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NEW CONSTRUCTIONS AND BOUNDS FOR WINKLER'S HAT GAME*

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Abstract. Hat problems have recently become a popular topic in combinatorics and discrete mathematics. These have been shown to be strongly related to coding theory, network coding, and auctions. We consider the following version of the hat game, introduced by Winkler and studied by Butler et al. A team is composed of several players; each player is assigned a hat of a given color; they do not see their own color but can see some other hats, according to a directed graph. The team wins if they have a strategy such that, for any possible assignment of colors to their hats, at least one player guesses their own hat color correctly. In this paper, we discover some new classes of graphs which allow a winning strategy, thus answering some of the open questions of Butler et al. We also derive upper bounds on the maximal number of possible hat colors that allow for a winning strategy for a given graph.

Key words. hat game, deterministic strategies, directed graphs

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1. Introduction.

Hat games are a popular topic in combinatorics. Typically, a hat game involves n players, each wearing a hat that can take a color from a given set of q colors. No player can see their own hat, but each player can see some subset of the other hats. All players are asked to guess the color of their own hat at the same time. For an extensive review of different hat games, see [1]. Different variations have been proposed: for instance, the players can be allowed to pass [2], or the players can guess their respective hat's color sequentially [3]. The variation in [2] mentioned above has been investigated further (see [1]) for it is strongly connected to coding theory via the concept of covering codes [4]; in particular, some optimal solutions for that variation involve the well-known Hamming codes [5]. In the variation called the "guessing game," players are not allowed to pass and must guess simultaneously [6]. The team wins if everyone has guessed their color correctly; the aim is to maximize the number of hat assignments which are correctly guessed by all players. This version of the hat game has been further studied in [7, 8] due to its relations to graph entropy, to circuit complexity, and to network coding, which is a means to transmit data through a network which allows the intermediate nodes to combine the packets they receive [9].

In this paper, we are interested in the following hat problem, a small variation to Winkler's hat game presented in [10]. We are given a directed graph D (without loops and repeated arcs, but possibly with bidirectional edges) on n vertices and a finite alphabet $[q] = \{0, \dots, q-1\}$ ($q \geq 2$). We say that $f = (f_1, \dots, f_n) : [q]^n \to [q]^n$ is a D-function if every local function $f_v : [q]^n \to [q]$ only depends on the values in the

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in-neighborhood of v in D: $f_v(x) = f_v(x_{N^-(x)})$. We ask whether there is a D-function over $[q]$ such that for any $x = (x_1, \dots, x_n) \in [q]^n$, $f_v(x) = x_v$ for some vertex v. In that case, we say that D is *q-solvable* and that f *solves* D.

In terms of the hat game, each vertex in the graph represents a player, and an arc from player u to v means that v can see u. The set $[q]$ then represents the possible colors of their hats and $x = (x_1, \dots, x_n) \in [q]^n$ represents a possible hat assignment. Each player v must guess the color of their hat according to some predetermined rule which can only depend on the hats that they see: $f_v(x_{N^-(x)})$. If one player guesses correctly, i.e., $x_v = f_v(x)$, then the team wins; if all guess incorrectly, the team loses. The question is then to come up with a winning strategy regardless of the hat assignment.

Clearly, if D is q-solvable, then it is also $(q-1)$-solvable. The clique K_q is q-solvable [10]: if we denote the players as elements in $[q]$, then v guesses that the sum of all hat assignments is equal to v modulo q: $f_v(x) = -\sum_{u \neq v} x_u + v$. More generally, if the players play on K_n, then there is a strategy which guarantees that at least $\lfloor n/q \rfloor$ players guess correctly (simply split K_n into $\lfloor n/q \rfloor$ cliques K_q). The case for K_n and $q = 2$ colors with unequal probabilities was further studied in [11, 12]; its relation to auctions has been revealed in [13] and developed in [14].

Results for other classes of graphs have been found in the literature. Butler et al. proved in [15] that for any q, there exists a q-solvable undirected bipartite graph. Unfortunately, that graph has a doubly exponential number of vertices. In the same paper, they also proved that undirected trees are not 3-solvable.

The main contributions of this paper are as follows. In [15], it is asked whether there exist K_q-free q-solvable undirected graphs with a polynomial number (in q) of vertices. We give an emphatic affirmative answer: for any ϵ, there exist $K_{\epsilon q}$-free q-solvable graphs with $n_\epsilon q$ vertices, where n_ϵ only depends on ϵ; moreover, we present a class of K_ω-free graphs with $\omega = o(q)$ which are q-solvable and have a polynomial number of vertices. We also refine the multiplicative constant for some values of ϵ by considering small undirected graphs or directed graphs. We also prove that a bipartite graph with parts of sizes m and n is not q-solvable if $q \geq m + 2$; more generally, we prove that the maximum q for which a directed graph D is q-solvable depends on the size of a maximum independent set in D. Another question asked in [15] concerns so-called edge-critical graphs, i.e., undirected graphs which are q-solvable but which have no q-solvable proper spanning subgraph. Clearly, the only edge-critical graph for $q = 2$ colors is K_2; [15] asks whether there exists an infinite family of edge-critical graphs for any other $q \geq 3$. By studying the solvability of cycles, we are able to show that the cycles whose length are a multiple of six form an infinite family of edge-critical graphs for 3 colors.

