, \tilde{\mathbf{y}}_0) \\ &= (1 - q^{-1}) q^{-7} \int_{\mathfrak{p}^{(2)}} |x|^\tau \|\tilde{y}_{11}, x\|^{2\rho} d\mu(x, \tilde{y}_1) \\ &= \frac{q^{-9-\tau-2\rho} (1 - q^{-2-\tau}) (1 - q^{-1})^2}{(1 - q^{-2-\tau-2\rho}) (1 - q^{-1-\tau})} =: I_{D_1}^2. \end{aligned}$$

4.1.3. Suppose finally that  $\bar{z} = 0$ . In this case,

$$I_{D_1, z} = (1 - q^{-1}) \int_{\mathfrak{p}^{(9)}} |x|^\tau \|h(\mathbf{y}_0), x\|^{2\rho} d\mu(x, \mathbf{y}_0).$$

Make the change of variables  $\mathbf{y}_0 = \varpi \mathbf{y}'_0$ , that is,  $y_{ij} = \varpi y'_{ij}$ , for  $y'_{ij} \in \mathfrak{o}$ . The norm of the Jacobian of this change of variables is  $q^{-8}$ . Since  $h$  is homogeneous of degree 2,

$$I_{D_1, z} = (1 - q^{-1}) \int_{\mathfrak{p} \times \mathfrak{o}^8} |x|^\tau \|\varpi^2 h(\mathbf{y}'_0), x\|^{2\rho} q^{-8} d\mu(x, \mathbf{y}'_0).$$

Write  $I_{D_1, z} = I_1 + I_2$ , where

$$\begin{aligned} I_1 &:= (1 - q^{-1}) q^{-8} \int_{\mathfrak{p}^2 \times \mathfrak{o}^8} |x|^\tau \|\varpi^2 h(\mathbf{y}'_0), x\|^{2\rho} d\mu(x, \mathbf{y}'_0), \\ I_2 &:= (1 - q^{-1}) q^{-8} \int_{\mathfrak{p} \setminus \mathfrak{p}^2 \times \mathfrak{o}^8} |x|^\tau \|\varpi^2 h(\mathbf{y}'_0), x\|^{2\rho} d\mu(x, \mathbf{y}'_0). \end{aligned}$$

Consider  $I_1$  and make the change of variables  $x = \varpi^2 x'$  for  $x' \in \mathfrak{o}$ . The norm of the Jacobian of this change of variables is  $q^{-2}$ . This yields

$$\begin{aligned} I_1/(1-q^{-1}) &= q^{-8} \int_{\mathfrak{o} \times \mathfrak{o}^8} |\varpi^2 x'|^\tau \|\varpi^2 h(\mathbf{y}'_0), \varpi^2 x'\|^2 q^{-2} d\mu(x', \mathbf{y}'_0) \\ &= q^{-10-2\tau-4\rho} \int_{\mathfrak{o} \times \mathfrak{o}^8} |x'|^\tau \|h(\mathbf{y}'_0), x'\|^2 d\mu(x', \mathbf{y}'_0) \\ &= q^{-10-2\tau-4\rho} \left( \int_{\mathfrak{o} \times \mathfrak{o}^8} d\mu(x', \mathbf{y}'_{ij}) + \frac{1}{1-q^{-1}} I_{D_1} \right) \\ &= q^{-10-2\tau-4\rho} \left( 1 - q^{-1} + \frac{1}{1-q^{-1}} I_{D_1} \right). \end{aligned}$$

Next, consider  $I_2$ . When  $x \in \mathfrak{p} \setminus \mathfrak{p}^2$ , we have  $\|\varpi^2 h(\mathbf{y}_0), x\|^2 = |x|^2 = q^{-2\rho}$ , so

$$I_2 = (1-q^{-1})q^{-8} \int_{\mathfrak{p} \setminus \mathfrak{p}^2 \times \mathfrak{o}^8} |x|^{\tau+2\rho} d\mu(x, \mathbf{y}'_0) = (1-q^{-1})^2 q^{-9-\tau-2\rho}.$$

We thus obtain

$$\begin{aligned} I_{D_1, z} &= I_1 + I_2 = \\ &= (1-q^{-1})q^{-10-2\tau-4\rho} \left( 1 - q^{-1} + \frac{1}{1-q^{-1}} I_{D_1} \right) + (1-q^{-1})^2 q^{-9-\tau-2\rho} =: I_{D_1}^3. \end{aligned}$$

4.1.4. *Conclusion.* Taking the above three cases together, (4.4) yields

(4.5)

$$\begin{aligned} I_{D_1} &= |\mathbb{A}^8(\mathbb{F}_q) \setminus V_3(\mathbb{F}_q)| I_{D_1}^1 + |V_3(\mathbb{F}_q) \setminus \{0\}| I_{D_1}^2 + I_{D_1}^3 \\ &= (q^8 - |V_3(\mathbb{F}_q)|) \frac{(1-q^{-1})^2 q^{-9-\tau}}{1-q^{-1-\tau}} + (|V_3(\mathbb{F}_q)| - 1) \frac{q^{-9-\tau-2\rho} (1-q^{-2-\tau}) (1-q^{-1})^2}{(1-q^{-2-\tau-2\rho}) (1-q^{-1-\tau})} \\ &\quad + (1-q^{-1}) q^{-10-2\tau-4\rho} \left( 1 - q^{-1} + \frac{1}{1-q^{-1}} I_{D_1} \right) + (1-q^{-1})^2 q^{-9-\tau-2\rho}. \end{aligned}$$

