

THE HOMOLOGY CORE OF MATCHBOX MANIFOLDS AND INVARIANT MEASURES

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ABSTRACT. We consider the topology and dynamics associated to a wide class of matchbox manifolds, including spaces of aperiodic tilings and suspensions of higher rank (potentially non-abelian) group actions on zero-dimensional spaces. For such a space we introduce a topological invariant, the homology core, built using an expansion of it as an inverse sequence of simplicial complexes. The invariant takes the form of a monoid equipped with a representation, which in many cases can be used to obtain a finer classification than is possible with the previously developed invariants. When the space is obtained by suspending a topologically transitive action of the fundamental group Γ of a closed orientable manifold on a zero-dimensional compact space Z , this invariant corresponds to the space of finite Borel measures on Z which are invariant under the action of Γ . This leads to connections between the rank of the core and the number of invariant, ergodic Borel probability measures for such actions. We illustrate with several examples how these invariants can be calculated and used for topological classification and how it leads to an understanding of the invariant measures.

1. INTRODUCTION

Given the action of the fundamental group $\Gamma = \pi_1(M)$ of a closed orientable manifold M of dimension d on the zero-dimensional compact metric space Z , one can suspend the Γ action over M to form a space \mathcal{M} . Provided this Γ action has a dense orbit (i.e., is topologically transitive), this space will have the structure of a *matchbox manifold*. More generally, a matchbox manifold is a compact, connected metric space that locally has the structure of $\mathbb{R}^d \times \mathcal{T}$, where \mathcal{T} is a zero-dimensional space. Such spaces occur naturally when considering minimal sets of foliations and hyperbolic attractors of diffeomorphisms of manifolds. We do not in general require any differentiable structure, but shall restrict our consideration to matchbox manifolds that admit an expansion as an inverse system (tower) of finite simplicial complexes with well-behaved projection and bonding maps; such expansions are known to exist for many classes, see for example, [14]. We detail the precise class of spaces we consider in Section 2.

Special and well studied examples of such objects include the so called tiling spaces arising from aperiodic tilings of a Euclidean space with finite (translational) local complexity; see Sadun's text [33] for a general introduction to such examples. These

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tiling spaces can be viewed as the suspension over a torus of a \mathbb{Z}^d action on a zero-dimensional space, as detailed by Sadun & Williams [34].

In this paper we introduce a homeomorphism invariant of oriented matchbox manifolds, the *homology core* of \mathcal{M} . In its strongest form, the core may be considered as a monoid equipped with a representation in a linear space, and it is this representation (as opposed to an abstract monoid) that can be used to make the finest distinctions. Our construction uses the top dimensional homology groups of an expansion of \mathcal{M} , and as such the invariance of the homology core can be considered a generalisation of the work of Barge & Diamond [3] and Swanson & Volkmer [36] on the weak equivalence of matrices related to a one dimensional substitution tiling system; we have however no reason to restrict attention to substitution tiling spaces, and we can often compute our invariant for oriented matchbox manifolds of any dimension to great advantage.

Our work may also be seen as providing a generalisation of invariants as introduced by Kellendonk [25] and Ormes, Radin & Sadun [30] which use oriented dimension or cohomology groups, applied to higher dimensional primitive substitution tiling spaces. For such tiling spaces a duality (clearest when working with coefficients in a field of characteristic zero) can be established with the homology core, but much more generally, the approach in this paper would also allow the construction and topological invariance of such cohomological objects to be established in any oriented matchbox manifold with a simplicial presentation, not just those defined by a single primitive matrix. Nevertheless, the homological approach appears to be the more natural place in which to do calculation, and in our final part we compute examples of spaces with various expansions in which, for example, the Perron-Frobenius theory heavily relied on by [30] is clearly not available.

While the homology core is constructed using expansions, as is usual in shape theory, the homology core is not a shape invariant. In fact, we show that, unlike Cech cohomology, the homology core can be used to distinguish examples of shape equivalent spaces, but at the same time, there are examples of spaces the homology core does not distinguish but which are distinguished by the authors' shape invariant L_1 defined in [12].

An intriguing and significant feature of the homology core we present appears when our underlying matchbox manifold \mathcal{M} is constructed by suspending a topologically transitive Γ action on a zero-dimensional compact space Z over the oriented manifold M . We show under these circumstances that the top Cech cohomology of such a matchbox manifold is tractable, and as a result in many natural cases the homology core can be identified with the space of finite Borel measures on the space Z that are invariant under the given action of Γ . A related result for tiling spaces is given by Bellissard, Benedetti & Gambaudo [7]. We note however that this result needs only part of the invariant, and is dependent only on the core as an abstract monoid, independent of its representation.

There is a connection with objects that have been previously used in the study of invariant measures in, for example, Bezuglyi, Kwiatkowski, Medynets, & Solomyak [9], Aliste-Prieto & D. Coronel [1], Petite [29] and Frank & Sadun [19]. In those constructions the number of invariant, ergodic, Borel probability measures is usually

found to be bounded above by the number of tile types, vertices in a related Bratteli diagram or similar information. From our viewpoint, in many cases we can directly compute the number of invariant, ergodic Borel measures in terms of extreme points in our potentially much smaller homology core of \mathcal{M} .

Furthermore, our result can be viewed as a refinement of the connection between the foliation cycles of a foliated space and the space of invariant measures discovered by Sullivan [35], see also [27]. The advantage of our approach is that one can often calculate the homology core in a direct and quite tangible way, capturing some of the geometric information lost in the other approaches.

A novel feature of our approach is that it is purely topological and makes no use of a smooth structure as in [35], [7]. At the same time, by considering the more general case we identify the key ingredients in the structure that make the argument go through. In particular, the fibred simplicial presentations as found in [14] are essential in the arguments of Theorem 5.7.

The paper is arranged as follows. In Sections 2 and 3 we specify the category of matchbox manifolds we consider, associated homology classes and their behaviour under homeomorphisms. In Section 4 we define the homology core, Definition 4.2, and prove its invariance under homeomorphism. We also introduce the properties of \mathbb{Z} and \mathbb{Q} -stability. In Section 5 we concentrate on those matchbox manifolds that are suspensions over manifolds, and relate the homology core to spaces of invariant measures. In this section we also sketch the relation to the ordered cohomology of [30], and indicate how the approach here could allow a generalisation of the construction in [30] and its properties to a wider class of objects. In the final section, 6, we detail a number of examples and computations. We recover and generalise, Theorem 6.2, a result of Cortez & Petite [16] on the unique ergodicity of certain solenoids and their associated odometers, provide examples, examples, 6, in which the core distinguishes between non-minimal tiling spaces with multiple invariant ergodic measures, examples 6.7, which demonstrate that the core can distinguish shape equivalent spaces, but also, 6.5, that the core will not fully distinguish between all shape inequivalent spaces.

2. BACKGROUND

In this section we shall present the preliminary results that allow us to obtain the topological invariance of the homology core we construct in Section 4. We begin by recalling the suspension construction, a special but important case of the matchbox manifolds we consider.

Let $\Gamma = \pi_1(M, m_0)$ be the fundamental group of a PL closed orientable manifold M of dimension d . Let Γ act on the left of the zero-dimensional compact space Z . We identify Γ with the deck transformations of the universal covering map $\widetilde{M} \rightarrow M$, and we consider Γ to act on the right of \widetilde{M} . This then leads to the *suspension* $\widetilde{M} \times_{\Gamma} Z$, which is the orbit space of the action of Γ on $\widetilde{M} \times Z$ given by

$$(\gamma, (m, c)) \mapsto (m \cdot \gamma^{-1}, \gamma \cdot c).$$

The space $\mathcal{M} := \widetilde{M} \times_{\pi_1(M)} Z$ thus constructed is a foliated space which is locally homeomorphic to $\mathbb{R}^d \times Z$. Provided that the action of Γ is topologically transitive, \mathcal{M} is connected and hence is an example of a matchbox manifold. All the standard examples of repetitive, aperiodic tiling spaces with finite local complexity are such suspensions with abelian group $\Gamma = \mathbb{Z}^d$ and M a d -torus [34].

DEFINITION 2.1. *A matchbox manifold \mathcal{M} is a compact, connected metric space with the structure of a foliated space, such that for each $x \in \mathcal{M}$, the transverse model space \mathcal{T}_x is totally disconnected.*

The topological dimension of a matchbox manifold of dimension d is the same as the dimension of its leaves, which coincide with the path components. In the case of a suspension over a manifold M , then d coincides with the dimension of M . The smoothness of a suspension $\widetilde{M} \times_{\pi_1(M)} Z$ along leaves in the case that M is smooth and its structure as a fiber bundle over M with fiber Z follow from general considerations, see [11, Chapt 3.1]. A matchbox manifold is *minimal* when each path component is dense. A suspension $\widetilde{M} \times_{\pi_1(M)} Z$ is minimal if the action of Γ on Z is minimal. We refer the reader to [13], [14] for a more detailed discussion.