The rest of the paper is organized as follows. In section 2, we prove the existence of bipartite or K_ω-free q-solvable undirected graphs with a relatively small number of vertices. In section 3, we refine some constructions by extending our consideration to directed graphs. We then derive some nonsolvability results in section 4. Finally, we prove the existence of a class of edge-critical 3-solvable graphs in section 5.

2. Undirected constructions.

In [15], it is proved that for any $q \geq 2$ there exists a q-solvable bipartite graph with a doubly exponential number of vertices ($q^{q^{q-1}} + q - 1$ vertices to be exact). We refine their argument to construct a q-solvable bipartite graph with only an exponential number of vertices.

We say that a set of words S in $[q]^m$ is *distinguishable* if there exists a word $x \in [q]^m$ such that $d_H(x,s) \le m-1$ for all $s \in S$, where d_H is the Hamming distance. Alternatively, using the terminology of [16], this is equivalent to S having remoteness at most $m-1$. The main reason we are interested in distinguishable sets is as follows. If in a graph there is an independent set M of cardinality m, and the vertices in M know that their hat assignment $x \in [q]^m$ is any possible element of a set $S \subseteq [q]^m$, then there exist guessing functions for the vertices of M achieving at least one correct guess if and only if S is distinguishable.

THEOREM 2.1. *The complete bipartite graph $K_{q-1,(q-1)^{q-1}}$ is q-solvable.*

Proof. Set $m = q-1$, and label the left vertices of $K_{q-1,(q-1)^{q-1}}$ by $v_1,\dots,v_m$. Write $[q]_+$ for the set $\{1,\dots,q-1\}$ (so $[q]_+ \subseteq [q]$) and label the right vertices of $K_{q-1,(q-1)^{q-1}}$ by w_z for $z \in [q]_+^m$. For each $z \in [q]_+^m$ define the guessing function $f_z : [q]^m \to [q]$ by

$$f_z(x) = \begin{cases} 0 & \text{if } d_H(x,z) = m, \\ \min\{i : x_i = z_i\} & \text{if } d_H(x,z) < m. \end{cases}$$

It is enough to show that for any hat configuration $(x,y) = (x_1,\dots,x_m, y_{(1,\dots,1)},\dots,y_{(q-1,\dots,q-1)})$ if all the vertices w_z guess incorrectly, then the vertices v_i know that the vector x lies in some distinguishable set.

That is, it is enough to show that for all y there exists $a \in [q]^m$ such that

$$\bigcap_{z\in[q]_+^m} f_z^{-1}(y_z)^{\mathrm{c}} \subseteq B_{m-1}(a),$$

where $B_{m-1}(a)$ is the ball around a of radius $m-1$ in the Hamming metric. (The m components of the vector a, which depends on y, are exactly the guessing functions for the vertices $v_1,\dots,v_m$.)

We prove by (reverse) induction on i the following.

Claim 1. Suppose $(x,y) \in [q]^m \times [q]^{[q]_+^m}$ is a configuration of hats guessed incorrectly by every vertex. Then, for every $i = 1,\dots,m$ and every $(z_1,\dots,z_{i-1}) \in [q]_+^{i-1}$ there exists $(z_i,\dots,z_m) \in [q]_+^{m-i+1}$ with $y_{(z_1,\dots,z_m)} \notin \{i,\dots,m\}$.

Proof of claim. Let $i = m$, and fix $z_1,\dots,z_{m-1}$. Consider the variables $y_{(z_1,\dots,z_{m-1},z)}$ for $z \in [q]_+$; if all are equal to m, then

$$X_m(z) := f^{-1}_{(z_1,\dots,z_{m-1},z)}(y_{(z_1,\dots,z_{m-1},z)}) = \{x \in [q]^m : x_i{\ne}z_i \text{ for all } i{<}m \text{ and } x_m{=}z\}.$$

Hence

$$\bigcup_{z\in[q]_+} X_m(z) = \{x \in [q]^m : x_i \ne z_i \text{ for all } i < m \text{ and } x_m \ne 0\},$$

implying that $\bigcap_{z\in[q]_+} X_m(z)^{\mathrm{c}} = B_{m-1}(z_1,\dots,z_{m-1},0)$, contradicting the fact that the vertices $v_1,\dots,v_m$ guess incorrectly. Therefore there exists some $z \in [q]_+$ with $y_{(z_1,\dots,z_{m-1},z)} \ne m$.