**Lemma 4.1.**

$$|V_2(\mathbb{F}_q)| = (q+1)(q^4-1) + 1, \quad |V_3(\mathbb{F}_q)| = q^3(q^4+q-1).$$

*Proof.* This can be easily proved directly or read off from [8, Proposition 3.1].  $\square$

Solving (4.5) for  $I_{D_1}$ , we obtain

$$(4.6) \quad I_{D_1} = \frac{(1-q^{-1})^2 q^{-9-\tau}}{1-q^{-10-2\tau-4\rho}} \cdot \left( \frac{(q^8 - q^3(q^4+q-1))}{1-q^{-1-\tau}} + \frac{q^{-2\rho}(q^3(q^4+q-1)-1)(1-q^{-2-\tau})}{(1-q^{-2-\tau-2\rho})(1-q^{-1-\tau})} + q^{-1-\tau-4\rho} + q^{-2\rho} \right).$$

4.2. **Computation of  $I_{D_2}$ .** We set, for  $z \in A_8$ ,

$$(4.7) \quad I_{D_2, z} = \int_{\mathfrak{p}^{(2)} \times (z + \mathfrak{p}^{(8)})} |x|^\tau \|F_2(\mathbf{y}), x^2\|^\rho \frac{\|F_3(\mathbf{y}) \cup F_2(\mathbf{y}) x^2\|^\rho}{\|F_2(\mathbf{y})\|^\rho} d\mu(x, \mathbf{y}).$$

Hence

$$I_{D_2} = \sum_{z \in A_8} I_{D_2, z}.$$

In the sequel we compute each of the integrals  $I_{D_2, z}$ . We will distinguish three cases, where the third one will be split further into seven subcases.

4.2.1. If  $\bar{z} \notin V_3(\mathbb{F}_q)$  and  $\mathbf{y} = (y, \mathbf{y}_0) \in \mathfrak{p} \times (z + \mathfrak{p}^{(8)})$ , then  $|m_{rs}(\mathbf{y}_0)| = 1$  for some  $r < s$ , and so  $\|F_2(\mathbf{y})\| = 1$ . Moreover,  $|h(z)| = 1$ , so  $\|F_3(\mathbf{y})\| = |y^2|$ . Thus, using Lemma 2.3,

$$\begin{aligned} I_{D_2, z} &= q^{-8} \int_{\mathfrak{p}^{(2)}} |x|^\tau \|x, y\|^{2\rho} d\mu(x, y) \\ &= \frac{q^{-10-2\rho}(1-q^{-2-\tau})(1-q^{-1})}{(1-q^{-2-\tau-2\rho})(1-q^{-1-\tau})} =: I_{D_2}^1. \end{aligned}$$

4.2.2. If  $\bar{z} \in V_3(\mathbb{F}_q) \setminus V_2(\mathbb{F}_q)$  and  $\mathbf{y} = (y, \mathbf{y}_0) \in \mathfrak{p} \times (z + \mathfrak{p}^{(8)})$ , then  $\|F_2(\mathbf{y})\| = 1$ , as above. Near any  $\mathfrak{o}$ -point of  $V_3$  which reduces to a smooth point mod  $\mathfrak{p}$  (i.e., any point away from the origin) we may replace  $h$  by the first, say, of eight coordinate functions  $\tilde{y}_{ij}$ , where  $i \in \{1, \dots, 4\}$ ,  $j \in \{1, 2\}$ , in the integral (4.7). Thus, using Lemma 2.4,

$$\begin{aligned} I_{D_2, z} &= \int_{\mathfrak{p}^{(2)} \times (z + \mathfrak{p}^{(8)})} |x|^\tau \|F_3(\mathbf{y}), x^2\|^\rho d\mu(x, \mathbf{y}) = \int_{\mathfrak{p}^{(10)}} |x|^\tau \|y\tilde{y}_1, x\|^{2\rho} d\mu(x, y, \tilde{\mathbf{y}}_0) \\ &= q^{-7} \int_{\mathfrak{p}^{(3)}} |x|^\tau \|y\tilde{y}_1, x\|^{2\rho} d\mu(x, y) \\ &= \frac{q^{-10-2\rho}(1-q^{-1})(1-q^{-1-\tau} + q^{-1-\tau-2\rho} - 2q^{-2-\tau-2\rho} + q^{-4-2\tau-2\rho})}{(1-q^{-2-\tau-2\rho})^2(1-q^{-1-\tau})} =: I_{D_2}^2. \end{aligned}$$

