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Equilibrium partition function for nonrelativistic fluids

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We construct an equilibrium partition function for a nonrelativistic fluid and use it to constrain the dynamics of the system. The construction is based on light cone reduction, which is known to reduce the Poincaré symmetry to Galilean in one lower dimension. We modify the constitutive relations of a relativistic fluid, and find that its symmetry broken phase—"null fluid" is equivalent to the nonrelativistic fluid. In particular, their symmetries, thermodynamics, constitutive relations, and equilibrium partition function match exactly to all orders in derivative expansion.

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I. INTRODUCTION

The constitutive relations of a relativistic fluid at local thermal equilibrium can be obtained from an equilibrium partition function up to some undetermined "transport coefficients" [1,2]. These coefficients can be determined either from experiments or through a microscopic calculation. Recently, nonrelativistic geometry and fluid are getting active attention [3–9]. If we think of nonrelativistic fluid as a limit of an underlying relativistic theory, we would expect its constitutive relations also to follow from an equilibrium partition function. The goal of this paper is to develop a formal and consistent way to compute partition function for a nonrelativistic fluid starting from a relativistic theory.

We start with a flat background $\mathrm{d}s^2 = -2\mathrm{d}x^-\mathrm{d}x^+ + \sum_{i=1}^{d}(\mathrm{d}x^i)^2$, which has $(d+2)$-dim Poincaré invariance. It is known that $(d+1)$-dim Galilean algebra sits inside Poincaré—all transformations that commute with $P_- = \partial_-$ (cf., [10]). Hence a theory on this background which respects x^- independent isometries $x^M \to x^M + \xi^M(x^+, \vec{x})$, $x^M = \{x^-, x^+, x^i\}$, enjoys Galilean invariance. Compactifying the x^- direction, we can realize this Galilean invariance as nonrelativistic invariance in $(d+1)$-dim [11]. This procedure to get nonrelativistic theories is known as *light cone reduction (LCR)*.

Turning on x^- independent fluctuations around the flat background,

$$\mathrm{d}s^2 = \mathrm{G}_{MN}\mathrm{d}x^M\mathrm{d}x^N = g_{ij}\mathrm{d}x^i\mathrm{d}x^j - 2e^{-\Phi}(\mathrm{d}x^+ + a_i\mathrm{d}x^i)(\mathrm{d}x^- - \mathcal{B}_+\mathrm{d}x^+ - \mathcal{B}_i\mathrm{d}x^i), \quad (1)$$

nonrelativistic theories can be described by a generating functional $\mathcal{Z}[\mathcal{B}_+, \mathcal{B}_i, \Phi, a_i, g_{ij}]$ written as a functional of background sources. Mapping Ward identities of this partition function to those of a Galilean theory, allows us to read out the Galilean currents and densities,

$$\rho \sim \frac{\delta W}{\delta \mathcal{B}_+}, \quad j^i_\rho \sim \frac{\delta W}{\delta \mathcal{B}_i}, \quad \epsilon \sim \frac{\delta W}{\delta \Phi}, \quad j^i_\epsilon \sim \frac{\delta W}{\delta a_i}, \quad t^{ij} \sim \frac{\delta W}{\delta g_{ij}}, \quad (2)$$

where $W = \ln \mathcal{Z}$, and ρ, j^i_ρ, ϵ, j^i_ϵ and t^{ij} are mass density, mass current, energy density, energy current, and stress tensor, respectively, of the nonrelativistic theory.

In an earlier work [12], following [13], we performed LCR of a relativistic fluid to obtain constitutive relations of a nonrelativistic fluid (at leading order in derivatives). While the reduced conservation equations agreed with their expected nonrelativistic form (in parity even sector), we observed that the nonrelativistic fluid gained by reduction is not the most generic one. In particular, thermodynamics that the reduced fluid follows is in some sense more restrictive than the most generic nonrelativistic fluid (chemical potential corresponding to particle number is not independent). We also found that the parity-odd sector of reduced fluid survives only for a special case of incompressible fluids. It strongly hints that to get the most generic nonrelativistic fluid via light cone reduction, we need to start with a modified relativistic system.