DEFINITION 2.2. *Let \mathcal{M} be a matchbox manifold of dimension d . A simplicial presentation of \mathcal{M} is an inverse sequence whose limit is homeomorphic to \mathcal{M}*

$$\mathcal{M} \approx \varprojlim \{M \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \dots\}$$

and is such that each X_n is a triangulated space and each bonding map f_n is surjective and simplicial. Additionally, we require for each n that:

- (i) each simplex in the triangulation of X_n is a face of a d -dimensional simplex and
- (ii) each d -dimensional simplex S of X_n pulls back in \mathcal{M} to a subset homeomorphic to $S \times K$ for some zero-dimensional compact K and that for each $k \in K$ the restriction of the projection $\mathcal{M} \rightarrow X_n$ to $S \times \{k\}$ is a homeomorphism onto its image.

Condition (ii) is similar to requiring the restrictions to leaves to be covering maps (as is the case of the fiber bundle projections in a suspension), only at the boundaries of the simplices S in the leaves (where there can be branching in X_n) the projections do not necessarily behave as covering maps. In addition to very general tiling spaces, according to the results of [14], a wide variety of minimal matchbox manifolds admit such a presentation. However, these presentations are only known to exist for matchbox manifolds with trivial holonomy. When the manifold is obtained from the suspension of the action of a group Γ , the condition of trivial holonomy implies that if $\gamma \in \Gamma$ fixes an element of the transverse space $z \in Z$, then γ must fix a neighbourhood of z . While it is not known the precise degree to which non-trivial holonomy provides an obstruction to the existence of simplicial expansions, this suggests a property of the group action that appears necessary for the connection between the Γ -invariant measures and the topologically invariant homology core. Since there are examples of group actions that do not admit non-trivial invariant Borel measures, this highlights a potential topological obstruction to the existence of such measures.

According to the definition, corresponding to the triangulation of X_n in a simplicial presentation which \mathcal{M} admits, there is a decomposition of \mathcal{M} into a finite number of sets of the form $S_i \times K_i$ that intersect only along sets of the form $\partial S_i \times K$, where ∂S_i is the boundary of S_i and K is a clopen subset of K_i . Thus, the leaves of \mathcal{M} can be given a simplicial structure induced by this decomposition. What is more, the leaves of \mathcal{M} can be considered as being tiled by finitely many tile types, one type corresponding to each simplex S_i in the triangulation of X_n . Given the nature of a triangulation, we also have that there are only finitely many ways that tiles may intersect in a leaf, which can be considered as a form of what is known as finite local complexity. Of course, unlike the case of the aperiodic tilings generally studied, there is no reason to assume the leaves are euclidean, or even contractible. Each of the successive approximating spaces X_n leads to a finer decomposition of \mathcal{M} and the fibers of the projection $\mathcal{M} \rightarrow X_{n+1}$ are contained in the fibers of the projection $\mathcal{M} \rightarrow X_n$ and the induced map $f_n : X_{n+1} \rightarrow X_n$ is simplicial in that it can be considered as the geometric realisation of a simplicial map of the complexes underlying the triangulations of X_{n+1} and X_n .

This special structure will allow us to apply a powerful result on the approximation of maps between inverse limits as described below.

DEFINITION 2.3. *For given inverse limits $\mathcal{M} = \varprojlim \{X_n, f_n\}$ and $\mathcal{N} = \varprojlim \{Y_n, g_n\}$, a map $h : \mathcal{M} \rightarrow \mathcal{N}$ is said to be induced if for a subsequence n_k of \mathbb{N} , there is for each $k \in \mathbb{N}$ a map $h_k : X_{n_k} \rightarrow Y_k$ such that the following diagram commutes*

$$\begin{array}{ccccccc}
 X_{n_1} & \xleftarrow{f_{n_1}^{n_2}} & X_{n_2} & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & X_{n_k} & \xleftarrow{f_{n_k}^{n_{k+1}}} & X_{n_{k+1}} & \xleftarrow{\quad} & \cdots \\
 h_1 \downarrow & & \downarrow h_2 & & & & \downarrow h_k & & \downarrow h_{k+1} & & \\
 Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & Y_k & \xleftarrow{g_k} & Y_{k+1} & \xleftarrow{\quad} & \cdots
 \end{array}$$

and the resulting map $\mathcal{M} \rightarrow \mathcal{N}$ given by $(x_i) \mapsto (h_i(x_{n_i}))$ is equal to h . Here, for $r < \ell$, we write f_r^ℓ for the composite $f_r \circ \cdots \circ f_{\ell-1}$.

We record the key result of Rogers [31, Thm 4] on the approximation of maps between inverse limits as maps between the factor spaces.

THEOREM 2.4. [31] *Given two matchbox manifolds with simplicial presentations $\mathcal{M} = \varprojlim \{X_n, f_n\}$ and $\mathcal{N} = \varprojlim \{Y_n, g_n\}$ and given any $\epsilon > 0$, any continuous map $f : \mathcal{M} \rightarrow \mathcal{N}$ is homotopic to an induced map f_ϵ in which points are moved no more than ϵ over the course of the homotopy from f to f_ϵ .*

3. ORIENTATION IN MATCHBOX MANIFOLDS

We now consider a matchbox manifold \mathcal{M} equipped with simplicial presentation $\mathcal{M} = \varprojlim \{X_n, f_n\}$. Recall that the leaf topology for a leaf L has a basis of open sets formed by intersecting L with open sets of plaques of the foliation charts, which gives L the structure of a connected manifold. A leaf can be orientable or not, and when L is orientable it admits one of two orientations. (A convenient way of considering orientations and orientability for a non-compact manifold admitting a

simplicial structure such as L is with the use of homology groups based on infinite chains, see, e.g., [28, p.33, 388].) If L has an orientation, each time it enters a subset of \mathcal{M} of the form $S \times K$, where S is a simplex of dimension d corresponding to a triangulation of some X_n , then L induces an orientation of S . It can happen that each time an oriented L enters $S \times K$ it induces the same orientation of S or it could induce different orientations. If L always induces the same orientation on S , we shall say L induces a *coherent* orientation on S , and if this is so for all simplices S in all X_n then we say L can be oriented *coherently*. In a minimal matchbox manifold \mathcal{M} , whether a given simplex S of X_n is coherently oriented is independent of the choice of orientable leaf L . (It should be borne in mind that for general matchbox manifolds not all leaves of a matchbox manifold need be homeomorphic and that it can even happen that some leaves are orientable while others not.)

DEFINITION 3.1. *A simplicial representation of a matchbox manifold $\mathcal{M} = \varprojlim \{X_n, f_n\}$ is orientable if the following conditions hold*

- (i) \mathcal{M} has an orientable dense leaf L and
- (ii) L can be oriented coherently with respect to the triangulations of all the X_n .

An orientation of an orientable simplicial presentation of \mathcal{M} is given by a choice of orientation of a dense leaf L as above and the corresponding induced orientation of each simplex occurring in the triangulations of the X_n .

From here on we shall only consider orientable presentations. While this originally seems quite restrictive, any non-orientable matchbox manifold has an orientable “double cover”, [11, p. 280]. As a basic example, the leaves of a tiling space arising from an aperiodic tiling of \mathbb{R}^d with finite translational local complexity admit a natural orientation induced by the translation action, and the various presentations that have been constructed using the structure of the tiles are coherent with this orientation provided one takes the extra step of introducing the simplicial structure on the complexes. Observe also that since we are endowing each X_n with the orientation induced by L and the bonding maps are simplicial, the bonding maps will preserve the orientation of each simplex.

DEFINITION 3.2. *A homeomorphism $h : \mathcal{M} \rightarrow \mathcal{N}$ of matchbox manifolds with corresponding oriented simplicial presentations $\mathcal{M} = \varprojlim \{X_n, f_n\}$ and $\mathcal{N} = \varprojlim \{Y_n, g_n\}$ with orientations induced by the leaf L of \mathcal{M} and $h(L)$ of \mathcal{N} is orientation preserving if h preserves the orientation of L and otherwise h is orientation reversing.*

The invariants we construct will be preserved by orientation preserving maps and are intimately related to how their homotopic induced maps act on the algebraic invariants of the approximating spaces X_n .

DEFINITION 3.3. *Given an oriented simplicial presentation $\mathcal{M} = \varprojlim \{X_n, f_n\}$, a positive homology class of X_n is a homology class in the top dimensional simplicial homology group $H_d(X_n)$ that can be represented as the positive integer combination of elementary chains of positively oriented d -simplices of some simplicial subdivision of X_n , and we denote the set of all positive homology classes as $H_d^+(X_n)$. Similarly, we define $H_d^-(X_n)$ as all the homology classes in the simplicial homology $H_d(X_n)$*

that can be represented as the negative integer combination of elementary chains of positively oriented d -simplices of some simplicial subdivision of X_n . (The zero class is considered to be in $H_d^+(X_n) \cap H_d^-(X_n)$.)

Observe that by our choices of orientation for the X_n and their common relation to a chosen leaf L , each bonding map $f_n : X_{n+1} \rightarrow X_n$ satisfies $(f_n)_*(H_d^+(X_{n+1})) \subset H_d^+(X_n)$ and similarly with $H_d^-(X_{n+1})$. The following result is key for the topological invariance of the homology core.