Now, suppose the statement is true for $i > 1$; we show it holds for $i-1$. Fix $z_1,\dots,z_{i-2}$; for each $a \in [q]_+$, by our inductive hypothesis there exist $z_i(a),\dots,z_m(a) \in [q]_+$ with

$$y_{(z_1,\dots,z_{i-2},a,z_i(a),\dots,z_m(a))} \notin \{i,\dots,m\}.$$

So, it is enough to show that for at least one $a \in [q]_+$ the variable $y_{(z_1,\dots,z_{i-2},a,z_i(a),\dots,z_m(a))}$ is not equal to $i-1$. For a contradiction, suppose not, so that all such variables equal $i-1$. Then,

$$\begin{aligned} X_{i-1}(a) &:= f^{-1}_{(z_1,\dots,z_{i-2},a,z_i(a),\dots,z_m(a))}(y_{(z_1,\dots,z_{i-2},a,z_i(a),\dots,z_m(a))}) \\ &= \{x \in [q]^m : x_j \neq z_j \text{ for all } j < i-1 \text{ and } x_{i-1} = a\}. \end{aligned}$$

Therefore,

$$\bigcup_{a\in[q]_+} X_{i-1}(a) = \{x \in [q]^m : x_j \neq z_j \text{ for all } j < i-1 \text{ and } x_{i-1} \neq 0\},$$

implying that $\bigcap_{a\in[q]_+} X_{i-1}(a)^{\mathrm{c}} \subseteq B_{m-1}(z_1,\dots,z_{i-2},0,\dots,0)$, contradicting the fact that $v_1,\dots,v_m$ guess incorrectly. ∎

Finally, applying the claim for $i=1$, we find a $z \in [q]_+^m$ where y_z cannot take any value in $\{1,\dots,m\}$. This implies that $y_z = 0$ and $f_z^{-1}(y_z)^{\mathrm{c}} = B_{m-1}(z)$, so that at least one of $v_1,\dots,v_m$ guesses correctly. ∎

For directed graphs, we identify the clique K_r with the so-called complete digraph $\overleftrightarrow{K_r}$ with arcs (i,j) for all $i \neq j$ [17]. The *clique number* of a directed graph D is the size of a largest induced clique in D; as such, a tournament has a clique number of one. The *lexicographic product* of a directed graph $D=(V,E)$ and a clique K_r, denoted as (D,r), is defined as the graph with vertex set $V \times [r]$, where $((u,a),(v,b))$ is an arc if and only if either $(u,v) \in E$ or $u=v$ and $a \neq b$. If D has n vertices and clique number ω, then the graph (D,r) has rn vertices and clique number $r\omega$.

LEMMA 2.2 (the blow-up lemma). *If G is a p-solvable directed graph, then (G,r) is a q-solvable graph, where $q=pr$.*

Proof. Let f be the corresponding guessing function that solves G over p colors. For any vertex (v,a) in (G,r), we denote the configuration as $(x_{(v,a)}, y_{(v,a)}) \in [p]\times[r]$ and we also denote $X_v = \sum_{a\in[r]} x_{(v,a)} \mod p$, $Y_v = \sum_{a\in[r]} y_{(v,a)} \mod r$ and write X for the vector $(X_v, v \in G)$. We claim that the (G,r)-function g, defined as follows for each (v,a), never fails:

$$g_{(v,a)}(x,y) = \left(f_v(X) - X_v + x_{(v,a)} \mod p, -Y_v + y_{(v,a)} - a \mod r\right).$$

Suppose (x,y) is guessed wrong by all vertices. In particular, it is guessed incorrectly by (v,a), hence either $f_v(X) \neq X_v$ or $Y_v \neq a$. Since this holds for all a, in particular this holds for $a = Y_v$; we conclude that $f_v(X) \neq X_v$. Since this holds for all v, this violates the fact that f is a solution for G. ∎

THEOREM 2.3. *For any $\epsilon > 0$, there exists n_ϵ such that the following holds. For any q, there exists a q-solvable undirected graph with at most $n_\epsilon q$ vertices and clique number ϵq.*

Proof. First, let $p = \lfloor 1/\epsilon \rfloor + 1$ and let q be divisible by p. Let G_p be the p-solvable bipartite graph in Theorem 2.1 and let g_p denote its size. Then by the blow-up lemma, $(G_p, q/p)$ is a q-solvable graph with $g_p q/p$ vertices and clique number $2q/p$. If q is not divisible by p, consider $q' = p\lceil q/p \rceil \leq q(1+1/p)$ and $n_\epsilon = (1+1/p)g_p/p$. ∎

THEOREM 2.4. *For any ω such that $\omega \geq \frac{q}{m}\frac{\log\log q}{\log q}$ holds for large enough q and some $m > 0$, there exists a q-solvable K_ω-free undirected graph with at most q^{2m+1} vertices for q large enough.*