We start with a relativistic "fluid" on a curved background which admits a null isometry [10,14–16]. Unlike the "usual" relativistic fluids, now isometry is also a background field and hence must be considered while writing the respective constitutive relations. We break the relativistic symmetry to Galilean by only considering diffeomorphisms which do not alter the isometry, and term

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the symmetry-broken relativistic fluid as "null fluid." This simple consideration happens to resolve all the issues we enlisted before. In fact it does much more than that; even before LCR, $(d+2)$-dim null fluid is essentially equivalent to a $(d+1)$-dim nonrelativistic fluid, as they have same symmetries. As we shall show, their constitutive relations, conservation equations, thermodynamics etc. match exactly to all orders in derivative expansion. In this paper we aim to use this equivalence to write an equilibrium partition function for Galilean fluids, and use Eq. (2) to constrain the dynamics of the system.

II. NULL BACKGROUNDS

We start with a $(d+2)$ dimensional spacetime $\mathcal{M}_{(d+2)}$ equipped with a metric G_{MN} and a metric compatible affine connection $\hat{\Gamma}^R{}_{MN}$ (with associated covariant derivative $\hat{\nabla}_M$). On this setup we define a null isometry V ($\Rightarrow \mathcal{L}_V G_{MN} = 0$) which satisfies $\hat{\nabla}_M V^N = 0$. We call this background "*null background*" [17]. On torsionless null backgrounds, which we shall consider in this work, the latter condition implies the former, and in addition: $\mathcal{H}_{MN} = 2\partial_{[M}V_{N]} = 0$. This is a dynamic constraint on the background, and can be violated by quantum fluctuations off-shell.

A physical theory living on a null background, can be characterized by (log of) a generating functional W, whose response to infinitesimal variation of metric is given by,

$$\delta W = \int \{\mathrm{d}x^M\}\sqrt{-\mathrm{G}}\frac{1}{2}T^{MN}\delta \mathrm{G}_{MN}. \tag{3}$$

T^{MN} is called the energy-momentum tensor. One can check that V being null allows for an arbitrary redefinition $T^{MN} \to T^{MN} + \theta V^M V^N$ for some scalar θ, which leaves Eq. (3) invariant. Demanding this partition function to be invariant under V preserving diffeomorphisms, we get the conservation law for energy-momentum,

$$\hat{\nabla}_M T^{MN} = 0. \tag{4}$$

A background is said to be in equilibrium configuration if it admits a timelike Killing vector K^M. We right away choose a basis $x^M = \{x^-, x^+, x^i\}$, such that $V = \partial_-$ and $K = \partial_+$. The most generic metric with this choice of basis is given by Eq. (1). It is easy to verify that under $(d+2)$-diffeomorphisms restricted to our choice of basis, Φ, $\mathcal{B}_+$ transform as scalars, $a_i, B_i (= \mathcal{B}_i - a_i \mathcal{B}_+)$ transform as Abelian vector gauge fields, and g_{ij} transforms as a metric in d-dim. Hence at equilibrium, the partition function W^{eqb} will be a gauge invariant scalar made out of these fundamental fields. Its variation can be worked out trivially from Eq. (3),

$$\delta W^{eqb} = \int \{\mathrm{d}x^i\}\sqrt{g}\frac{1}{\vartheta_o}\bigg[\mathrm{e}^{\Phi}(T_{+-} + T_{--}\mathcal{B}_+)\frac{1}{\vartheta_o}\delta\vartheta_o + T^i{}_+\delta a_i + \frac{1}{2}T^{ij}\delta g_{ij} + \vartheta_o T_{--}\delta\varpi_o - T^i{}_-\delta B_i\bigg]. \tag{5}$$

In getting this we have identified inverse temperature of the Euclidean field theory to be $1/\tilde{\vartheta} = \tilde{\beta}\tilde{R}$, where $\tilde{\beta}$ is the radius of the Euclidean time circle, and $\tilde{R}$ is the radius of the compactified null direction. Further we have defined the redshifted equilibrium temperature $\vartheta_o = \tilde{\vartheta}\mathrm{e}^{\Phi}$ and mass chemical potential $\vartheta_o\varpi_o = \mathcal{B}_+\mathrm{e}^{\Phi}$.

