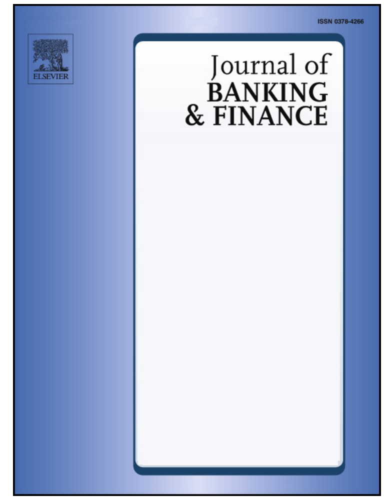

Reducing the Impact of Real Estate Foreclosures with Amortizing Participation Mortgages

Rafal M. Wojakowski, M. Shahid Ebrahim, Mark B. Shackleton

PII: S0378-4266(16)30076-0
DOI: http://dx.doi.org/10.1016/j.jbankfin.2016.05.005
Reference: JBF 4960

To appear in: *Journal of Banking & Finance*

Reducing the Impact of Real Estate Foreclosures with Amortizing Participation Mortgages[1]

Rafal M. Wojakowski[2] M. Shahid Ebrahim[3] Mark B. Shackleton[4]

June 21, 2016

[1]We thank Brent Ambrose, John Duca, Robert Edelstein, Douglas McManus, Rahim Munshi, Radu Tunaru, Tu Yong, participants of the FEBS Guildford and ASSA-AREUEA San Francisco as well as Carol Alexander, the Editor, and two anonymous Referees for their helpful comments on our paper.

[2]**Corresponding author:** Faculty of Business, Economics and Law, University of Surrey, Guildford GU2 7XH, UK. Phone: +44(1483)683477; `http://www.wojakowski.net` ; Email: `r.wojakowski@surrey.ac.uk`

[3]Durham University Business School, Durham, UK. Phone: +44(191)3345405, Email: `m.s.ebrahim@durham.ac.uk`

[4]Lancaster University Management School, Lancaster, UK. Phone: +44(1524)594131, Email: `m.shackleton@lancaster.ac.uk`

Abstract

We employ Amortizing Participation Mortgage (APM) to offer a novel *ex post* renegotiation method of a foreclosure. APM belongs to the family of home loan credit facilities advocated in the *Dodd-Frank Wall Street Reform and Consumer Protection Act 2010*. In our framework, APMs reduce the endemic agency costs of debt by improving affordability. These benefits increase the demand for real estate in bust times and reduce fragility of the financial system thereby preventing foreclosures. We evaluate APMs in a stochastic control framework and provide solutions for an optimal amortization schedule. We generalize our approach to partially amortizing and commercial mortgages which encompass balloon payments. Finally, we provide concrete numerical examples of home loan modifications. We also offer detailed sensitivity analysis to market parameters such as house price volatility and interest rates.

Keywords: Dodd-Frank Act, Foreclosure, Home loan modifications, Prepayment, Default, Partial amortization, Commercial mortgage, Residential mortgage, Shared income mortgage, Indexation.

JEL: R31, R38, G21

"Mathematical finance does show a way towards reducing risks and [...] help prevent [...] crisis."

Robert Shiller, London School of Economics, 20 May 2009

1 Introduction

Problems in the mortgage lending sector of the mid 2000's resulted in individual borrower's defaults, liquidity issues, runs on banks, capital adequacy measures and, because of the securitization, were felt in places located a long way from the US. The ensuing crisis of confidence, going down all the way from banks to individual customers, amplified in Autumn 2008. In 2010 the *Dodd-Frank Wall Street Reform and Consumer Protection Act* called for a *"widespread use of shared appreciation mortgages (SAMs) to strengthen local housing markets, provide new opportunities for affordable homeownership, and enable homeowners at risk of foreclosure to refinance or modify their mortgages."*[1] The alternative mortgage scheme we structure in this paper, called Amortizing Participation Mortgage (APM), is an example of such innovative facility. It belongs in fact to the same product family as the SAMs advocated by Dodd-Frank.