Proof. Let $p = \lfloor \frac{2q}{\omega} \rfloor + 1$. According to Theorem 2.1, the graph $K_{p-1,(p-1)^{p-1}}$ is p-solvable. Then by the blow-up lemma, there exists a q-solvable graph with $n := \frac{q}{p}\left((p-1)^{p-1} + p - 1\right)$ vertices and clique number $2\frac{q}{p} < \omega$. We have $n \leq q(p-1)^{p-1}$, and hence for q large enough

$$p-1 \leq \frac{2q}{\omega} \leq 2m\frac{\log q}{\log\log q} \quad \text{and}$$
$$\log n \leq \log q + 2m\frac{\log q}{\log\log q}\{\log(2m)+\log\log q - \log\log\log q\} \leq (2m+1)\log q,$$

and hence $n \leq q^{2m+1}$. ∎

In general, the constant n_ϵ obtained from Theorem 2.1 decreases rapidly with ϵ. We refine it below for $\epsilon = 2/3$.

PROPOSITION 2.5. *The complete bipartite graph $K_{2,2}$ is 3-solvable.*

Proof. Denote the bipartition as $\{v_1, v_2\} \cup \{v_3, v_4\}$. With

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

the guessing function is given by

$$(f_1, f_2) = (x_3, x_4)A, \qquad (f_3, f_4) = (x_1, x_2)A^{-1}.$$

Suppose x is guessed wrong by all vertices. The vertices v_3 and v_4 guess wrong, hence we have

$$(x_3, x_4) = (x_1, x_2)A^{-1} + w$$

for some $w = (w_1, w_2) \in S := \{(1,1), (1,2), (2,1), (2,2)\}$. Similarly, we have

$$(x_1, x_2) = (x_3, x_4)A + u = (x_1, x_2) + wA + u$$

for some $u \in S$. However, it can be shown that for any $(w_1, w_2) \in S$, $wA \notin S$ and hence $wA + u \neq (0,0)$. We thus obtain the contradiction $(x_1, x_2) \neq (x_1, x_2)$. ∎

COROLLARY 2.6. *For any q divisible by 3, there exists a q-solvable graph on $4q/3$ vertices with clique number $2q/3$.*

3. Directed constructions. If we allow directed graphs, then we can further refine the constants obtained in section 2.

THEOREM 3.1. *If q is divisible by 2, there exists a q-solvable directed graph with $3q/2$ vertices and clique number $q/2$. If q is divisible by 3, there exists a q-solvable directed graph on $4q$ vertices of clique number $q/3$. If q is divisible by 4, there exists a q-solvable directed graph on $10q$ vertices and with clique number $q/4$.*

The main strategy to produce a p-solvable oriented graph is by using a gadget, defined below. We can then apply the blow-up lemma to the oriented graphs we build in Lemma 3.3 using gadgets.

DEFINITION 3.2. *An oriented graph D on n vertices is called a q-*gadget *if it is not q-solvable but if there exists a D-function f over $[q]$ such that any configuration x guessed incorrectly by f satisfies an equality of the form $x_1 = \phi(x_2, \ldots, x_n)$ for some $\phi : [q]^{n-1} \to [q]$.*

LEMMA 3.3 (the gadget lemma). *If there exists a p-gadget on n vertices, then there exists a p-solvable oriented graph on $n\binom{p}{2} + p$ vertices.*

Proof. Start with a transitive tournament on p vertices with arcs (i,j) for all $i < j$. For every non-arc (i,j) with $i > j$, add a gadget $D_{i,j}$ and arcs from i to all vertices in $D_{i,j}$ and whence to j. This yields an oriented graph G on $n\binom{p}{2} + p$ vertices; we claim that G is p-solvable.

We denote the vertices of the original tournament as $0, 1, \dots, p-1$ and for each $i > j$, the vertices of the gadget $D_{i,j}$ are $1_{i,j}, \dots, n_{i,j}$.

Let f be the function on the gadget D with corresponding ϕ. The corresponding function g for G is as follows:

$$\begin{aligned} g_j(x) &= -\sum_{k<j} x_k - \sum_{k>j} \left[\phi(x_{2_{k,j}}, \dots, x_{n_{k,j}}) - x_{1_{k,j}}\right] + j, \\ g_{1_{i,j}}(x) &= f_1(x_{2_{i,j}}, \dots, x_{n_{i,j}}) - x_i, \\ g_{v_{i,j}}(x) &= f_v(x_{1_{i,j}} + x_i, x_{2_{i,j}}, \dots, x_{n_{i,j}}), \qquad v = 2, \dots, n. \end{aligned}$$

Suppose that x is guessed incorrectly by all vertices. First, all vertices in $D_{i,j}$ guess wrong; we then have

$$\begin{aligned} f_1(x_{2_{i,j}}, \dots, x_{n_{i,j}}) &\neq x_{1_{i,j}} + x_i, \\ f_v(x_{1_{i,j}} + x_i, x_{2_{i,j}}, \dots, x_{n_{i,j}}) &\neq x_{v_{i,j}}, \qquad v = 2, \dots, n, \end{aligned}$$

hence

$$x_i = \phi(x_{2_{i,j}}, \dots, x_{n_{i,j}}) - x_{1_{i,j}}$$

for all $i > j$.