4.2.3. Finally we consider the case  $\bar{z} \in V_2(\mathbb{F}_q)$ , which will itself split into seven subcases.

If  $\bar{z} \in V_2(\mathbb{F}_q)$  and  $\mathbf{y} = (y, \mathbf{y}_0) \in \mathfrak{p} \times (z + \mathfrak{p}^{(8)})$ , then  $|m_{rs}(\mathbf{y}_0)| < 1$  for all  $r < s$ . Since  $\bar{z}$  is a smooth point, we may introduce coordinates  $\tilde{\mathbf{y}}_0 = (\tilde{y}_{ij})$ , for  $i \in \{1, \dots, 4\}$ ,  $j \in \{1, 2\}$ , such that the  $\mathfrak{o}$ -points  $V_3(\mathfrak{o})$  are given by the vanishing of the coordinate function  $\tilde{y}_1 := \tilde{y}_{11}$ , and  $V_2(\mathfrak{o})$  is given by the vanishing of the coordinate function  $\tilde{y}_1$ ,  $\tilde{y}_2 := \tilde{y}_{12}$  and  $\tilde{y}_3 := \tilde{y}_{13}$ . Note that  $V_3$  has codimension one in  $\mathbb{A}^8$  and that  $V_2$  has codimension two in  $V_3$ . Since  $\bar{z} \neq 0$  as  $z \in W_8(\mathfrak{o})$ , there exists at least one  $y_{ij}$  such that  $|y_{ij}| = 1$ . Thus

$$\begin{aligned} I_{D_2, z} &= \int_{\mathfrak{p}^{(2)} \times (z + \mathfrak{p}^{(8)})} |x|^\tau \|F_2(\mathbf{y}), x^2\|^\rho \frac{\|F_3(\mathbf{y}) \cup F_2(\mathbf{y})x^2\|^\rho}{\|F_2(\mathbf{y})\|^\rho} d\mu(x, \mathbf{y}) \\ &= \int_{\mathfrak{p}^{(10)}} |x|^\tau \|x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|^{2\rho} \frac{\|xy, x\tilde{y}_1, x\tilde{y}_2, x\tilde{y}_3, y\tilde{y}_1\|^{2\rho}}{\|y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|^{2\rho}} d\mu(x, y, \tilde{\mathbf{y}}) \\ &= q^{-5} \int_{\mathfrak{p}^{(5)}} |x|^\tau \|x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|^{2\rho} \frac{\|xy, x\tilde{y}_1, x\tilde{y}_2, x\tilde{y}_3, y\tilde{y}_1\|^{2\rho}}{\|y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|^{2\rho}} d\mu(x, y, \tilde{\mathbf{y}}) =: I_{D_2}^3. \end{aligned}$$

We partition the domain of integration  $\mathfrak{p}^{(5)}$  into pieces on which the above integral simplifies. For  $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathfrak{p}^{(5)}$ , define

$$M(\mathbf{a}) = \{i \in \{1, 2, 3, 4, 5\} \mid |a_i| = \|\mathbf{a}\|\}.$$

That is,  $M(\mathbf{a})$  comprises those coordinates where  $\|\mathbf{a}\| = \max\{|a_i| \mid i \in \{1, \dots, 5\}\}$  is attained. Define the following mutually disjoint subsets of  $\mathfrak{p}^{(5)}$ :

$$\begin{aligned}\Omega_1 &= \{\mathbf{a} \in \mathfrak{p}^{(5)} \mid 1 \in M(\mathbf{a})\}, \\ \Omega_2 &= \{\mathbf{a} \in \mathfrak{p}^{(5)} \mid 1 \notin M(\mathbf{a}), M(\mathbf{a}) \cap \{2, 3\} = \{2\}\}, \\ \Omega_3 &= \{\mathbf{a} \in \mathfrak{p}^{(5)} \mid 1 \notin M(\mathbf{a}), M(\mathbf{a}) \cap \{2, 3\} = \{3\}\}, \\ \Omega_4 &= \{\mathbf{a} \in \mathfrak{p}^{(5)} \mid 1 \notin M(\mathbf{a}), M(\mathbf{a}) \cap \{2, 3\} = \{2, 3\}\}, \\ \Omega_5 &= \{\mathbf{a} \in \mathfrak{p}^{(5)} \mid 1 \notin M(\mathbf{a}), M(\mathbf{a}) \cap \{2, 3\} = \emptyset, M(\mathbf{a}) \cap \{4, 5\} = \{4\}\}, \\ \Omega_6 &= \{\mathbf{a} \in \mathfrak{p}^{(5)} \mid 1 \notin M(\mathbf{a}), M(\mathbf{a}) \cap \{2, 3\} = \emptyset, M(\mathbf{a}) \cap \{4, 5\} = \{5\}\}, \\ \Omega_7 &= \{\mathbf{a} \in \mathfrak{p}^{(5)} \mid 1 \notin M(\mathbf{a}), M(\mathbf{a}) \cap \{2, 3\} = \emptyset, M(\mathbf{a}) \cap \{4, 5\} = \{4, 5\}\}.\end{aligned}$$

We obtain a partition  $\mathfrak{p}^{(5)} = \bigcup_{i=1}^7 \Omega_i$  and a corresponding decomposition

$$I_{D_2}^3 = q^{-5} \sum_{i=1}^7 I_{D_2}^{3,i},$$

where, for  $i = 1, \dots, 7$ , we set

$$I_{D_2}^{3,i} := \int_{\Omega_i} |x|^\tau \|x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|^{2\rho} \frac{\|xy, x\tilde{y}_1, x\tilde{y}_2, x\tilde{y}_3, y\tilde{y}_1\|^{2\rho}}{\|y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|^{2\rho}} d\mu(x, y, \tilde{\mathbf{y}}).$$

We compute each of the integrals  $I_{D_2}^{3,i}$  in turn.