PROPOSITION 3.4. *Given a homeomorphism $h : \mathcal{M} \rightarrow \mathcal{N}$ of d -dimensional matchbox manifolds with corresponding oriented simplicial presentations $\mathcal{M} = \varprojlim \{X_n, f_n\}$ and $\mathcal{N} = \varprojlim \{Y_n, g_n\}$ let $h' : \mathcal{M} \rightarrow \mathcal{N}$ be any induced homotopic map corresponding to the following commutative diagram*

$$\begin{array}{ccccccc}
 X_{n_1} & \xleftarrow{f_{n_1}^{n_2}} & X_{n_2} & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & X_{n_k} & \xleftarrow{f_{n_k}^{n_{k+1}}} & X_{n_{k+1}} & \xleftarrow{\quad} & \cdots \\
 h_1 \downarrow & & \downarrow h_2 & & & & \downarrow h_k & & \downarrow h_{k+1} & & \\
 Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & Y_k & \xleftarrow{g_k} & Y_{k+1} & \xleftarrow{\quad} & \cdots
 \end{array}$$

Then for each $i \in \mathbb{N}$, either

- (i) $(h_i)_*(H_d^+(X_{n_i})) \subset H_d^+(Y_i)$ and $(h_i)_*(H_d^-(X_{n_i})) \subset H_d^-(Y_i)$ or
- (ii) $(h_i)_*(H_d^+(X_{n_i})) \subset H_d^-(Y_i)$ and $(h_i)_*(H_d^-(X_{n_i})) \subset H_d^+(Y_i)$

according as h is orientation preserving (i), or reversing (ii).

Proof. Suppose then that we have an orientation preserving homeomorphism $h : \mathcal{M} \rightarrow \mathcal{N}$ with homotopic induced map $h' : \mathcal{M} \rightarrow \mathcal{N}$ as above, and let $i \in \mathbb{N}$. To calculate the map induced on homology $(h_i)_* : H_d^+(X_{n_i}) \rightarrow H_d^+(Y_i)$, one first finds a simplicial approximation $H : X_{n_i} \rightarrow Y_i$ to h_i . Notice that this simplicial approximation also induces a simplicial map $H_L : L \rightarrow h'(L)$. As the path components coincide with the leaves of these spaces and h' is homotopic to h , we have $h'(L) = h(L)$. By hypothesis, h preserves the orientation and maps the positive generator of $H_d(L)$, which is the class formed by the sum of all the elementary chains of positively oriented simplices of dimension d , to the positive generator of $H_d(h(L))$. The same is true then for the homotopic map h' (and the map it induces on leaves) and so also for the simplicial approximation H_L . But that means that H_L must map positively oriented simplices to positively oriented simplices or degenerate simplices. The other cases are similar. \square

It is important to realise that even when the underlying map h is an induced homeomorphism, the maps h_n are often not homeomorphisms.

4. HOMOLOGY CORE AND HOMEOMORPHISM INVARIANCE

In this section we introduce the homology core and show the subtle ways it is preserved by homeomorphism, depending on the precise nature of the space in question. Unlike (Cech) cohomology, homology does not generally behave well with respect to

inverse limits, and so some care is needed in discussing this variance. The notion of the homology core is two-fold: it is a sub-object of the inverse limit $\varprojlim \{H_d(X_n), (f_n)_*\}$ built from the positive homology classes of X_n , and it is equipped with a representation, a map to linear space (in fact, a family of such representations). By virtue of the nature of inverse limits, there are obvious natural maps to the homology groups $H_d(X_n, \mathbb{R})$, and it is these which provide our representations.

Throughout this section, we assume a simplicial presentation $\mathcal{M} = \varprojlim \{X_n, f_n\}$ of our matchbox manifold which is oriented. Observe that, as the X_n contain no $(d+1)$ -simplices, the groups $H_d(X_n)$ are free abelian of some finite rank. Consider the subgroup P_n of $H_d(X_n)$ generated by $H_d^+(X_n)$, which will then also be a free abelian group, say of rank r_n . Define V_n as $P_n \otimes \mathbb{R}$, an \mathbb{R} -vector space also of dimension r_n . As previously observed, $(f_n)_*$ maps $H_d^+(X_{n+1})$ to $H_d^+(X_n)$, and so $(f_n)_* \otimes id_{\mathbb{R}}$ yields a linear map $L_n : V_{n+1} \rightarrow V_n$. (When needing to distinguish these vector spaces or maps for different spaces we add a superscript, e.g. $L_n^{\mathcal{M}}$.)

DEFINITION 4.1. *The positive and negative cone in V_n is*

$$\mathcal{C}_n := \left\{ \sum x_i \otimes r_i \mid r_i \geq 0, x_i \in H_d^+(X_n) \right\} \cup \left\{ \sum x_i \otimes r_i \mid r_i \leq 0, x_i \in H_d^+(X_n) \right\}$$

and the positive cone in V_n is

$$\mathcal{C}_n^+ := \left\{ \sum x_i \otimes r_i \mid r_i \geq 0, x_i \in H_d^+(X_n) \right\}.$$

Our previous observations can then be rephrased as $L_n(\mathcal{C}_{n+1}) \subset \mathcal{C}_n$. However, this inclusion will often be strict. This leads us to the following.

DEFINITION 4.2. *We define the homology core at place k of the oriented presentation $\mathcal{M} = \varprojlim \{X_n, f_n\}$ and linear maps $L_n : V_{n+1} \rightarrow V_n$ as above by*

$$\mathcal{C}_{\mathcal{M}}(k) := \bigcap_{n>k} L_k^n(\mathcal{C}_n),$$

where $L_k^n = L_k \circ \cdots \circ L_{n-2} \circ L_{n-1}$. Similarly, the positive homology core $\mathcal{C}_{\mathcal{M}}^+(k)$ is given by

$$\mathcal{C}_{\mathcal{M}}^+(k) := \bigcap_{n>k} L_k^n(\mathcal{C}_n^+).$$

The fact that we must consider the core at various ‘‘places’’ k is a reflection of the fact that induced maps of towers homotopic to a given map do not have to respect the places in the two corresponding towers.

THEOREM 4.3. *Suppose we have a homeomorphism $h : \mathcal{M} \rightarrow \mathcal{N}$ of d -dimensional matchbox manifolds with corresponding oriented simplicial presentations $\mathcal{M} = \varprojlim \{X_n, f_n\}$ and $\mathcal{N} = \varprojlim \{Y_n, g_n\}$.*

- (i) *Then there are subsequences m_i, n_i and linear maps K_i with J_i that map the cones in the following diagram surjectively.*

$$(1) \quad \begin{array}{ccccccccccc} \mathcal{C}_{\mathcal{M}}(n_1) & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_2) & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_3) & \longleftarrow & \cdots & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_k) & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_{k+1}) & \longleftarrow & \cdots \\ \downarrow K_1 & \nearrow J_1 & \downarrow K_2 & \nearrow J_2 & \downarrow K_3 & & & & \downarrow K_k & \nearrow J_k & \downarrow K_{k+1} & & \\ \mathcal{C}_{\mathcal{N}}(m_1) & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_2) & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_3) & \longleftarrow & \cdots & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_k) & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_{k+1}) & \longleftarrow & \cdots \end{array}$$

- (ii) If the linear maps $L_n^{\mathcal{N}}$ are eventually injective, then there is an ℓ such that each linear map J_i ($i \geq \ell$) as in the above Diagram 1 maps $\mathcal{C}_{\mathcal{N}}(m_{i+1})$ isomorphically onto $\mathcal{C}_{\mathcal{M}}(n_i)$.
- (iii) Moreover, if there is a uniform (for all n) bound to $\dim V_n^{\mathcal{N}}$, then there is an ℓ such that each linear map J_i ($i \geq \ell$) as in the above Diagram 1 maps $\mathcal{C}_{\mathcal{N}}(m_{i+1})$ isomorphically onto $\mathcal{C}_{\mathcal{M}}(n_i)$.

Proof. (i) Let us write ℓ for the inverse of the homeomorphism h , and suppose h' and ℓ' are induced maps as in Theorem 2.4 corresponding to h and ℓ . Then we have the following diagram between subtowers after reindexing:

$$\begin{array}{ccccccccccc} X_{n_1} & \xleftarrow{f_{n_1}^{m_2}} & X_{n_2} & \xleftarrow{f_{n_2}^{m_3}} & X_{n_3} & \longleftarrow & \cdots & \longleftarrow & X_{n_k} & \xleftarrow{f_{n_k}^{m_{k+1}}} & X_{n_{k+1}} & \longleftarrow & \cdots \\ \downarrow h'_1 & \nearrow \ell'_1 & \downarrow h'_2 & \nearrow \ell'_2 & \downarrow h'_3 & & & & \downarrow h'_k & \nearrow \ell'_k & \downarrow h'_{k+1} & & \\ Y_{m_1} & \xleftarrow{g_{m_1}^{m_2}} & Y_{m_2} & \xleftarrow{g_{m_2}^{m_3}} & Y_{m_3} & \longleftarrow & \cdots & \longleftarrow & Y_{m_k} & \xleftarrow{g_{m_k}^{m_{k+1}}} & Y_{m_{k+1}} & \longleftarrow & \cdots \end{array}$$

In general this diagram will *not* be commutative, but the maps induced on homology are commutative since the compositions of h' and ℓ' are homotopic to the respective identities. By Proposition 3.4 we are then led to the following commutative diagram of (restrictions of) linear maps.

$$(2) \quad \begin{array}{ccccccccccc} \mathcal{C}_{\mathcal{M}}(n_1) & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_2) & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_3) & \longleftarrow & \cdots & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_k) & \longleftarrow & \mathcal{C}_{\mathcal{M}}(n_{k+1}) & \longleftarrow & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & & & \downarrow & \nearrow & \downarrow & & \\ \mathcal{C}_{\mathcal{N}}(m_1) & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_2) & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_3) & \longleftarrow & \cdots & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_k) & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_{k+1}) & \longleftarrow & \cdots \end{array}$$

By construction, each horizontal map in the diagram is surjective. Moreover, each of the vertical maps K_i is part of a commutative triangle

$$(3) \quad \begin{array}{ccc} \mathcal{C}_{\mathcal{M}}(n_i) & & \\ \downarrow K_i & \nearrow J_i & \\ \mathcal{C}_{\mathcal{N}}(m_i) & \longleftarrow & \mathcal{C}_{\mathcal{N}}(m_{i+1}) \end{array}$$

and as the horizontal map is surjective, K_i is surjective as required in Diagram 1. Similar arguments apply to the J_i .

