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Lengths of words in transformation semigroups generated by digraphs

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Abstract Given a simple digraph D on n vertices (with $n \geq 2$), there is a natural construction of a semigroup of transformations $\langle D \rangle$. For any edge (a, b) of D, let $a \to b$ be the idempotent of rank $n-1$ mapping a to b and fixing all vertices other than a; then, define $\langle D \rangle$ to be the semigroup generated by $a \to b$ for all $(a, b) \in E(D)$. For $\alpha \in \langle D \rangle$, let $\ell(D, \alpha)$ be the minimal length of a word in $E(D)$ expressing α. It is well known that the semigroup Sing_n of all transformations of rank at most $n-1$ is generated by its idempotents of rank $n-1$. When $D = K_n$ is the complete undirected graph, Howie and Iwahori, independently, obtained a formula to calculate $\ell(K_n, \alpha)$, for any $\alpha \in \langle K_n \rangle = \mathrm{Sing}_n$; however, no analogous non-trivial results are known when $D \neq K_n$. In this paper, we characterise all simple digraphs D such that either $\ell(D, \alpha)$ is equal to Howie–Iwahori's formula for all $\alpha \in \langle D \rangle$, or $\ell(D, \alpha) = n - \mathrm{fix}(\alpha)$ for all $\alpha \in \langle D \rangle$, or $\ell(D, \alpha) = n - \mathrm{rk}(\alpha)$ for all $\alpha \in \langle D \rangle$. We also obtain bounds for $\ell(D, \alpha)$ when D is an acyclic digraph or a strong tournament (the latter case corresponds to a smallest generating set of idempotents of rank $n-1$ of Sing_n). We finish the paper with a list of conjectures and open problems.

Keywords Transformation semigroup · Simple digraph · Word length

1 Introduction

For any $n \in \mathbb{N}$, $n \geq 2$, let Sing_n be the semigroup of all singular (i.e. non-invertible) transformations on $[n] := \{1, \ldots, n\}$. It is well known (see [2]) that Sing_n is generated

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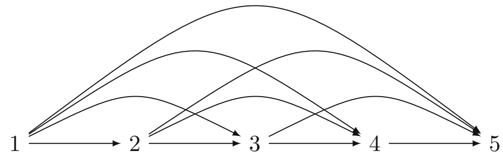
Fig. 1 $\vec{T}_5$



by its idempotents of defect 1 (i.e. the transformations $\alpha \in \mathrm{Sing}_n$ such that $\alpha^2 = \alpha$ and $\mathrm{rk}(\alpha) := |\mathrm{Im}(\alpha)| = n - 1$). There are exactly $n(n-1)$ such idempotents, and each one of them may be written as $(a \to b)$, for $a, b \in [n]$, $a \neq b$, where, for any $v \in [n]$,

$$(v)(a \to b) := \begin{cases} b & \text{if } v = a, \\ v & \text{otherwise.} \end{cases}$$

Motivated by this notation, we refer to these idempotents as *arcs*.

In this paper, we explore the natural connections between simple digraphs on $[n]$ and subsemigroups of Sing_n. For any subset $U \subseteq \mathrm{Sing}_n$, denote by $\langle U \rangle$ the semigroup generated by U. For any simple digraph D with vertex set $V(D) = [n]$ and edge set $E(D)$, we associate the semigroup

$$\langle D \rangle := \big\langle (a \to b) \in \mathrm{Sing}_n : (a, b) \in E(D) \big\rangle .$$

We say that a subsemigroup S of Sing_n is *arc-generated* by a simple digraph D if $S = \langle D \rangle$.

For the rest of the paper, we use the term 'digraph' to mean 'simple digraph' (i.e. a digraph with no loops or multiple edges). A digraph D is *undirected* if its edge set is a symmetric relation on $V(D)$, and it is *transitive* if its edge set is a transitive relation on $V(D)$. We shall always assume that D is *connected* (i.e. for every pair $u, v \in V(D)$ there is either a path from u to v, or a path from v to u) because otherwise $\langle D \rangle \cong \langle D_1 \rangle \times \cdots \times \langle D_k \rangle$, where $D_1, \ldots, D_k$ are the connected components of D. We say that D is *strong* (or *strongly connected*) if for every pair $u, v \in V(D)$, there is a directed path from u to v. We say that D is a *tournament* if for every pair $u, v \in V(D)$ we have $(u, v) \in E(D)$ or $(v, u) \in E(D)$, but not both.

Many famous examples of semigroups are arc-generated. Clearly, by the discussion of the first paragraph, Sing_n is arc-generated by the complete undirected graph K_n. In fact, for $n \geq 3$, Sing_n is arc-generated by D if and only if D contains a strong tournament (see [3]). The semigroup of order-preserving transformations $\mathrm{O}_n := \{\alpha \in \mathrm{Sing}_n : u \leq v \Rightarrow u\alpha \leq v\alpha\}$ is arc-generated by an undirected path P_n on $[n]$, while the Catalan semigroup $\mathrm{C}_n := \{\alpha \in \mathrm{Sing}_n : v \leq v\alpha, u \leq v \Rightarrow u\alpha \leq v\alpha\}$ is arc-generated by a directed path $\vec{P}_n$ on $[n]$ (see [9, Corollary 4.11]). The semigroup of non-decreasing transformations $\mathrm{OI}_n := \{\alpha \in \mathrm{Sing}_n : v \leq v\alpha\}$ is arc-generated by the transitive tournament $\vec{T}_n$ on $[n]$ (Fig. 1 illustrates $\vec{T}_5$).

Connections between subsemigroups of Sing_n and digraphs have been studied before (see [9–12]). The following definition, which we shall adopt in the following sections, appeared in [12]:

Definition 1 For a digraph D, the *closure* $\bar{D}$ of D is the digraph with vertex set $V(\bar{D}) := V(D)$ and edge set $E(\bar{D}) := E(D) \cup \{(a, b) : (b, a) \in E(D)$ is in a directed cycle of $D\}$.

Say that D is *closed* if $D = \bar{D}$. Observe that $\langle D \rangle = \langle \bar{D} \rangle$ for any digraph D.

Recall that the *orbits* of $\alpha \in \mathrm{Sing}_n$ are the connected components of the digraph on $[n]$ with edges $\{(x, x\alpha) : x \in [n]\}$. In particular, an orbit Ω of α is called *cyclic* if it is a cycle with at least two vertices. An element $x \in [n]$ is a *fixed point* of α if $x\alpha = x$. Denote by $\mathrm{cycl}(\alpha)$ and $\mathrm{fix}(\alpha)$ the number of cyclic orbits and fixed points of α, respectively. Denote by $\ker(\alpha)$ the partition of $[n]$ induced by the *kernel* of α (i.e. the equivalence relation $\{(x, y) \in [n]^2 : x\alpha = y\alpha\}$).

We introduce some further notation. For any digraph D and $v \in V(D)$, define the *in-neighbourhood* and the *out-neighbourhood* of v by

$$N^-(v) := \{u \in V(D) : (u, v) \in E(D)\} \text{ and } N^+(v) := \{u \in V(D) : (v, u) \in E(D)\},$$

respectively. We extend these definitions to any subset $C \subseteq V(D)$ by letting $N^\epsilon(C) := \bigcup_{c\in C} N^\epsilon(c)$, where $\epsilon \in \{+, -\}$. The *in-degree* and *out-degree* of v are $\deg^-(v) := |N^-(v)|$ and $\deg^+(v) := |N^+(v)|$, respectively, while the *degree* of v is $\deg(v) := |N^-(v) \cup N^+(v)|$. For any two vertices $u, v \in V(D)$, the *D-distance* from u to v, denoted by $d_D(u, v)$, is the length of a shortest path from u to v in D, provided that such a path exists. The *diameter* of D is $\mathrm{diam}(D) := \max\{d_D(u, v) : u, v \in V(D),\ d_D(u, v) \text{ is defined}\}$.

Let D be any digraph on $[n]$. We are interested in the lengths of transformations of $\langle D \rangle$ viewed as words in the free monoid $D^* := \{(a \to b) : (a, b) \in E(D)\}^*$. Say that a word $\omega \in D^*$ *expresses* (or *evaluates to*) $\alpha \in \langle D \rangle$ if $\alpha = \omega\phi$, where $\phi : D^* \to \langle D \rangle$ is the evaluation semigroup morphism. For any $\alpha \in \langle D \rangle$, let $\ell(D, \alpha)$ be the minimum length of a word in D^* expressing α. For $r \in [n - 1]$, denote

$$\ell(D, r) := \max \{\ell(D, \alpha) : \alpha \in \langle D \rangle, \mathrm{rk}(\alpha) = r\},$$
$$\ell(D) := \max \{\ell(D, \alpha) : \alpha \in \langle D \rangle\}.$$

The main result in the literature in the study of $\ell(D, \alpha)$ was obtained by Howie and Iwahori, independently, when $D = K_n$.

