Population Growth and human capital: a Welfarist Approach

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Abstract

In this paper we investigate the relationship between economic and population growth in an endogenous growth model driven by human capital accumulation à la Lucas (1988). Since we allow for endogenous population growth, we adopt the population criterion Relative Critical Level Utilitarianism (an extension of Critical Level Utilitarianism, Blackorby et al. 1995) which allows axiomatically founded welfare orderings under variable population. Under this extension the Critical Level Utility is dependent on parents' wellbeing. In this scenario we investigate the equilibrium relation between economic growth and population growth as functions of the underlying parameters and we provide the conditions for the economic take-off to occur. A simulation analysis calibrated on Developing Countries shows that the model has the potential to explain the divergent dynamics of GDP per capita growth and population growth experienced by those countries.

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1. Introduction

Endogenous growth models have been flourishing over the last three decades. Starting from the pioneering work by Romer (1986) such a strand of literature has provided new insights on the relationship between human capital accumulation, technological progress and economic growth.

These contributions have clarified that long-run per capita growth, in the absence of exogenous technological progress, can only be achieved if returns to capital are constant asymptotically (see Barro and Sala-i-Martin 2004, ch. 5). There are various approaches to this, such as Romer (1986), in which externalities deriving from existing capital (spillovers as "learning by doing") are introduced, Lucas (1988), where growth is due to linearity in human capital accumulation,[1] and the R&D/quality ladder models, following Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992).

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[1] Lucas (1988) building on a framework developed by Uzawa (1964) showed that decreasing returns to capital could be avoided by adopting a broad view of capital itself that entails human capital as well (externalities from "human capital"). Under the Uzawa-Lucas model, human capital accumulation is in fact the engine of growth that can avoid diminishing returns to capital.

A strand of this literature has focused on the interaction between long-run (endogenous) economic growth and population growth. The question of population growth comes naturally in the R&D models and the human capital models. In the human capital model a larger population growth rate makes per-capita physical capital accumulation more costly, and consequently more resources are devoted to human capital accumulation, and thereby increasing the economic growth rate. In the R&D framework there is the potential scale effect, in that larger populations produce more ideas/innovations, and therefore would grow faster. If the scale effect is removed by assuming diminishing returns in creation of ideas (as in Jones 1995, Kortum 1997, Segerstrom 1998) then the long-run economic growth rate is positively related only to the population growth rate.

However, there seems to be little or no evidence at all, in postwar data or from historical data of the past 200 years, that faster population growth leads to a higher equilibrium growth rate (once other factors such as education, quality of institutions, country size and openness to trade have been taken into account), (see Acemoglu 2009, p. 448, Bloom et. al. 2003, p. 17).

In light of this, models have been developed to allow for a richer relationship between economic growth and population growth.[2] Dalgaard and Kreiner (2001) introduce human capital accumulation in the R&D model. A fraction of output is used in the production of human capital (i.e. not time). This produces congestion in human capital accumulation, making per capita human capital more costly to sustain at higher population sizes. In this model economic growth and population growth are negatively related. Strulik (2005) proposes a two-sector research model where human capital is used in discovery of new varieties and in quality improvement of existing ones. Here, the effect of population growth on economic growth is ambiguous.

Analogously, Bucci (2008, 2013), in a model with human capital accumulation and expanding varieties, shows that the relation between population growth and economic growth can be ambiguous.

A key feature of the combined R&D and human capital models above is that, by specification, the growth rate of per capita human capital is negative in the population growth rate,[3] and is central in producing the negative relationship between economic growth and population growth.

There is also a strand of literature, seeking to analyse the relationship between economic growth and population growth when both are endogenous. In doing so, all articles we have come across, use a version of the objective function proposed by Barro and Becker (1989), specifying utility of parents as a function of the number of children (an exception is Tournemaine and Luangaram, 2012, who specify utility as a function of population size). The question is then how both growth rates vary with underlying parameters of the model.

Connolly and Peretto (2003) investigate endogenous fertility in the presence of horizontal and vertical R&D. Under this scenario they show that in the long run, growth and fertility may diverge due to exogenous shocks (policies) affecting vertical R&D costs or horizontal innovation costs, while, again in the long run, they still move in the same direction under demographic shocks concerning the mortality rate or the cost of reproduction. However, such results are obtained through simulations.

In the model by Tournemaine and Luangaram (2012), economic growth is driven by two sectors: R&D (which increases total factor productivity) and human capital accumulation (which increases labour productivity). Preferences are specified over per capita consumption and population size. Policy, acting as a subsidy to the R&D sector, induces a move from human capital to R&D, affecting both economic growth and population growth, which both go in opposite directions (apart from a razors edge case when policy is neutral).

[2] Some authors in removing the scale effect, obtain the result that the economic growth rate is independent of the population growth rate, see Dinopoulos and Thompson (1998) and Peretto (1998) (though in the latter, per-capita consumption growth is positively related to the population growth rate.

[3] Bucci (2008) also allows for the case where the population growth rate does not enter the per capital human capital accumulation equation. In that case, under perfect altruism, the population growth rate has no effect on the economic growth rate.

Chu et al. (2013) propose a quality-ladder model with human capital. Again, by specification, the growth rate of per capita human capital is negative in the population growth rate. Among their results are that population growth is increasing in patent protection, while consumption growth is ambiguous. Consequently the relationship between population growth and economic growth is ambiguous, when patent protection is the deep parameter being varied.

Prettner (2013) introduces a Blanchard type overlapping generations model, where each individual is facing a constant death probability at each date, in order to study the effect of birth rates and longevity. Nesting the frameworks of Romer (1990) and Jones (1995), depending on parameters, three cases are found. In one case population growth is negative and economic growth is zero and in the second case population growth is zero and economic growth is exponential in the Romer framework and zero in the Jones framework. In the final case, population growth is positive and economic growth is hyper exponential in the Romer framework and exponential in the framework of Jones. Consequently, varying the spillover parameter, the model suggests a positive relationship between population growth and economic growth.

Previous literature has relied on Barro-Becker population preferences.[4] However, there is a parallel literature on objective functions for population decisions (for an overview see Blackorby et al 2005). This research puts particular emphasis on the fact that, in the presence of endogenous population, welfare evaluations typically imply the comparisons between states of the world in which the size of population is different. This observation has two consequences: first, one needs Social Welfare Orderings that are axiomatically founded also in presence of variable population. Second, one also aims at avoiding undesirable outcomes such as the so-called Repugnant Conclusion (RC henceforth; see Parfit 1976, 1984, Blackorby et al. 2002). According to the RC, any state in which each member of the population enjoys a life above neutrality is declared inferior to a state in which each member of a larger population lives a life with lower utility (Blackorby et al. 1995, 2002). In particular, in an economic growth setting, the RC takes the form of an upper-corner solution for the population growth rate (i.e. society reproduces at its physical maximum rate, see Renström and Spataro 2011 for a discussion).

To cope with these problems we adopt a population criterion proposed in Renström and Spataro (2012), referred to as Relative Critical Level Utilitarianism (RCLU), which is in the spirit of the Critical Level Utilitarianism[5], in that it is axiomatically founded. Under RCLU the judgment (the critical level of utility for life worth living) is relative to the existing generation's level of wellbeing. In other words, according to such a criterion a society or an individual household[6] at low level of utility will set a lower threshold of utility for the next generation, and a society or an individual household with high living standard will set a higher level. So if parents had a good life, they require their children to have a good life as well, and vice versa.

In the light of this background, in the present paper we aim at taking a step further by addressing the following question: what does the relationship between endogenous economic growth, human capital and endogenous fertility look like under Relative Critical Utilitarianism? To the best of our knowledge, this has not been done before.

We have seen that previous literature has been successful in generating a negative relationship between economic growth and population growth, in models where an R&D sector and human

[4] There is also a literature on (optimal) endogenous fertility, although without endogenous economic growth, see, for example, de la Croix et al. (2012) and Pestieau and Ponthiere (2014), as well as a literature on endogenous fertility and endogenous growth in non-dynastic economies, where the planning horizon for each agent is 2 or 3 periods, see Blackburn and Cipriani (2002). In the latter, two regimes are generated, one with low economic growth and high population growth, and on with the opposite.

[5] Critical Level Utilitarianism (see Blackorby et al. 1995), is an axiomatically founded population principle where the Critical Level is defined as a utility value (α) of an extra person, who if added to the (unaffected) population, would make society as well off as without that person.

[6] One can apply the RCLU criterion at either social level (for normative analysis) or at individual family level (for positive analysis) by aggregating over individuals that have preferences with both intergenerational altruism and reference point represented by previous generation's welfare. See next footnote.

capital are combined, and where per-capita human capital accumulation is specified to be negatively related to the population growth rate. To highlight the role of endogenous population growth under Relative Critical Level Utilitarianism, we will rely on only one growth generating sector (human capital), and we also propose a specification of per-capita human capital accumulation where the population growth rate does not enter (we derive such a formulation in appendix).

To summarize, we explore the relationship between endogenous growth and population growth under RCLU, when economic growth is driven by human capital accumulation, by pinning down the conditions under which sustained long-run growth occurs and by unveiling the circumstances under which population growth and economic growth are positively or negatively correlated.

The paper is organized as follows: in section 2 we lay out the model; in section 3 we characterize the solution; in section 4 we prove local stability of the endogenous growth path; in section 5 we carry out comparative statics; and in section 6 we perform calibration and numerical simulations. Section 7 concludes.

2. The model

2.1 Preferences

Following Renström and Spataro (2012), we focus on a single dynasty (household) or a policymaker choosing consumption and population growth over time, so as to maximize:

$$W(u_{t-1}, N_t, u_{t+1}, N_{t+1}, \ldots) = \sum_{s=0}^{\infty} \beta^s N_{t+s} \left[u_{t+s} - \alpha u_{t-1+s} \right] \quad (1)$$

where N_t is the population (family) size of generation *t*, $u_t = u(c_t) \geq 0$ is the utility function of an individual of generation *t*, with $u(0) = 0$, $u' > 0, u'' < 0$, $\beta = \frac{1}{1+\rho}, \beta \in (0,1)$ the intergenerational discount factor and $\rho > 0$ the intergenerational discount rate; αu_{t-1} is the Critical Level Utility, with $\alpha \in (0,1)$ applied to generation *t*. Such a critical value is a positive function of previous generation's utility (only if αu_{t-1} is a constant, this social ordering would coincide with CLU). Renström and Spataro (2012) refer to this population criterion as "Relative CLU" (RCLU) [7].