The purpose of this paper is to illustrate how the APM can serve as a workout loan in the aftermath of a default. An APM is an extension of the Participation Mortgage (PM) facility with the added feature of amortization to lessen the agency issues. We extend the approach initiated in Ebrahim, Shackleton, and Wojakowski (2011)[2] to value APMs with help of *profit caps* closed-form formulae of Shackleton and Wojakowski (2007).

Our first contribution in this paper is to illustrate that unlike interest bearing mortgages, the optimal structure of Participation Mortgages (PMs) requires managing a stochas-

[1]Dodd-Frank Wall Street Reform and Consumer Protection Act (2010), Title XIV: Mortgage Reform and Anti-Predatory Lending Act, Sec. 1406 Study of shared appreciation mortgages.

[2]See also Shiller, Wojakowski, Ebrahim, and Shackleton (2013).

tic amortization schedule. Indeed, because a PMs' repayment flow is composed of two, deterministic and stochastic flows, optimal participation ratios are a solution to a class of intertemporal optimal control problems.[3] We illustrate this specificity of participating mortgages by formulating an *optimal stochastic control* problem. We then provide the corresponding Hamilton-Jacobi-Bellman equation and discuss the implications of the optimal control, which we obtain in closed form. We show that the participating variants of repayment mortgages reduce agency costs leading to an increase in the value of property.

The second contribution of this paper is to recommend the employment of the Amortizing Participation Mortgages and its various offshoots in the form of ASAMs (Amortizing Shared Appreciation Mortgages) and Amortizing Shared Income Mortgages (ASIMs) to revive the real sector of the economy in the aftermath of the subprime meltdown. This is consistent with the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010. We follow this up by conducting comparative statics.

Our assertions are corroborated by Shiller (2014), who states that a "one-size fit all solution" provided by the standard Fixed Rate Mortgage (FRM) is not the optimal solution. He questions why housing finance is still stuck in this primitive state. As roughly 10.7% of mortgage loans encompassing $345.1 billion in homes are in negative equity at the second quarter of 2014 (see CoreLogic, 2014), his views are substantiated by empirical data. A similar view is affirmed by Real Capital Analytics, a New York real estate research firm for commercial mortgages. They point out that more than $160 billion of commercial properties are either in default, foreclosed or bankrupt (see Real Capital Analytics, 2014). A hike in interest rates during the economic recovery is expected to put pressure on valuations, complicate refinancing, and hinder debt servicing. This would cause further dislocations in real estate markets.

[3]Stochastic optimal control techniques were originally introduced in finance by Merton (1969) and Merton (1971) in the context of optimal consumption and portfolio problems. For an application to optimal mortgage refinancing see Lee and Rosenfield (2005).

Designing an innovative financial product is not easy. Since the benefits of innovations, in due course, become public goods available to other mortgage originators, it may not pay innovators to bear the costs of creating. The only option available is for academics to serve as drivers for their advancement.

Our paper is organized as follows. In the next section we discuss the participating mortgage literature. In the following sections we introduce assumptions of our participating mortgage valuation model and we price the APMs and its variants in the form of ASAMs and ASIMs. We focus on the problem of optimally controlling repayment of various amortizing participating mortgages. We then conduct comparative statics along with illustrative examples and extensions. The final section concludes.

2 Participation Mortgages

Shared Appreciation Mortgages (SAMs) advocated in the Dodd-Frank are in fact only one of the many Participation Mortgages (PMs), which also include shared income and shared ownership mortgages. A PM[4] allows a financier to get a fraction of cash flows generated by the underlying collateral (i.e., the property). If the facility is of commercial type, possibilities include taking a portion of the operating income, keeping a fraction of cash flows after senior debt service and/or sharing with the borrower (and owner) the profits from selling off the property. The borrower benefits from a higher loan to value ratio and/or has the contract interest rate lowered. PMs improve valuation and sharing rules in the real estate sector. These products allow mortgage borrowers to remain property owners of their homes by sharing ownership, appreciation or rental income of their property with the lender.