Now, j guesses wrong; therefore

$$\sum_{k<j} x_k + \sum_{k>j} \left[\phi(x_{2_{k,j}}, \dots, x_{n_{k,j}}) - x_{1_{k,j}}\right] + x_j \neq j,$$

which combined with the above yields

$$\sum_{k \in [p]} x_k \neq j.$$

Since this holds for all $j \in [p]$, this leads to a contradiction. ∎

PROPOSITION 3.4. *The following graphs are gadgets:*

1. *The graph with a single vertex and no arc is a* 2*-gadget.*
2. *The directed cycle on three vertices is a* 3*-gadget.*
3. *The graph D on six vertices in Figure* 1 *is a* 4*-gadget.*

Proof. The first graph is trivial. For the directed cycle on vertices $1, 2, 3$ and arcs $(1, 2), (2, 3), (3, 1)$, the function f is

$$\begin{aligned} f_1(x) &= x_3, \\ f_2(x) &= x_1, \\ f_3(x) &= x_2. \end{aligned}$$

Therefore, x is not guessed correctly by any vertex if and only if x_1, x_2, and x_3 are all distinct. Thus we have $\{x_1, x_2, x_3\} = [3]$ and hence $x_1 + x_2 + x_3 = 0$.

For D, first note that the transpose of its adjacency matrix (i.e., the matrix $A_D^\top$, where $A_{i,j}^\top = 1$ if and only if (j, i) is an arc in D) is given by

$$A_D^\top = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

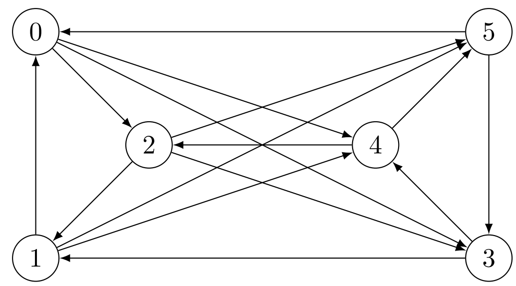

FIG. 1. *The* 4*-gadget* D *in Proposition* 3.4.

For ease of presentation we shall write the hat configuration x as a column vector; we let $f(x) = Mx$, where

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

Then x is guessed wrong by all vertices if and only if Lx is nowhere zero, where

$$L = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}.$$

Denoting the rows of L as $L_0, \dots, L_5$, we see that $L_3 = L_0 - L_1$, $L_4 = L_1 - L_2$, $L_5 = L_2 - L_0$. Therefore, x is not guessed right if and only if L_0x, L_1x, and L_2x are all distinct and nonzero. Therefore, $\{L_0x, L_1x, L_2x\} = \{1, 2, 3\}$ and x must satisfy

$$2x_0 + 2x_1 + 2x_2 + x_3 + x_4 + x_5 = 2.$$

Renaming the vertices such that the fifth vertex becomes first, we obtain the desired equality. ∎

However, it is still unknown whether there exist gadgets for more than four colors.

4. Nonsolvability results. In section 2 we showed that a complete bipartite graph with one part of size $q-1$ was q-solvable. In contrast, in this section we show that any bipartite graph that has a partition with one part of size at most $q-2$ is not q-solvable. To do this we consider the following nondistinguishable set in $[q]^m$ (in other words, a subset of $[q]^m$ with remoteness m). Set $m = q-2$, and denote the words $w_a = (a, \dots, a) \in [q]^m$ for all $a \in [q]\backslash\{0\}$; then $W = \{w_a : a \in [q]\backslash\{0\}\}$ is nondistinguishable. Indeed, for any $x \in [q]^m$, let $X = \{b \in [q] : x_i = b \,\text{for some}\, i\}$ denote the set of values taken by the coordinates of x; then $|X| \le m < |W|$ and hence there exists $a \in ([q]\backslash\{0\})\backslash X$ and thus $d_H(x, w_a) = m$.

In fact, our proof applies to a larger class of graphs than bipartite graphs, defined as follows.

DEFINITION 4.1. *We say a directed graph D is (m,s)-*semibipartite *if its vertex set can be partitioned into $V = L \cup R$, where $|L| = m$, $|R| = s$, and $D[L]$ is an independent set and $D[R]$ is acyclic.*

THEOREM 4.2. *Any (m,s)-semibipartite graph is not $(m+2)$-solvable.*

Proof. Let $q = m+2$ and denote the vertices of R as $r_1, \dots, r_s$. Let $y \in [q]^s$ such that

$$\begin{aligned} &y_1 \notin \{f_{r_1}(w_a) : a \in [q]\}, \\ &y_2 \notin \{f_{r_2}(w_a, y_1) : a \in [q]\}, \\ &\quad \vdots \\ &y_s \notin \{f_{r_s}(w_a, y_1, \dots, y_{s-1}) : a \in [q]\}; \end{aligned}$$

such y exists for each set on the right-hand side has cardinality at most $|W| = q-1$. Furthermore, let $b \in [q]\backslash\{f_{l_1}(y), \dots, f_{l_m}(y)\}$ (where $l_1, \dots, l_m$ are the vertices of L); then all vertices guess (w_b, y) incorrectly. ◻

This theorem is best possible, for Theorem 2.1 indicates that there are q-solvable bipartite graphs with left part of size $q-1$.