4.2.4. *Subcase 1.* Assume that  $(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \Omega_1$ . Then  $|x| \geq \|y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|$ , so  $|x\tilde{y}_1| \geq |y\tilde{y}_1|$ , and

$$\|xy, x\tilde{y}_1, x\tilde{y}_2, x\tilde{y}_3, y\tilde{y}_1\| = \|xy, x\tilde{y}_1, x\tilde{y}_2, x\tilde{y}_3\| = |x| \cdot \|y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|.$$

Thus

$$I_{D_2}^{3,1} = \int_{\Omega_1} |x|^{\tau+4\rho} d\mu(x, y, \tilde{\mathbf{y}}).$$

For  $X \in \mathbb{N}$ ,

$$\begin{aligned}& \mu(\{(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \Omega_1 \mid v(x) = X\}) \\ &= \mu(\{(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \mathfrak{p}^{(5)} \mid v(x) = X, X \leq \min\{v(y), v(\tilde{y}_1), v(\tilde{y}_2), v(\tilde{y}_3)\}\}) \\ &= \mu(\{(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \mathfrak{p}^X \setminus \mathfrak{p}^{X+1} \times (\mathfrak{p}^X)^4\}) = (1 - q^{-1})q^{-5X}.\end{aligned}$$

Thus

$$I_{D_2}^{3,1} = \sum_{X \in \mathbb{N}} (1 - q^{-1})q^{-5X} q^{-(\tau+4\rho)X} = \boxed{\frac{q^{-5-\tau-4\rho}(1 - q^{-1})}{1 - q^{-5-\tau-4\rho}}}.$$

Assume now that  $(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \notin \Omega_1$ , that is,  $x \notin M$ . Then  $|x| < \|y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\|$ , and so

$$I_{D_2}^{3,i} = \int_{\Omega_i} |x|^\tau \|xy, x\tilde{y}_1, x\tilde{y}_2, x\tilde{y}_3, y\tilde{y}_1\|^{2\rho} d\mu(x, \tilde{\mathbf{y}}),$$

for all  $2 \leq i \leq 7$ . We treat these six remaining cases in what follows.



4.2.5. *Subcase 2.* We have

$$I_{D_2}^{3,2} = \int_{\Omega_2} |x|^\tau |y|^{2\rho} \|x, \tilde{y}_1\|^{2\rho} d\mu(x, y, \tilde{\mathbf{y}})$$

and, for fixed  $(X, Y, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) \in \mathbb{N}^5$ ,

$$\begin{aligned} & \mu(\{(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \Omega_2 \mid v(x) = X, v(y) = Y, v(\tilde{y}_i) = \tilde{Y}_i, \text{ for } i = 1, 2, 3\}) \\ &= \begin{cases} (1 - q^{-1})^5 q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} & \text{if } Y < X, Y < \tilde{Y}_1, Y \leq \tilde{Y}_2, Y \leq \tilde{Y}_3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We thus get

$$\begin{aligned} I_{D_2}^{3,2} &= (1 - q^{-1})^5 \sum_{\substack{(X,Y,\tilde{Y}_i) \in \mathbb{N}^5 \\ Y < X, Y < \tilde{Y}_1, Y \leq \tilde{Y}_2, Y \leq \tilde{Y}_3}} q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} q^{-\tau X} q^{-2\rho Y} q^{-2\rho \min\{X, \tilde{Y}_1\}} \\ &= (1 - q^{-1})^5 \sum_{\substack{(X,Y,\tilde{Y}_i) \in \mathbb{N}^5 \\ Y < X, Y < \tilde{Y}_1, Y \leq \tilde{Y}_2, Y \leq \tilde{Y}_3}} q^{-(1+\tau)X} q^{-(1+2\rho)Y} q^{-\tilde{Y}_1} q^{-\tilde{Y}_2} q^{-\tilde{Y}_3} q^{-2\rho \min\{X, \tilde{Y}_1\}}. \end{aligned}$$

Set  $X = Y + X'$ ,  $\tilde{Y}_1 = Y + \tilde{Y}'_1$ ,  $\tilde{Y}_2 = Y + \tilde{Y}'_2$ ,  $\tilde{Y}_3 = Y + \tilde{Y}'_3$ , where  $X', \tilde{Y}'_1 \in \mathbb{N}$  and  $\tilde{Y}'_2, \tilde{Y}'_3 \in \mathbb{N}_0$ . Using Lemma 2.2 (2), we then get