Theorem 1.1 [4,5] *For any* $\alpha \in \mathrm{Sing}_n$,

$$\ell(K_n, \alpha) = n + \mathrm{cycl}(\alpha) - \mathrm{fix}(\alpha).$$

Therefore, $\ell(K_n, r) = n + \left\lfloor \frac{1}{2}(r - 2) \right\rfloor$, *for any* $r \in [n - 1]$, *and* $\ell(K_n) = \ell(K_n, n - 1) = \left\lfloor \frac{3}{2}(n - 1) \right\rfloor$.

In the following sections, we study $\ell(D, \alpha)$, $\ell(D, r)$, and $\ell(D)$, for various classes of digraphs. In Sect. 2, we characterise all digraphs D on $[n]$ such that either $\ell(D, \alpha) = n + \mathrm{cycl}(\alpha) - \mathrm{fix}(\alpha)$ for all $\alpha \in \langle D \rangle$, or $\ell(D, \alpha) = n - \mathrm{fix}(\alpha)$ for all $\alpha \in \langle D \rangle$, or $\ell(D, \alpha) = n - \mathrm{rk}(\alpha)$ for all $\alpha \in \langle D \rangle$. In Sect. 3, we are interested in the maximal possible length of a transformation in $\langle D \rangle$ of rank r among all digraphs D on $[n]$ of certain class $\mathcal{C}$; we denote this number by $\ell^{\mathcal{C}}_{\max}(n,r)$. In particular, when $\mathcal{C}$ is the class of acyclic digraphs, we find an explicit formula for $\ell^{\mathcal{C}}_{\max}(n,r)$. When $\mathcal{C}$ is the class of strong tournaments, we find upper and lower bounds for $\ell^{\mathcal{C}}_{\max}(n,r)$ (and for the analogously defined $\ell^{\mathcal{C}}_{\min}(n,r)$). Finally, in Sect. 4 we provide a list of conjectures and open problems.

2 Arc-generated semigroups with short words

Let D be a digraph on $[n]$, $n \geq 3$, and $\alpha \in \langle D \rangle$. Theorem 1.1 implies the following three bounds:

$$\ell(D,\alpha) \geq n + \mathrm{cycl}(\alpha) - \mathrm{fix}(\alpha) \geq n - \mathrm{fix}(\alpha) \geq n - \mathrm{rk}(\alpha). \tag{1}$$

The lowest bound is always achieved for constant transformations (i.e. transformations of rank 1).

Lemma 2.1 *For any digraph D on $[n]$, if $\alpha \in \langle D \rangle$ has rank 1, then $\ell(D,\alpha) = n-1$.*

Proof It is clear that $\ell(D,\alpha) \geq n-1$ because α has $n-1$ non-fixed points. Let $\mathrm{Im}(\alpha) = \{v_0\} \subseteq [n]$. Note that, for any $v \in [n]$, there is a directed path in D from v to v_0 (as otherwise, $\alpha \notin \langle D \rangle$). For any $d \geq 1$, let

$$C_d := \{v \in [n] : d_D(v, v_0) = d\}.$$

Clearly, $[n]\backslash\{v_0\} = \bigcup_{d=1}^{m} C_d$, where $m := \max_{v\in[n]}\{d_D(v,v_0)\}$ and the union is disjoint. For any $v \in C_d$, let v' be a vertex in C_{d-1} such that $(v \to v') \in D$. For any distinct $v, u \in C_d$ and any choice of $v', u' \in C_{d-1}$, the arcs $(v \to v')$ and $(u \to u')$ commute; hence, we can decompose α as

$$\alpha = \bigcirc_{d=m}^{1} \bigcirc_{v\in C_d} (v \to v'),$$

where the composition of arcs is done from m down to 1. □

Remark 1 Using a similar argument as in the previous proof, we may show that $\langle D \rangle$ contains all constant transformations if and only if D is strongly connected.

Inspired by the bounds given in (1), we characterise all the connected digraphs D on $[n]$ satisfying the following conditions:

$$\forall \alpha \in \langle D \rangle,\ \ \ell(D,\alpha) = n + \mathrm{cycl}(\alpha) - \mathrm{fix}(\alpha); \tag{C1}$$

$$\forall \alpha \in \langle D \rangle,\ \ \ell(D,\alpha) = n - \mathrm{fix}(\alpha); \tag{C2}$$

$$\forall \alpha \in \langle D \rangle,\ \ \ell(D,\alpha) = n - \mathrm{rk}(\alpha). \tag{C3}$$

2.1 Digraphs satisfying condition (C1)

Theorem 1.1 says that K_n satisfies (**C1**). In order to characterise all digraphs satisfying (**C1**), we introduce the following property on a digraph D:

($\star$) If $d_D(v_0, v_2) = 2$ and v_0, v_1, v_2 is a directed path in D, then $N^+(\{v_1, v_2\}) \subseteq \{v_0, v_1, v_2\}$.

We shall study the strong components of digraphs satisfying property ($\star$). We state few observations that we use repeatedly in this section.

Remark 2 Suppose that D satisfies property ($\star$). If v_0, v_1, v_2 is a directed path in D and $\deg^+(v_1) > 2$, or $\deg^+(v_2) > 2$, then $(v_0, v_2) \in E(D)$. Indeed, if $(v_0, v_2) \notin E(D)$, then $d_D(v_0, v_2) = 2$, so, by property ($\star$), $N^+(\{v_1, v_2\}) \subseteq \{v_0, v_1, v_2\}$; this contradicts that $\deg^+(v_1) > 2$, or $\deg^+(v_2) > 2$.

Remark 3 Suppose that D satisfies property ($\star$). If v_0, v_1, v_2 is a directed path in D and either v_1 or v_2 has an out-neighbour not in $\{v_0, v_1, v_2\}$, then $(v_0, v_1) \in E(D)$.

Remark 4 If D satisfies property ($\star$), then $\operatorname{diam}(D) \leq 2$. Indeed, if $v_0, v_1, \ldots, v_k$ is a directed path in D with $d_D(v_0, v_k) = k \geq 3$, then v_0, v_1, v_2 is a directed path in D and v_2 has an out-neighbour $v_3 \notin \{v_0, v_1, v_2\}$; by Remark 3, $(v_0, v_2) \in E(D)$, which contradicts that $d_D(v_0, v_k) = k$.

Note that digraphs satisfying property ($\star$) are a slight generalisation of transitive digraphs.

Let D be a digraph and let C_1 and C_2 of be two strong components of D. We say that C_1 *connects* to C_2 if $(v_1, v_2) \in E(D)$ for some $v_1 \in C_1$, $v_2 \in C_2$; similarly, we say that C_1 *fully connects* to C_2 if $(v_1, v_2) \in E(D)$ for all $v_1 \in C_1$, $v_2 \in C_2$. The strong component C_1 is called *terminal* if there is no strong component $C \neq C_1$ of D such that C_1 connects to C.

Lemma 2.2 *Let D be a closed digraph satisfying property ($\star$). Then, any strong component of D is either an undirected path P_3 or complete. Furthermore, P_3 may only appear as a terminal strong component of D.*

Proof Let C be a strong component of D. Since D is closed, C must be undirected. The lemma is clear if $|C| \leq 3$, so assume that $|C| \geq 4$. We have two cases:

Case 1 Every vertex in C has degree at most 2. Then C is a path or a cycle. Since $|C| \geq 4$ and $\operatorname{diam}(D) \leq 2$, then C is a cycle of length 4 or 5; however, these cycles do not satisfy property ($\star$).

Case 2 There exists a vertex $a \in C$ of degree 3 or more. Any two neighbours of a are adjacent: indeed, for any $u, v \in N(a)$, u, a, v is a path and $\deg^+(a) > 2$, so $(u, v) \in E(D)$ by Remark 2. Hence, the neighbourhood of a is complete and every neighbour of a has degree 3 or more. Applying this rule recursively, we obtain that every vertex in C has degree 3 or more, and the neighbourhood of every vertex is complete. Therefore, C is complete because $\operatorname{diam}(D) \leq 2$.