W^{eqb} can be written order by order in derivatives of the source fields. While the partition function is itself gauge and diffeo invariant, it is not true for the integrand. In fact, *Chern-Simons* terms can be added to it whose variation is gauge invariant only up to some boundary terms. We shall consider such terms when required. At ideal order, W^{eqb} is given by a function of ϑ_o, ϖ_o,

$$W_o^{eqb} = \int \{\mathrm{d}x^i\}\sqrt{g}\frac{1}{\vartheta_o}P_o(\vartheta_o, \varpi_o), \tag{6}$$

where P_o is the local thermodynamic *pressure* at equilibrium. Varying it and using Eq. (5), we will get,

$$T^{ij} = P_o g^{ij}, \qquad T_{--} = R_o, \qquad \mathrm{e}^{\Phi}(T_{+-} + T_{--}\mathcal{B}_o) = E_o, \tag{7}$$

where considering ϑ_o, ϖ_o as thermodynamic variables at equilibrium, we have defined the first law of thermodynamics and Gibbs-Duhem equation,

$$\mathrm{d}E = \vartheta \mathrm{d}S + \vartheta\varpi \mathrm{d}R, \qquad E = S\vartheta + \vartheta R\varpi - P. \tag{8}$$

Defining a null field $\bar{V}^M_{(K)} = \mathrm{e}^{\Phi}(K^M + \mathcal{B}_+ V^M)$ normal to V (i.e., $V^M\bar{V}_{(K)M} = -1$), and a projection operator $P^{MN}_{(K)} = \mathrm{G}^{MN} + 2V^{(M}\bar{V}^{N)}_{(K)}$ transverse to V, $\bar{V}_{(K)}$, Eq. (7) can be covariantly repackaged into,

$$T^{MN} = R_o\bar{V}^M_{(K)}\bar{V}^N_{(K)} + 2E_o\bar{V}^{(M}_{(K)}V^{N)} + P_o P^{MN}_{(K)}. \tag{9}$$

III. NULL FLUID

Having constructed null backgrounds, we proceed to define hydrodynamics on this setup. This is the essence of our work; we claim that this "modified fluid" is just a different representation of the nonrelativistic fluid, with exact (yet trivial) mapping between the two facilitated by light cone reduction. Note that conservation laws (4) are $(d+2)$ equations, so any system with $(d+2)$ independent variables would be exactly solvable on this background. We choose to describe our system by a fluid, with a null velocity field u^M normalized as $u^M u_M = 0$, $u^M V_M = -1$ and two thermodynamic variables, temperature ϑ, and mass chemical potential ϖ. The most generic constitutive relations (after using the T^{MN} redefinition freedom) are given in terms of fluid variables and background quantities as,

$$T^{MN} = \mathcal{R} u^M u^N + 2\mathcal{E} u^{(M} V^{N)} + \mathcal{P} P^{MN}_{(u)} + 2\mathbb{R}^{(M} u^{N)} + 2\mathbb{E}^{(M} V^{N)} + \mathbb{T}^{MN}, \tag{10}$$

where $\mathcal{R}$, $\mathcal{E}$, $\mathcal{P}$ are some arbitrary functions of ϑ, ϖ. The tensors $\mathbb{R}^M$, $\mathbb{E}^M$, $\mathbb{T}^{MN}$ (traceless) contain derivative corrections and are transverse to u^M and V^M through projection operator: $P^{MN}_{(u)} = \mathrm{G}^{MN} + 2V^{(M} u^{N)}$.

From Eq. (9) we can deduce that at equilibrium (ideal order), $\mathcal{R}$, $\mathcal{E}$, $\mathcal{P}$ and u^M boil down to the thermodynamic functions R, E, P and equilibrium null vector $\bar{V}^M_{(K)}$. Outside equilibrium however, none of the fluid variables are uniquely defined, and are subjected to arbitrary redefinitions. These are two scalars and a vector worth of freedom, which we fix by working in the "mass frame," i.e., we identify $\mathcal{E}$, $\mathcal{R}$ with E, R dumping all the dissipation into $\mathcal{P}$, and set $\mathbb{R}^M = 0$,

$$T^{MN} = R u^M u^N + 2E u^{(M} V^{N)} + P P^{MN}_{(u)} + 2\mathcal{E}^{(M} V^{N)} + \Pi^{MN}. \tag{11}$$