$$\begin{aligned} I_{D_2}^{3,2} &= (1 - q^{-1})^5 \sum_{(X', Y, \tilde{Y}'_1, \tilde{Y}'_2, \tilde{Y}'_3) \in \mathbb{N}^3 \times \mathbb{N}_0^2} q^{-(1+\tau)(Y+X')} q^{-(1+2\rho)Y} q^{-(Y+\tilde{Y}'_1)} q^{-(Y+\tilde{Y}'_2)} q^{-(Y+\tilde{Y}'_3)} \\ &\quad \cdot q^{-2\rho \min\{Y+X', Y+\tilde{Y}'_1\}} \\ &= (1 - q^{-1})^5 \sum_{(X', Y, \tilde{Y}'_1, \tilde{Y}'_2, \tilde{Y}'_3) \in \mathbb{N}^3 \times \mathbb{N}_0^2} q^{-(1+\tau)X'} q^{-(5+\tau+4\rho)Y} q^{-\tilde{Y}'_1} q^{-\tilde{Y}'_2} q^{-\tilde{Y}'_3} q^{-2\rho \min\{X', \tilde{Y}'_1\}} \\ &= (1 - q^{-1})^5 \frac{1}{(1 - q^{-1})^2} \frac{q^{-5-\tau-4\rho}}{1 - q^{-5-\tau-4\rho}} \sum_{(X', \tilde{Y}'_1) \in \mathbb{N}^2} q^{-(1+\tau)X'} q^{-\tilde{Y}'_1} q^{-2\rho \min\{X', \tilde{Y}'_1\}} \\ &= (1 - q^{-1})^5 \frac{1}{(1 - q^{-1})^2} \frac{q^{-5-\tau-4\rho}}{1 - q^{-5-\tau-4\rho}} \frac{q^{-2-\tau-2\rho}(1 - q^{-2-\tau})}{(1 - q^{-2-\tau-2\rho})(1 - q^{-1-\tau})(1 - q^{-1})} \\ &= \boxed{\frac{q^{-7-2\tau-6\rho}(1 - q^{-1})^2(1 - q^{-2-\tau})}{(1 - q^{-5-\tau-4\rho})(1 - q^{-2-\tau-2\rho})(1 - q^{-1-\tau})}}. \end{aligned}$$

4.2.6. *Subcase 3.* We have

$$I_{D_2}^{3,3} = \int_{\Omega_3} |x|^\tau |\tilde{y}_1|^{2\rho} \|x, y\|^{2\rho} d\mu(x, y, \tilde{\mathbf{y}}).$$

Since this integral is given by that in Section 4.2.5 by permuting the variables  $y$  and  $\tilde{y}_1$ , the two integrals coincide, that is,  $I_{D_2}^{3,3} = I_{D_2}^{3,2}$ .

4.2.7. *Subcase 4.* We have

$$I_{D_2}^{3,4} = \int_{\Omega_4} |x|^\tau |\tilde{y}_1|^{2\rho} \|x, y\|^{2\rho} d\mu(x, y, \tilde{\mathbf{y}})$$

and, for fixed  $(X, Y, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) \in \mathbb{N}^5$ ,

$$\begin{aligned} & \mu(\{(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \Omega_4 \mid v(x) = X, v(y) = Y, v(\tilde{y}_i) = \tilde{Y}_i, \text{ for } i = 1, 2, 3\}) \\ &= \begin{cases} (1 - q^{-1})^5 q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} & \text{if } Y < X, Y = \tilde{Y}_1, Y \leq \tilde{Y}_2, Y \leq \tilde{Y}_3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We thus get

$$\begin{aligned} I_{D_2}^{3,4} &= (1 - q^{-1})^5 \sum_{\substack{(X, Y, \tilde{Y}_i) \in \mathbb{N}^5 \\ Y < X, Y = \tilde{Y}_1, Y \leq \tilde{Y}_2, Y \leq \tilde{Y}_3}} q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} q^{-\tau X} q^{-2\rho \tilde{Y}_1} q^{-2\rho \min\{X, Y\}} \\ &= (1 - q^{-1})^5 \sum_{\substack{(X, Y, \tilde{Y}_2, \tilde{Y}_3) \in \mathbb{N}^4 \\ Y < X, Y \leq \tilde{Y}_2, Y \leq \tilde{Y}_3}} q^{-(1+\tau)X} q^{-2(1+\rho)Y} q^{-\tilde{Y}_2} q^{-\tilde{Y}_3} q^{-2\rho \min\{X, Y\}}. \end{aligned}$$

Set  $X = Y + X'$ ,  $\tilde{Y}_2 = Y + \tilde{Y}'_2$ ,  $\tilde{Y}_3 = Y + \tilde{Y}'_3$ , where  $X' \in \mathbb{N}$ ,  $\tilde{Y}'_2, \tilde{Y}'_3 \in \mathbb{N}_0$ . We then get

$$\begin{aligned} & I_{D_2}^{3,4} \\ &= (1 - q^{-1})^5 \sum_{(X', Y, \tilde{Y}'_2, \tilde{Y}'_3) \in \mathbb{N}^2 \times \mathbb{N}_0^2} q^{-(1+\tau)(Y+X')} q^{-2(1+\rho)Y} q^{-(Y+\tilde{Y}'_2)} q^{-(Y+\tilde{Y}'_3)} \\ & \quad \cdot q^{-2\rho \min\{Y+X', Y\}} \\ &= (1 - q^{-1})^5 \sum_{(X', Y, \tilde{Y}'_2, \tilde{Y}'_3) \in \mathbb{N}^2 \times \mathbb{N}_0^2} q^{-(1+\tau)X'} q^{-(5+\tau+4\rho)Y} q^{-\tilde{Y}'_2} q^{-\tilde{Y}'_3} \\ &= (1 - q^{-1})^5 \frac{q^{-1-\tau}}{1 - q^{-1-\tau}} \frac{q^{-5-\tau-4\rho}}{1 - q^{-5-\tau-4\rho}} \frac{1}{(1 - q^{-1})^2} = \boxed{\frac{q^{-6-2\tau-4\rho}(1 - q^{-1})^3}{(1 - q^{-1-\tau})(1 - q^{-5-\tau-4\rho})}}. \end{aligned}$$