COROLLARY 4.3. *The complete bipartite graph $K_{m,n}$ is not $(m+2)$-solvable.*

A *feedback vertex set* of a directed graph D is a set of vertices of D whose complement induces an acyclic subgraph.

COROLLARY 4.4. *Any graph with a minimum feedback vertex set of cardinality one is q-solvable if and only if $q = 2$.*

Proof. By Theorem 4.2, such a graph is not 3-solvable. Conversely, it is not acyclic, hence it contains a directed cycle as a subgraph: let us prove that the directed cycle C_n on n vertices is 2-solvable. Let the function be $f_1(x) = x_n$ and $f_i(x) = x_{i-1} + 1$ for $2 \le i \le n$; then x is guessed incorrectly by all vertices if and only if $x_1 = x_2 = \cdots = x_n = x_1 + 1$, which is clearly impossible. ◻

THEOREM 4.5. *Let D be a directed graph on n vertices with an acyclic induced subgraph of size I. If*

$$(n-I)\left(\frac{q}{q-1}\right)^I < q,$$

then D is not q-solvable.

Proof. We denote the set of vertices inducing an acyclic subgraph of cardinality I as A; we also denote a guessing function as f. Let $x \in [q]^I$ be the hat assignment on A and $y \in [q]^{n-I}$ be the assignment on the rest of the vertices. For each choice of y, denote by $S_d(y)$ the set of choices for x such that exactly d vertices in A guess correctly for all $0 \le d \le I$. It is easy to prove by induction on I that $N_d := |S_d(y)| = \binom{I}{d}(q-1)^{I-d}$. We shall consider the situation when $x \in S_0(y)$, i.e., when no vertex in A guesses correctly; given y, there are $N_0 = (q-1)^I$ such assignments.

For any y, let G denote the number of times the vertices in A guess their colors correctly when $x \notin S_0(y)$:

$$G := \sum_{x \in [q]^I} \sum_{i=1}^{I} \mathbf{1}\{f_{a_i}(x,y) = x_i\} = \sum_{d=1}^{I} dN_d = Iq^{I-1}.$$

The total number of correct guesses, over all assignments (x, y), is of course equal to nq^{n-1}. Therefore, there are at most

$$H := nq^{n-1} - q^{n-I}G = (n - I)q^{n-1}$$

correct guesses over the whole graph for any (x, y) where $x \in S_0(y)$. On average, such an assignment is guessed correctly

$$\frac{H}{q^{n-I}N_0} = \frac{(n-I)q^{I-1}}{(q-1)^I} < 1$$

times, and hence one hat assignment is never guessed correctly. ∎

COROLLARY 4.6. *A graph with an acyclic induced subgraph of size I is q-solvable only if it has at least $I + q(1 - \frac{1}{q})^I$ vertices in total.*

COROLLARY 4.7. *If a graph on n vertices has an acyclic induced subgraph of cardinality at least $n/2$, then it is q-solvable only if $n \geq 2\alpha(q-1)$, where $\alpha \sim 0.5675$ satisfies $\alpha + \log \alpha = 0$.*

Proof. Suppose $n < 2\alpha(q-1)$, and let $i = n/(2q) < \alpha(q-1)/q$; then $\log i + i\frac{q}{q-1} < \log \frac{q-1}{q}$ and hence

$$\begin{aligned} 0 &> \log i + i\frac{q}{q-1} \\ &> \log i + iq \log\left(1 + \frac{1}{q-1}\right), \\ 1 &> i\left(1 + \frac{1}{q-1}\right)^{iq}, \\ q &> \frac{n}{2}\left(\frac{q}{q-1}\right)^{\frac{n}{2}}, \end{aligned}$$

which, by Theorem 4.5, shows that the graph is not q-solvable. ∎

5. Even cycles. In this section we show that a cycle whose length is a multiple of 6 is 3-solvable. In fact, we can define guessing functions for any even cycle which have the property that at most 3 hat configurations are not guessed correctly by any vertex.