Finally, if P_3 is a strong component of D, there cannot be any edge coming out of it because of property ($\star$), so it must be a terminal component. $\square$

Lemma 2.3 *Let D be a closed digraph satisfying property $(\star)$. Let C_1 and C_2 be strong components of D, and suppose that C_1 connects to C_2.*

(i) *If C_2 is non-terminal, then C_1 fully connects to C_2.*
(ii) *Let $|C_2| = 1$. If either $|C_1| \neq 2$, or the vertex in C_1 that connects to C_2 has out-degree at least 3, then C_1 fully connects to C_2.*
(iii) *Let $|C_2| = 2$. If not all vertices in C_1 connect to the same vertex in C_2, then C_1 fully connects to C_2.*
(iv) *If $|C_2| \geq 3$, then C_1 fully connects to C_2.*

Proof Recall that C_1 and C_2 are undirected because D is closed. If $|C_1| = 1$ and $|C_2| = 1$, clearly C_1 fully connects to C_2. Henceforth, we assume $|C_1| \geq 2$ or $|C_2| \geq 2$. Let $c_1 \in C_1$ and $c_2 \in C_2$ be such that $(c_1, c_2) \in E(D)$. As C_1 is a non-terminal, Lemma 2.2 implies that C_1 is complete.

(i) As C_2 is non-terminal, there exists $d \in D\backslash(C_1 \cup C_2)$ such that $(c_2, d) \in E(D)$. Suppose that $|C_1| \geq 2$. Then, for any $c_1' \in C_1\backslash\{c_1\}$, c_1', c_1, c_2 is a directed path in D with $d \in N^+(c_2)$, so Remark 3 implies $(c_1', c_2) \in E(D)$. Suppose now that $|C_2| \geq 2$. Then, for any $c_2' \in C_2\backslash\{c_2\}$, c_1, c_2, c_2' is a directed path in D with $d \in N^+(c_2)$, so again $(c_1, c_2') \in E(D)$. Therefore, C_1 fully connects to C_2.
(ii) Suppose that $|C_1| \geq 2$. If $|C_1| > 2$, then $\deg^+(c_1) > 2$, because C_1 is complete. Thus, for each $c_1' \in C_1\backslash\{c_1\}$, c_1', c_1, c_2 is a directed path in D with $\deg^+(c_1) > 2$, so $(c_1', c_2) \in E(D)$ by Remark 2. As $|C_2| = 1$, this shows that C_1 fully connects to C_2.
(iii) Let $C_2 = \{c_2, c_2'\}$ and let $c_1' \in C_1\backslash\{c_1\}$ be such that $(c_1', c_2') \in E(D)$. For any $b, d \in C_1$, $b \neq c_1$, $d \neq c_1'$, both b, c_1, c_2 and d, c_1', c_2' are directed paths in D with $c_2' \in N^+(c_2)$ and $c_2 \in N^+(c_2')$; hence, $(b, c_2), (d, c_2') \in E(D)$ by Remark 3.
(iv) Suppose that $C_2 = P_3$. Say $C_2 = \{c_2, c_2', c_2''\}$ with either $d_D(c_2, c_2'') = 2$ or $d_D(c_2', c_2'') = 2$. In any case, c_1, c_2, c_2' is a directed path in D with $c_2'' \in N^+(\{c_2, c_2'\})$, so $(c_1, c_2') \in E(D)$ by Remark 3; now, c_1, c_2', c_2'' is a directed path in D with $c_2 \in N^+(\{c_2', c_2''\})$, so $(c_1, c_2'') \in E(D)$. Hence, c_1 is connected to all vertices of C_2. As C_1 is complete, a similar argument shows that every $c_1' \in C_1\backslash\{c_1\}$ connects to every vertex in C_2.
Suppose now that $C_2 = K_m$ for $m \geq 3$. By a similar reasoning as the previous paragraph, we show that $(c_1, v) \in E(D)$ for all $v \in C_2$. Now, for any $c_1' \in C_1\backslash\{c_1\}$, $v \in C_2$, c_1', c_1, v is a directed path in D so $(c_1', v) \in E(D)$ by Remark 3. □

Lemma 2.4 *Let D be a closed digraph satisfying property $(\star)$. Let C_i, $i = 1, 2, 3$, be strong components of D, and suppose that C_1 connects to C_2 and C_2 connects to C_3. If C_1 does not connect to C_3, then $|C_2| = |C_3| = 1$, C_3 is terminal in D, and C_2 is terminal in $D\backslash C_3$.*

Proof By Lemma 2.3 (i), C_1 fully connects to C_2. Assume that C_1 does not connect to C_3. Let $c_i \in C_i$, $i = 1, 2, 3$, be such that $(c_1, c_2), (c_2, c_3) \in E(D)$. If C_2 has a vertex different from c_2, Remark 3 ensures that $(c_1, c_3) \in E(D)$, which contradicts our hypothesis. Then $|C_2| = 1$. The same argument applies if C_3 has a vertex different from c_3, so $|C_3| = 1$. Finally, Remark 3 applied to the path c_1, c_2, c_3 also implies that C_3 is terminal in D and C_2 is terminal in $D \backslash C_3$. □

The following result characterises all digraphs satisfying condition (**C1**).

Theorem 2.5 *Let D be a connected digraph on $[n]$. The following are equivalent:*

(i) *For all $\alpha \in \langle D \rangle$, $\ell(D, \alpha) = n + \mathrm{cycl}(\alpha) - \mathrm{fix}(\alpha)$.*
(ii) *D is closed satisfying property* $(\star)$.

Proof In order to simplify notation, denote

$$g(\alpha) := n + \mathrm{cycl}(\alpha) - \mathrm{fix}(\alpha).$$

First, we show that (i) implies (ii). Suppose $\ell(D, \alpha) = g(\alpha)$ for all $\alpha \in \langle D \rangle$. We use the one-line notation for transformations: $\alpha = (1)\alpha\ (2)\alpha\ \dots\ (k)\alpha$, where $x = (x)\alpha$ for all $x > k$, $x \in [n]$. Clearly, if D is not closed, there exists an arc $a \in \langle D \rangle \backslash D$, so $1 < \ell(D, a) \neq g(a) = 1$. In order to prove that property $(\star)$ holds, let $1, 2, 3$ be a shortest path in D. If $(2 \to v) \in \langle D \rangle$, for some $v \in [n] \backslash \{1, 2, 3\}$, then $\alpha = 3v3v \in \langle D \rangle$, but $g(\alpha) = 2 \neq \ell(D, \alpha) = 3$. If $(3 \to v) \in \langle D \rangle$, then $\alpha = 3vvv \in \langle D \rangle$, but $g(\alpha) = 3 \neq \ell(D, \alpha) = 4$. Therefore, $N^+(\{2, 3\}) \subseteq \{1, 2, 3\}$, and $(\star)$ holds.

Conversely, we show that (ii) implies (i). Let $\alpha \in \langle D \rangle$. We remark that any cycle of α belongs to a strong component of D.

Claim 2.6 *Let C be a strong component of D. Then either α fixes all vertices of C or $|(C\alpha) \cap C| < |C|$.*

Proof Suppose that $\alpha|_C$, the restriction of α to C, is non-trivial and $|(C\alpha) \cap C| = |C|$. Then $\alpha|_C$ is a permutation of C. Let $u \in C$ and suppose that $(u \to v)$ is the first arc moving u in a word expressing α in D^*. If $v \in C$, we have $u\alpha = v\alpha$, which contradicts that $\alpha|_C$ is a permutation. If $v \in C'$ for some other strong component C' of D, then $u\alpha \notin C$ which again contradicts our assumption. □

Claim 2.7 *Let $u, v \in [n]$ be such that $u\alpha = v$. If $d_D(u, v) = 2$, then:*

1. *v is in a terminal component of D.*
2. *There is a path u, w, v of length 2 in D such that $w\alpha = v\alpha = v$; for any other path u, x, v of length 2 in D, we have $x\alpha \in \{x, v\}$.*

Proof Let C_1 and C_2 be strong components of D such that $u \in C_1$ and $v \in C_2$. We analyse the four possible cases in which $d_D(u, v) = 2$. In the first three cases, we use the fact that $\langle P_3 \rangle \cong \mathrm{O}_3$, hence we can order $u < w < v$ and α is an increasing transformation of the ordered set $\{u, w, v\}$; thus $u\alpha = w\alpha = v\alpha = v$.

Case 1 $C_1 = C_2$. By Lemma 2.2, $C_1 \cong P_3$ and it is a terminal component. Therefore, 2. holds as there is a unique path from u to v.

Case 2 C_1 connects to C_2 and $|C_2| \neq 2$. As $d_D(u, v) = 2$, C_1 does not fully connect C_2, so, by Lemma 2.3, $|C_2| = 1$, C_2 is terminal, $|C_1| = 2$, and the vertex $w \in C_1$ connecting to $C_2 = \{v\}$ has out-degree 2. Then, by property $(\star)$, u, w, v is the unique path from u to v.