A PM can be employed to resolve the classic mortgage tilt problem. Where high infla-

[4]See Ebrahim (1996) and Ebrahim and Hussain (2010) for discussions of: a) Pareto-superiority of PMs and, b) adequacy of PMs and convertible securities for a developed financial market economy, respectively.

tion rates embedded in the mortgage financing rates impact affordability, an APM helps to revitalize the economy and offers borrowers better alternatives to interest-based products. An APM: (i) improves affordability; and (ii) reduces the endemic agency costs of debt. Resolving these twin issues simultaneously: (i) increases the demand for real estate; while (ii) reducing the fragility of the financial intermediation system. The end result is an improvement in the value of real assets via reduction in deadweight costs of foreclosure, in conjunction with a resilient financial architecture promoting economic growth, as described in our results.

Assets such as undeveloped land or properties, which possess growth options, may yield low or negligible income in the initial stage of their life cycle (see Titman, 1985). The negligible initial yield is due to the huge front end costs of constructing roads, utilities, sidewalks etc. while the generation of cash flows is deferred to later in life cycle stages. In such a context, PMs help reconcile diverging interests of lenders and investors. Lenders are included by sharing the upside potential of a project. Borrowers obtain financing they would otherwise not obtain, because of initially very low level of generated cash flows.

Shiller (2014) states that Participation Mortgages were adopted in a limited way in the form of Shared Appreciation Mortgages (SAMs) for residential real estate in the 1990s by the Bank of Scotland and Bear Stearns in the United Kingdom (see again Shiller, 2014). The character of SAMs was tarnished when home-owners, who borrowed against homes, lost out to the financial intermediary the bulk of the appreciation. Some found they were "locked" in their property and could not downsize after prices moved up without giving up large gains to the mortgage provider. The gains conveyed to lenders were too high but the residents (who were near retirement age), would not have lost financially had they not wanted to sell their homes 5-10 years after the (zero coupon) mortgage had its conversion strike and sharing price fixed. This incensed homeowners and the media, leading to a class action suit against the issuers. The law suit was eventually settled out of court for an undisclosed sum but the reputational damage of the facility was permanent.

2.1 The original model of a non-amortizing PM: Assumptions

We assume the existence of a complete market, reliable economic environment where mortgage contracts can be priced via risk-neutral methods. In particular, if real estate is to serve as collateral, it requires the presence of foreclosure procedures, property rights and reliable valuation methods (see Levine, Loayza, and Beck, 2000). This assumes that information costs for underwriting mortgages are not binding for the following reasons:

1. Mortgage underwriters (as principals in a debt contract) can costlessly decipher any proprietary (i.e., ex-ante) information held by the prospective real estate owners (the agents in the debt contract) for the following reasons. First, the funds are not directly transferred to the borrower. It is rather transferred to the seller of the property after a very elaborate due diligence method, where the title of the property is confirmed in the escrow process,[5,6] its structural soundness is verified by an engineer, and its value estimated through an appraisal process. Second, a financier can evaluate the ex-ante probability distribution of payoffs of a property from its ex-post risk and return information. This is accomplished by trading financial claims over a multi-period horizon (see Hosios and Peters, 1989).

2. Underwriters can also deter moral hazard, ensuing from ex-post change in borrower behavior, by undertaking the following preventive measures. First, since the un-

[5] An escrow process is a contractual (or monitoring) arrangement whereby a third party receives documentation and funds on behalf of transacting parties and assists in the transfer of title and the disbursement of the funds for closing the transaction. The presence of a non-related third party to verify the title of the property (or asset) and to facilitate its exchange for an agreed upon fee helps mitigate adverse selection. This is because the third party ensures that the asset (or property) that the buyer has offered to buy is exactly the same and that the amount agreed upon is transferred to the legal owner of the asset.