The continuous time version of (1) can be written as (see Appendix A1):

$$U = \int_0^{\infty} e^{-\rho t} N_t u(c_t) \left[1 - \alpha (n_t - \rho) \right] dt . \quad (2)$$

[7] In fact, if one assumes that individuals are entailed with both intergenerational altruism and relative-consumption (or relative-welfare) preferences, with the reference group being the previous generation's consumption (or welfare), then an individual's preferences could be written as:

$$U_t = (u_t - \alpha u_{t-1}) + \beta \frac{N_{t+1}}{N_t} U_{t+1} ,$$

such that, aggregating over individuals, we obtain:

$$W_t = N_t U_t = N_t (u_t - \alpha u_{t-1}) + \beta N_{t+1} U_{t+1}$$

which coincides with eq. (1) in the text. Hence, as shown in Renström and Spataro (2012), the current analysis can be interpreted as being either normative or positive.

n_t is the (endogenous) population growth rate, i.e.

$$\frac{\dot{N}_t}{N_t} = n_t \tag{3}$$

with $n_t \in [\underline{n}, \overline{n}]$. The integral is finite only if $(\rho - \overline{n}) > 0$, which we assume throughout the paper, implying that $1 - \alpha(n_t - \rho)$>0.

2.2. Manufacturing sector

As in Lucas (1988) we assume a Cobb-Douglas constant returns-to-scale (CRS) production technology:

$$Y_t = F(K_t, L_t) = AK_t^{\gamma} L_t^{1-\gamma} \tag{4}$$

where A is the parameter representing total factor productivity, $\gamma \in (0,1)$ is the output elasticity with respect to capital, K_t. $L_t = h_t v_t N_t$ is effective labour, h_t is the human capital stock, v_t is the fraction of time dedicated to work (and 1-v_t the time dedicated to education), such that $v_t N_t$ is the number of individuals that are at work in each instant t. The capital accumulation equation is:

$$\dot{K}_t = F(K_t, L_t) - c_t N_t \tag{5}$$

Note that in eq. (5) there is no explicit cost for raising children. This is done for two reasons: first, we aim to keep or analysis in the spirit of Renström and Spataro (2011), where the same assumption is posed, and look at the very consequences of introducing RCLU in an endogenous growth model, other things being equal. Second, for the sake of tractability, we aim to be as parsimonious as possible in terms of parameters: as it will be clear in the next section, a trade-off in the choice of giving birth to an extra child clearly emerges in our model, which allows us to avoid the adoption of explicit childbearing costs of any particular form, without loss of generality[8].

2.3 The human capital sector

We specify the accumulation of human capital, h_t, in per-capita terms. In Appendix E we show how it can be derived, by disaggregating human capital accumulation into a research sector and an education sector. With congestion in education, as the population grows larger, more individuals are needed to educate the larger population the scale effect is neutralised. The increase in human capital is then a function of the fraction of the population allocated to

$$\dot{h}_t = \theta(1 - v_t)h_t \tag{6}$$

Notice that we have neither positive, nor negative scale effect.

[8] For a model with CLU and taxation in the presence of childbearing costs, see Spataro and Renström (2012).

3. Solution

Since the dynastic objective (1) is also the social welfare function, and since there are no externalities, a decentralized version of the model (where firms and individuals are price takers) would yield exactly the same equilibrium (i.e. the First Welfare Theorem applies). This means we can interpret our equilibrium as either a decentralized one or as a socially optimal one (i.e. positive and normative analysis coincide). We may therefore use the terminology "socially optimal" or "individually optimal" interchangeably.[9]

The current-value Hamiltonian of the household's problem is the following:

$$H_t = N_t u(c_t)[1-\alpha(n_t-\rho)] + q_t[F(K_t, L_t) - c_t N_t] + \lambda_t n_t N_t + p_t(1-v_t)\theta_t h_t \tag{7}$$

The term $\lambda_t n_t N_t$ in the Hamiltonian associated with eq. (3) (the law of motion for population size) and captures the fact that at each instant of time the population size is given (and thus is a state variable) and can only be controlled by the choice of n_t (which is a control variable). Hence, λ_t can be interpreted as the shadow value of population. q_t and p_t are the usual shadow prices of physical and human capital respectively.

From the first-order conditions (see Appendix A.2), we obtain four dynamic equations that, together with the transversality conditions (see Appendix B), fully characterise our dynamic system (from now on, we omit time subscripts for the sake of notation):

$$\frac{\dot{c}}{c} = -\frac{u}{\alpha u' c} + \frac{G}{\alpha}\left[1 - \frac{F_N}{c}\right] \tag{8}$$

$$\frac{\dot{k}}{k} = \frac{F_K}{\gamma} - \frac{c}{k} - n \tag{9}$$

$$\frac{\dot{h}}{h} = \theta(1-v) \tag{10}$$

$$\dot{n} = \frac{G}{\alpha}\left[\frac{u''}{u'}\dot{c} - (\rho - F_K)\right] = G\left\{\frac{u''}{u'}\left[-\frac{u}{\alpha u'} + \frac{G}{\alpha}(c - F_N)\right] - (\rho - F_K)\right\} \tag{11}$$

where $G \equiv 1 + \alpha(\rho - n) > 0$. Eq. (9) is (5) in per capita terms, where $k \equiv K/N$.

We now briefly comment on aspects of the solution. In particular eq. (9) states that, along the transition path, both the of growth of population, n, and consumption, c, must satisfy the resources available for the economy, while. eq. (10) recovers the law of accumulation of human capital. Moreover, eq. (8) states that at the optimum both consumption and fertility should be chosen in such a way the rate of growth of consumption is proportional to the difference between the increase of social welfare due to an extra individual at the margin, u, and the marginal value (in utility units) of what a newborn takes out of society, $Gu'[c-F_N]$, which is positive due to the presence of a positive capital stock. This is a consequence of RCLU.

In order to simplifying the analysis of the dynamic system, let us define:

[9] We show in Appendix E that the economy can be decentralised.

.

$$\tilde{h} \equiv \frac{vh}{k};\tag{12}$$

Then we have (see Appendix A.2):

$$\frac{\dot{\tilde{h}}}{\tilde{h}} = \frac{\theta + n - F_K}{\gamma}.\tag{13}$$

3.1. Balanced growth path (BGP)

Along the balanced-growth path $\dot{\tilde{h}}$ =0, such that, from (13):

$$F_K = \theta + n\tag{14}$$

Moreover, by equating (9) and (10) and using (14) we get

$$\frac{c}{k} = v\theta + \frac{1-\gamma}{\gamma}(\theta + n)\tag{15}$$

giving us the consumption-capital ratio, along the BGP. Finally, from eq. (11), a BGP, where $\dot{n} = 0$, implies

$$\frac{\dot{c}}{c} = \frac{F_K - \rho}{\sigma(c)}\tag{16}$$

where $\sigma(c) = -\frac{u''c}{u'} > 0$ is the inverse of the intertemporal elasticity of substitution (IES) of consumption. Throughout the paper we will assume the constancy of such a IES, by adopting a CES function for utility.

Moreover, by equating (16) and (10) it follows:

$$F_K = \sigma\theta(1-v) + \rho\tag{17}$$

and from (14) and (17):

$$n = \sigma\theta(1-v) + \rho - \theta.\tag{18}$$

At first sight eq. (18), combined with eq. (10), indicates a positive relationship between population growth and economic growth. However, such a relationship is more complex, given that *v* enters both equations and is, in turn, endogenous. We derive the BGP value for *v* in Appendix B. Substituting this solution for *v* into (17), (18), (15) and (10), gives the balanced-growth values for F_K, *n*, *c/k* and the growth rate of the economy, respectively.

Note that, in order to have interior solution for *v*, it must be that *v*<1. This implies, a set of restrictions on the parameters that insures interiority of the solution for *v*:

Proposition 1: *Necessary and sufficient for having v<1 is*:

$$\theta \geq \theta^* = \frac{\sigma}{2\alpha(1-\sigma)} + \frac{1}{2\alpha}\sqrt{\left(\frac{\sigma}{1-\sigma}\right)^2 + 4\alpha\rho\frac{1-\gamma}{1-\sigma}} \tag{19}$$

Proof: See Appendix B.[10] □

Proposition 1 states that some economies can be trapped at zero (per-capita) growth when educational efficiency is too low (θ too low): in that case society (or households) finds it convenient to employ all labour force (or work) in the manufacturing sector (with *v*=1 and, thus, $\frac{\dot{h}}{h} = \theta(1-v) = 0$.

3.2. Zero growth steady state

If *v* is at its upper corner (*v*=1), i.e. (19) is violated, balanced growth is not achievable. In this case the growth rates are zero

$$\frac{\dot{c}}{c} = \frac{\dot{h}}{h} = \theta(1-v) = 0. \tag{20}$$

Hence, by (16) (or 28)

$$F_K = \rho \tag{21}$$

which univocally pins down the steady-state capital intensity. Moreover, the derivative of the Hamiltonian with respect to *v* (eq. (A.7) in Appendix A.2) now becomes:

$$\frac{\partial H}{\partial v_t} = q_t F_{v_t} - p_t \theta_t h_t > 0 \tag{22}$$

In this case the steady state values of *c* and $\rho - n$ are provided by the system of equations (8) and (9), both being equal to zero, which yield:

$$\rho - n = \frac{c - F_N}{k} = \frac{\sigma}{2\alpha(1-\sigma)} + \frac{1}{2\alpha}\sqrt{\left(\frac{\sigma}{1-\sigma}\right)^2 + 4\frac{\alpha\rho(1-\gamma)}{(1-\sigma)}} \tag{23}$$

Note that since (19) is violated, $\theta^* > \theta$, and eq. (14) is now an inequality:

$$\rho - n > \theta. \tag{24}$$

[10] We have written inequality (19) as a restriction of θ, as a function of the other parameters. Of course we could have rewritten the inequality to provide restrictions on another parameter, as a function of θ.