Here Π^{MN} is not traceless. To leading derivative order [one derivative in parity-even sector and $(n-1)$ derivative in parity-odd sector for $d = 2n-1$], constitutive relations in the mass frame are given as:

$$\Pi^{MN} = -\eta\sigma^{MN} - P^{MN}_{(u)} \zeta\Theta,$$
$$\mathcal{E}^M = P^{MN}_{(u)} [\kappa\partial_N \vartheta + \lambda\partial_N \varpi] + \tilde{\omega} l^M, \tag{12}$$

where $\sigma^{MN} = 2P^{\langle MR}_{(u)} P^{N\rangle S}_{(u)} \hat{\nabla}_R u_S$ is symmetric traceless, $\Theta = \hat{\nabla}_M u^M$ and $l^M = \star[V \wedge u \wedge (\mathrm{d}u)^{\wedge(n-1)}]^M$.

A. Equilibrium partition function

Similar to relativistic fluids in [1,2], equilibrium partition function gives equality type constraints among various transport coefficients appearing in the null fluid constitutive relations. Away from ideal order we can construct the partition function W^{eqb} order by order in derivatives of the background fields. It is easy to see that at leading derivative order, there are no scalars at equilibrium and the partition function is trivially zero up to some Chern-Simons terms:

$$W^{eqb} = W^{eqb}_o - \int (nC_1 \tilde{\vartheta} a + C_2 B) \wedge (\mathrm{d}B)^{\wedge(n-1)}. \tag{13}$$

The term coupling to C_1 goes as $B \wedge \mathrm{d}a \wedge (\mathrm{d}B)^{\wedge(n-2)}$ up to a total derivative term, which vanishes on-shell as $\mathcal{H}_{ij} = 0 \Rightarrow \mathrm{d}a = 0$. But since partition functions are to be written off-shell, we must include this term. On the other hand, in constitutive relations Eq. (12), only terms coupling to λ and $\tilde{\omega}$ survive at equilibrium. Comparing these to the constitutive relations generated by partition function variation defined in Eq. (5), one can easily get the constraints:

$$\lambda = 0, \qquad \tilde{\omega} = n\vartheta\left(\vartheta C_1 + \frac{E + P - \vartheta\varpi R}{R} C_2\right). \tag{14}$$

Fluid variables ϑ, ϖ do not get corrections at leading order, while velocity gets a parity-odd correction:

$$u^M = \bar{V}^M_{(K)} - \frac{\vartheta_o n}{R_o} C_2 \star[(\mathrm{d}B)^{\wedge(n-1)}]^M. \tag{15}$$

B. Entropy current

The second law of thermodynamics demands that there must exist an entropy current J^M_s, whose divergence is positive semidefinite $\hat{\nabla}_M J^M_s \geq 0$. The most generic form of entropy current is given as, $J^M_s = J^M_{s(\mathrm{can})} + \Upsilon^M_s$, where Υ^M_s are arbitrary derivative corrections (not necessarily projected), and,

$$J^M_{s(\mathrm{can})} = \frac{1}{\vartheta} P u^M - \frac{1}{\vartheta} T^{MN} u_N + \varpi T^{MN} V_N, \tag{16}$$

is the canonical entropy current. Its divergence can be computed to be,

$$\vartheta \hat{\nabla}_M J^M_{s(\mathrm{can})} = -\Pi^{MN} \hat{\nabla}_M u_N - \frac{1}{\vartheta} \mathcal{E}^M \hat{\nabla}_M \vartheta. \tag{17}$$

Plugging in the constitutive relations Eq. (12) and only allowed derivative correction $\Upsilon^M_s = \tilde{\omega}_s l^M$ (other vectors give pure derivative terms in divergence and hence must vanish), one can find that the second law of thermodynamics gives the same constraints Eq. (14) (in parity-even sector) and in addition: $\eta, \zeta \geq 0$ and $\kappa \leq 0$. In the parity-odd sector however, it sets $C_2 = 0$ and $C_1 = C_1(\vartheta)$ which unexpectedly is weaker than the partition function constraints. This discrepancy can be accounted to the fact that in this computation we have missed constraint(s) coupling to $\mathcal{H} = \mathrm{d}V$, which is set to zero by a torsionlessness requirement. This condition can however be violated off-shell and hence the coupled constraint(s) are visible to the equilibrium partition function. In a companion paper [18], we will show that on introducing just enough torsion to allow for nonzero values of $\mathcal{H}$, and setting it to zero after the entropy current computation, we will recover the missed constraints.