[6] An escrow also includes a trust account held in the borrower's name to pay his/her obligations such as property taxes and insurance premiums. This alleviates moral hazard. This is because the presence of the third party (or monitoring agent) reduces the risk of the financier.

derlying collateral is immobile (i.e., fixed), the borrower cannot run away with it. Second, the borrower cannot easily dispose of the assets without paying off the loan against it as the title of the property has a lien of the bank against it. Third, financier can add iron-clad covenants to the mortgage contract to preserve the value of the underlying collateral (Smith, Jr. and Warner, 1979). These include: (i) adequate maintenance of the property; (ii) payment of taxes; and (iii) minimum insurance coverage.[7]

In what follows we assume that:

1. Implicitly, any monitoring costs are added ex-ante into the loan contract (along with any mandatory property maintenance, tax payments, and insurance coverage, etc.) and are borne by the borrower.

2. Financing the business activity (prerequisite for generating the profit flow, which we assume is present at the valuation time $t = 0$) has already been sorted out, e.g. by issuing shares, and does not interfere with financing the building (e.g. an office or a warehouse) via mortgage.

2.2 Mortgage loans' identity

In a mortgage financing situation, a business or a prospective homeowner needs to borrow an amount Q_0 to acquire a property valued at H_0 (where subscripts denote initial time $t = 0$). Typically, the lender imposes a maximum loan to value ratio Q_0/H_0 lower than 100% and the difference is made up by the buyer's deposit. The mortgage loan

[7] We are grateful to an anonymous referee, who asked us to contrast real estate mortgages with consumer loans generally made to an individual on a non-secured basis for personal, family or household purposes. These unsecured facilities have a high chance of misappropriation due to both adverse selection as well as moral hazard. This is because there may not be any escrow process and the funds may be transferred directly to the borrower. The facility may also not be backed by real (or durable) assets making it easier to deceive the lender ex-post. The costs here are higher because of high default rates due to their unsecured nature of the facility.

contract then defines the time to maturity T (also known as tenure) as well as a coupon schedule $\{x_t : t \in [0, T]\}$. Depending on the type of contract, future coupons x_t may not be known at the time of origination. Their computation often requires information which will become known in the future. The contract must nevertheless specify rules as to how time-dependent coupons x_t should be computed. A Fixed Rate Mortgage (FRM), for example, will assume fixed coupons $x_t = x$; an Adjustable Rate Mortgage (ARM) will specify a premium π to be added to the riskless interest rate[8] r_t to compute the interest due $(\pi + r_t)\, Q_t$, where Q_t is the balance outstanding at time t. In case of *amortizing* loans a fraction of x_t systematically repays[9] a portion of initial capital, reducing default risk over time.

In arbitrage-free complete financial markets, there exists a secondary market for coupons x_t, for example in the form of securitized mortgage pools. Finance theory then tells us that the relationship between present values and future cash flows takes the following form

$$Q_0 = E\left[\int_0^T e^{-rt} x_t \, dt + e^{-rT} Q_T\right] \,, \tag{1}$$

where Q_T is the final *balloon payment* and $E[\cdot]$ is the risk-neutral expectation taken at time $t = 0$ under the unique equivalent martingale measure[10] $\mathcal{Q}$ i.e. $E[z] = E^{\mathcal{Q}}[z | \mathcal{F}_0]$. This identity is a very general expression. Note that we assumed a continuous time and continuous coupon structure, where the infinitesimal coupon *amount* paid during time dt is $x_t dt$, i.e. the coupon *flow rate* (per unit of time) is x_t. To keep things simple and focus on shared income features of APMs, which are the topic of this paper, we assume a flat

[8]During the lifetime of the contract the risk premium π will not change much, compared to the short term interest rate r_t. Therefore, as a first approximation, it can be assumed constant.