To summarise the analysis carried out so far, we can state that in the present model a crucial condition for sustained long-run growth to emerge is that the efficiency of human capital production, represented by θ, is sufficiently high. In fact, if returns to human capital investment are too low, income per capita will be constant and the economy will be entrapped in a zero per-capita-growth regime, where aggregate quantities are driven by population growth. On the other hand, if educational efficiency is large enough, then society (or household) will find it convenient to invest resources in the education sector, so that a BGP regime will emerge.

In fact, in the BGP regime the relationship between the economic growth and population growth is nontrivial, and will be analyzed in the section that follows.

Finally, it can be shown that necessary for avoiding the RC is that $\alpha > 0$ (i.e. the proof provided in Renström and Spataro (2012) applies also our model with human capital accumulation).

4. Stability

We now analyze the local stability of the BGP equilibrium[11]. First, let us define $\tilde{c} \equiv \dfrac{c}{k}$, which is constant along the BGP. We can reformulate the dynamic equations characterizing our economy as follows:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} \equiv -\frac{1}{\alpha(1-\sigma)} + \frac{G}{\alpha}\left(1-\frac{F_N}{k\tilde{c}}\right) - \frac{F_K}{\gamma} + \tilde{c} + n = \frac{(1+\alpha\rho)(1-\sigma)-1}{\alpha(1-\sigma)} - \frac{F_K}{\gamma}\left[\frac{G}{\alpha\tilde{c}}(1-\gamma)+1\right] + \tilde{c} \tag{25}$$

$$\frac{\dot{\tilde{h}}}{\tilde{h}} = \frac{\theta + n - F_K}{\gamma} \tag{26}$$

$$\dot{n} = -\frac{G}{\alpha}\left[\frac{-\sigma+\rho\alpha(1-\sigma)}{\alpha(1-\sigma)} - F_K\left[\frac{(1-\gamma)}{\gamma}\frac{G\sigma}{\alpha\tilde{c}}+1\right] + \sigma\frac{G}{\alpha}\right] \tag{27}$$

The associated Jacobian matrix is:

$$J = \begin{bmatrix} \omega + \tilde{c} & -\tilde{c}\,\omega\dfrac{(1-\gamma)G + \tilde{c}\,\alpha}{\tilde{h}G} & \dfrac{\tilde{c}\,\omega\alpha}{G} \\ 0 & -\dfrac{\tilde{c}\,\omega\alpha}{G} & \dfrac{\tilde{h}}{\gamma} \\ -\dfrac{\omega G\sigma}{\alpha\tilde{c}} & \omega\dfrac{G\sigma(1-\gamma)+\gamma\alpha\tilde{c}}{\alpha\tilde{h}} & \sigma\dfrac{G-\omega\alpha}{\alpha} \end{bmatrix}$$

where $\omega \equiv \dfrac{(1-\gamma)}{\gamma}\dfrac{F_K G}{\alpha\tilde{c}} > 0$, $\omega < \dfrac{G}{\alpha}$, $F_{K\tilde{h}} = (1-\gamma)\dfrac{F_K}{\tilde{h}}$.

Hence, we can provide the following Proposition:

Proposition 2: *The balanced growth path is locally stable*.

Proof: See Appendix C. □

[11] As for the local stability of the zero-growth equilibrium, it has been analyzed in Renström and Spataro (2012).

5. Comparative statics

In this section we carry out some comparative statics analyses in order to characterize the role of the models' parameters in affecting the economic growth rate and the population growth rate.

5.1. The role of deep parameters of the model

We first focus on parameters for which we obtain a negative relationship between economic growth and population growth.

The results of the analysis can be summarised through the following Proposition:

Proposition 3: *The balanced growth-rate is increasing in* θ *(human capital production efficiency), and decreasing in* ρ *(the intertemporal discount rate). The rate of growth of population is increasing in* ρ *and decreasing in* θ.

Proof: See Appendix D. □

Consequently, economic growth and population growth move in opposite directions when the human capital production parameter or the intertemporal discount rate changes. As for the other parameters we have:

Proposition 4: *The balanced growth-rate is increasing in* γ *(capital's share) and* α *(the critical level parameter) and decreasing in* σ *(the inverse of consumption elasticity of substitution). The rate of growth of population is increasing in* γ *and* α *and ambiguous in* σ.

Proof: See Appendix D. □

Consequently, for some parameter, a positive relation between economic growth and population growth is obtained.

We summarise Proposition 3 in Table 1

Table 1. The effect of parameters on equilibrium growth

Parameters/Variables	***v*** **(Proportion of individuals allocated to production)**	***g*** **(Balanced growth rate)**	***n*** **(Population growth rate)**
θ (Efficiency in human capital production)	-	+	-
γ (Capital share)	-	+	+
α (Critical level parameter)	-	+	+
ρ (Intertemporal discount rate)	+	-	+
σ (Inverse of consumption elasticity of substitution)	+	-	+/-

The above propositions have some remarkable implications:

First, according to our model, differences in long-run growth rates among countries may depend on differences in both 1) preferences and 2) technology, the latter being concerned with both i) the manufacturing sector and ii) the education sector.

1) As for preferences, an increase in the critical level parameter, α, will make consumption more costly for society (in that future generations wellbeing will be more demanding in terms of resources required to current generations). This will induce individuals to devote more resources to the accumulation of both physical and human capital. As a consequence, economic growth will increase. As a consequence, economic growth will increase. However, since output per capita will be higher, there is room for a higher number of individuals to share the increased amount of resources, so that the optimal rate of growth of population will be higher.

On the other hand, an increase of the discount rate, ρ, by making wellbeing of future generations less relevant, will make it less costly for society to reduce future consumption. This will be possible by increasing the number of children per household on the one hand (increase in *n;* recall that utility function is linear in the population size) and by reducing the pace of accumulation of both human and physical capital on the other hand (i.e. reducing future per-capita consumption; recall that social welfare is concave in per-capita consumption).

2) As for technology:

i) in the manufacturing sector, an increase of γ will cause an increase in the factor price ratio between capital and labour, such that individuals will tend to move from the manufacturing sector to the education sector. This will imply higher accumulation of human capital and higher per-capita growth, which leaves room for more individuals to be brought into life.

ii) as for the education sector, as already stressed in the previous section, the parameter measuring the efficiency of the production function of human capital, θ, is capable to produce a qualitative switch in the growth regime. In fact, under the zero-growth rate regime ($\theta < \theta^*$), changes in the latter parameter do not produce any effect on the economy's equilibrium. However, increases of θ beyond such a threshold, by making investment in human capital more attractive, will produce an increase of the latter and thus, of per-capita income growth rate. Since such a shift will generate a less than proportional increase in the steady state value of marginal productivity of capital (see eq. 28), according to eq. (14) the population growth rate must necessarily decrease.

5.2. The relationship between balanced growth and population dynamics

In this section we illustrate our results, by comparing them with the ones delivered by the existing literature.

Recall that in most previous work (Uzawa-Lucas and the semi-endogenous growth models a la Jones 1995), fertility affects positively the economic growth and, moreover, is the only engine of economic growth. However, our model tells a somehow different story. In fact, by eqs. (10) and (18) we get that the economic growth rate associated with the BGP is:

$$g \equiv \theta(1-v) = \frac{n}{\sigma} + \frac{\theta - \rho}{\sigma} \tag{28}$$

which is positive even if population growth is null. In fact, in our model the engine of growth is the accumulation of human capital, and, in particular, by the effectiveness of such a process, whose returns are measured by θ.

Furthermore, according to the above expression, it is still possible that higher fertility is associated with higher growth. The most straightforward example is the case of higher critical level α that positively affects *g* only through *n*.

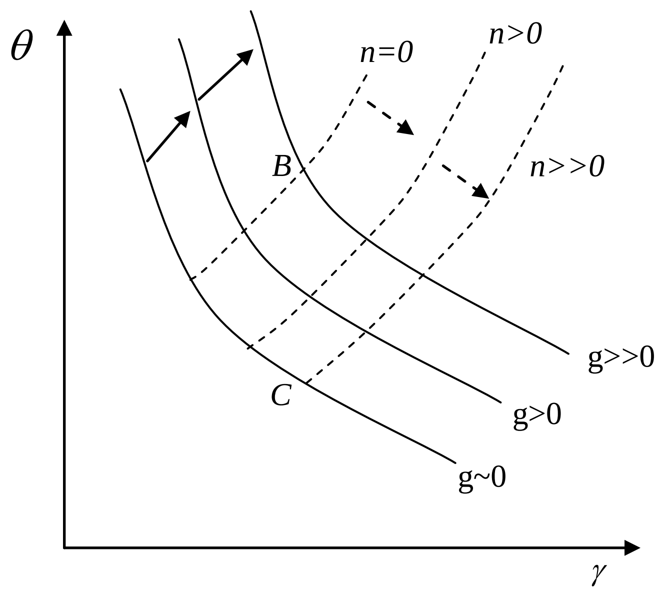
However, we have also shown that lower population growth does not necessarily imply lower economic growth, in that *n* is also a function of parameters that affect *g*. Similarly, according to our results higher population growth rates can be associated with lower economic growth rates.

To make our point clearer, we depict our results in Figure 1.

The solid lines are "iso-g" lines, that is, the loci of all combinations of the parameters (in this case θ, γ) that provide the same economic growth rate *g*. According to Proposition 3, such locus is negatively sloped and the associated *g* increases pointing north-east. The dotted lines are the "iso-n" lines, that is the loci of all combinations of parameters that entail the same rate of growth of population. Our results imply that these loci are positively sloped and the associated *n* increases going south-east. Take for example two points in this Figure (B and C): in the first case low population growth is associated with high levels of economic growth, while in the latter a high level of population growth is associated with low economic growth.