(ii) Assume then that the horizontal maps are additionally injective from some point in the tower associated to \mathcal{N} . Then for sufficiently large i in the triangle (3), we see J_i must also be injective on $\mathcal{C}(m_{i+1})$.

(iii) Assume then that there is a uniform (for all n) bound to $\dim V_n^{\mathcal{N}}$. This then implies a uniform bound for the *topological* dimension of $\mathcal{C}_{\mathcal{N}}(m)$. As the maps $(L_n^m)^{\mathcal{N}}$ are linear, they cannot raise (topological) dimension. Hence, for sufficiently large values (say, $m \geq W$) the topological dimension of $\mathcal{C}_{\mathcal{N}}(m)$ must have the same value D . Then, for all $k > W$ the restriction of the maps $(L_W^k)^{\mathcal{N}}$ to $\mathcal{C}_{\mathcal{N}}(k)$ must be injective and we have the hypothesis for (ii). \square

The spaces which are best understood are those for which there is a uniform bound on $\text{rank } H_d(X_n)$ for a presentation. In such cases, by telescoping the given presentation, one can obtain a presentation for which $\text{rank } H_d(X_n)$ is constant. We can already see from the above the homology core yields a good deal of information for such spaces. However, depending on the exact conditions we can say much more in special cases.

DEFINITION 4.4. *An oriented simplicial presentation $\mathcal{M} = \varprojlim\{X_n, f_n\}$ is said to be homologically \mathbb{Z} -stable if for each n the functions $(f_n)_* : H_d(X_{n+1}) \rightarrow H_d(X_n)$ and $L_n : V_{n+1} \rightarrow V_n$ are isomorphisms, and we say the presentation is homologically \mathbb{Q} -stable if for each n the functions $(f_n)_* : H_d(X_{n+1}, \mathbb{Q}) \rightarrow H_d(X_n, \mathbb{Q})$ and $L_n : V_{n+1} \rightarrow V_n$ are isomorphisms.*

REMARK 4.5. Observe that in the case of a \mathbb{Q} -stable presentation, each core $\mathcal{C}_{\mathcal{M}}(k)$ can be identified as the image in $H^*(X_k, \mathbb{R})$ of the limit of the inverse sequence of the cones \mathcal{C}_n ($n \geq k$) with bonding maps the restrictions of the L_n . The core $\mathcal{C}_{\mathcal{M}}(k)$, however, retains geometric information lost in the abstract limit $\varprojlim\{\mathcal{C}_n, L_n\}$ as it includes a specific embedding in V_k , a point that will be demonstrated in the examples of the final section. However, this identification with the inverse limit alone will still be significant when relating the cores to their dual counterparts and spaces of measures in the next section.

We can now state and prove our key invariance result.

THEOREM 4.6. *For $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}\}$, suppose we have homologically \mathbb{X} -stable simplicial presentations $\mathcal{M} = \varprojlim\{X_n, f_n\}$ and $\mathcal{N} = \varprojlim\{Y_n, g_n\}$ and a homeomorphism $h : \mathcal{M} \rightarrow \mathcal{N}$. Choose a basis for each $V_n^{\mathcal{M}}, V_n^{\mathcal{N}}$ consisting of elements of $H_d^+(X_n), H_d^+(Y_n)$ so that the corresponding linear maps $L_n^{\mathcal{M}}, L_n^{\mathcal{N}}$ are represented by elements of $GL(D, \mathbb{X})$, where D is the common dimension of the $V_n^{\mathcal{M}}, V_n^{\mathcal{N}}$. Then, with respect to these bases, all the homology cores $\mathcal{C}_{\mathcal{M}}(m)$ and $\mathcal{C}_{\mathcal{N}}(n)$ are in the same $GL(D, \mathbb{X})$ -orbit.*

Proof. Consider now diagram 1 as before and the associated diagram on homology groups with \mathbb{X} coefficients. Under our new hypotheses, all the horizontal maps are isomorphisms and hence all the vertical and diagonal maps are also isomorphisms as well. With the bases we have chosen, the result follows directly. \square

REMARK 4.7. Some relaxation of the stability assumptions in this theorem is possible, and we really only need there to be a uniform bound on the ranks of the

groups $H_d(X_n)$. We can apply similar arguments to the general case as Theorem 4.3 to obtain that for homeomorphic matchbox manifolds \mathcal{M}, \mathcal{N} (not necessarily enjoying stability properties) that the cores $\mathcal{C}_{\mathcal{M}}(m)$ and $\mathcal{C}_{\mathcal{N}}(n)$ are images of matrices with integer entries for *restricted* choices of m, n as indicated in the theorem. Observe that if there is a uniform bound on rank $H_d(X_n)$, then we can find an inverse sequence of groups which is \mathbb{Q} -stable and which is pro-equivalent to the inverse sequence of the V_n and L_n , and this is sufficient to draw the same conclusions as in the theorem above.

5. HOMOLOGY CORE AND INVARIANT MEASURES

In this section we relate part of our invariant (the monoid, without the representation) to sets of transverse invariant measures, paralleling and extending a number of earlier works, as discussed in the remarks below, and as mentioned in the introduction. We consider matchbox manifolds that are given as suspensions $\mathcal{M} := \widetilde{M} \times_{\Gamma} Z$ over a closed PL manifold M with fundamental group $\Gamma = \pi_1(M)$. We denote the associated bundle projection by $\pi : \mathcal{M} \rightarrow M$. Explicitly, in this setting we directly equate the homology core with the space of Γ -invariant, finite Borel measures for a topologically transitive Γ -action on the zero dimensional space Z . We shall only consider finite measures and henceforth will assume all measures are finite.

DEFINITION 5.1. *Let us say that the matchbox manifold $\mathcal{M} = \widetilde{M} \times_{\Gamma} Z$ has a consistent presentation if it has an oriented simplicial presentation $\mathcal{M} = \varprojlim \{X_n, f_n\}$ for which the fibers of the projection $\mathcal{M} \rightarrow X_1$ are subsets of the fibers of the bundle map $\pi : \mathcal{M} \rightarrow M$.*

The class of matchbox manifolds obtained by such a suspension construction is quite large and includes among others all translational tiling spaces of finite local complexity [34], but not all possible orientable examples, [17]. We see below, Proposition 5.4, that this class admits a simple description of its top cohomology $H^d(\mathcal{M})$.

DEFINITION 5.2. *Denote by $\mathfrak{M}(Z)$ the set of all Borel measures on the space Z . For a ring $R = \mathbb{Z}$ or \mathbb{R} , denote by $C(Z; R)$ the R -module of continuous R -valued functions on Z . A positive element of $C(Z; R)$ is a function that takes only non-negative values. A positive homomorphism $C(Z; R) \rightarrow \mathbb{R}$ is an R -linear map which takes positive elements to non-negative numbers. We write $\text{Phom}_R(C(Z; R); \mathbb{R})$ for the set of positive homomorphisms $C(Z; R) \rightarrow \mathbb{R}$.*

LEMMA 5.3.

$$\mathfrak{M}(Z) = \text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z}); \mathbb{R}).$$

Proof. The Riesz Representation theorem tells us that the set of measures $\mathfrak{M}(Z)$ can be identified with $\text{Phom}_{\mathbb{R}}(C(Z; \mathbb{R}); \mathbb{R})$, where $\mu \in \mathfrak{M}(Z)$ corresponds to the positive homomorphism $f \mapsto \int_{\mathbb{Z}} f d\mu$. Any functional $\int_{\mathbb{Z}} - d\mu$ is however determined by its values on \mathbb{R} -valued step functions taking finitely many values; this set of functions can be equated with $C(Z; \mathbb{Z}) \otimes \mathbb{R}$. The lemma follows by noting the equivalence

$$\text{Phom}_{\mathbb{R}}(C(Z; \mathbb{Z}) \otimes \mathbb{R}; \mathbb{R}) \equiv \text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z}); \mathbb{R}).$$

□

PROPOSITION 5.4. *Suppose the d -dimensional matchbox manifold $\mathcal{M} = \widetilde{M} \times_{\Gamma} Z$ has a consistent presentation. Then the top dimension Cech cohomology, $H^d(\mathcal{M})$ can be identified with $C(Z; \mathbb{Z})_{\Gamma}$, the Γ -coinvariants of $C(Z; \mathbb{Z})$.*

Proof. A Serre spectral sequence for the cohomology of \mathcal{M} using the bundle structure

$$Z \longrightarrow \widetilde{M} \times_{\Gamma} Z = \mathcal{M} \xrightarrow{\pi} M$$

yields an E_2 page

$$E_2^{p,q} = \begin{cases} H^p(M; H^0(Z)) = H^p(M; C(Z; \mathbb{Z})) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

This follows from the fact that the Cech cohomology of a totally disconnected space Z is $C(Z; \mathbb{Z})$ in dimension 0 and is trivial in all higher dimensions. The spectral sequence thus collapses, with no extension problems, giving $H^p(\mathcal{M}) = H^p(M; C(Z; \mathbb{Z}))$. To conclude the proof, we show that in general, a group cohomology $H^d(M; A)$ for a closed, orientable triangulated d -manifold with fundamental group Γ and coefficients A with (potentially non-trivial) Γ -action can be identified with the coinvariants A_{Γ} .