[9]NB: this paper works in continuous time, e.g. $r = \ln(1 + R)$, etc. where r, R are continuously and discretely compounded interest rates, respectively.

[10]For expectations taken at general time $t \geq 0$ we use the notation $E_t[x] = E^{\mathcal{Q}}[x | \mathcal{F}_t]$, where $\mathcal{F}_t$ is information available at time t.

(constant) term structure at r. At this stage it can be seen, that under absence of other risks (e.g. default risk) a *non-amortizing* mortgage with contractually constant coupons will have them set at $x = rQ_0$ and the contractual balloon payment will be $Q_T = Q_0$. This is a consequence of assumed risk-neutrality. In reality, the presence of risks will command an additional risk premium π, resulting in $x = iQ_0$ where $i = r + \pi$.

2.3 Non-amortizing participating mortgages and the Dodd-Frank Act

Specific forms of non-amortizing participating mortgages have been discussed in detail by Ebrahim et al. (2011). As an introduction to the more interesting, *amortizing* form, we therefore provide here only a brief re-cap of pertinent features and most illustrative results. We start with definitions needed in analyzing *any* form of participating mortgages, including the amortizing ones.

Assuming as before that trading in underlying assets is possible[11] so that hedging is possible, the *incoming cash flow* process P_t can be written as e.g. in He (2009)

$$dP_t = (r - \delta)\, P_t dt + \sigma P_t dZ_t \,, \tag{2}$$

where Z_t is the corresponding Brownian motion under the risk-neutral measure $\mathcal{Q}$ and r is the risk-free interest rate. The *cash yield* δ is typical to the given type of business activity or employment type and is analogous to the dividend rate of a stock. Incoming cash flow dynamics, i.e. the geometric Brownian motion (2), is adequate to describe, for example, the operating profit of a commercial property. In case of income flow from employment,

[11] Our assumption is analogous to that underlying the literature on real options which also requires complete and arbitrage-free markets. The lender offering the borrower to effectively exchange part of its obligation for a share in profits effectively sells a call option on the borrower's income. In order to delta-hedge, the lender will have to partially sell some quantity of traded claims on borrower's income in a sufficiently liquid market. This de facto assumes existence of such "macro markets," e.g. real estate futures, on macroeconomic variables (individual income indexes, real estate indexes, etc.) as stipulated e.g. in Shiller (1993).

e.g. wages or salary, δ can be interpreted as cash outflow summarizing various professional costs which must be covered to manage the asset. Note also that the total current value of risky profit flow can be computed as a discounted risk-neutral expectation and is given by $A_0 = \int_0^\infty e^{-rt} E\left[P_t\right] dt = P_0/\delta$. In particular, the total current value A is driven by the same dynamics as P i.e. (2). Furthermore, because $P_0 = \delta A_0$, the current profit cash flow P_0 can be represented as a constant proportion δ ("dividend" or cash flow yield) of it's present value.[12]

A commercial property usually represents an asset in its own right, even without hosting a business. We therefore assume that real estate prices H are well described by the following risk-neutral dynamics

$$dH_t = \left(r - \delta_H\right) H_t dt + \sigma_H H_t dZ_t^H \ , \tag{3}$$

where δ_H is the "rental rate" or "service flow" (see e.g. Kau, Keenan, Muller, and Epperson, 1992). Parameter σ_H is the real estate volatility and Z_t^H is the standard Brownian motion driving real estate values. For simplicity, we assume zero correlation of house price returns with profit cash flows, i.e. $\rho = 0$. Consequently, the risk-neutral expectation of $dZ_t dZ_t^H = \rho dt$ also equals zero.[13]

A general Participation Mortgage (PM) can now be characterized by further specifying

[12] If cash flows are zero in some periods known in advance our model can be split into finite sub-periods and our pricing methods applied within those sub-periods where cash flows are positive. The resulting present value would then be the sum of sub-period's present values.