4.2.8. *Subcase 5.* We have

$$I_{D_2}^{3,5} = \int_{\Omega_5} |x|^\tau \|x\tilde{y}_2, y\tilde{y}_1\|^{2\rho} d\mu(x, y, \tilde{\mathbf{y}})$$

and, for fixed  $(X, Y, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) \in \mathbb{N}^5$ ,

$$\begin{aligned} & \mu(\{(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \Omega_5 \mid v(x) = X, v(y) = Y, v(\tilde{y}_i) = \tilde{Y}_i, \text{ for } i = 1, 2, 3\}) \\ &= \begin{cases} (1 - q^{-1})^5 q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} & \text{if } \tilde{Y}_2 < X, \tilde{Y}_2 < Y, \tilde{Y}_2 < \tilde{Y}_1, \tilde{Y}_2 < \tilde{Y}_3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We thus get

$$\begin{aligned} I_{D_2}^{3,5} &= (1 - q^{-1})^5 \sum_{\substack{(X, Y, \tilde{Y}_i) \in \mathbb{N}^5 \\ \tilde{Y}_2 < X, \tilde{Y}_2 < Y, \tilde{Y}_2 < \tilde{Y}_1, \tilde{Y}_2 < \tilde{Y}_3}} q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} q^{-\tau X} q^{-2\rho \min\{X+\tilde{Y}_2, Y+\tilde{Y}_1\}} \\ &= (1 - q^{-1})^5 \sum_{\substack{(X, Y, \tilde{Y}_i) \in \mathbb{N}^5 \\ \tilde{Y}_2 < X, \tilde{Y}_2 < Y, \tilde{Y}_2 < \tilde{Y}_1, \tilde{Y}_2 < \tilde{Y}_3}} q^{-(1+\tau)X} q^{-Y} q^{-\tilde{Y}_1} q^{-\tilde{Y}_2} q^{-\tilde{Y}_3} q^{-2\rho \min\{X+\tilde{Y}_2, Y+\tilde{Y}_1\}}. \end{aligned}$$

Set  $X = \tilde{Y}_2 + X'$ ,  $Y = \tilde{Y}_2 + Y'$ ,  $\tilde{Y}_1 = \tilde{Y}_2 + \tilde{Y}'_1$ ,  $\tilde{Y}_3 = \tilde{Y}_2 + \tilde{Y}'_3$ , for  $X', Y', \tilde{Y}'_1, \tilde{Y}'_3 \in \mathbb{N}$ . Then

$$\begin{aligned}
& I_{D_2}^{3,5} \\
&= (1 - q^{-1})^5 \sum_{(X', Y', \tilde{Y}'_1, \tilde{Y}'_2, \tilde{Y}'_3) \in \mathbb{N}^5} q^{-(1+\tau)(\tilde{Y}_2+X')} q^{-(\tilde{Y}_2+Y')} q^{-(\tilde{Y}_2+\tilde{Y}'_1)} q^{-\tilde{Y}_2} q^{-(\tilde{Y}_2+\tilde{Y}'_3)} \\
&\quad \cdot q^{-2\rho \min\{\tilde{Y}_2+X'+\tilde{Y}_2, \tilde{Y}_2+Y'+\tilde{Y}_2+\tilde{Y}'_1\}} \\
&= (1 - q^{-1})^5 \sum_{(X', Y', \tilde{Y}'_1, \tilde{Y}'_2, \tilde{Y}'_3) \in \mathbb{N}^5} q^{-(5+\tau+4\rho)\tilde{Y}_2} q^{-(1+\tau)X'} q^{-Y'} q^{-\tilde{Y}'_1} q^{-\tilde{Y}'_3} q^{-2\rho \min\{X', Y'+\tilde{Y}'_1\}} \\
&= (1 - q^{-1})^5 \frac{q^{-5-\tau-4\rho}}{1 - q^{-5-\tau-4\rho}} \frac{q^{-1}}{1 - q^{-1}} \sum_{(X', Y', \tilde{Y}'_1) \in \mathbb{N}^3} q^{-(1+\tau)X'} q^{-(Y'+\tilde{Y}'_1)} q^{-2\rho \min\{X', Y'+\tilde{Y}'_1\}}.
\end{aligned}$$