IV. LIGHT CONE REDUCTION

To give null backgrounds a Galilean interpretation, we need to get rid of the isometry direction V. To do so, we curl up the V direction into an infinitesimal circle, reducing the effective dimensions by one, where the nonrelativistic theory can live. To make this more precise, we pick up an arbitrary vector field $T(\neq V)$ on $\mathcal{M}_{(d+2)}$, and use it to define a unique foliation of $\mathcal{M}_{(d+2)} = S^1_V \times R^1_T \times \mathcal{M}^T_{(d)}$, where,

$$\mathcal{M}^T_{(d)} \coloneqq \{v^M : v^M V_M = v^M T_M = 0\}. \tag{18}$$

After this point one just has to choose a basis on this foliation and read out the Galilean results, i.e., $x^M = \{x^-, x^+, x^i\}$ such that $V = \partial_-$, $T = \partial_+$, and $\vec{x} = \{x^i\}$ span the spatial manifold $\mathcal{M}^T_{(d)}$. Note that this basis depends on the choice of "Galilean frame" T, which we will refer to as *local rest* of a frame T. One could also work in a frame independent *Newton-Cartan* formalism, which we discuss in a followup paper [18].

One can check that with this choice of basis, metric G_{MN} decomposes as Eq. (1), with all its components being independent of x^-. Using this decomposition, partition function variation Eq. (3) can be reduced to,

$$\delta W = \tilde{R} \int \mathrm{d}x^+ \{\mathrm{d}x^i\} \sqrt{g} \mathrm{e}^{-\Phi} [-(\mathrm{e}^{-\Phi} j^i_\epsilon - j^i_\rho \mathcal{B}_+) \delta a_i + \hat{\epsilon}_{\mathrm{tot}} \delta\Phi + \frac{1}{2} t^{ij} \delta g_{ij} + (\mathrm{e}^{\Phi} \hat{\rho} \delta \mathcal{B}_+ + j^i_\rho \delta B_i)], \tag{19}$$

where $\tilde{R}$ is the radius of compactified x^-, and we have identified,

$$\hat{\rho} = T_{--}, \qquad \hat{\epsilon}_{\mathrm{tot}} = \mathrm{e}^{\Phi}(T_{-+} + T_{--}\mathcal{B}_+), \qquad t^{ij} = T^{ij}$$
$$j^i_\rho = -T^i{}_-, \qquad j^i_\epsilon = -\mathrm{e}^{\Phi}(T^i{}_+ + T^i{}_- \mathcal{B}_+). \tag{20}$$

Using these identifications, we can reduce the Ward identities Eq. (4) to get,

$$\begin{aligned}
&\frac{1}{\sqrt{g}} \partial_+(\sqrt{g}\rho) + \nabla_i(\mathrm{e}^{-\Phi} j^i_\rho) = 0, \\
&\frac{1}{\sqrt{g}} \partial_+(\sqrt{g}\epsilon_{\mathrm{tot}}) + \nabla_i(\mathrm{e}^{-\Phi} j^i_\epsilon) = -\frac{1}{2} t^{ij} \partial_+ g_{ij} - \mathrm{e}^{-\Phi} j^i_\rho \alpha_i, \\
&\frac{1}{\sqrt{g}} \partial_+(\sqrt{g} p_i) + \nabla_j(\mathrm{e}^{-\Phi} t^j{}_i) = -\frac{1}{2} \mathrm{e}^{-\Phi} a_i t^{jk} \partial_+ g_{jk}, \\
&\qquad - \mathrm{e}^{-\Phi}(\hat{\rho} \mathrm{e}^{-\Phi} \alpha_i + j^j_\rho \omega_{ji}),
\end{aligned} \tag{21}$$