We can conclude that in our paper, the link between population dynamics and economic development is weakened, on one hand, because the former is no longer essential for the latter, and enriched, on the other hand, since more combinations are possible between the two variables, depending on the fundamental parameters of the economy. Thus we argue that the different combinations of such fundamental parameters might be at the origin of the observed cross-countries differences in long-run performances. The analysis of this argument is left for future research.

Figure 1: Iso-growth curves as functions of parameters



6. Simulation exercise

In order to assess the properties of the dynamics of the model and to provide a quantitative assessment of the effects of the main parameters, in this section we describe results obtained from some numerical simulations.

6.1. Calibration

The exercise has been carried out by calibrating the parameters in order to match some macroeconomic variables, as for years 1970-1979, of Developing Countries, as classified by World Bank (2015), and making use of the equilibrium equations of our model. The group of Developing Countries is of particular interest in that, especially since the early 1980s, it has been experiencing a clear divergent dynamics between population growth and GDP per capita growth, which is precisely the phenomenon addressed in this work.

Data have been obtained from World Bank database and Penn World Table 8. Given that in our model one period is about 50 years (life-length of an adult individual), all variables used in the simulations have been computed accordingly. Table 2 reports the precise values of the parameters.

As for the individual parameters, the choice of the inverse of the elasticity of substituion, σ, of the intertemporal discount rate ρ, and of the critical level α, although not based on observed data, was not completely arbitrary.

Table 2: Parameters calibration for Developing Countries (1970-1979)

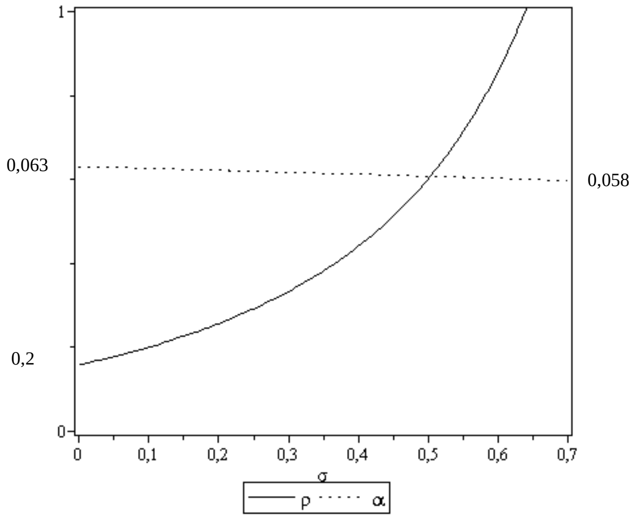
Observed data		**Estimated from BGP equilibrium equations**		**Calibrated parameteres**
Capital intensity $k = 9621$		Efficiency of human capital production $\theta = 18.24$		Critical level utility parameter $\alpha = 0.928$
Human capital per capita $h = 4.08$		Per worker human capital intensity $\tilde{h} = 3.21 \cdot 10^{-4}$		Intertermporal discount rate $\rho = 6\%$
GDP per capita growth rate $g = 3.43\%$		TFP $A = 1670.44$		CIES parameter $\sigma = 0.62$
Population growth rate $n = 2.15\%$		Proportion of individuals allocated to production $v = 0.76$		
Capital share $\gamma = 0.53$				

Data source: our calculations on World Bank (2015) and Penn World Table 8.

In fact, on the one hand, we were constrained by the fact that all the three parameters above must be positive and lower than 1. Moreover, after fixing all the other parameters and , two equilibrium equations of our model provide the relationships for these three variables (namely, eqs. 17 and B.4).

Figure 2 shows such equilibrium relationships, with all other parameters calibrated according to the data and equilibrium equations, as reported in the first two columns of Table 2.

Figure 2: BGP Equilibrium relations from our model between intertemporal discount rate ρ, critical level parameter α and the inverse of the elasticity of substituion, σ.



All other parameters have been fixed at values presented in Table 2

In fact, as shown in the Figure, the range of variation in ρ is very small (it stands around 6% for any possible values of σ below unity). Hence, we decided to fix ρ at 6%, that is the lowest level compatible with acceptable values of α; such a choice also allowed for a sufficiently high value for σ. More precisely, σ and α were fixed at 0.62 and 0.928 respectively.

6.2. Simulation results

We now present the results of the simulation exercise, by showing the impulse functions for the population growth rate and the per-capita GDP growth rate resulting from exogenous shocks hitting the model parameters. The system has been linearised around the BGP equilibrium presented in Table 2.

It is worth noting that g presents no dynamics, in that it jumps immediately on its new BGP level. On the other hand, n takes, on average, about 0.2 periods (i.e. 10 years) to reach the new equilibrium level, depending on the size of the shock and on which parameter has changed. [12]

As for the long run changes, all the simulated variations confirm the signs of the derivatives of *g* and *n* provided in section 5; moreover, recall that the sign of the effect of a change in σ, the inverse of the consumption elasticity of substituion, on the population growth rate, is in principle ambiguous. However, under the parameters specification used in this calibrated exercise, such a sign is positive, pointing to the fact that a decrease in σ, that is an increase in the IES of consumption, will decrease both GDP per capita growth and population growth rate, both in the short and in the long run.

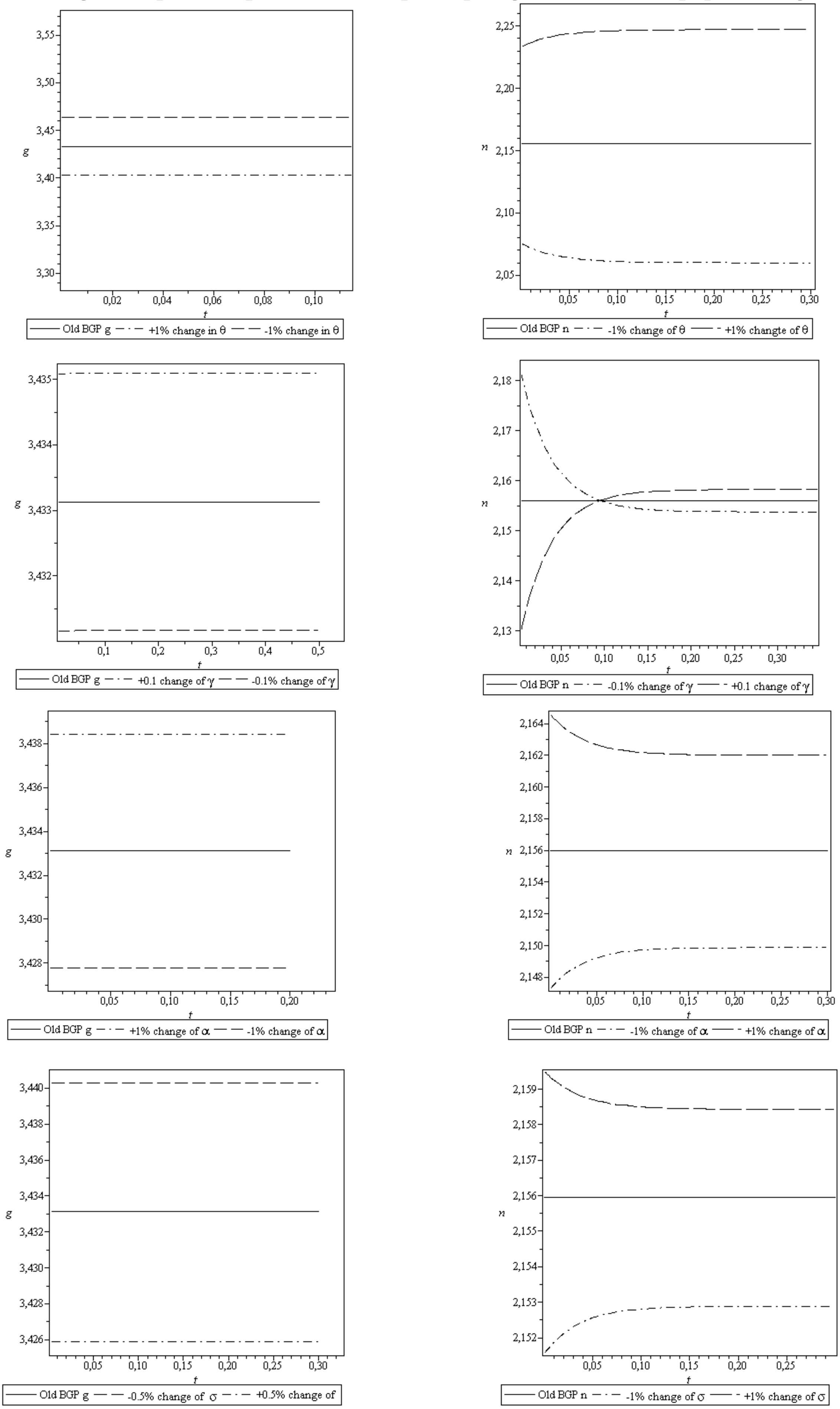
As for the quantitative effect of parameter changes, under our specification it turns out that long run responses of both g and n to changes in θ are larger than the responses occurring due to changes in γ. In fact, a 1% change of the former parameter produces a 0.87% increase in the equilibrium value of g and about 3.23% decrease in the equilibrium value of *n*, while the same percentage change in the parameter measuring the share of capital increases both equilibrium *g* and *n* by 0.56% and 1% respectively. However, the sizes of the short run responses are reversed.