Applying Lemma 2.2 (3) we get

$$\begin{aligned}
& I_{D_2}^{3,5} \\
&= (1 - q^{-1})^4 \frac{q^{-6-\tau-4\rho}}{1 - q^{-5-\tau-4\rho}} \frac{q^{-3-\tau-2\rho}(1 - q^{-1-\tau} + q^{-1-\tau-2\rho} - 2q^{-2-\tau-2\rho} + q^{-4-2\tau-2\rho})}{(1 - q^{-2-\tau-2\rho})^2(1 - q^{-1-\tau})(1 - q^{-1})^2} \\
&= \boxed{\frac{q^{-9-2\tau-6\rho}(1 - q^{-1})^2(1 - q^{-1-\tau} + q^{-1-\tau-2\rho} - 2q^{-2-\tau-2\rho} + q^{-4-2\tau-2\rho})}{(1 - q^{-5-\tau-4\rho})(1 - q^{-2-\tau-2\rho})^2(1 - q^{-1-\tau})}}.
\end{aligned}$$

4.2.9. *Subcase 6.* We have

$$I_{D_2}^{3,6} = \int_{\Omega_6} |x|^\tau \|x\tilde{y}_3, y\tilde{y}_1\|^{2\rho} d\mu(x, y, \tilde{y}).$$

Since this integral is given by that in Section 4.2.8 by permuting the variables  $\tilde{y}_2$  and  $\tilde{y}_3$ , the two integrals coincide, that is,  $I_{D_2}^{3,6} = I_{D_2}^{3,5}$ .

4.2.10. *Subcase 7.* We have

$$I_{D_2}^{3,7} = \int_{\Omega_7} |x|^\tau \|x\tilde{y}_2, y\tilde{y}_1\|^{2\rho} d\mu(x, y, \tilde{y}).$$

and, for fixed  $(X, Y, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) \in \mathbb{N}^5$ ,

$$\begin{aligned}
& \mu(\{(x, y, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \Omega_7 \mid v(x) = X, v(y) = Y, v(\tilde{y}_i) = \tilde{Y}_i, \text{ for } i = 1, 2, 3\}) \\
&= \begin{cases} (1 - q^{-1})^5 q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} & \text{if } \tilde{Y}_2 < X, \tilde{Y}_2 < Y, \tilde{Y}_2 < \tilde{Y}_1, \tilde{Y}_2 = \tilde{Y}_3, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We thus get

$$\begin{aligned}
I_{D_2}^{3,7} &= (1 - q^{-1})^5 \sum_{\substack{(X, Y, \tilde{Y}_i) \in \mathbb{N}^5 \\ \tilde{Y}_2 < X, \tilde{Y}_2 < Y, \tilde{Y}_2 < \tilde{Y}_1, \tilde{Y}_2 = \tilde{Y}_3}} q^{-X-Y-\tilde{Y}_1-\tilde{Y}_2-\tilde{Y}_3} q^{-\tau X} q^{-2\rho \min\{X+\tilde{Y}_2, Y+\tilde{Y}_1\}} \\
&= (1 - q^{-1})^5 \sum_{\substack{(X, Y, \tilde{Y}_i) \in \mathbb{N}^5 \\ \tilde{Y}_2 < X, \tilde{Y}_2 < Y, \tilde{Y}_2 < \tilde{Y}_1, \tilde{Y}_2 = \tilde{Y}_3}} q^{-(1+\tau)X} q^{-Y} q^{-\tilde{Y}_1} q^{-2\tilde{Y}_2} q^{-2\rho \min\{X+\tilde{Y}_2, Y+\tilde{Y}_1\}}.
\end{aligned}$$

Set  $X = \tilde{Y}_2 + X'$ ,  $Y = \tilde{Y}_2 + Y'$ ,  $\tilde{Y}_1 = \tilde{Y}_2 + \tilde{Y}'_1$ , for  $X', Y', \tilde{Y}'_1 \in \mathbb{N}$ . We then get

$$\begin{aligned}
& (1 - q^{-1})^{-5} I_{D_2}^{3,7} \\
&= \sum_{(X', Y', \tilde{Y}'_1, \tilde{Y}_2) \in \mathbb{N}^4} q^{-(1+\tau)(\tilde{Y}_2+X')} q^{-(\tilde{Y}_2+Y')} q^{-(\tilde{Y}_2+\tilde{Y}'_1)} q^{-2\tilde{Y}_2} q^{-2\rho \min\{\tilde{Y}_2+X'+\tilde{Y}_2, \tilde{Y}_2+Y'+\tilde{Y}_2+\tilde{Y}'_1\}} \\
&= \sum_{(X', Y', \tilde{Y}'_1, \tilde{Y}_2) \in \mathbb{N}^4} q^{-(5+\tau+4\rho)\tilde{Y}_2} q^{-(1+\tau)X'} q^{-Y'} q^{-\tilde{Y}'_1} q^{-2\rho \min\{X', Y'+\tilde{Y}'_1\}} \\
&= \frac{q^{-5-\tau-4\rho}}{1 - q^{-5-\tau-4\rho}} \sum_{(X', Y', \tilde{Y}'_1) \in \mathbb{N}^3} q^{-(1+\tau)X'} q^{-(Y'+\tilde{Y}'_1)} q^{-2\rho \min\{X', Y'+\tilde{Y}'_1\}}.
\end{aligned}$$

Applying Lemma 2.2 (3) we get  $I_{D_2}^{3,7} = \frac{(1-q^{-1})}{q^{-1}} I_{D_2}^{3,5}$ .