Case 3 C_1 connects to C_2 and $|C_2| = 2$. As $d_D(u, v) = 2$, C_1 does not fully connect C_2, so, by Lemma 2.3, C_2 is terminal and u, w, v is the unique path of length two from u to v, where w is the other vertex of C_2.

Case 4 C_1 does not connect to C_2. Since $d_D(u, v) = 2$, there exist strong components $C^{(1)}, \dots, C^{(k)}$ such that C_1 connects to $C^{(i)}$ and $C^{(i)}$ connects to C_2, for all $1 \leq i \leq k$. By Lemma 2.4, $C^{(i)} = \{x_i\}$, $C_2 = \{v\}$ is terminal and $N^+(x_i) = \{v\}$ for all i. Thus u, x_i, v are the only paths of length two from u to v; in particular, $x_i\alpha \in \{x_i, v\}$ for all x_i. As $u\alpha = v$, there must exist $1 \leq j \leq k$ such that $w := x_j$ is mapped to v. □

Now we produce a word $\omega \in D^*$ expressing α of length $g(\alpha)$. Define

$$U := \{u \in D : d_D(u, u\alpha) = 2\}.$$

For every $u \in U$, let u' be a vertex in D such that $u, u', u\alpha$ is a path and $u'\alpha = u\alpha$. The existence of u' is guaranteed by Claim 2.7. Define a word $\omega_0 \in D^*$ by

$$\omega_0 := \bigcirc_{u \in U} (u \to u')(u' \to u\alpha).$$

Sort the strong components of D in topological order: $C_1, \dots, C_k$, i.e. for $i \neq j$, C_i connects to C_j only if $j > i$. For each $1 \leq i \leq k$, define

$$S_i := \{v \in C_i \backslash (U \cup U') : v\alpha \in C_i\},$$

where $U' := \{u' : u \in U\}$, and consider the transformation $\beta_i : C_i \to C_i$ defined by

$$x\beta_i = \begin{cases} x\alpha & \text{if } x \in S_i \\ x & \text{otherwise.} \end{cases}$$

If $|C_i| \leq 2$ or $C_i \cong P_3$, then $\mathrm{cycl}(\beta_i) = 0$ and β_i can be computed with $|C_i| - \mathrm{fix}(\beta_i)$ arcs. Otherwise, C_i is a complete undirected graph. If $\beta_i \in \mathrm{Sing}(C_i)$, then by Theorem 1.1, there is a word $\omega_i \in C_i^* \subseteq D^*$ of length $|C_i| + \mathrm{cycl}(\beta_i) - \mathrm{fix}(\beta_i)$ expressing β_i. Suppose now that β_i is a non-identity permutation of C_i. By Claim 2.6, α does not permute C_i and there exists $h_i \in C_i \backslash (C_i\alpha)$. Note that $h_i \in C_i \backslash S_i$. Define $\hat{\beta}_i \in \mathrm{Sing}(C_i)$ by

$$x\hat{\beta}_i = \begin{cases} x\alpha & \text{if } x \in S_i \\ a_i & \text{if } x = h_i \\ x & \text{otherwise,} \end{cases}$$

where a_i is any vertex in S_i. Then $\alpha|_{S_i} = \hat{\beta}|_{S_i}$. Again by Theorem 1.1, there is a word $\omega_i \in C_i^* \subseteq D^*$ of length $|C_i| + \mathrm{cycl}(\hat{\beta}_i) - \mathrm{fix}(\hat{\beta}) = |C_i| + \mathrm{cycl}(\beta_i) - \mathrm{fix}(\beta_i)$ expressing $\hat{\beta}_i$.

The following word maps all the vertices in $[n]\backslash(U \cup U' \cup C_i)$ that have image in C_i:

$$\omega_i' = \bigcirc \left\{(a \to a\alpha) : a \in [n]\backslash(U \cup U' \cup C_i), a\alpha \in C_i\right\}.$$

Finally, let

$$\omega := \omega_0 \omega_k \omega_k' \ldots \omega_1 \omega_1' \in D^*.$$

It is easy to check that ω indeed expresses α. Since $\sum_{i=1}^{k} \operatorname{fix}(\beta_i) = \operatorname{fix}(\alpha) + \sum_{i=1}^{k} |C_i \backslash S_i|$ and $\sum_{i=1}^{k} \ell(\omega_i') = \sum_{i=1}^{k} |C_i \backslash (U \cup U' \cup S_i)|$, we have

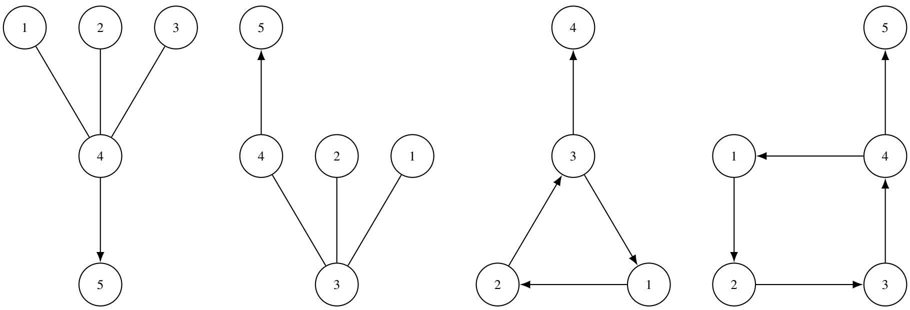
$$\ell(\omega) = 2|U| + \sum_{i=1}^{k} (\ell(\omega_i) + \ell(\omega_i')) = n + \sum_{i=1}^{k} \operatorname{cycl}(\beta_i) - \operatorname{fix}(\alpha) = g(\alpha).$$

□

2.2 Digraphs satisfying condition (C2)

The characterisation of connected digraphs satisfying condition (**C2**) is based on the classification of connected digraphs D such that $\operatorname{cycl}(\alpha) = 0$, for all $\alpha \in \langle D \rangle$.

For $k \geq 3$, let Θ_k be the directed cycle of length k. Consider the digraphs Γ_1, Γ_2, Γ_3 and Γ_4 as illustrated below:



Γ_1 Γ_2 Γ_3 Γ_4

Lemma 2.8 *Let D be a connected digraph on $[n]$. The following are equivalent:*

(i) *For all $\alpha \in \langle D \rangle$,* $\operatorname{cycl}(\alpha) = 0$.
(ii) *D has no subdigraph isomorphic to Γ_1, Γ_2, Γ_3, Γ_4, or Θ_k, for all $k \geq 5$.*

Proof In order to prove that (i) implies (ii), we show that if Γ is equal to Γ_i or Θ_k, for $i \in [4]$, $k \geq 5$, then there exists $\alpha \in \langle \Gamma \rangle$ such that $\operatorname{cycl}(\alpha) \neq 0$.

- If $\Gamma = \Gamma_1$, take

$$\alpha := (3 \to 4)(4 \to 5)(1 \to 4)(4 \to 3)(2 \to 4)(4 \to 1)(3 \to 4)(4 \to 2)\\ = 21555.$$

- If $\Gamma = \Gamma_2$, take

$$\alpha := (3 \to 4)(4 \to 5)(1 \to 3)(3 \to 4)(2 \to 3)(3 \to 1)(4 \to 3)(3 \to 2)\\ = 21555.$$

- If $\Gamma = \Gamma_3$, take

$$\alpha := (3 \to 4)(2 \to 3)(1 \to 2)(3 \to 1) = 2144.$$

- If $\Gamma = \Gamma_4$, take

$$\alpha = (3 \to 4)(4 \to 5)(2 \to 3)(3 \to 4)(1 \to 2)(4 \to 1) = 21555.$$

- Assume $\Gamma = \Theta_k$ for $k \geq 5$. Consider the following transformation of $[k]$:

$$(u \Rightarrow v) := (u \to u_1) \dots (u_{d-1} \to v),$$

where $u, u_1, \dots, u_{d-1}, v$ is the unique path from u to v on the cycle Θ_k. Take

$$\alpha := (1 \Rightarrow k-3)(k \Rightarrow k-4)(k-1 \Rightarrow 1)(k-2 \Rightarrow k)\\ (k-3 \Rightarrow k-1)(k-4 \Rightarrow k-2).$$

Then, $\alpha = (k-1)(k-1)\dots(k-1)\ k\ 1\ (k-2)$, where $(k-1)$ appears $k-3$ times, has the cyclic component $(k-2, k)$.

Conversely, assume that D satisfies (ii). If $n \leq 3$, it is clear that $\mathrm{cycl}(\alpha) = 0$, for all $\alpha \in \langle D \rangle$, so suppose $n \geq 4$. We first obtain some key properties about the strong components of $\bar{D}$.