For $n > 1$, let C_{2n} be the cycle of length $2n$ and let $V = \{v_1, v_2, \ldots, v_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a partition of the vertices of C_{2n} into independent sets, with v_i adjacent to w_i and w_{i-1} for all $i = 1, \ldots, n$ (index arithmetic taken modulo n). Denote the hat color of v_i by x_i and its guessing function by f_i. Similarly, for w_i, denote its hat color by y_i and its guessing function by g_i. We define the guessing functions to be

$$f_i(y_{i-1}, y_i) = \begin{cases} y_i - 1 & \text{if } y_i \neq y_{i-1} + 1 \\ y_i + 1 & \text{if } y_i = y_{i-1} + 1 \end{cases} \quad \text{for } i \neq 1; \tag{5.1}$$

$$f_1(y_n, y_1) = \begin{cases} y_1 - 1 & \text{if } y_1 \neq y_n - 1, \\ y_1 + 1 & \text{if } y_1 = y_n - 1, \end{cases} \tag{5.2}$$

$$g_i(x_i, x_{i+1}) = \begin{cases} x_i & \text{if } x_i \neq x_{i+1} + 1 \\ x_i - 1 & \text{if } x_i = x_{i+1} + 1 \end{cases} \quad \text{for } i \neq n; \tag{5.3}$$

$$g_n(x_n, x_1) = \begin{cases} x_n & \text{if } x_n \neq x_1, \\ x_n - 1 & \text{if } x_n = x_1. \end{cases} \tag{5.4}$$

Graphically, we have

$$f_i : y_i \begin{array}{|ccc|}\hline 1 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \\ \hline \end{array} \qquad\qquad f_1 : y_1 \begin{array}{|ccc|}\hline 0 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 2 \\ \hline \end{array}$$
$$\qquad y_{i-1} \qquad\qquad\qquad\qquad\qquad y_n$$

$$g_i : x_{i+1} \begin{array}{|ccc|}\hline 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ \hline \end{array} \qquad\qquad g_n : x_1 \begin{array}{|ccc|}\hline 0 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 2 \\ \hline \end{array}$$
$$\qquad x_i \qquad\qquad\qquad\qquad\qquad x_n$$

the sets $f_i^{-1}(x_i)$ and $g_i^{-1}(y_i)$ forming L-shaped regions of $[3]^n$.

THEOREM 5.1. *The cycle C_{2n} is 3-solvable for $n \equiv 0 \pmod 3$. Using the guessing functions as defined above, when $n \equiv 1 \pmod 3$, the only configurations (x, y) that all vertices guess incorrectly are*

$$x = (a, a+2, a+1, a, \dots, a), y = (a, a+2, a+1, a, \dots, a) \textit{ for some } a \in [3],$$

and when $n \equiv 2 \pmod 3$, the only configurations (x, y) that all vertices guess incorrectly are

$$x = (a+2, a, a+1, a+2, \dots, a), y = (a, a+1, a+2, a, \dots, a+1) \textit{ for some } a \in [3].$$

Proof. Suppose $y = (y_1, \dots, y_n) \in [3]^n$ is the configuration of hat colors for the vertices in W and that each vertex in W guesses incorrectly. Then $x \in \bigcap_{i=1}^n g_i^{-1}(y_i)^{\mathrm{c}}$, where

$$\bigcap_{i=1}^n g_i^{-1}(y_i)^{\mathrm{c}} = \bigcap_{i<n} \{x : x_i = y_i - 1 \text{ or } x_{i+1} = y_i - 1 \text{ or } (x_i, x_{i+1}) = (y_i + 1, y_i + 1)\}$$
$$\cap \{x : x_n = y_n - 1 \text{ or } x_1 = y_n \text{ or } (x_n, x_1) = (y_n + 1, y_n - 1)\}.$$

Suppose further that each vertex in V guesses incorrectly. We claim the following implications are true.

Claim 2. If (x, y) is guessed incorrectly by all vertices, then the following hold. For all $i \neq 1$,

- A_i: if $x_i = y_i - 1$, then $y_i = y_{i-1} + 1$ and $x_{i-1} = y_{i-1} - 1$;
- B_i: if $x_i = y_i + 1$, then either
  1. $y_i = y_{i-1}$ and $x_{i-1} = y_{i-1} + 1$, or
  2. $y_i \neq y_{i-1} + 1$ and $x_{i-1} = y_{i-1} - 1$;

and for all $i \neq n$,

- C_i: if $x_i = y_i$, then $y_{i+1} = y_i - 1$ and $x_{i+1} = y_{i+1}$.

Proof of claim. Take $i \neq 1$, and suppose $x_i = y_i - 1$. Since v_i guesses incorrectly, we must have $y_i = y_{i-1} + 1$, so that $x_i = y_{i-1}$. But $x \in g_{i-1}^{-1}(y_{i-1})^{\mathrm{c}}$, which implies that $x_{i-1} = y_{i-1} - 1$, establishing A_i.

Now suppose $x_i = y_i + 1$. Since v_i guesses incorrectly, we must have $y_i \neq y_{i-1} + 1$, so that $x_i \neq y_{i-1} - 1$. But $x \in g_{i-1}^{-1}(y_{i-1})^{\mathrm{c}}$, which implies that either $x_{i-1} = y_{i-1} - 1$ or $(x_{i-1}, x_i) = (y_{i-1} + 1, y_{i-1} + 1)$, the latter implying that $y_i = y_{i-1}$, which establishes B_i.