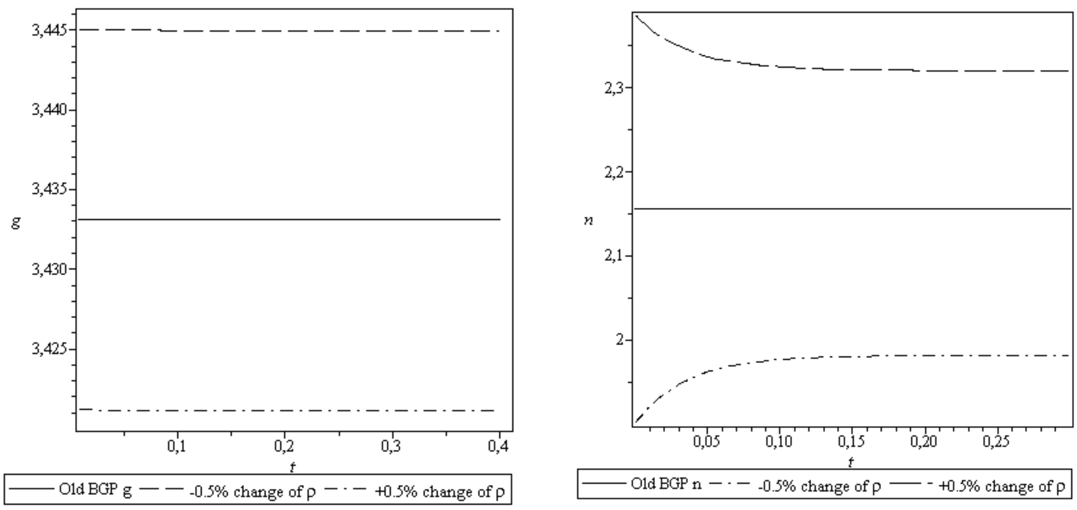
As for the quantitative size of parameters changes, under our specification it turns out that long run responses of both *g* and *n* to changes in θ are wider than the responses occurring to changes in γ. In fact, a 1% change of the former parameter produces a 0.87% increase in the equilibrium value of *g* and about 3.23% decrease in the equilibrium value of *n*, while the same percentage change in the parameter measuring the share of capital increases both equilibrium *g* and *n* by 0.56% and 1% respectively. However, the sizes of the short run responses are reversed.

Although the simulation exercise was carried out for illustrative purposes, one may wonder if the numerical simulations could somehow help interpreting any episodes of the past or recent economic history of some countries.

[12] In fact our model presents some properties of the AK model, in that, on the one hand, after that an exogenous shock has hit the economy, the per capita GDP growth rate, θ (1-*v*) jumps immediately to its equilibrium value. This is due to the fact that the variable *v*, the fraction of individuals employed in the manufacturing sector, is a flow variable, while the population growth rate, although jumping itself, has a nonzero dynamics.
The explanation of the different dynamic behaviour of the latter two variables (*v* and *n*) is that *v* is "costless" to the household, while *n* is not (it enters directly the utility function). Hence, while the former can optimally jump to its new BGP value, following the change in the relative prices (marginal costs and benefits) caused by the exogenous shock, the latter variables jumps at an intermediate level and moves steadily towards its new equilibrium value. Finally, $\tilde{c}$, while jumping itself after the exogenous shock, moves accordingly with *n* to satisfy the feasibility constraint.

Fig. 3: Impulse responses of GDP per capita growth rate and population growth rate

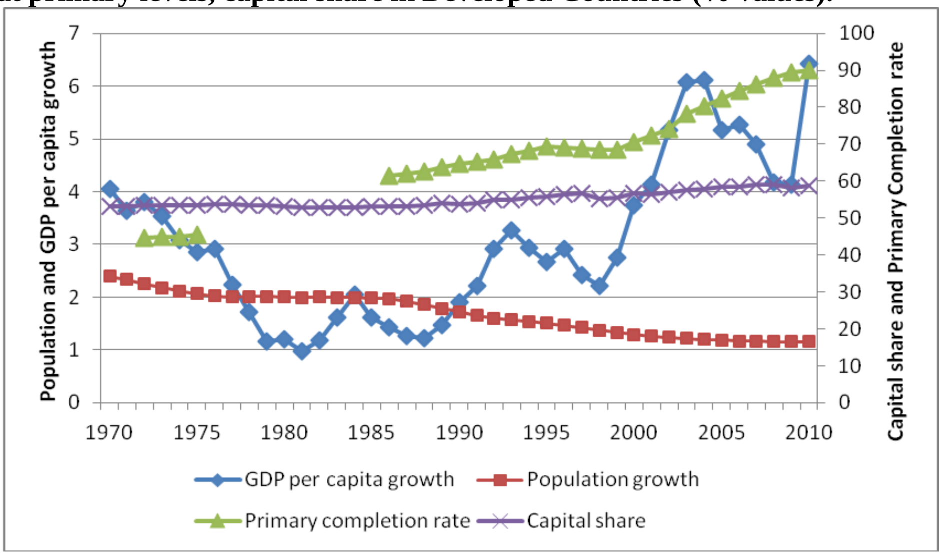




Trying to address this, in Figure 3 we report some data from Developing Countries (source: World Bank 2015 and Penn World Table 8.), which show that, over the last 40 years, the dynamics of g and n previously mentioned has been accompanied by a steady increase of the capital share (from 0.53% of the 1970s to 0.58% in the period 2000-2010), together with a 25% increase of human capital per worker in the same period (see Feenstra et al. 2013). Finally, the indexes of human capital production show a sharp improvement (such as the decreasing pupil-teacher ratio, and the increase in the completion rates of both primary and secondary education).

If we interpret the changes in g and n as responses to increases in θ and γ, other things being equal, we can try verify how our model would perform in replicating these data. Notice that according to our model, an increase in the capital share parameter, as the one documented by the data, would unambiguously increase both equilibrium g and n in the long run, which is at odds with the decrease in n documented by the empirical evidence. Hence, in order to match these trends, we should also allow for an increase of θ. Unfortunately, although there is evidence of improvement in education systems in developed countries, there is no clear index providing an estimate for θ, the parameter measuring the efficiency in the production of the human capital. Hence, in our long-run simulation exercise we tried different changes in θ.

Figure 4: population growth (5 year moving average) GDP per capita growth (5 year moving average), completion rates at primary levels, capital share in Developed Countries (% values).



Source: our computations on World Bank (2015) and Feenstra et al. (2013)

More precisely, by using data for the share of capital for the period 1995-2010 for Developing Countries (equal to 0.5733), we also simulated different increases in θ. The best results we obtained were those occurring with an increase of 10% of the parameter measuring the efficiency in the production of human capital: as shown in Table 3, we would obtain long run values of g and n that are not far from the real values, although underestimating the yearly rate of growth of GDP per capita by 0.15%.

Table 3 Long run simulation for Developing Countries (1995-2010)

Observed Values (%)		**Simulated Values (%)**	
g	*n*	*g*	*n*
4%	1,31	3.85	1.30

γ =0.57 (observed values), θ = 20.64 All other parameters were set at values shown in Table 2.

Although not conclusive, these numerical computations show that both the qualitative and quantitative results of our model are not at odds with some recent empirical evidence of Developing Countries. Consequently, our framework could be further developed for carrying out robust empirical analyses. In particular, we are aware of the fact that, for applicability to real data, we should have a better estimate of θ, the parameter measuring the efficiency of the human capital production. This task is left for future research.

7. Conclusions

In this work we have analysed the long-run relationship between population growth, human capital accumulation and economic growth. For doing this we have adopted a framework that entails both endogenous economic growth and endogenous fertility. Moreover, we have assumed a Social Welfare function that is axiomatically founded, purely welfarist and that allows avoidance of the Repugnant Conclusion (that is, upper-corner solutions for population growth).

Under these assumptions we have shown that the take-off regime of sustained long-run economic growth can only take place when the efficiency of human capital accumulation (the education sector) reaches a certain threshold. Below such a level, increases of the efficiency in the education system produce no effect on the economy, which will continue to be stuck at its zero per-capita-growth regime. On the other hand, beyond such a threshold, sustained growth does occur, and further increases in efficiency in the education sector will generate an increase of the economic growth rate, an increase in human capital accumulation, and a decrease in the population growth rate. The latter results seem in line with the empirical findings concerning the co-movements of the above mentioned variables in the last decades (see, for example, Galor 2005).

Moreover, we have shown that in the long run, population growth and economic growth can diverge, in that positive and high economic growth rates can be associated with low levels of population growth, and vice-versa. Since both variables depend on the parameters of the underlying economy, the exact shape of their relationship is ultimately an empirical matter.

Finally, we have also performed some simulation exercises showing the impulse-response functions for the variables, when exogenous shocks hit the models parameters. The results have shown that in the long run, the per-capita GDP growth rate and the population growth rate are more sensitive to changes in human capital production efficiency, relative to capital share, while in the short run such degree of sensitivity is reversed.

As for policy implications, according to our model any policy aimed at producing the conditions for underdeveloped countries to escape from poverty traps and to enter the regime of sustained growth should be focused on the enhancement of human capital accumulation, that is, on the development of the education sector of such countries. The analysis of the effects of public expenditure on the education sector is left for future research.

References

Acemoglu D (2009) *Introduction to Modern Economic Growth*. Princeton Economic Press. Princeton and Oxford.

Aghion P, Howitt P (1992) A model of growth through creative destruction. Econometrica 60:323–351

Barro RJ, Becker GS (1989) Fertility Choice in a Model of Economic Growth. *Econometrica* 57: 481-501.

Barro, RJ, Sala-i-Martin X (2004) *Economic Growth.* Cambridge, Mass.: MIT Press.

Blackorby C, Bossert W, and Donaldson D (1995) Intertemporal population ethics: critical-level utilitarian principles. *Econometrica* 63(6): 1303-20.

Blackorby C, Bossert W, and Donaldson, D (2002) Critical –lelvel population principles and the repugnant conclusion. *Department of Economics, University of British Columbia Discussion paper*, 02-16.

Blackbun, K., Cipriani, G. (2002) A model of longevity, fertility and growth. Journal of Economic Dynamics and Control, 26(2), 187-204.

Blackorby C, Bossert W, and Donaldson D (2005) Population Issues in Social Choice Theory, Welfare Economics, and Ethics, Cambridge University Press, Cambridge.

Bloom DE, Canning D, Sevilla J (2003). The Demographic Dividend: A New Perspective on the Economic Consequences of Population Change. *RAND*, Santa Monica, CA, MR-1274.

Bucci A (2008) Population growth in a model of economic growth with human capital accumulation and horizontal R&D. *Journal of Macroeconomics* 30(3): 1124-1147.