Lift the triangulation of M to a triangulation on the universal cover \widetilde{M} , and consider $C_{\Gamma}^d(\widetilde{M}; A)$, the Γ -equivariant d -cochains on \widetilde{M} with values in A . As M is compact these form a free, finite dimensional A -module. As we can find a path from the interior of one d -simplex to that of any other, passing only through $d - 1$ simplices, the cohomology

$$(4) \quad H^d(M; A) = \frac{C_{\Gamma}^d(\widetilde{M}; A)}{\text{Im} \left(\delta^d: C_{\Gamma}^{d-1}(\widetilde{M}; A) \rightarrow C_{\Gamma}^d(\widetilde{M}; A) \right)}$$

is generated by a single copy of A . However, for each $\gamma \in \Gamma$ and each d -simplex S , there is a path from the interior of S to itself which represents γ , and crosses only codimension one simplices. The sum of the coboundaries of these $d - 1$ simplices, taken over all $\gamma \in \Gamma$, show that the quotient (4) is the full coinvariants A_{Γ} . \square

REMARK 5.5. In the situation where the manifold M is also aspherical, we can prove more. This case includes any d -torus, as is the case when \mathcal{M} is a tiling space for a d -dimensional tiling of finite local complexity, and also the case when M is any Riemannian manifold of non-positive curvature. If M is aspherical (so, $\pi_n(M) = 0$ for all $n > 1$), then M is a model for the classifying space $B\Gamma$. The cohomology $H^p(M; C(Z; \mathbb{Z}))$ can thus be identified with the group cohomology $H^p(\Gamma; C(Z; \mathbb{Z}))$. Moreover, the Poincaré duality of the manifold M tells us that Γ is a Poincaré duality group, and this latter property implies that, for any Γ -module A , the group homology and cohomology of Γ with coefficients in A are related by the isomorphism

$$(5) \quad H_n(\Gamma; A) \cong H^{d-n}(\Gamma; A).$$

The conclusion of Proposition 5.4 for these M now follows since

$$H^d(\mathcal{M}) = H^d(M; C(Z; \mathbb{Z})) = H_0(\Gamma; C(Z; \mathbb{Z})) = C(Z; \mathbb{Z})_{\Gamma}$$

where the last equivalence can be taken as the definition of group homology (i.e., that for a given group Γ , the group homologies $H_*(\Gamma; -)$ are the left derived functors of the coinvariant functor $A \mapsto A_{\Gamma}$; see, for example, [10] section II.3). Clearly though,

for such manifolds M more is true and the intermediate dimensional cohomology can be described in a fashion similar to that used in [23] section 3.

COROLLARY 5.6. *Suppose \mathcal{M} is an oriented matchbox manifold of dimension d with a consistent presentation. Then $\mathfrak{M}^\Gamma(Z)$, the Γ -invariant measures on Z , can be identified*

$$\mathfrak{M}^\Gamma(Z) = \text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z})_{\Gamma}; \mathbb{R}) = \text{Phom}_{\mathbb{Z}}(H^d(\mathcal{M}); \mathbb{R}).$$

Proof. As the Γ action on Z induces an action on $C(Z; \mathbb{Z})$ which takes positive elements to positive elements, Lemma 5.3 and a simple adjunction yields the identification

$$\begin{aligned} \mathfrak{M}^\Gamma(Z) &= \text{Positive } \Gamma\text{-invariant } \mathbb{Z}\text{-linear homomorphisms } C(Z; \mathbb{Z}) \rightarrow \mathbb{R} \\ &= \text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z})_{\Gamma}; \mathbb{R}) \\ &= \text{Phom}_{\mathbb{Z}}(H^d(\mathcal{M}); \mathbb{R}), \quad \text{by Lemma 5.4.} \end{aligned}$$

□

The set of positive homomorphisms $\text{Phom}_{\mathbb{Z}}(H^d(\mathcal{M}); \mathbb{R})$, being dual to a cohomological gadget, has a natural homological interpretation, the homology core of Section 4.

THEOREM 5.7. *Let \mathcal{M} be an oriented matchbox manifold of dimension d with a consistent presentation, and assume also that the presentation is homologically \mathbb{Q} -stable. Then for any n , the space of Γ -invariant, Borel measures on Z can be identified with the positive homology core*

$$\mathfrak{M}^\Gamma(Z) = C_{\mathcal{M}}^+(n).$$

Proof. First observe that the usual pairing of homology and cohomology gives a pairing between $H^d(\mathcal{M}) = H^d(\lim_{\leftarrow} X_n) = \lim_{\rightarrow} H^d(X_n)$ and $\lim_{\leftarrow} H_d(X_n)$: denote an element of $\lim_{\rightarrow} H^d(X_n)$ as a sequence $b = (b_n \in H^d(X_n) \mid (f_n)^*(b_n) = b_{n+1})$ and one of $\lim_{\leftarrow} H_d(X_n)$ as a sequence $a = (a_n \in H_d(X_n) \mid (f_n)_*(a_{n+1}) = a_n)$. Then define

$$\langle b, a \rangle = \langle b_n, a_n \rangle.$$

This is well defined, i.e., independent of n , since

$$\langle b_n, a_n \rangle = \langle b_n, (f_n)_*(a_{n+1}) \rangle = \langle (f_n)^*(b_n), a_{n+1} \rangle = \langle b_{n+1}, a_{n+1} \rangle.$$

By stability and finite dimensionality, this pairing gives a duality between $H^d(\mathcal{M}; \mathbb{R})$ and $\lim_{\leftarrow} H_d(X_n; \mathbb{R})$, and so each \mathbb{R} -linear homomorphism $H^d(\mathcal{M}; \mathbb{R}) \rightarrow \mathbb{R}$ is given by a pairing $\langle -, a \rangle$ for some $a \in \lim_{\leftarrow} H_d(X_n; \mathbb{R})$. Hence it suffices to check that the isomorphism of Proposition 5.4 identifies the positive homomorphisms $\text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z})_{\Gamma}; \mathbb{R})$ with those $H^d(\mathcal{M}; \mathbb{R}) \rightarrow \mathbb{R}$ given by elements of the core $C_{\mathcal{M}}^+(n) \subset \lim_{\leftarrow} H_d(X_n; \mathbb{R})$.

Let S be a d -simplex in M , and pick an interior point z . Regard Z as the fibre in \mathcal{M} over z , and let Z_n be the finite, discrete space, the image of Z in X_n . Let β be a continuous function $Z \rightarrow \mathbb{Z}$, representing a free abelian class $[\beta] \in C(Z; \mathbb{Z})_{\Gamma} = H^d(\mathcal{M})$ (we are not interested in torsion classes since these pass to zero after taking \mathbb{R} coefficients). As in the construction of the proof of Proposition 5.4, β can be interpreted as a cocycle on \mathcal{M} by first mapping $Z \times S \rightarrow \mathbb{Z}$ using β on Z (and constant on the S component), extended trivially to the rest of \mathcal{M} . Any such cocycle

is the pullback of a cocycle on X_n , for some sufficiently large n , defined similarly using some function $\beta_n: Z_n \rightarrow \mathbb{Z}$. Then β is a positive function if and only if β_m is, for all $m \geq n$, and if β_m is a positive function, the cohomology class $[\beta_m] \in H^d(X_n)$ is positive in the sense of pairing non-negatively with elements in the positive homology core. Thus any element of $\mathcal{C}_{\mathcal{M}}^+(n)$, considered as a function $\langle -, a \rangle$, restricts to an element of $\text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z})_{\Gamma}; \mathbb{R})$, giving the inclusion $\mathcal{C}_{\mathcal{M}}^+(n) \subset \mathfrak{M}^{\Gamma}(Z)$.