Claim 2.9 *Any strong component of $\bar{D}$ is an undirected path, an undirected cycle of length 3 or 4, or a claw $K_{3,1}$ (i.e. a bipartite undirected graph on $[4] = [3] \cup \{4\}$). Moreover, if a strong component of D is not an undirected path, then it is terminal.*

Proof Let C be a strong component of $\bar{D}$. Clearly, C is undirected and, by (ii), it cannot contain a cycle of length at least 5. If C has a cycle of length 3 or 4, then the whole of C must be that cycle and C is terminal (otherwise, it would contain Γ_3 or Γ_4, respectively). If C has no cycle of length 3 and 4, then C is a tree. It can only be a path or $K_{3,1}$, for otherwise it would contain Γ_1 or Γ_2; clearly, $K_{3,1}$ may only appear as a terminal component. ◻

Suppose there is $\alpha \in \langle D \rangle$ that has a cyclic orbit (so $\mathrm{cycl}(\alpha) \neq 0$). This cyclic orbit must be contained in a strong component C of $\bar{D}$, and Claim 2.9 implies that $C \cong \Gamma$, where $\Gamma \in \{K_{3,1}, \bar{\Theta}_s, P_r : s \in \{3,4\}, r \in \mathbb{N}\}$. If $\Gamma = K_{3,1}$ or $\Gamma = \bar{\Theta}_s$, then C is a terminal component, so α acts on C as some transformation $\beta \in \langle \Gamma \rangle$; however, it is easy to check that no transformation in $\langle \Gamma \rangle$ has a cyclic orbit. If $\Gamma = P_r$, for some r, then α acts on C as a partial transformation β of P_r. Since $\langle P_r \rangle = \mathrm{O}_r$, β has no cyclic orbit. □

We introduce a new property of a connected digraph D:

($\star\star$) For every strong component C of D, $|C| \leq 2$ if C is non-terminal, and $|C| \leq 3$ if C is terminal.

Lemma 2.10 *Let D be a closed connected digraph on $[n]$ satisfying property ($\star$). The following are equivalent:*

(i) *D satisfies property ($\star\star$).*
(ii) *D has no subdigraph isomorphic to Γ_1, Γ_2, Γ_3, Γ_4, or Θ_k, for some $k \geq 5$.*

Proof If (i) holds, it is easy to check that D does not contain any subdigraphs isomorphic to Γ_1, Γ_2, Γ_3, Γ_4, or Θ_k for some $k \geq 5$.

Conversely, suppose that (ii) holds. Let C be a strong component of D. If C is non-terminal, Lemma 2.2 implies that C is complete; hence, $|C| \leq 2$ as otherwise D would contain Γ_4 as a subdigraph. If C is terminal, Lemma 2.2 implies that C is complete or P_3; hence, $|C| \leq 3$ as otherwise D would contain Γ_3 as a subdigraph. □

Theorem 2.11 *Let D be a connected digraph on $[n]$. The following are equivalent:*

(i) *For all $\alpha \in \langle D \rangle$, $\ell(D, \alpha) = n - \mathrm{fix}(\alpha)$.*
(ii) *D is closed satisfying properties ($\star$) and ($\star\star$).*

Proof Clearly, D satisfies (i) if and only if it satisfies condition (**C1**) and $\mathrm{cycl}(\alpha) = 0$, for all $\alpha \in \langle D \rangle$. By Theorem 2.5 and Lemmas 2.8 and 2.10, D satisfies (i) if and only if D satisfies (ii). □

2.3 Digraphs satisfying condition (C3)

The following result characterises digraphs satisfying condition (**C3**).

Theorem 2.12 *Let D be a connected digraph on $[n]$. The following are equivalent:*

(i) *For every $\alpha \in \langle D \rangle$, $\ell(D, \alpha) = n - \mathrm{rk}(\alpha)$.*
(ii) *$\langle D \rangle$ is a band, i.e. every $\alpha \in \langle D \rangle$ is idempotent.*
(iii) *Either $n = 2$ and $D \cong K_2$, or there exists a bipartition $V_1 \cup V_2$ of $[n]$ such that $(i_1, i_2) \in E(D)$ only if $i_1 \in V_1$, $i_2 \in V_2$.*

Proof Clearly (i) implies (ii): if $\ell(D, \alpha) = n - \mathrm{rk}(\alpha)$, then $\mathrm{rk}(\alpha) = \mathrm{fix}(\alpha)$ by inequality (1), so α is idempotent.

Now we prove that (ii) implies (iii). If there exist $u, v, w \in [n]$ pairwise distinct such that $(u, v), (v, w) \in E(D)$, then $\alpha = (v \to w)(u \to v)$ is not an idempotent.

Therefore, for $n \geq 3$, if every $\alpha \in \langle D \rangle$ is idempotent, then a vertex in D either has in-degree zero or out-degree zero: this corresponds to the bipartition of $[n]$ into V_1 and V_2.

We finally prove that (iii) implies (i). Let $n \geq 3$ and suppose that there exists a bipartition $V_1 \cup V_2$ of $[n]$ such that $(i_1, i_2) \in E(D)$ only if $i_1 \in V_1$, $i_2 \in V_2$. Then for any $\alpha \in \langle D \rangle$, all elements of V_2 are fixed by α and $i_1\alpha \in \{i_1\} \cup N^+(i_1)$ for any $i_1 \in V_1$. In particular, any non-fixed point of α is mapped to a fixed point, so $r := \mathrm{rk}(\alpha) = \mathrm{fix}(\alpha)$. Let $J := \{v_1, \dots, v_{n-r}\} \subseteq V_1$ be the set of non-fixed points of α; therefore

$$\alpha = (v_1 \to v_1\alpha) \dots (v_{n-r} \to v_{n-r}\alpha),$$

where each one of the $n - r$ arcs above belongs to $\langle D \rangle$. The result follows by inequality (1). □

3 Arc-generated semigroups with long words

Fix $n \geq 2$. In this section, we consider digraphs D that maximise $\ell(D, r)$ and $\ell(D)$. For $r \in [n-1]$, define

$$\ell_{\max}(n, r) := \max \{\ell(D, r) : V(D) = [n]\},$$
$$\ell_{\max}(n) := \max \{\ell(D) : V(D) = [n]\}.$$

The first few values of $\ell_{\max}(n, r)$, calculated with the GAP package *Semigroups* [7], are given in Table 1. By Lemma 2.1, $\ell_{\max}(n, 1) = n - 1$ for all $n \geq 2$; henceforth, we shall always assume that $n \geq 3$ and $r \in [n-1]\setminus\{1\}$.

In the following sections, we restrict the class of digraphs that we consider in the definition of $\ell_{\max}(n, r)$ and $\ell_{\max}(n)$ to two important cases: acyclic digraphs and strong tournaments.

3.1 Acyclic digraphs

For any $n \geq 3$, let $\mathrm{Acyclic}_n$ be the set of all acyclic digraphs on $[n]$, and, for any $r \in [n-1]$, define

Table 1 First values of $\ell_{\max}(n, r)$

n	r 1	2	3	4	5
2	1				
3	2	6			
4	3	11	13		
5	4	18	24	33	
6	5	26	42	51	66

$$\ell_{\max}^{\text{Acyclic}}(n, r) := \max\left\{\ell(A, r) : A \in \text{Acyclic}_n\right\},$$
$$\ell_{\max}^{\text{Acyclic}}(n) := \max\left\{\ell(A) : A \in \text{Acyclic}_n\right\}.$$

Without loss of generality, we assume that any acyclic digraph A on $[n]$ is topologically sorted, i.e. $(u, v) \in E(A)$ only if $v > u$.

In this section, we establish the following theorem.

Theorem 3.1 *For any $n \geq 3$ and $r \in [n-1]\backslash\{1\}$,*

$$\ell_{\max}^{\text{Acyclic}}(n, r) = \frac{(n-r)(n+r-3)}{2} + 1,$$
$$\ell_{\max}^{\text{Acyclic}}(n) = \ell_{\max}^{\text{Acyclic}}(n, 2) = \frac{1}{2}(n^2 - 3n + 4).$$

First of all, we settle the case $r = n - 1$, for which we have a finer result.

Lemma 3.2 *Let $n \geq 3$ and $A \in \text{Acyclic}_n$. Then, $\ell(A, n-1)$ is equal to the length of a longest path in A. Therefore,*

$$\ell_{\max}^{\text{Acyclic}}(n, n-1) = n - 1.$$

Proof Let $v_1, \ldots, v_{l+1}$ be a longest path in A. Then $\alpha \in \langle A \rangle$ defined by

$$v\alpha := \begin{cases} v_{i+1} & \text{if } v = v_i,\ i \in [l], \\ v & \text{otherwise}, \end{cases}$$

has rank $n - 1$ and requires at least l arcs, since it moves l vertices.