Bucci, A (2013) Returns to Specialization, Competition, Population, and Growth. *Journal of Economic Dynamics and Control*, 37(10), 2023-40.

Connolly M, Peretto P (2003) Industry and the family: two engines of growth. *Journal of Economic Growth* 8: 115–148.

Chu, A. C., Cozzi, G. and Chih-Hsing Liao (2013) Endogenous Fertility and Human Capital in a Schumpeterian Growth Model, Journal of Population Economics, 26, 181-202

Dalgaard C, Kreiner C (2001) Is declining productivity inevitable? *Journal of Economic Growth* 6(3): 187-203.

de la Croix D, Pestieau P, and Ponthiere G (2012) How powerful is demography? The Serendipity Theorem revisited. *Journal of Population Economics*, 25(3): 899-922.

Dinopoulos E, Thompson P (1998) Schumpterian Growth without Scale Effects. *Journal of Economic Growth* 3: 313–335.

Feenstr, RC, Inklaar R and Timmer MP (2013) The Next Generation of the Penn World Table, available for download at www.ggdc.net/pwt.

Galor O (2005) From Stagnation To Growth: Unfied Growth Theory. In Aghion P and Durlauf SN (eds) *Handbook of Economic Growth*, Vol. 1A, Elsevier.

Grossman G, Helpman E (1991) Quality ladders in the theory of growth. Reiew of Economic Studies 58:43–61.

Jones CI (1995) R&D-Based Models of Economic Growth. *Journal of Political Economics* 103: 759–784.

Kortum S (1997) Research, patenting and technological change. *Econometrica* 65(6): 1389-1419.

Lucas RE (1988) On the Mechanics of Economic Development. *Journal of Monetary Economics* 22: 3–42.

Parfit D (1976) On Doing the Best for Our Children. In *Ethics and Populations*, ed. By M. Bayles. Cambridge: Schenkman.

Parfit D (1984) *Reasons and Persons*. Oxford/New York: Oxford University Press.

Peretto PF (1998) Technological change and population growth. *Journal of Economic Growth* 3(4): 283–311.

Pestieau P, Ponthiere G (2014) Optimal fertility along the life cycle. *Economic Theory*, 55: 185-

224;
Prettner K (2013) Population ageing and endogenous economic growth. *Journal of Population Economics* 26(2): 811-834.
Romer PM (1986) Increasing Returns and Long-Run Growth. *Journal of Political Economy* 94: 1002–1037.
Romer PM (1990) Endogenous Technological Change. *Journal of Political Economy* 98(5): 71–102
Renström TI, Spataro L (2011) The Optimum Growth Rate for Population under Critical-Level Utilitarism. *Journal of Population Economics* 24(3): 1181-1201.
Renström T, Spataro L (2012) Population Growth and Technological Change: a Pure Welfarist Approach. *Durham University Business School Working Papers Series*, n. 8.
Segerstrom PS (1998) Endogenous Growth without Scale Effects. *American Economic Review* 88: 1290–1310.
Spataro L, Renström TI (2012) Optimal taxation, critical-level utilitarianism and economic growth. *Journal of Public Economics*, 96: 727-738.
Strulik H (2005) The Role of human capital and population growth in R&D based models of economic growth. *Review of International Economics* 13(1): 129-145.
Tournemaine, F., Luangaram, P. (2012) R&D, human capital, fertility and growth. Journal of Population Economics, 25: 923-953.
Uzawa H (1964) Optimal Growth in a Two-Sector Model of Capital Accumulation. *Review of Economic Studies* 31: 1–24.
World Bank (2015), “World Development Indicators (WDI)” Accessed on 12 March 2015, available at http://data.worldbank.org/data-catalog/world-development-indicators.

Appendix A.1: The form of eq. (2) (drawn from Renström and Spataro 2012).

By starting from eq. (1) and collecting utility terms of the same date, the welfare function W can be written as:

$$W = \sum_{t=0}^{\infty} \beta^t N_t u(c_t)[1-\alpha\beta(1+n_t)] - \alpha N_0 u(c_{-1}) \qquad \text{(A.1)}$$

By ignoring c_{-1} as it is irrelevant for the planning horizon, and defining $\beta = \frac{1}{1+\rho}$ we get: $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t N_t u(c_t)\left(1-\alpha\frac{1+n_t}{1+\rho}\right)$. In continuous time, by approximating $\frac{1+n_t}{1+\rho} \approx -(\rho - n_t)$ the latter expression can be written as follows:

$$U = \int_0^{\infty} e^{-\rho t} N_t u(c_t)[1-\alpha(n_t-\rho)]dt \qquad \square \qquad \text{(A.2)}$$

Appendix A.2: The first-order conditions and the solution

The first order conditions to the problem imply:

$$\frac{\partial H_t}{\partial c_t} = N_t u'_t\,[1-\alpha(n_t-\rho)] - N_t q_t = 0 \Rightarrow u'_t\,[1-\alpha(n_t-\rho)] = q_t \qquad \text{(A.3)}$$

$$\frac{\partial H_t}{\partial n_t} = N_t(-\alpha u_t + \lambda_t) = 0 \Rightarrow \alpha u_t = \lambda_t \qquad \text{(A.4)}$$

$$\frac{\partial H_t}{\partial N_t} = \rho\lambda_t - \dot{\lambda}_t \Rightarrow \dot{\lambda}_t = (\rho - n_t)\lambda_t - u_t[1-\alpha(n_t-\rho)] - q_t\left[F_{N_t} - c_t\right] \qquad \text{(A.5)}$$

$$\frac{\partial H_t}{\partial K_t} = \rho q_t - \dot{q}_t \Rightarrow \dot{q}_t = q_t\left[\rho - F_{K_t}\right] \qquad \text{(A.6)}$$

$$\frac{\partial H_t}{\partial v_t} = q_t F_{v_t} - p_t\theta_t h_t = 0 \qquad \text{(A.7)}$$

$$\frac{\partial H_t}{\partial h_t} = q_t F_{h_t} + p_t\theta_t(1-v_t) = \rho p_t - \dot{p}_t \qquad \text{(A.8)}$$

plus eqs. (3) and (5) and the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t} q_t K_t = 0, \ \lim_{t\to\infty} e^{-\rho t}\lambda_t N_t = 0. \ \ \lim_{t\to\infty} e^{-\rho t} p_t h_t = 0 \qquad \text{(A.9)}$$

In what follows we assume interiority of the solution for n_t, such that eq. (A.4) holds along the transition path.[13] By substituting for eqs. (A.3) and (A.4) into (A.5) we get:

$$\dot{\lambda}_t = -u_t - u'_t \left[1-\alpha(n_t-\rho)\right]\left[F_{N_t} - c_t\right]. \tag{A.10}$$

Eq. (8) is obtained by taking the time derivative of eq. (A.4) and combining with eq. (A.10), eq. (9) is (5) in per capita terms, where $k \equiv K/N$, (10) stems from eq. (6) and (11) stems from the time derivative of (A.3) and combining with eq. (A.6), (A.3) and (8)**.** Next, differentiating (12) with respect to time, the from (A.7) and (A.8) it follows:

$$\frac{\dot{p}}{p} = \rho + \theta(1-v) - \theta h \frac{F_h}{F_v} = \rho - \theta \tag{A.11}$$

where we have exploited $\frac{F_h}{F_v} = \frac{v}{h}$. Moreover, substituting for $F_v = F_L hN$ into (A.7) and exploiting $F_L = (1-\gamma)A\tilde{h}^{-\gamma}$, time derivative of (A.7) is:

$$\frac{\dot{q}}{q} + \frac{\dot{N}}{N} - \gamma\frac{\dot{\tilde{h}}}{\tilde{h}} = \frac{\dot{p}}{p} \tag{A.12}$$

such that, by exploiting (3), (A.6), and (A.11), we obtain(13).

Appendix B: The BGP value of *v*

By equating eq. (8) to (10) we get:

$$\theta(1-v) = -\frac{1}{\alpha(1-\sigma)} + \frac{\left[1+\alpha(\rho-n)\right]}{\alpha}\left[1-\frac{F_N}{c}\right]. \tag{B.1}$$

Equation (B.1), using (14) and recalling that $\frac{F_N}{c} = \frac{1-\gamma}{\gamma}\frac{k}{c}F_K$, becomes:

$$\theta v = \theta + \frac{1}{\alpha(1-\sigma)} - \frac{\left[1+\alpha(-\sigma\theta(1-v)+\theta)\right]}{\alpha}\left[1-\frac{1-\gamma}{\gamma}\frac{k}{c}\left(\sigma\theta(1-v)+\rho\right)\right] \tag{B.2}$$

Finally, by using (15) and (18), eq. (B.2) can be written as:

$$\theta - \theta v + \frac{1}{\alpha(1-\sigma)} = (v\theta)\frac{\frac{1}{\alpha} + \theta(1-\sigma) + \sigma\theta v}{v\theta\left(1-\frac{1-\gamma}{\gamma}\sigma\right) + \frac{(1-\gamma)}{\gamma}(\sigma\theta+\rho)}, \tag{B.3}$$

[13] For corner n_t the economy would behave as in Lucas (1988).

which provides the following second order equation for *v*:

$$(\theta v)^2 + \frac{(ab+m)}{1-a}\theta v - \frac{bm}{1-a} = 0 \tag{B.4}$$

with $a \equiv \frac{1-2\gamma}{\gamma}\sigma$, $b \equiv \theta + \frac{1}{\alpha}\frac{1}{1-\sigma}$, $m \equiv \frac{1-\gamma}{\gamma}(\rho + \sigma\theta)$.