Conversely, by the same construction, any positive element of $C(Z; \mathbb{Z})_{\Gamma}$ can be represented as a positive element of the cochain group $C^d(\mathcal{M}; \mathbb{Z})$, and so any element of $\text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z})_{\Gamma}; \mathbb{R})$ is represented by an element of $\text{Phom}_{\mathbb{Z}}(C^d(\mathcal{M}; \mathbb{Z}), \mathbb{R})$, and moreover one that vanishes on coboundaries, i.e., is a cycle in $\lim_{\leftarrow} H_d(X_n; \mathbb{R})$. Hence $\mathfrak{M}^{\Gamma}(Z) \subset \mathcal{C}_{\mathcal{M}}^+(n)$. \square

REMARK 5.8. Recall that the set of Γ -invariant probability measures on Z can be identified with the convex set in $\text{Phom}_{\mathbb{Z}}(C(Z; \mathbb{Z})_{\Gamma}; \mathbb{R})$ of functionals satisfying $\int_Z \mathbf{1}_Z d\mu = 1$, and the ergodic ones can be identified with the extreme points of this set. Thus, when the conditions of the theorem are met this allows us to identify the set of invariant ergodic probability measures directly.

REMARK 5.9. As noted in the introduction, results of the form of Theorem 5.7 have appeared elsewhere. In particular we highlight the work of Bellissard, Benedetti, and Gambaudo [7] who have a corresponding result for the special cases of spaces of *perfect aperiodic tilings*, i.e., repetitive, aperiodic tilings of euclidean space \mathbb{R}^d of finite local complexity, and thus are suspension matchbox manifolds with underlying manifold M a d -torus. Though it makes use of smoothness assumptions on the underlying manifold, the proof given in [7] might extend to more general settings. The approach here however, purely in the PL category, is greatly aided by establishing our explicit description of top cohomology, Proposition 5.4, of the underlying objects.

REMARK 5.10. Ormes, Radin and Sadun [30] define a notion of positive cohomology cone ('ordered cohomology') in the specific case of spaces of primitive substitution tilings. These correspond to matchbox manifolds with constant expansions

$$\mathcal{M} = \lim_{\leftarrow} \left\{ X \xleftarrow{A} X \xleftarrow{A} X \xleftarrow{A} \dots \right\}$$

and where (some power of) the matrix A^* on d -dimensional cochains $C^d(X, \mathbb{Z}) \rightarrow C^d(X, \mathbb{Z})$ is strictly positive. The construction and proof of the homeomorphism invariance of the resulting cohomology cone is heavily dependent on the application of Perron Frobenius theory, but the duality used in the proof of Theorem 5.7 shows that for these tiling spaces, and with \mathbb{R} coefficients, there is a straightforward duality between the ordered cohomology cone and the homology core. Indeed for this case, these gadgets can be described very explicitly: the homology core is the subspace of $H_d(X; \mathbb{R})$ spanned by the Perron Frobenius eigenvector, and the positive cone in cohomology is the dual cone to this, the half space of vectors which pair non-negatively with the positive PF elements. Integrally however the relationship is potentially a lot more complicated, and depends in particular to questions of the \mathbb{Z} -stability of the expansion.

Nevertheless, the work of this paper shows a way of generalising the construction of [30], defining the ordered cohomology for general matchbox manifolds in terms of dual

cones to the positive homology core; homeomorphism invariance in the general case, at least for examples with uniformly bounded homology ranks of the X_n could then be proved from the invariance of the homology core. Computation in the general case still presents considerable difficulties, especially in the non-constant presentations, examples of which are discussed in the next section, and we suspect that the homology variant, with its natural representation in the vector spaces $H_d(X_n; \mathbb{R})$ is nevertheless the most convenient place for computation.

REMARK 5.11. Theorem 5.7 makes use only of part of the paraphernalia of the homology core, namely the cone of positive elements in $\varprojlim H_d(X_n; \mathbb{R})$, and not the specific representation in the vector space $H_d(X_n; \mathbb{R})$; we shall see the importance of the additional structure in the results of the next section.

6. APPLICATIONS AND EXAMPLES

We begin by considering an example that exploits the connection between the homology core and the structure of the invariant measures of the underlying action. This first example has some overlap with the results of Cortez and Petite [16].

EXAMPLE 6.1. *Solenoids and Γ odometers*

Let M be a PL orientable d -dimensional manifold with fundamental group $\Gamma = \pi_1(M, m_0)$. Consider then a chain of (not necessarily normal) subgroups of finite index greater than 1:

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots \supset \Gamma_i \supset \cdots$$

and the associated Cantor set

$$C = \varprojlim \left\{ \Gamma/\Gamma_1 \leftarrow \Gamma/\Gamma_2 \leftarrow \cdots \leftarrow \Gamma/\Gamma_i \leftarrow \cdots \right\}.$$

There is a natural minimal action of Γ on C given by translation in each factor. The suspension of this action over M then yields a minimal matchbox manifold $\mathcal{M} = \widetilde{M} \times_{\Gamma} C$ which has a consistent presentation in which $X_1 = M$ and $X_n = \widetilde{M}/\Gamma_n$. This presentation can be made simplicial by taking a simplicial structure for M that is then lifted to \widetilde{M} , which in turn pushes down to a simplicial structure for the leaves and the quotients X_n .

THEOREM 6.2. *The action of Γ on C as above is uniquely ergodic.*

Proof. In this case $H_d(X_n)$ is isomorphic to \mathbb{Z} for each n . The induced homology maps are multiplication by the degrees of the corresponding covering maps, which in turn are given by the indices of the subgroups. Thus, this presentation is \mathbb{Q} -stable and each core $\mathcal{C}_{\mathcal{M}}(n)$ and each vector space V_n can be identified with \mathbb{R} . By Theorem 5.7 the action of Γ is uniquely ergodic. \square

We now begin an investigation of how to calculate the homology core for \mathbb{Q} and \mathbb{Z} -stable presentations.

DEFINITION 6.3. *A sequence of matrices of constant rank d with non-negative entries $(M_n)_{n \in \mathbb{N}}$, is recurrent if there are indices $k_1 < \ell_1 \leq k_2 < \ell_2 \leq \dots$ and a matrix B with positive entries satisfying for all n*

$$B = M_{k_n}^{\ell_n - 1}$$

It is known, see e.g. [20, pp. 91–95], that if $(M_n)_{n \in \mathbb{N}}$, is recurrent then there is a $v \in \mathbb{R}^d$ with positive entries satisfying

$$\text{span } v = \bigcap_{n \in \mathbb{N}} M_0^n(\mathcal{C}^d),$$

where \mathcal{C}^d denotes the positive and negative cone in \mathbb{R}^d . Recurrent sequences have been important in the study of S -adic systems, see e.g. [8].

It then follows that if the sequence of matrices $(M_n)_{n \in \mathbb{N}}$ representing the linear maps as described in Theorem 4.6 is recurrent, then $\mathcal{C}_{\mathcal{M}}(n)$ will be a single line for each n . We shall see below in Example 6.7 that this condition is however not necessary for the core to be a single line in each place. In the special case $(M_n)_{n \in \mathbb{N}}$ is a sequence each term of which is the same positive matrix, $\mathcal{C}_{\mathcal{M}}(n)$ is a single line formed by the span of the Perron–Frobenius right eigenvector.

EXAMPLE 6.4. *Substitution tiling spaces*

We illustrate how to distinguish two substitution tiling spaces of dimension one using their homology cores.

$$\begin{aligned} \sigma_1 : \{a, b\} \rightarrow \{a, b\}^* \text{ is given by } & \begin{cases} a \mapsto a^{10}b^7 \\ b \mapsto a^3b^2 \end{cases} \text{ and} \\ \sigma_2 : \{a, b\} \rightarrow \{a, b\}^* \text{ is given by } & \begin{cases} a \mapsto a^{11}b^4 \\ b \mapsto a^3b \end{cases}. \end{aligned}$$

Each substitution σ_i is primitive, aperiodic and is proper (see [4] for the role properness plays in expansions as inverse limits). Thus, the corresponding tiling spaces \mathcal{T}_i (formed by suspending the associated substitution subshift on $\{a, b\}^{\mathbb{Z}}$ over the circle) admit the following presentations [4]

$$(6) \quad \mathcal{T}_i \approx \varprojlim \{X \xleftarrow{f_i} X \xleftarrow{f_i} X \xleftarrow{f_i} \dots\}$$

where X is the wedge of two circles in both cases and the map f_i is the natural one induced by the corresponding substitution σ_i . This can be easily adjusted to yield an oriented simplicial presentation by introducing vertices in X (progressively more as one passes down the sequence). For each copy of X in the two towers we take as a basis for $H_1(X) \approx \mathbb{Z} \oplus \mathbb{Z}$ the cycle corresponding to the a -circle $\approx (1, 0)$ and the cycle corresponding to the b -circle $\approx (0, 1)$. Then each V_n in the two sequences is isomorphic to \mathbb{R}^2 with the corresponding bases. We then have the corresponding towers of the V_n and L_n .

$$(7) \quad \mathbb{R}^2 \xleftarrow{M_1} \mathbb{R}^2 \xleftarrow{M_2} \mathbb{R}^2 \xleftarrow{M_1} \dots$$

where $M_1 = \begin{pmatrix} 10 & 7 \\ 3 & 2 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 11 & 4 \\ 3 & 1 \end{pmatrix}$ represent the corresponding linear transformations with respect to the chosen bases. Observe then that both presentations are \mathbb{Z} -stable, and so by Theorem 4.6 the two tiling spaces are homeomorphic only if their homology cores are in the same $GL(2, \mathbb{Z})$ orbit. As the matrices are positive, by our above remarks the cores at all places are given by the span of the Perron-Frobenius right eigenvector of the corresponding matrix. Such eigenvectors are given by $v_1 := \begin{pmatrix} \frac{1}{7}(4 + \sqrt{37}) \\ 1 \end{pmatrix}$ for M_1 and $v_2 := \begin{pmatrix} \frac{1}{4}(5 + \sqrt{37}) \\ 1 \end{pmatrix}$ for M_2 . As the continued fraction expansions of $\frac{1}{7}(4 + \sqrt{37})$ and $\frac{1}{4}(5 + \sqrt{37})$ are not tail equivalent, the vectors v_1 and v_2 cannot be in the same $GL(2, \mathbb{Z})$ orbit [22, Thm. 174].