**4.3. Putting the pieces together.** The computations in Section 4.2 combine to an explicit formula for

$$I_{D_2} = |\mathbb{A}^8(\mathbb{F}_q) \setminus V_3(\mathbb{F}_q)| I_{D_2}^1 + (|V_3(\mathbb{F}_q)| - |V_2(\mathbb{F}_q)|) I_{D_2}^2 + |V_2(\mathbb{F}_q) \setminus \{0\}| I_{D_2}^3,$$

with  $I_{D_2}^3 = p^{-5} \sum_{i=1}^7 I_{D_2}^{3,i}$ . Using Lemma 4.1 and equations (4.2) and (4.6), we obtain, with  $a = q^{-1}$ ,  $b = q^{-\tau}$ ,  $c = q^{-\rho}$ , that

$$\begin{aligned}
\mathcal{Z}_o(\rho, \tau) &= \frac{ab(a-1)^2}{(1-a^5bc^4)(1-a^5bc^2)(1-a^2bc^2)^2(1-ab)} \\
&\left( a^{22}b^4c^{10} + a^{21}b^4c^{10} + a^{20}b^4c^{10} + a^{19}b^4c^{10} + a^{18}b^4c^{10} \right. \\
&+ a^{17}b^4c^{10} + a^{16}b^4c^{10} + a^{15}b^4c^{10} + a^{14}b^4c^{10} - 2a^{20}b^3c^{10} - a^{19}b^3c^{10} - a^{18}b^3c^{10} \\
&- 2a^{17}b^3c^{10} - a^{15}b^3c^{10} - a^{14}b^3c^{10} - a^{13}b^3c^{10} - a^{19}b^3c^8 - a^{18}b^3c^8 - 2a^{17}b^3c^8 \\
&- 5a^{16}b^3c^8 - 4a^{15}b^3c^8 - 4a^{14}b^3c^8 - 3a^{13}b^3c^8 - 3a^{12}b^3c^8 - a^{11}b^3c^8 - a^{10}b^3c^8 \\
&- a^9b^3c^8 + a^{18}b^2c^8 + a^{17}b^2c^8 + 2a^{16}b^2c^8 + 5a^{15}b^2c^8 + 4a^{14}b^2c^8 + 4a^{13}b^2c^8 \\
&+ 3a^{12}b^2c^8 + 3a^{11}b^2c^8 + a^{10}b^2c^8 + a^9b^2c^8 + a^8b^2c^8 + a^{16}b^3c^6 + a^{15}b^3c^6 + a^{14}b^3c^6 \\
&- a^{12}b^3c^6 - a^{11}b^3c^6 - a^{10}b^3c^6 - a^9b^3c^6 - a^{16}b^2c^6 + a^{14}b^2c^6 + 2a^{13}b^2c^6 + 5a^{12}b^2c^6 \\
&+ 4a^{11}b^2c^6 + 6a^{10}b^2c^6 + 4a^9b^2c^6 + 3a^8b^2c^6 + 2a^7b^2c^6 - 2a^{13}bc^6 - 3a^{12}bc^6 \\
&- 4a^{11}bc^6 - 4a^{10}bc^6 - 5a^9bc^6 - 3a^8bc^6 - 2a^7bc^6 - 2a^6bc^6 - a^{13}b^2c^4 - 2a^{12}b^2c^4 \\
&- a^{11}b^2c^4 - a^{10}b^2c^4 + a^8b^2c^4 + 3a^7b^2c^4 + a^6b^2c^4 + a^5b^2c^4 + a^4b^2c^4 + a^{12}bc^4 + 2a^{11}bc^4 \\
&+ a^9bc^4 - 2a^8bc^4 - 3a^7bc^4 - 4a^6bc^4 - 3a^5bc^4 - a^4bc^4 - a^3bc^4 + a^8c^4 + 2a^7c^4 + 2a^6c^4 \\
&+ 2a^5c^4 + a^4c^4 + a^8bc^2 + a^7bc^2 + 2a^6bc^2 - a^5bc^2 - a^4bc^2 - a^3bc^2 - 2a^2bc^2 - a^7c^2 \\
&\left. - a^6c^2 - 2a^5c^2 + a^4c^2 + a^3c^2 + a^2c^2 + 2ac^2 + a^5 - a^4 - a + 1 \right)
\end{aligned}$$

By (4.1) and a direct computation, this completes the proof of Theorem 1.3.

Corollary 1.4 follows, for instance, along the lines of [5, Lemma 5.5]. Indeed, the invariant  $\beta$  defined on [5, p. 124] pertinent to the Euler product

$$\prod_{\mathfrak{p}} (q^8 t^3 - q^7 t^2 + q^6 t^2 - q^5 t^2 - q^3 t + q^2 t - qt + 1)$$

equals  $\beta = \max\{8/3, 7/2, 3/1\} = 7/2$ .

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ALEXANDER STASINSKI, DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, DURHAM, DH1 3LE, UK

*E-mail address:* [alexander.stasinski@durham.ac.uk](mailto:alexander.stasinski@durham.ac.uk)

CHRISTOPHER VOLL, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, POSTFACH 100131, D-33501 BIELEFELD, GERMANY

*E-mail address:* [voll@math.uni-bielefeld.de](mailto:voll@math.uni-bielefeld.de)