For a<1, the (strictly) positive root of (B.4) is:

$$v_1 = -\frac{1}{2\theta}\frac{ab+m}{1-a} + \frac{1}{\theta}\sqrt{\frac{1}{4}\frac{(ab+m)^2}{(1-a)^2} + \frac{bm}{1-a}}\,. \tag{B.5}$$

For a>1, eq. (B.4) has two positive roots; however, since the argmax ($\hat{v}$) of the parabola in eq. (B.4) in *v* is bigger than 1, i.e.

$$\hat{v} = -\frac{1}{2\theta}\frac{ab+m}{1-a} = \frac{\frac{1-2\gamma}{\gamma}\sigma\left(\theta + \frac{1}{\alpha}\frac{1}{1-\sigma}\right) + \frac{1-\gamma}{\gamma}(\rho+\sigma\theta)}{2\theta\left(\frac{1-2\gamma}{\gamma}\sigma - 1\right)} \geq 1 \Leftrightarrow \theta(2+\sigma) + \frac{1-\gamma}{\gamma}\rho + \frac{1-2\gamma}{\gamma}\frac{\sigma}{\alpha}\frac{1}{1-\sigma} \geq 0$$

then the only (strictly positive) root of eq. (B.4) that can be lower than 1 is

$$v_2 = \frac{1}{2\theta}\frac{ab+m}{a-1} - \frac{1}{\theta}\sqrt{\frac{1}{4}\frac{(ab+m)^2}{(1-a)^2} + \frac{bm}{1-a}}\,. \tag{B.6}$$

Note that, in order to have interior solution for *v*, it must be that *v*<1. This implies, both for v_1 and v_2 that the following inequality, stemming from (B.4), must be satisfied:

$$(1-a)\theta^2 + (ab+m)\theta - bm \geq 0 \tag{B.7}$$

The latter condition identifies a set of restrictions on the parameters that insures interiority of the solution for *v*, as reported in Proposition 1. The tranversality conditions are also satisfied.[14]

[14] The transversality conditions (A.9) hold under the endogenous growth path. To see this, let *g* denote the balanced growth rate, then (using (A.6) and (3)) we have $\frac{d}{dt}(qK) = (\rho - F_K + n + g)qK$, which when integrated, together with (A.9) gives $e^{-\rho t}q_t K_t = q_0 K_0 e^{-(F_K - n - g)t} = q_0 K_0 e^{-\theta v t}$ (where the last equality follows from (10) and (14)). Next, by (A.4), we have $\lambda N = \alpha u N$, thus $\frac{d}{dt}(\lambda N) = \left(\frac{u'}{u}\frac{\dot{c}}{c} + n\right)\lambda N = [(1-\sigma)g + n]\lambda N$, where the last equality follows from the iso-elastic utility function. Integrating, together with (A.9), gives $e^{-\rho t}\lambda_t N_t = \lambda_0 N_0 e^{-[\rho - n - (1-\sigma)g]t} = \lambda_0 N_0 e^{-\theta v t}$ (where the last equality follows from (10) and (18)). Finally, using (A.11), we have $\frac{d}{dt}(ph) = (\rho - \theta + g)ph$, which when integrated, together with (A.9), gives $e^{-\rho t}p_t h_t = p_0 h_0 e^{-(\theta - g)t} = p_0 h_0 e^{-\theta v t}$ (where the last equality follows from (10)). Thus all three terms in (A.9) go to zero as $t \to \infty$.

Appendix C: Proof of Proposition 2

Let ϕ be the eigenvalues, then the characteristic equation associated with the Jacobian matrix above is:

$$-\phi^3 + \phi^2\Omega_1 + \phi\Omega_2 + \Omega_3 \equiv \Pi(\phi) = 0 \qquad \text{(C.1)}$$

where

$$\Omega_1 = tr(J) = \frac{1}{G\alpha}\left\{\alpha\tilde{c}(G-\alpha\omega) + G\alpha\omega(1-\sigma) + \sigma G^2\right\} > 0 \qquad \text{(C.2)}$$

$$\Omega_2 = \frac{\alpha\omega\tilde{c}}{G}\left[(1-\sigma)\omega + \tilde{c}\right] + 2\tilde{c}\,\omega\left(\frac{1}{2}+\sigma\right) + \frac{G\sigma}{\alpha\gamma}\left[\omega(1-\gamma) - \tilde{c}\right] \qquad \text{(C.3)}$$

$$\Omega_3 = \det(J) = -\frac{\tilde{c}\,\omega}{G\alpha\gamma}\left\{\alpha G\omega\left[(1+\sigma)\gamma - \sigma\right] + \sigma G^2(1-\gamma) + \tilde{c}\,\gamma\alpha\left[G + \sigma(G-\alpha\omega)\right]\right\} \qquad \text{(C.4)}$$

Since we do not obtain closed solutions to eq. (C.1), we characterize the shape of $\Pi(\phi)$.
First of all, $\lim_{\phi\to\pm\infty}\Pi(\phi) = \mp\infty$ and $\Pi(0) = \Omega_3$. Moreover, first derivative:

$$\Pi'(\phi) = -3\phi^2 + 2\Omega_1\phi + \Omega_2 \qquad \text{(C.5)}$$

has a positive argmax in ϕ and $\lim_{\phi\to\pm\infty}\Pi(\phi)' = -\infty$. Hence, $\Pi(\phi)$ is either always decreasing (if (C.5) has no real roots then $\Pi'(\phi)$ is always negative) or it is increasing in the interval, (ϕ_1, ϕ_2), (which are the smaller and the larger roots to (C.5) respectively). In either cases, we recall that stability is ensured by the existence of one negative root to eq. (C.1). The analysis carried out so far implies that sufficient for eq. (C.1) to have only one negative root is that Ω_3 <0. We now show that Ω_3 <0.

By exploiting the definition of ω and rearranging terms, (C.4) becomes:

$$\Omega_3 = -(1+\sigma)\omega\left[F_K\frac{G}{\alpha}\tau + \delta\left(\frac{G}{\alpha} - F_K\right)\tilde{c} + \tilde{c}^2\right] \qquad \text{(C.6)}$$

with $\tau \equiv \left(\frac{1-\gamma}{\gamma}\right)\left(1 - \frac{1}{\gamma}\frac{\sigma}{1+\sigma}\right)$ and $\delta \equiv \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\sigma}{1+\sigma}\right)$. Next, substituting for $\tilde{c}^2$ in (C.6) from the identity $\tilde{c}^2 \equiv \left(\tilde{c} - \frac{1-\gamma}{\gamma}F_K\right)^2 + 2\frac{1-\gamma}{\gamma}F_K\left(\tilde{c} - \frac{1-\gamma}{\gamma}F_K\right) - \left(\frac{1-\gamma}{\gamma}F_K\right)^2$, where $\left(\tilde{c} - \frac{1-\gamma}{\gamma}F_K\right) = F_N > 0$ and rearranging terms, we get:

$$\Omega_3 = -(1+\sigma)\left\{\omega\left(\frac{1-\gamma}{\gamma}\right)F_K\left[\frac{G}{\alpha}\tau + \delta\left(\frac{G}{\alpha} - F_K\right) + \left(\frac{1-\gamma}{\gamma}\right)F_K\right] + \right.$$
$$\left. + \left[\delta\left(\frac{G}{\alpha} - F_K\right) + 2\left(\frac{1-\gamma}{\gamma}\right)F_K\right]\left(\tilde{c} - \frac{1-\gamma}{\gamma}F_K\right) + \left(\tilde{c} - \frac{1-\gamma}{\gamma}F_K\right)^2\right\}.$$

Since $\tau+\delta>0$ and $\left(\frac{1-\gamma}{\gamma}\right)-\delta>0$, the first expression in square brackets is positive. Moreover, since $2\left(\frac{1-\gamma}{\gamma}\right)-\delta$ >0, also the expression in the second square brackets is positive. Hence, we can conclude that Ω_3 <0 □

Appendix D: Proof of Proposition 3

Preliminarily, let us rewrite eq. (B.4) as follows:

$$(1-a)(\theta v)^2+(ab+m)\theta v-bm=0 \tag{D.1}$$

and differentiate it with respect to the parameters, such that:

$$[2(1-a)(\theta v)+ab+m]d(\theta v)+\theta v(b-\theta v)da+(a\theta v-m)db+(\theta v-b)dm=0 \tag{D.2}$$

by recognizing from (D.1) that $ab+m=\frac{bd-(1-a)(\theta v)^2}{\theta v}$ and plugging it into the first term in square brackets of (D.2) we get that the latter term is:

$$\Delta\equiv\left[(1-a)(\theta v)+\frac{bm}{\theta v}\right]=\frac{1}{\theta v}[m(b-\theta v)+b(m-a\theta v)] \tag{D.3}$$

Given that $(b-\theta v)=\theta(1-v)+\frac{1}{\alpha}\frac{1}{1-\sigma}>0$ (D.4)

and $(a\theta v-m)=-\left\{\left(\frac{1-\gamma}{\gamma}\right)[\rho+\sigma\theta(1-v)]+\sigma\theta v\right\}<0$ (D.5)

It also follows that Δ >0.

D.1. The effect of θ.

When θ varies, we get that:

$$da=0,\ db=d\theta,\ dm=\left(\frac{1-\gamma}{\gamma}\right)\sigma d\theta.$$

Preliminarily, note that, from eq. (D.2):

$$\frac{d(\theta v)}{d\theta}=\frac{m-a\theta v+\left(\frac{1-\gamma}{\gamma}\right)\sigma(b-\theta v)}{\Delta} \tag{D.6}$$

which, by eqs. (D.3), (D.4) and (D.5) is positive.

Next, we can write:

$$\theta\frac{dv}{d\theta}=\frac{d(\theta v)}{d\theta}-v=\frac{m-a\theta v+\left(\frac{1-\gamma}{\gamma}\right)\sigma(b-\theta v)}{\Delta}-v,$$

which yields:

$$\theta\frac{dv}{d\theta}=\frac{1}{\Delta}\left\{\left(\frac{1-\gamma}{\gamma}\right)\left[\rho\left(1-2\frac{b}{\theta}\right)+(\theta-b)\sigma\right]+\left[a(\theta-b)+\left(\frac{1-\gamma}{\gamma}\right)\rho\right]v\right\} \tag{D.7}$$

Note that, while the sign of the first term in square brackets is negative, the one of the second term in square brackets can be ambiguous. If it is negative, then $\frac{dv}{d\theta}$ >0; if it is positive, then the sign of (D.7) is ambiguous; in the latter case we can check whether, for v=1, the whole expression in (D.7) can be positive as well. In fact, we get that, for v=1, (D.7) becomes:

$$\theta\frac{dv}{d\theta}=\left(\frac{\theta-b}{\theta}\right)\frac{2\left(\frac{1-\gamma}{\gamma}\right)\rho+\theta\sigma}{\Delta}<0. \tag{D.8}$$

Hence, we can conclude that $\frac{dv}{d\theta}$ <0.