EXAMPLE 6.5. *Relation to shape type*

We shall see in Example 6.7 an entire class of shape equivalent spaces that the homology core can distinguish topologically, but here we supplement the pair σ_1, σ_2 from Example 6 with a third substitution that demonstrates a limitation of the homology core for the purposes of topological classification, showing that the core cannot distinguish all shape *inequivalent* spaces.

$$\sigma_3 : \{a, b\} \rightarrow \{a, b\}^* \text{ is given by } \begin{cases} a & \mapsto ababaababaababab \\ b & \mapsto ababa \end{cases}$$

This substitution is not proper, but its square $(\sigma_3)^2$ is proper. The tiling space \mathcal{T}_3 corresponding to σ_3 is the same as the tiling space corresponding to $(\sigma_3)^2$ in the sense that the subshifts of $\{a, b\}^{\mathbb{Z}}$ determined by these substitutions are the same. Thus, again \mathcal{T}_3 admits an oriented simplicial presentation as in Equation 6, where X is again the wedge of two circles, but the map f_3 is induced by the substitution $(\sigma_3)^2$. With respect to the bases as before, the homology tower for \mathcal{T}_3 is as in Equation 7 with M_i replaced by $(M_1)^2$. Thus, the homology core of at all places is identical to that of \mathcal{T}_1 . Observe that the bonding map f_3 yields an automorphism of $\pi_1(K)$ (with base point the point common to both circles) whose inverse can be represented by the automorphism of the free group generated by $\{a, b\}$ given by the square of the following:

$$\begin{aligned} a & \mapsto a^{-1}b^3a^{-1}b^4 \\ b & \mapsto b^{-4}ab^{-3}ab^{-3}a \end{aligned}$$

It follows that the L_1 invariant (see [12]) vanishes for \mathcal{T}_3 . However, the L_1 invariant does not vanish for \mathcal{T}_1 , as can be seen by an application of the folding lemma of Stallings. (See [12] for similar examples.) Thus, although these spaces are not homeomorphic or even shape equivalent, the homology core does not detect this.

In general, once an appropriate presentation has been found as indicated in [2], one can calculate the homology core of a substitution tiling space of higher dimension in a similar way using a single matrix.

While there are many techniques that have been developed to topologically classify minimal substitution tiling spaces as above, the homology core can be applied equally well to non-minimal substitution tiling spaces as studied in [26]. In [26] an inverse limit presentation for non-minimal substitution tiling spaces is developed that allows us to find the homology core.

EXAMPLE 6.6. *Non-minimal substitution tiling spaces*

We now show how we can similarly use the homology core to distinguish two non-minimal substitution tiling spaces of dimension one using their homology cores.

$$\tau_1 : \{a, b, \bar{a}, \bar{b}, x\} \rightarrow \{a, b, \bar{a}, \bar{b}, x\}^* \text{ is given by } \begin{cases} a & \mapsto aba \\ b & \mapsto a \\ \bar{a} & \mapsto \bar{a}\bar{b}\bar{a} \\ \bar{b} & \mapsto \bar{a} \\ x & \mapsto a\bar{a} \end{cases} \text{ and}$$

$$\tau_2 : \{a, b, \bar{a}, \bar{b}, x\} \rightarrow \{a, b, \bar{a}, \bar{b}, x\}^* \text{ is given by } \begin{cases} a & \mapsto aab \\ b & \mapsto ab \\ \bar{a} & \mapsto \bar{a}\bar{a}\bar{b} \\ \bar{b} & \mapsto \bar{a} \\ x & \mapsto b\bar{a} \end{cases}$$

Substitutions of this type are considered in [26, Section 6], where the tiling space has two disjoint minimal sets that are both in the closure of a single orbit (the “bridge”) that results from the sequences stemming from the substitution applied to x , although no x appears in the ultimate subshift resulting from the substitution. In these cases the tiling space has the form

$$\mathcal{T}_i \approx \varprojlim \{X_i \xleftarrow{f_i} X_i \xleftarrow{f_i} X_i \xleftarrow{f_i} \dots\}$$

where X_i is formed by joining the complexes associated to the two minimal sets by an arc. The bonding map f_1 induces an isomorphism $(f_1)_*$ on $H_1(X_1, \mathbb{Z}) \approx \mathbb{Z}^4$

represented by $M_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. As $(f_1)_*$ is an isomorphism, we see that this

presentation is \mathbb{Z} -stable. Direct analysis of the matrix M_1 reveals that the homology

core at any place is the union of the two sectors bounded by the spans of $\begin{pmatrix} 1 + \sqrt{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}$

and $\begin{pmatrix} 0 \\ 0 \\ 1 + \sqrt{2} \\ 1 \end{pmatrix}$. Similarly the bonding map f_2 induces an isomorphism $(f_2)_*$ on

$H_1(X_2, \mathbb{Z}) \approx \mathbb{Z}^4$ represented by $M_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ with resulting homology core

at any place the union of the two sectors bounded by the spans of $\begin{pmatrix} \frac{1}{2}(1 + \sqrt{5}) \\ 1 \\ 0 \\ 0 \end{pmatrix}$

and $\begin{pmatrix} 0 \\ 0 \\ 1 + \sqrt{2} \\ 1 \end{pmatrix}$. Due to the \mathbb{Z} -stability of these presentations, by Theorem 4.6 \mathcal{T}_1

and \mathcal{T}_2 are homeomorphic only if there there is a matrix in $GL(4, \mathbb{Z})$ that permutes their homology cores, which is clearly impossible since $\sqrt{2}$ is not rationally related to $\sqrt{5}$, and we see directly that \mathcal{T}_1 and \mathcal{T}_2 are not homeomorphic. By Theorem 5.7 each of the subshifts X_{τ_i} admit two invariant Borel ergodic, probability measures.

General homeomorphisms of the Cantor set with multiple invariant ergodic measures are studied in detail in [9], and while our techniques could be applied equally well to suspensions of the homeomorphisms described there, to effectively calculate homology cores one would first need to develop the theory of their inverse limit presentations and their relation to the incidence matrices they consider.

We now see how one can apply Theorem 4.6 to great advantage to topologically classify some natural classes of spaces that are not substitution tiling spaces but matchbox manifolds of dimension one.

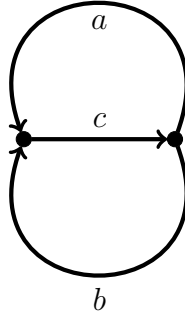
For convenience to make indices match their usual interpretations, in the following two examples we will index inverse sequences starting with index 0.

EXAMPLE 6.7. *Continued fractions*

For simplicity (as it does not affect the homology calculations) we represent K as the CW complex depicted in Figure 1 with three one cells with the indicated orientations and two vertices.

For each positive integer n let $f_n : K \rightarrow K$ be the map defined by

$$\begin{array}{l} a \longrightarrow c \\ b \longrightarrow ca \overbrace{cb \cdots cb}^{n-1 \text{ copies}} c \\ c \longrightarrow b \end{array}$$

FIGURE 1. The complex K

where f_n maps each cell onto cells in the indicated order from left to right, preserving orientation. For each sequence of positive integers $N = (n_0, n_1, \dots)$, we define the orientable matchbox manifold

$$(8) \quad X_N := \varprojlim \{K \xleftarrow{f_{n_0}} K \xleftarrow{f_{n_1}} K \xleftarrow{f_{n_2}} K \xleftarrow{f_{n_3}} \dots\}.$$

For homology calculations, we use the classes $[z_1]$ and $[z_2]$ of the cycles $z_1 \sim cb$ and $z_2 \sim ca$ as generators of $H_1(K, \mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}$. With respect to these generators, we have the induced homomorphism on $H_1(K, \mathbb{Z})$ given by

$$(f_n)_* \sim \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}.$$

Notice that $(f_n)_*$ is an isomorphism for each n , and so each presentation as given in Equation 8 is \mathbb{Z} -stable, and so we may apply Theorem 4.6 to the family of spaces

$$\mathfrak{X} := \{X_N \mid N \text{ is a sequence of positive integers}\}.$$

Observe that

$$\begin{pmatrix} n_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} n_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$$

where $\frac{p_k}{q_k} = [n_0, n_1, n_2, \dots, n_k]$ in continued fraction notation. Observe that with $\alpha_N := [n_0, n_1, n_2, \dots]$, we have that $\lim_{k \rightarrow \infty} \frac{p_k}{q_k} = \alpha_N$. Now $\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$ maps the positive and negative cones in V_k to the sectors in V_0 bounded by the lines spanned by $\begin{pmatrix} p_k \\ q_k \end{pmatrix}$ and $\begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix}$. Hence, we have that the homology core of X_N at place zero is given by

$$\text{span} \begin{pmatrix} \alpha_N \\ 1 \end{pmatrix} = \bigcap_{n \in \mathbb{N}} M_0^n(\mathcal{C}_n).$$