where we have defined corrected densities due to time not being "flat," $\rho = \hat{\rho} - \mathrm{e}^{-\Phi} j^i_\rho a_i$, $\epsilon_{\mathrm{tot}} = \hat{\epsilon}_{\mathrm{tot}} - \mathrm{e}^{-\Phi} j^i_\epsilon a_i$, and $p^i = j^i_\rho - \mathrm{e}^{-\Phi} t^{ij} a_j$. Identifying x^+ with the Galilean time, these equations can be realized as mass, energy, and momentum conservation laws, respectively, of a Galilean theory. Mass is exactly conserved, while energy/momentum are being sourced due to time-dependence of background metric. Further, energy/momentum are also being sourced due to pseudoenergy/force caused by acceleration $\alpha_i = -\mathrm{e}^{\Phi}[\mathrm{d}\mathcal{B}]_{i+}$ and vorticity $\omega^{ij} = [\mathrm{d}\mathcal{B}]^{ij}$ of the Galilean frame.

V. NONRELATIVISTIC FLUID

It is interesting to see how the dynamics of a nonrelativistic fluid emerges from the aforementioned reduction. The conservation equations (21) are in one-to-one correspondence with the dynamical equations of a nonrelativistic fluid as given in [12], generalized to curved space. This motivates us to interpret null fluid on $\mathcal{M}_{(d+2)}$ as a Galilean fluid on $\mathbb{R}^1_T \times \mathcal{M}^T_{(d)}$, as seen by some reference frame T. We can right away use Eq. (10) as an ansatz for T^{MN} and perform the reduction as suggested by Eq. (20),

$$\begin{aligned}
&\hat{\epsilon}_{\mathrm{tot}} = \hat{\epsilon} + \frac{1}{2}\hat{\rho} v^i v_i + \varsigma^i_\rho v_i, \qquad j^i_\rho = \hat{\rho} v^i + \varsigma^i_\rho, \\
&t^{ij} = P g^{ij} + \hat{\rho} v^i v^j + 2 v^{(i} \varsigma^{j)}_\rho + \pi^{ij}, \\
&j^i_\epsilon = (\hat{\epsilon}_{\mathrm{tot}} + P) v^i + \varsigma^i_\epsilon + \pi^{ik} v_k + \frac{1}{2} \varsigma^i_\rho v^j v_j,
\end{aligned} \tag{22}$$

with identifications:

$$\hat{\rho} = \mathcal{R}, \qquad \hat{\epsilon} = \mathcal{E}, \qquad \varsigma^i_\rho = \mathbb{R}^i, \qquad \varsigma^i_\epsilon = \mathbb{E}^i,$$
$$\pi^{ij} = (\mathcal{P} - P) g^{ij} + \mathbb{T}^{ij}.$$

Dynamics of these densities/currents is governed by the conservation laws equation (21). The entropy current equation (23) on the other hand reduces to:

$$\hat{s} = \frac{\hat{\epsilon} + P}{\vartheta} - \varpi\hat{\rho} + \Upsilon_{s-}, \qquad j^i_s = \hat{s} v^i + \frac{1}{\vartheta} \varsigma^i_\epsilon - \varpi \varsigma^i_\rho + \Upsilon^i_s. \tag{23}$$

$\hat{s} = S$ at ideal order in equilibrium. The statement of second law of thermodynamics then becomes $\partial_+(\sqrt{g}s) + \partial_i(\sqrt{g}\mathrm{e}^{-\Phi} j^i_s) \geq 0$, where $s = \hat{s} - \mathrm{e}^{-\Phi} j^i_s a_i$.