As for the economy's rate of growth $\frac{\dot{h}}{h}=\theta(1-v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\theta}=(1-v)-\theta\frac{dv}{d\theta}>0 \tag{D.9}$$

Finally, as for $n=\theta(\sigma-1)-\sigma\theta v+\rho$, one gets:

$$\frac{dn}{d\theta}=(\sigma-1)-\sigma v-\sigma\frac{d(\theta v)}{d\theta}<0 \tag{D.10}$$

by eq. (D.6).

D.2. The effect of γ.

When γ varies, we get that:

$da = -\frac{\sigma}{\gamma^2} d\gamma$, $db = 0$, $dm = -\frac{\rho + \theta\sigma}{\gamma^2} d\gamma$. Substituting in to (B.2) and collecting terms, we get:

$$\theta \frac{dv}{d\gamma} = -\frac{1}{\Delta} \frac{(b - \theta v)}{\gamma^2} [\rho + \sigma\theta(1 - v)] < 0 \tag{D.11}$$

As for the economy's rate of growth $\frac{\dot{h}}{h} = \theta(1 - v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\gamma} = -\theta \frac{dv}{d\gamma} > 0 \tag{D.12}$$

Finally, as for $n = \theta(\sigma - 1) - \sigma\theta v + \rho$, one gets:

$$\frac{dn}{d\gamma} = -\theta\sigma \frac{dv}{d\gamma} > 0. \tag{D.13}$$

D.3. The effect of α.

When α varies, we get that:

$da = 0$, $db = -\frac{1}{\alpha^2}\left(\frac{1}{1 - \sigma}\right) d\alpha$, $dm = 0$. Substituting in to (B.2) and collecting terms, we get:

$$[2(1 - a)(\theta v) + ab + m] d(\theta v) + (a\theta v - m) db = 0 \tag{D.14}$$

such that,

$$\theta \frac{dv}{d\alpha} = \frac{1}{\Delta}(a\theta v - m)\frac{1}{\alpha^2}\left(\frac{1}{1 - \sigma}\right) < 0 \tag{D.15}$$

As for the economy's rate of growth $\frac{\dot{h}}{h} = \theta(1 - v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\alpha} = -\theta \frac{dv}{d\alpha} > 0 \tag{D.16}$$

Finally, as for $n = \theta(\sigma - 1) - \sigma\theta v + \rho$, one gets:

$$\frac{dn}{d\alpha} = -\sigma\theta \frac{dv}{d\alpha} > 0 . \qquad \text{(D.17)}$$

D.4. The effect of ρ.

As for the effect of ρ**,** when the latter parameter changes one obtains:

$$da = 0 ,\ db = 0 ,\ dm = \left(\frac{1-\gamma}{\gamma}\right) d\rho .$$

Substituting into (D.2) it descends:

$$\theta \frac{dv}{d\rho} = -\frac{1}{\Delta}(\theta v - b)\left(\frac{1-\gamma}{\gamma}\right) > 0 \qquad \text{(D.18)}$$

As for the economy's rate of growth $\frac{\dot{h}}{h} = \theta(1-v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\rho} = -\theta \frac{dv}{d\rho} < 0 \qquad \text{(D.19)}$$

Finally, as for $n = \theta(\sigma - 1) - \sigma\theta v + \rho$, one gets:

$$\frac{dn}{d\rho} = -\theta\sigma \frac{dv}{d\rho} + 1 . \qquad \text{(D.20)}$$

which, in principle, could take any sign. However by expanding it we get:

$$\frac{dn}{d\rho} = \frac{1}{\Delta}\left[2bm - (ab + m)\theta v - \sigma(b - \theta v)\left(\frac{1-\gamma}{\gamma}\right)\theta v\right] . \qquad \text{(D.21)}$$

The source of ambiguity of the sign of (D.21) are the second and the third term in the square brackets. However, by checking whether, for *v*=1, (D.21) is positive or negative, we get:

$$\frac{dn}{d\rho} = \frac{1}{\Delta}\left\{\left[b\left(\frac{1-\gamma}{\gamma}\right)\rho + \sigma\theta\right] + \left(\frac{1-\gamma}{\gamma}\right)\rho(b - \theta)\right\} > 0 . \qquad \text{(D.22)}$$

Hence, we can conclude that $\frac{dn}{d\rho}$>0 for any level of *v*.

D.5. The effect of σ .

When σ varies, we get that:

$da = \frac{1-2\gamma}{\gamma} d\sigma$, $db = \frac{1}{\alpha}\left(\frac{1}{1-\sigma}\right)^2 d\sigma$, $dm = \theta\left(\frac{1-\gamma}{\gamma}\right) d\sigma$. Substituting in to (D.2) and collecting terms, we get:

$$\theta \frac{dv}{d\sigma} = \frac{1}{\Delta}\left\{(b-\theta v)\frac{\theta}{\gamma}\left[(1-\gamma)(1-v)+\gamma v\right]+(m-a\theta v)\frac{1}{\alpha}\left(\frac{1}{1-\sigma}\right)^2\right\} > 0 \tag{D.23}$$

As for the economy's rate of growth $\frac{\dot{h}}{h} = \theta(1-v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\sigma} = -\theta \frac{dv}{d\sigma} < 0 \tag{D.24}$$

Finally, as for $n = \theta(\sigma-1) - \sigma\theta v + \rho$, one gets:

$$\frac{dn}{d\sigma} = \theta(1-v) - \theta\sigma \frac{dv}{d\sigma} \tag{D.25}$$

which is ambiguous. In fact, we can evaluate such a derivative at the extremes of the existence interval. Such an interval is defined by $\sigma = 0$ and σ^* (where v=1):

$$\left.\frac{dn}{d\sigma}\right|_{\sigma=0} = \theta(1-v|_{\sigma=0}) > 0$$

where $v|_{\sigma=0} = -\frac{m}{2\theta} + \frac{1}{\theta}\sqrt{\frac{m^2}{4} + bm}$

and $\left.\frac{dn}{d\sigma}\right|_{\sigma=\sigma^*} = -\sigma\theta \left.\frac{dv}{d\sigma}\right|_{v=1} < 0$

Hence, we can conclude that the derivative of n with respect to σ can change sign.

Appendix E: The human capital sector

Consider a research sector and an education sector (dissemination of knowledge). The population not working in manufacturing, $(1-v_t)N_t$, is divided between research and schooling, denoted N_t^h and N_t^s, respectively. Each researcher discovers new knowledge, δ_t, proportional to existing human capital, h_t, implying total research discovery is:

$$\delta_t N_t^h = \mu h_t N_t^h \tag{E.1}$$

where μ is a research-productivity parameter. Let Δ_t be the addition to knowledge per person through education. The entire population is educated according to a function of research discovery and educators (whose productivity is proportional to existing human capital):

$$\Delta_t N_t = \omega\left(\mu h_t N_t^h\right)^{\varepsilon}\left(h_t N_t^s\right)^{1-\varepsilon} \quad \text{(E.2)}$$

where ω is a schooling-productivity parameter and $\varepsilon \in (0,1)$ the elasticity of the additional knowledge with respect to research activity. Notice that the entire population is educated, including researchers and educators (to account for the fact that research and teaching is specialised, and researchers and teachers need to learn from other fields).

Efficiency in education requires allocation between researchers and educators to maximise (E.2), subject to $(1-v_t)N_t = N_t^h + N_t^s$. The first-order conditions (w.r.t N_t^h and N_t^s) give

$$N_t^h = \varepsilon(1-v_t)N_t \quad \text{(E.3)}$$

$$N_t^s = (1-\varepsilon)(1-v_t)N_t \quad \text{(E.4)}$$

Substituting (E.3) and (E.4) into (E.2) gives

$$\Delta_t N_t = \theta(1-v_t)h_t N_t \quad \text{(E.5)}$$

where $\theta \equiv \omega(\mu\varepsilon)^{\varepsilon}(1-\varepsilon)^{1-\varepsilon}$, which is a measure of research efficiency and schooling efficiency, which in reality may differ across countries and also time.

Dividing (E.5) by N_t we have the per-person addition to knowledge, which in continuous time is $\dot{h}_t$, (equation (6)).[15]

[15] The economy can also be decentralised as follows. Let us suppose for simplicity that the government produces education, hiring $N_t^g = N_t^h + N_t^s$. Efficiency for the government implies (by using eq. (E.2)): *max* $\omega\left(\mu h_t N_t^h\right)^{\varepsilon}\left(h_t N_t^s\right)^{1-\varepsilon} - w^g h_t\left(N_t^h + N_t^s\right)$, which gives $N_t^h = \varepsilon N_t^g$, $N_t^s = (1-\varepsilon)N_t^g$ and $\Delta_t N_t = \theta h_t N_t^g$. Let us assume that the cost to the government, given by $w_t^g N_t^g$, is financed by a lump-sum tax, T_t. Manufacturing firms hire capital and labour services by solving: *max* $F(K_t, h_t N_t^y) - w^y h_t N_t^y - r_t K_t$. Under this scenario the household sector faces the following constraint: $\dot{K}_t = r_t K_t + w_t^y h_t v_t N_t + w_t^g h_t (1-v_t) N_t - T_t - c_t N_t$. Market clearing conditions give: $w_t^y = w_t^g = w_t$, $v_t N_t = N_t^y$, $(1-v_t)N_t = N_t^g$, $T_t = w_t h_t N_t^g$, $F_{K_t} = r_t$, $F_{L_t} = w_t$. Finally, substituting the latter conditions into the household sector budget constraint and exploiting CRS of the manufacturing production function, we get eq. (5).