Conversely, let $\alpha \in A$ be a transformation of rank $n - 1$, and consider a word expressing α in A^*:

$$\alpha = (u_1 \to v_1)(u_2 \to v_2) \ldots (u_s \to v_s).$$

Since α has rank $n - 1$, we must have $v_2 = u_1$ and by induction $v_i = u_{i-1}$ for $2 \leq i \leq s$. As A is acyclic, $u_s, u_{s-1}, \ldots, u_1, v_1$ forms a path in A, so $s \leq l$. □

The following lemma shows that the formula of Theorem 3.1 is an upper bound for $\ell_{\max}^{\text{Acyclic}}(n, r)$.

Lemma 3.3 *For any $n \geq 3$ and $r \in [n-1]\backslash\{1\}$,*

$$\ell_{\max}^{\text{Acyclic}}(n, r) \leq \frac{(n-r)(n+r-3)}{2} + 1.$$

Proof Let A be an acyclic digraph on $[n]$, let $\alpha \in \langle A \rangle$ be a transformation of rank $r \geq 2$, and let $L \subset V(A)$ be the set of terminal vertices of A. For any $u, v \in [n]$, denote the length of a longest path from u to v in A as $\psi_A(u, v)$.

Claim 3.4 $\ell(A,\alpha) \le \sum_{v\in[n]} \psi_A(v, v\alpha)$.

Proof Let $\omega = (a_1 \to b_1)\dots(a_l \to b_l)$ be a shortest word expressing α in A^*, with $l = \ell(A,\alpha)$. Say that the arc $(a_i \to b_i)$, $i \ge 2$, *carries* $v \in [n]$ if $v(a_1 \to b_1)\dots(a_{i-1} \to b_{i-1}) = a_i$ (assume that $a_1 \to b_1$ only carries a_1). Every arc $(a_i \to b_i)$ carries at least one vertex, for otherwise we could remove that arc form the word ω and obtain a shorter word still expressing α. Let $v \in [n]$, and denote $v_0 = v$ and $v_i = v(a_1 \to b_1)\dots(a_i \to b_i)$ (and hence $v_l = v\alpha$). Let us remove the repetitions in this sequence: let $j_0 = 0$ and for $i \ge 1$, $j_i = \min\{j : v_j \ne v_{j_{i-1}}\}$. Then the sequence $v = v_{j_0}, v_{j_1}, \dots, v_{j_{l(v)}} = v\alpha$ forms a path in A of length $l(v)$, and hence $l(v) \le \psi(v, v\alpha)$. For each $v \in [n]$, there are $l(v)$ arcs in ω carrying v, so the length of ω satisfies

$$l \le \sum_{v=1}^{n} l(v) \le \sum_{v\in[n]} \psi_A(v, v\alpha).$$

$\square$

Claim 3.5 *If* $|L| \ge 2$, *then* $\sum_{v\in[n]} \psi_A(v, v\alpha) \le \frac{(n-r)(n+r-3)}{2}$.

Proof As $|L| \ge 2$, and A is topologically sorted, we have $\{n, n-1\} \subseteq L$, and any $\alpha \in \langle A\rangle$ fixes both $n-1$ and n, i.e. $\psi_A(v, v\alpha) = 0$ for $v \in \{n-1, n\}$. For any $v \in [n-2]$, we have

$$\psi_A(v, v\alpha) \le \min\{n-1, v\alpha\} - v.$$

Hence

$$\begin{aligned}\sum_{v\in[n]} \psi_A(v, v\alpha) &= \sum_{v\in[n-2]} \psi_A(v, v\alpha)\\ &\le \sum_{v\in[n-2]} (\min\{n-1, v\alpha\} - v)\\ &= \sum_{w\in[n-2]\alpha} \Big(\min\{n-1, w\}|w\alpha^{-1}|\Big) - T_{n-2},\end{aligned}$$

where $T_k = \frac{k(k+1)}{2}$. The summation is maximised when $|n\alpha^{-1}| = n-r$ and $|w\alpha^{-1}| = 1$ for $n-r+1 \le w \le n-2$, thus yielding

$$\begin{aligned}\sum_{v\in[n]} \psi_A(v, v\alpha) &\le (n-1)(n-r) + (T_{n-2} - T_{n-r}) - T_{n-2}\\ &= \frac{(n-r)(n+r-3)}{2}.\end{aligned}$$

$\square$

Claim 3.6 *If* $|L| = 1$, *then* $\ell(A,\alpha) \le \frac{(n-r)(n+r-3)}{2} + 1$.

Proof As A is topologically sorted, $L = \{n\}$. We use the notation from the proof of Claim 3.4. We then have $l(n) = 0$. We have three cases:

Case 1 $(n-1)$ is fixed by α. Then, $l(n-1) = 0$ and $l(v) \leq \min\{n-1, v\alpha\} - v$ for all $v \in [n-2]$. By the same reasoning as in Claim 3.5, we obtain $\ell(A, \alpha) \leq \frac{(n-r)(n+r-3)}{2}$.

Case 2 $(n-1)\alpha = n$ and $v\alpha \leq n-1$ for every $v \in [n-2]$. Then again $l(v) \leq \min\{n-1, v\alpha\} - v$, for all $v \in [n-2]$, and $\ell(A, \alpha) \leq \frac{(n-r)(n+r-3)}{2}$.

Case 3 n has at least two pre-images under α. Let $\omega = (a_1 \to b_1) \dots (a_l \to b_l)$ be a shortest word expressing α in A^*, and denote $\alpha_0 = \mathrm{id}$ and $\epsilon_i = (a_i \to b_i)$, $\alpha_i = \epsilon_1 \dots \epsilon_i$ for $i \in [l]$. We partition $n\alpha^{-1}$ into two parts S and T:

$$S = \{v \in n\alpha^{-1} : v_{l(v)-1} = n-1\}, \quad T = n\alpha^{-1} \backslash S.$$

For all $v \in S$, if the arc carrying v to $n-1$ is ϵ_j, then $(n-1)\alpha_{j-1}^{-1} \subseteq S$ (v can only collapse with other pre-images of α). Then the arc $(n-1 \to n)$ occurs only once in the word ω (if it occurs multiple times, then remove all but the last occurrence of that arc to obtain a shorter word expressing α). If we do not count that arc, we have $l'(v) \leq n-1-v$ arcs carrying v if $v \in S$, $l(v) \leq n-1-v$ arcs carrying v if $v \in T$, and $l(v) \leq v\alpha - v$ if $v\alpha \neq n$. Again, we obtain $\ell(A, \alpha) \leq \frac{(n-r)(n+r-3)}{2} + 1$. □

Lemma 3.3 follows by the previous claims. □

The following lemma completes the proof of Theorem 3.1.

Lemma 3.7 *For any $n \geq 3$ and $r \in [n-1]\backslash\{1\}$, there exists an acyclic digraph Q_n on $[n]$ and a transformation $\beta_r \in \langle Q_n \rangle$ of rank r such that*

$$\ell(Q_n, \beta_r) \geq \frac{(n-r)(n+r-3)}{2} + 1.$$

Proof Let Q_n be the acyclic digraph on $[n]$ with edge set

$$E(Q_n) := \{(u, u+1) : u \in [n-1]\} \cup \{(n-2, n)\}.$$

For any $r \in [n-1]\backslash\{1\}$, define $\beta_r \in \langle Q_n \rangle$ by

$$v\beta_r := \begin{cases} n-r+v & \text{if } v \in [r-2], \\ n-1 & \text{if } v \in [n-1]\backslash[r-2],\ n-v \equiv 0 \mod 2, \\ n & \text{if } v \in [n-1]\backslash[r-2],\ n-v \equiv 1 \mod 2, \\ n & \text{if } v = n. \end{cases}$$

Let β_r be expressed as a word in Q_n^* of minimum length as

$$\beta_r = (a_1 \to b_1) \dots (a_l \to b_l),$$

where $l = \ell(Q_n, \beta_r)$. Denote $\alpha_0 := \mathrm{id}$, $\epsilon_i := (a_i \to b_i)$, and $\alpha_i := \epsilon_1 \dots \epsilon_i$, for $i \in [l]$. Say that ϵ_i carries $u \in [n]$ if $u\alpha_{i-1} = a_i$ and hence $u\alpha_i \neq u\alpha_{i-1}$.