As an example we can consider leading order fluid on flat background in three spatial dimensions ($d = 3$) expressed in mass frame ($\varsigma^i_\rho = 0$), as given in [19],

$$\begin{aligned}
&j^i_\rho = R v^i, \qquad \hat{\epsilon}_{\mathrm{tot}} = E + \frac{1}{2} R v^i v_i, \\
&t^{ij} = R v^i v^j + P g^{ij} - 2\eta \partial^{\langle i} v^{j\rangle} - \zeta g^{ij} \partial_k v^k, \\
&j^i_\epsilon = \left(E + P + \frac{1}{2} R v^i v_i\right) v^i - 2\eta v_j \partial^{\langle i} v^{j\rangle} - \zeta v^i \partial_k v^k \\
&\qquad + \kappa \partial^i \vartheta + \lambda \partial^i \varpi + \tilde{\omega} \epsilon^{ijk} \partial_j v_k, \\
&j^i_s = S v^i + \frac{1}{\vartheta} [\kappa \partial^i \vartheta + \lambda \partial^i \varpi + (\tilde{\omega} + \vartheta \tilde{\omega}_s) \epsilon^{ijk} \partial_j v_k].
\end{aligned} \tag{24}$$

Demanding the second law of thermodynamics to hold will give same constraints as the null fluid; in particular $\lambda = 0$. As expected, parity-even sector contains a bulk viscosity, a shear viscosity, and a thermal conductivity term. Parity-odd sector however has a thermal Hall conductivity term coupled to fluid vorticity. These constitutive relations follow the conservation laws equation (21) restricted to flat space. Here we explicitly chose to work in mass frame, any other choice of frame in the null fluid will follow trivially to the nonrelativistic fluid due to trivial mapping of currents.

A. Equilibrium partition function

From the perspective of a nonrelativistic fluid, equilibrium is defined by a preferred reference frame K with respect to which the system does not evolve in time. The variation of the equilibrium partition function in local rest of reference frame K is essentially same as the null fluid equation (5) written in terms of Galilean quantities, and hence,

$$\hat{\rho}_o = \frac{\delta W^{eqb}}{\delta \varpi_o}, \qquad j^i_{o\rho} = \vartheta_o \frac{\delta W^{eqb}}{\delta B_i}, \qquad t^{ij}_o = 2\vartheta_o \frac{\delta W^{eqb}}{\delta g_{ij}},$$
$$\hat{\epsilon}_o = \vartheta_o^2 \frac{\delta W^{eqb}}{\delta \vartheta_o}, \qquad j^i_{o\epsilon} - \varpi_o \vartheta_o j^i_{o\rho} = -\vartheta_o^2 e^{\Phi} \frac{\delta W^{eqb}}{\delta a_i}. \tag{25}$$

These will reduce to the expected relations Eq. (2) in flat space $\vartheta_o = 1$, $g_{ij} = \delta_{ij}$, $\varpi_o = B_i = a_i = 0$. In equilibrium, null-fluid and Galilean fluid have the same field content and symmetries, so we expect the equilibrium partition function to also be the same, i.e., Eqs. (6) and (13). To ideal order it will identify $\hat{\rho}_o$, $\hat{\epsilon}_o$ with the thermodynamic functions R, E, and hence will give physical interpretation to the thermodynamics of null fluids equation (8) in terms of nonrelativistic physics. At leading derivative order it will give the constraints equation (14).

We would like to note here that [6] also constructed an equilibrium partition function and entropy current for an uncharged Galilean fluid up to leading order in derivatives, but purely from a Galilean perspective, without invoking null reduction or a relativistic system. As expected, we find our constitutive relations and equilibrium partition function to be in exact agreement with [6].

VI. DISCUSSION

One of the most striking features of our construction is that the relativistic null fluid is equivalent to a Galilean fluid, and is related just by a choice of basis $V = \partial_-$. This gives us a new and rather simplified way to look at Galilean fluids altogether, since we have all the machinery of relativistic hydrodynamics at our disposal. In the current work we have used it to construct an equilibrium partition function and an entropy current for torsionless Galilean fluids at leading order in derivatives, and to find constraints on various transport coefficients appearing in the fluid constitutive relations. The procedure can also be extended to include an (anomalous) U(1) current which we consider in a companion paper [18]. In another paper [20] we use this idea of null backgrounds to study (non-Abelian) gauge and gravitational anomalies in (torsional) Galilean theories, and in particular their effect on hydrodynamics. Literature on nonrelativistic hydrodynamics is vast and is increasing every day, thus it is not possible to give an exhaustive list of references. We plan to include a more complete list of references in a follow-up work.

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