Finally, take $i \neq n$ and suppose $x_i = y_i$. Since $x \in g_i^{-1}(y_i)^c$, we must have $x_{i+1} = y_i - 1$. But v_{i+1} guesses incorrectly, which implies that $y_{i+1} = y_i - 1$. To see this, use the fact that the function f_i, for $i \neq 1$, can also be written as

$$f_i(y_{i-1}, y_i) = \begin{cases} y_{i-1} - 1 & \text{if } y_i \neq y_{i-1} - 1, \\ y_{i-1} + 1 & \text{if } y_i = y_{i-1} - 1. \end{cases}$$

Therefore $x_{i+1} = y_{i+1}$, establishing C_i. $\square$

We use the implications A_i, B_i, and C_i as follows. First, suppose $x_n = y_n - 1$. Then using the chain of implications $A_n, A_{n-1}, \ldots, A_2$ we find that $x_i = y_i - 1$ for all i and $y_i = y_{i-1} + 1$ for $i \neq 1$, so $y_n = y_1 + (n-1) \pmod 3$. Since $x_1 = y_1 - 1$ and v_1 also guesses incorrectly, we must have $y_1 = y_n - 1$, a contradiction unless $n \equiv 2 \pmod 3$. When $n \equiv 2 \pmod 3$, we discover that the configurations $x = (a+2, a, a+1, a+2, \ldots, a), y = (a, a+1, a+2, a, \ldots, a+1)$ for $a \in [3]$ are guessed incorrectly by all vertices.

Now suppose $x_n = y_n + 1$. Since $x \in g_n^{-1}(y_n)^c$ we have that $x_1 \neq y_n + 1$. We consider the chain of implications $B_n, B_{n-1}, \ldots$ for as far as possible and note that case 1 of B_i cannot occur for all $i \neq 1$, for then $x_i = y_i + 1$ for all i and $y_i = y_{i-1}$ for $i \neq 1$, contradicting the fact that $x_1 \neq y_n + 1$. This means that for some $k > 1$ case 2 of B_k occurs, so that $x_{k-1} = y_{k-1} - 1$. We then apply the chain of implications $A_{k-1}, A_{k-2}, \ldots, A_2$ to find that $x_i = y_i - 1$ for all $i < k$, so in particular $x_1 = y_1 - 1$. Since v_1 guesses incorrectly, we have that $y_1 = y_n - 1$, which contradicts the fact that $x_1 \neq y_n + 1$. Hence for any configuration with $x_n = y_n + 1$ there must be some vertex that guesses correctly.

Finally, suppose $x_n = y_n$. Since $x \in g_n^{-1}(y_n)^c$ we have that $x_1 = y_n$. Since v_1 guesses incorrectly, we must have that $y_1 = y_n$. To see this use the fact that f_1 can be also written as

$$f_1(y_n, y_1) = \begin{cases} y_n & \text{if } y_1 \neq y_n, \\ y_n - 1 & \text{if } y_1 = y_n. \end{cases}$$

Therefore $x_1 = y_1$. We now apply the chain of implications $C_1, C_2, \ldots, C_{n-1}$ to find that $x_i = y_i$ for all i, and $y_{i+1} = y_i - 1$ for $i \neq n$. Therefore $y_n = y_1 - (n-1) \pmod 3$, which is a contradiction unless $n \equiv 1 \pmod 3$. When $n \equiv 1 \pmod 3$, we discover that the configurations $x = (a, a+2, a+1, a, \ldots, a), y = (a, a+2, a+1, a, \ldots, a)$ for $a \in [3]$ are guessed incorrectly by all vertices. $\square$

Unfortunately, this L-shaped construction falls just short of proving 3-solvability when $n \not\equiv 0 \pmod 3$; indeed, of the 3^{2n} possible hat configurations, there are only 3 where all vertices guess incorrectly!

In any case, the family of cycles of length a multiple of 6 gives an answer to a question of Butler et al. about edge-critical graphs. A graph G is called *edge-critical for q colors* if G is q-solvable, but $G - e$ is not q-solvable for any edge $e \in G$. For $q = 2$ the only edge-critial graph is the graph of a single edge, and for $q > 2$ there are at least two distinct edge-critical graphs, namely, K_q and some subgraph of the bipartite graph $K_{q-1,(q-1)^{q-1}}$ presented earlier. Butler et al. ask whether there are infinitely many graphs which are edge-critical for q colors, for $q > 2$. Since trees are known not to be 3-solvable, the cycles of length a multiple of 6 form such an infinite family for $q = 3$.

THEOREM 5.2. *The family $\{C_{6k} : k \in \mathbb{N}\}$ is an infinite family of edge-critical graphs for 3 colors.*

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