Claim 3.8 *For each $i \in [l]$, the arc ϵ_i carries exactly one vertex.*

Proof First, $(a_1, b_1) \in E(Q_n)$ and $a_1\beta_r = b_1\beta_r$ imply that $a_1 = n-1$ and $b_1 = n$. Suppose that there is an arc ϵ_j, $j \in [l]$, that carries two vertices $u < v$; take j to be minimal index with this property. We remark that $v \le n-2$ and $u\alpha_{j-1} = v\alpha_{j-1}$ imply $u\beta_r = v\beta_r$. Then $w := u+1$ satisfies $w\beta_r \ne u\beta_r$, so w is not carried by ϵ_j. If $w\alpha_{j-1} \le n-2$, then $u\alpha_{j-1} < w\alpha_{j-1} < v\alpha_{j-1}$ since $u < w < v$ and the graph induced by $[n-2]$ in Q_n is the directed path $\vec{P}_{n-2}$; this contradicts that $u\alpha_{j-1} = v\alpha_{j-1}$. Hence $w\alpha_{j-1} \ge n-1$ and $v\alpha_{j-1} \ge n-1$. If $v\alpha_{j-1} = n$ or $v\beta_r = n-1$, then ϵ_j does not carry v. Thus, $v\alpha_{j-1} = n-1$ and $v\beta_r = n$. Then, in order to carry v to $n-1$, we have $\epsilon_s = (n-2 \to n-1)$ for at least one $s \in [l]$, and $\epsilon_j = (n-1 \to n)$. For $s \in [j-1]$, replace all occurrences $\epsilon_s = (n-2 \to n-1)$ with $\epsilon_s' := (n-2 \to n)$ and delete ϵ_j: this yields a word in Q_n^* of length $l' < l$ expressing β_r, which is a contradiction. □

For all $i \in [l]$, denote $\delta(i) := \sum_{v\in[n]} d_{Q_n}(v\alpha_i, v\beta_r)$. We then have $\delta(l) = 0$, and by the claim, $\delta(i) \ge \delta(i-1) - 1$ for all $i \in [l]$. Thus $l \ge \delta(0)$, where

$$\begin{aligned}\delta(0) &= \sum_{v\in[n]} d_{Q_n}(v, v\beta_r)\\ &= \sum_{v=1}^{r-2}(n-r) + \sum_{v=r-1}^{n-2}(n-1-v) + 1\\ &= \frac{(n-r)(n+r-3)}{2} + 1.\end{aligned}$$

□

3.2 Strong tournaments

Let $n \ge 3$. Recall that if T is a strong tournament on $[n]$, then $\{a \to b : (a, b) \in E(T)\}$ is a minimal generating set of Sing_n. Let Tour_n denote the set of all strong tournaments on $[n]$. For $r \in [n-1]$, define

$$\ell_{\max}^{\mathrm{Tour}}(n, r) := \max\{\ell(T, r) : T \in \mathrm{Tour}_n\},$$
$$\ell_{\max}^{\mathrm{Tour}}(n) := \max\{\ell(T) : T \in \mathrm{Tour}_n\}.$$

Define analogously $\ell_{\min}^{\mathrm{Tour}}(n, r)$ and $\ell_{\min}^{\mathrm{Tour}}(n)$. The first few values of $\ell_{\min}^{\mathrm{Tour}}(n, r)$ and $\ell_{\max}^{\mathrm{Tour}}(n, r)$, calculated with the GAP package *Semigroups* [7] using data from [6], are given in Table 2. The calculation of these values has been the inspiration for the results of this section and the conjectures of the next one.

Lemma 3.9 *Let $n \ge 3$ and $T \in \mathrm{Tour}_n$.*

1. *For any partition P of $[n]$ into r parts, there exists an idempotent $\alpha \in \mathrm{Sing}_n$ with $\ker(\alpha) = P$ such that $\ell(T, \alpha) = n - r$.*

Table 2 First values of $\left(\ell^{\text{Tour}}_{\min}(n,r), \ell^{\text{Tour}}_{\max}(n,r)\right)$

n	r 2	3	4	5	6
3	(6, 6)				
4	(8, 8)	(11, 11)			
5	(6, 11)	(8, 14)	(10, 17)		
6	(8, 13)	(10, 18)	(11, 21)	(13, 24)	
7	(8, 16)	(10, 22)	(11, 26)	(13, 29)	(15, 32)

2. *For any r-subset S of $[n]$, there exists an idempotent $\alpha \in \mathrm{Sing}_n$ with $\mathrm{Im}(\alpha) = S$ such that $\ell(T, \alpha) = n - r$.*

Proof 1. Let $P = \{P_1, \dots, P_r\}$. For all $1 \le i \le r$, the digraph $T[P_i]$ induced by P_i is a tournament, so it is connected and there exists a vertex v_i reachable by any other vertex in P_i: let α map the whole of P_i to v_i. Then α, when restricted to P_i, is a constant map, which can be computed using $|P_i| - 1$ arcs. Summing for i from 1 to r, we obtain that $\ell(T, \alpha) = n - r$.

2. Without loss of generality, let $S = [r] \subseteq [n]$. For every $v \in [n]$, define

$$s(v) := \min\{s \in S : d_T(s', v) \ge d_T(s, v), \forall s' \in S\}.$$

In particular, if $v \in S$, then $s(v) = v$. Moreover, if $v = v_0, v_1, \dots, v_d = s(v)$ is a shortest path from v to $s(v)$, with $d = d_T(v, s(v))$, then $s(v_i) = s(v)$ for all $0 \le i \le d$. For each $v \in [n]$, fix a shortest path P_v from v to $s(v)$, and consider the digraph D on $[n]$ with edges

$$E(D) := \{(a, b) : (a, b) \in E(P_v) \text{ for some } v \in [n]\}.$$

Then, D is acyclic and the set of vertices with out-degree zero in D is exactly S. Let sort $[n]$ so that D has reverse topological order: $(a, b) \in E(D)$ only if $a > b$. Note that S is fixed by this sorting. Let α be given by $v\alpha := s(v)$; hence, with the above sorting

$$\alpha = \bigcirc_{v=n}^{r+1} (v \to v_1).$$

□

Lemma 3.10 *Let $n \ge 3$, $T \in \mathrm{Tour}_n$, and $\alpha := (u \to v) \in \mathrm{Sing}_n$, for $(u, v) \notin E(T)$. Then*

$$\ell(T, \alpha) = 4d_T(u, v) - 2.$$

Proof Let $u = v_0, v_1, \dots, v_d = v$ be a shortest path from u to v in T, where $d := d_T(u, v)$. As $(u, v) \notin E(T)$ and T is a tournament, we must have $(v, u) \in E(T)$.

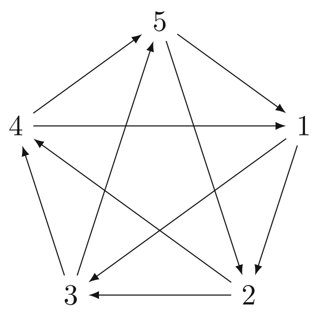


Fig. 2 Circulant tournament κ_5

By the minimality of the path, for any $j+1<i$, we have $(v_j, v_i) \notin E(T)$, so $(v_i, v_j) \in E(T)$. Then, the following expresses α with arcs in T^*:

$$\begin{aligned}(v_0 \to v_d) = {} & (v_d \to v_0)(v_{d-1} \to v_d)(v_{d-2} \to v_{d-1}) \cdots (v_1 \to v_2)(v_0 \to v_1)\\ & ((v_2 \to v_0)(v_1 \to v_2))\,((v_3 \to v_1)(v_2 \to v_3))\\ & \cdots ((v_d \to v_{d-2})(v_{d-1} \to v_d))\\ & (v_{d-2} \to v_{d-1}) \cdots (v_0 \to v_1).\end{aligned}$$

So $\ell(T,\alpha) \leq 4d-2$. For the lower bound, we note that any word in T^* expressing $(u \to v)$ must begin with $(v \to u)$. Then, u has to follow a walk in T towards v; say this walk has length $l \geq d$. All the vertices on the walk must be moved away (as otherwise they would collapse with u) and have to come back to their original position (since α fixes them all); as the shortest cycle in a tournament has length 3, this process adds at least $3(l-1)$ symbols to the word. Altogether, this yields a word of length at least

$$1 + l + 3(l-1) = 4l - 2 \geq 4d - 2.$$

$\square$

Let $n = 2m+1 \geq 3$ be odd, and let κ_n be the *circulant tournament* on $[n]$ with edges $E(\kappa_n) := \{(i, (i+j) \mod n) : i \in [n], j \in [m]\}$. Figure 2 illustrates κ_5. In the following theorem, we use κ_n to provide upper and lower bounds for $\ell^{\mathrm{Tour}}_{\min}(n,r)$ and $\ell^{\mathrm{Tour}}_{\max}(n,r)$ when n is odd.