Hence, the corresponding \mathbb{Z} action on the Cantor set is uniquely ergodic by Theorem 5.7, and by Theorem 4.6 the spaces X_N and X_M are homeomorphic only if there is a matrix in $GL(2, \mathbb{Z})$ that maps $\begin{pmatrix} \alpha_N \\ 1 \end{pmatrix}$ into $\text{span} \begin{pmatrix} \alpha_M \\ 1 \end{pmatrix}$. By the classical theorem

on the classification of numbers by their continued fraction expansions, [22, Thm. 174], this happens precisely when the continued fraction expansions for α_N and α_M share a common tail: there exist k and l such that for all positive integers i we have $m_{k+i} = n_{l+i}$. When this happens, the inverse sequences defining X_N and X_M have equal cofinal subsequences and so are clearly homeomorphic. Thus we obtain the following classification of the spaces in \mathfrak{X} , c.f. [6],[18].

THEOREM 6.8. *X_M and X_N are homeomorphic if and only if the sequences M and N share a common tail.*

Observe that *all* the spaces in \mathfrak{X} are shape equivalent to K and hence to a wedge of two circles. Hence, while the Example 6.5 illustrates that there are spaces for which shape invariants (such as the L_1 invariant) can distinguish spaces that are not distinguished by their homology cores, there are also large classes of shape equivalent spaces that the homology core can distinguish.

Similar examples of families with matrices of larger size can be obtained using the matrices corresponding to higher dimensional versions of continued fractions, see, e.g., [21].

EXAMPLE 6.9. *Generalised continued fractions*

With K as before, for each pair of positive integers m, n let $f_{m,n} : K \rightarrow K$ be the map defined by

$$\begin{aligned} a &\rightarrow c \\ b &\rightarrow \underbrace{ca \cdots ca}_{m \text{ copies}} \underbrace{cb \cdots cb}_{n-1 \text{ copies}} c \\ c &\rightarrow b \end{aligned}$$

With respect to the same generators as before, we have the induced isomorphism on $H_1(K, \mathbb{Q})$ given by

$$(f_{m,n})_* \sim \begin{pmatrix} n & 1 \\ m & 0 \end{pmatrix}.$$

Thus while our presentation of $X(\alpha, \beta)$ is not generally \mathbb{Z} -stable, it is \mathbb{Q} -stable. For given sequences of positive integers $\alpha = (a_1, a_2, \dots), \beta = (b_0, b_1, \dots)$ we will then have the orientable matchbox manifold given by

$$X(\alpha, \beta) := \varprojlim \{K \xleftarrow{f_{1,b_0}} K \xleftarrow{f_{a_1,b_1}} K \xleftarrow{f_{a_2,b_2}} K \xleftarrow{f_{a_3,b_3}} \dots\}.$$

For given sequences α and β we then have

$$\begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ a_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_k & 1 \\ a_k & 0 \end{pmatrix} = \begin{pmatrix} A_k & A_{k-1} \\ B_k & B_{k-1} \end{pmatrix}$$

where $\frac{A_k}{B_k}$ is the k -th convergent of the generalised continued fraction

$$b_0 + \mathbf{K}_{n=1}^{\infty} \frac{a_n}{b_n},$$

where

$$b_0 + \mathbf{K}_{n=1}^{\infty} \frac{a_n}{b_n}$$

is used to denote

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \dots}}}}$$

see, e.g., [24]. Now $\begin{pmatrix} A_k & A_{k-1} \\ B_k & B_{k-1} \end{pmatrix}$ maps the positive and negative cones in V_k to the sectors in V_0 bounded by the lines spanned by $\begin{pmatrix} \frac{A_k}{B_k} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{A_{k-1}}{B_{k-1}} \\ 1 \end{pmatrix}$. Hence, the key to understanding the cores for these spaces is the limiting behaviour of these convergents.

We can rewrite our general continued fraction as an equivalent one (one with the same convergents) as follows [24, 2.3.24]

$$b_0 + \mathbf{K}_{n=1}^{\infty} \frac{a_n}{b_n} \approx b_0 + \frac{1}{b_1/a_1 + \frac{1}{a_1 b_2/a_2 + \frac{1}{b_3 a_2/a_1 a_3 + \frac{1}{a_1 a_2 b_4/a_2 a_4 + \dots}}}}$$

where now we have positive rational entries. By the theorem of Van Vleck [24, Thm. 4.29], for a continued fraction with positive entries of the form

$$k_0 + \mathbf{K}_{n=1}^{\infty} \frac{1}{k_n},$$

we have that if $\sum_{i=1}^{\infty} k_i$ converges, then the even and odd convergents converge monotonically to different values (the larger/smaller terms decrease/increase), and if $\sum_{i=1}^{\infty} k_i$ diverges, then the convergents converge to a single value.

In what follows, for a given continued fraction $b_0 + \mathbf{K}_{n=1}^{\infty} \frac{a_n}{b_n}$ we use the notation $k_0 = b_0$, $k_1 = b_1/a_1$ and recursively $k_n = \frac{b_n b_{n-1}}{a_n k_{n-1}}$ for $n > 1$. These k_i are then the values in the equivalent continued fraction $k_0 + \mathbf{K}_{n=1}^{\infty} \frac{1}{k_n}$. With this notation, we have the following conclusions.

PROPOSITION 6.10.

For $X(\alpha, \beta)$ associated to the continued fraction $b_0 + \mathbf{K}_{n=1}^{\infty} \frac{a_n}{b_n}$

- (I) If $\sum_n k_n$ diverges, then with $\ell = \lim \frac{A_n}{B_n}$, we have the homology core at place zero given by

$$\text{span} \begin{pmatrix} \ell \\ 1 \end{pmatrix} = \bigcap_{n \in \mathbb{N}} M_0^n(\mathcal{C}_n).$$

- (II) If $\sum_n^\infty k_n$ converges, then with $\ell_E = \lim \frac{A_{2n}}{B_{2n}}$ and $\ell_O = \lim \frac{A_{2n-1}}{B_{2n-1}}$ we have the homology core at place zero $\bigcap_{n \in \mathbb{N}} M_0^n(\mathcal{C}_n)$ is the union of the two sectors in V_0 bounded by $\text{span} \begin{pmatrix} \ell_E \\ 1 \end{pmatrix}$ and $\text{span} \begin{pmatrix} \ell_O \\ 1 \end{pmatrix}$.

□

Observe that two spaces $X(\alpha, \beta)$ and $X(\alpha', \beta')$ are homeomorphic only if they both correspond to the same case (I) or (II) as above as the dimension of the homology core depends on which of the two cases we are in.

We then can directly apply Theorem 4.6 and Theorem 5.7 to obtain the following.

PROPOSITION 6.11.

- (1) If $X(\alpha, \beta)$ and $X(\alpha', \beta')$ both satisfy the condition in (I) above, then $X(\alpha, \beta)$ and $X(\alpha', \beta')$ are homeomorphic only if the vectors $\begin{pmatrix} \ell \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \ell' \\ 1 \end{pmatrix}$ are in the same $GL(2, \mathbb{Q})$ orbit, and the corresponding \mathbb{Z} actions are uniquely ergodic.
- (2) If both $X(\alpha, \beta)$ and $X(\alpha', \beta')$ satisfy the conditions in (II) above, then $X(\alpha, \beta)$ and $X(\alpha', \beta')$ are homeomorphic only if $\begin{pmatrix} \ell_E \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \ell_O \\ 1 \end{pmatrix}$ are (as a pair) in the same $GL(2, \mathbb{Q})$ orbit as $\begin{pmatrix} \ell'_E \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \ell'_O \\ 1 \end{pmatrix}$, and in this case the corresponding \mathbb{Z} actions have two invariant ergodic probability measures.

□

The spaces in Example 6.7 provide a class of spaces that all satisfy condition (I), but there are many other spaces as well. Besides the recurrence of the sequence of matrices, a simple criterion that guarantees that we are in case (I) is given by $b_n > a_n$ for sufficiently large n , and a simple example of case (II) is given by $\alpha = (2^{2n-1})_{n \in \mathbb{Z}^+}$, $\beta = (1, 1, 1, \dots)$. It appears to be quite difficult to calculate the values of $\begin{pmatrix} \ell_E \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \ell_O \\ 1 \end{pmatrix}$ in case (II), which makes it quite difficult to give a natural classification.

PROBLEM 6.12. *Identify a natural class of spaces $X(\alpha, \beta)$ of class (II) for which the corresponding homology cores lead to a classification.*

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