Theorem 3.11 *For any n odd, we have*

$$n + r - 2 \leq \ell^{\mathrm{Tour}}_{\min}(n,r) \leq n + 8r,$$
$$(\hat{r}+1)(n-\hat{r}) - 1 \leq \ell^{\mathrm{Tour}}_{\max}(n,r) \leq 6rn + n - 10r.$$

where $\hat{r} = \min\{r-1, \lfloor n/2 \rfloor\}$.

Proof Let $T \in \mathrm{Tour}_n$ and $2 \leq r \leq n-1$. We introduce the following notation:

$$[n]_r := \{\mathbf{u} := (u_1, \dots, u_r) : u_i \neq u_j, \forall i, j\},$$
$$\Delta(T, r) := \max \left\{ \sum_{i=1}^{r} d_T(u_i, v_i) : \mathbf{u}, \mathbf{v} \in [n]_r \right\}.$$

The result follows by the next claims.

Claim 3.12 *$r'(\mathrm{diam}(T) - r' + 1) + r - r' \leq \Delta(T, r) \leq r\mathrm{diam}(T)$, where $r' = \min\{r, \lfloor (\mathrm{diam}(T) + 1)/2 \rfloor\}$.*

Proof The upper bound is clear. For the lower bound, let $u, v \in [n]$ be such that $d_T(u, v) = \mathrm{diam}(T)$, and let $u = v_0, v_1, \dots, v_d = v$ be a shortest path from u to v, where $d = \mathrm{diam}(T)$. Then, $d_T(v_i, v_j) = j - i$, for all $0 \leq i \leq j \leq D$. If $1 \leq r \leq \lfloor (d+1)/2 \rfloor$, consider $\mathbf{u}' = (v_0, \dots, v_{r-1})$ and $\mathbf{v}' = (v_{d-r+1}, \dots, v_d)$, so we obtain $\Delta(T, r) \geq r(d - r + 1)$. If $r \geq \lfloor (d+1)/2 \rfloor$, simply add vertices u'_j and v'_j such that $(u'_j, v'_j) \notin T$. □

Claim 3.13 $\min\{\Delta(T, r) : T \in \mathrm{Tour}(n)\} = \Delta(\kappa_n, r) = 2r$.

Proof Let $\mathbf{u} = (u_1, \dots, u_n)$ form a Hamiltonian cycle, and choose $\mathbf{v} = (u_n, u_1, \dots, u_{n-1})$. Then $d_T(u_i, v_i) \geq 2$ for all i. Conversely, since $\mathrm{diam}(\kappa_n) = 2$, we have $\Delta(\kappa_n, r) = 2r$. □

Claim 3.14 $n - r + \Delta(T, r-1) \leq \ell(T, r) \leq n + 6r\mathrm{diam}(T) - 4r$.

Proof For the lower bound, consider $\alpha \in \mathrm{Sing}_n$ as follows. Let $\mathbf{u} = (u_1, \dots, u_{r-1})$ and $\mathbf{v} = (v_1, \dots, v_{r-1})$ achieve $\Delta(T, r-1)$, and let $v \notin \{v_1, \dots, v_{r-1}\}$; define

$$x\alpha = \begin{cases} v_i & \text{if } x = u_i, \\ v & \text{otherwise.} \end{cases}$$

Let $\omega = e_1 \dots e_l$ (where $e_i = (a_i \to b_i)$) be a shortest word expressing α, where $l := \ell(T, \alpha)$. Recall that an arc e_i carries a vertex c if $ce_1 \dots e_{i-1} = a_i$. By the minimality of ω, every arc carries at least one vertex. Moreover, if c and d are carried by e_i, then $c\alpha = d\alpha$; therefore, we can label every arc e_i of ω by an element $c(e_i) \in \mathrm{Im}(\alpha)$ if e_i carries vertices eventually mapping to $c(e_i)$. Denote the number of arcs labelled c as $l(c)$, we then have $l = \sum_{c \in \mathrm{Im}(\alpha)} l(c)$. For any $u \in V$, there are at least $d_T(u, u\alpha)$ arcs carrying u. Therefore,

$$l = \sum_{c \in \mathrm{Im}(\alpha)} l(c) \geq \sum_{i=1}^{r-1} d_T(u_i, v_i) + \sum_{a \notin \mathbf{u}} d_T(a, v) \geq \Delta(T, r-1) + n - r.$$

For the upper bound, we can express any $\alpha \in \mathrm{Sing}_n$ of rank r in the following fashion. By Lemma 3.9, there exists $\beta \in \mathrm{Sing}_n$ with the same kernel as α such that

$\ell(T,\beta) = n - r$. Suppose that $\mathrm{Im}(\alpha) = \{v_1, \dots, v_r\}$ and $\mathrm{Im}(\beta) = \{u_1, \dots, u_r\}$, where $u_i\beta^{-1} = v_i\alpha^{-1}$, for $i \in [r]$. Let $h \in [n]\backslash\mathrm{Im}(\beta)$. Define a transformation γ of $[n]$ by

$$x\gamma = \begin{cases} v_i & \text{if } x = u_i, \\ v_1 & \text{if } x = h, \\ x & \text{otherwise.} \end{cases}$$

Then $\alpha = \beta\gamma$, where $\gamma \in \mathrm{Sing}_n$, and by Theorem 1.1

$$\ell(K_n, \gamma) = n - \mathrm{fix}(\gamma) + \mathrm{cycl}(\gamma) \le r + \frac{r}{2} = \frac{3r}{2}.$$

By Lemma 3.10, each arc associated with K_n may be expressed in at most $4\mathrm{diam}(T) - 2$ arcs associated with T; therefore,

$$\ell(T, \gamma) \le \frac{3r}{2}(4\mathrm{diam}(T) - 2) = 6r\mathrm{diam}(T) - 3r.$$

Thus,

$$\ell(T, \alpha) \le \ell(T, \beta) + \ell(T, \gamma) \le n + 6r\mathrm{diam}(T) - 4r.$$

□

□

4 Conjectures and open problems

We finish the paper by proposing few conjectures and open problems.

Let π_n be the tournament on $[n]$ with edges $E(\pi_n) := \{(i, (i+1) \mod n) : i \in [n]\} \cup \{(i, j) : j + 1 < i\}$. Figure 3 illustrates π_5.

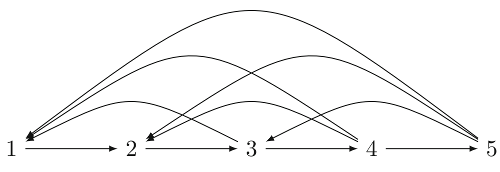
Conjecture 4.1 *For every $n \ge 3$, $r \in [n-1]$, and $T \in \mathrm{Tour}_n$, we have*

$$\ell(T, r) \le \ell(\pi_n, r) = \ell^{\mathrm{Tour}}_{\max}(n, r),$$

with equality if and only if $T \cong \pi_n$. Furthermore,

$$\ell(\pi_n) = \ell^{\mathrm{Tour}}_{\max}(n) = \frac{n^2 + 3n - 6}{2},$$

Fig. 3 π_5



which is achieved for $\alpha := n\ (n-1)\ \ldots\ 2\ n$.

Tournament π_n has appeared in the literature before: it is shown in [8] that π_n has the minimum number of strong subtournaments among all strong tournaments on $[n]$. On the other hand, it was shown in [1] that, for n odd, the circulant tournament κ_n has the maximal number of strong subtournaments among all strong tournaments on $[n]$.

Conjecture 4.2 *For every* $n \geq 3$ *odd,* $r \in [n-1]$, *and* $T \in \mathrm{Tour}_n$, *we have*

$$\ell_{\min}^{\mathrm{Tour}}(n, r) = \ell(\kappa_n, r).$$

Furthermore,

$$\ell_{\min}^{\mathrm{Tour}}(n, 2) = n + 1 \ \text{ and } \ \ell_{\min}^{\mathrm{Tour}}(n, r) = n + r,$$

for all $3 \leq r \leq \frac{n+1}{2}$.

Conjecture 4.3 *There exists* $c > 0$ *such that for every simple digraph* D *on* $[n]$, $\ell(D) = O(n^c)$.

The referee of this paper noted that the automorphism groups of K_n and $\langle K_n \rangle = \mathrm{Sing}_n$ are both isomorphic to Sym_n and proposed the following problems.

Problem 1 Investigate connections between the automorphism groups of D and $\langle D \rangle$. Is it possible to classify all digraphs D such that the automorphism group of D and of $\langle D \rangle$ are isomorphic?

Problem 2 Generalise the ideas of this paper to oriented matroids. Is there a natural way to associate (not necessarily idempotent) transformations to each signed circuit of an oriented matroid?

In a forthcoming paper, we investigate the relationship between the graph theoretic properties of D and the semigroup properties of $\langle D \rangle$.

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