[13] See also footnote 17.

the condition (1) to

$$Q_0 = E\left[\int_0^T e^{-rt} x_t dt + e^{-rT} Q_T\right] \tag{4}$$

where (5)

$$x_t = \underbrace{i_t Q_t}_{\text{interest flow}} + \underbrace{\theta_{P,t}\,(P_t - K)^+}_{\text{share in income flow}} \tag{6}$$

$$Q_T = \underbrace{\theta_{H,T}\,(H_T - H_0)^+}_{\text{share in appreciation}} + \underbrace{Q_{T,\theta}}_{\text{balloon payment}} \quad . \tag{7}$$

The first (second) term within the square brackets in expression (4) is the present value of intermediate (terminal) payments, where K is a fixed *profit threshold* above which income participation is payable and $z^+ = \max\{z, 0\}$ is the *positive part* function, equal to z if $z > 0$ and equal to zero otherwise. The *appreciation threshold* need not be set to the initial real estate value H_0. However, it is typically expected (but cannot be guaranteed) that property will appreciate until maturity, so that $H_T > H_0$. In contrast, participation in the income flow starts from time $t = 0$, so K would typically be set below P_0. A contract then further characterizes a PM by specifying the three remaining parameters $\{i_t, \theta_{P,t}, \theta_{H,T}\}$ i.e. the contract rate schedule i_t, the income participation schedule $\theta_{P,t}$ and the appreciation proportion $\theta_{H,T}$. In practice, this should be the result of negotiating between borrower and lender at the onset of the contract. In principle, these three quantities can be defined so as to depend on current time t, tenure T or be indexed on some economic indicator such as a central bank base rate or a consumer price index. In (7) notation $Q_{T,\theta}$ has been used to reflect dependence of the terminal payment on participation $\theta_{H,T}$. Note also that, for a given Q_0, the terminal balloon $Q_{T,\theta}$ is implicitly dependent on the choice of the income participation policy $\{\theta_{P,t} : t \in [0, T]\}$ as well as on other parameters of the model. For what follows, however, we will focus on the simple time-homogeneous case where parameters are all contractually set at origination and do not change in time (i.e. no tenure effects, etc.). Consequently, we drop irrelevant subscripts in our notation. For example we will now use notation θ_P and θ_H for income and appreciation participation parameters, respectively.

More importantly, for the equality to hold in (4), it is impossible to set i, θ_P and θ_H independently. This feature has been exploited by mortgage originators to big advantage and extent to create new types of mortgages. We enumerate several particular cases of interest below.

Case 1 *If the income participation is set to zero ($\theta_P = 0$), the property appreciation participation ratio can be increased away from zero ($\theta_H > 0$) and, simultaneously, the contract rate i can be reduced. This is the well established **Shared Appreciation Mortgage (SAM)**, which has recently been advocated in the Dodd-Frank act to deal with foreclosure threat ex ante.*

Case 2 *If the property appreciation participation ratio is set to zero ($\theta_H = 0$), a positive income participation fraction $\theta_P > 0$ can be agreed and the required interest rate i can be reduced. This is the **Shared Income Mortgage (SIM)** discussed in the present paper.*

2.4 Relationship to FRMs, ARMs, PLAMs and DIMs

When both θ_P, θ_H are set to zero we obtain the particular case of a standard mortgage, either FRM or ARM. In the case of ARMs, i is indexed on some benchmark interest rate. Price Level Adjusted Mortgages (PLAMs) are obtained if the contractual interest flow iQ is indexed (via Q) on a consumer price index (see Modigliani, 1974 and Lessard and Modigliani, 1975). Dual Indexed Mortgages (DIMs), advocated by World Bank[14] and deployed in 1990's to stimulate house ownership in developing economies such as Mexico,

[14]See Chiquier (1998), Chiquier and Lea (2009), Buckley, Lipman, and Persaud (1989) and Buckley, Lipman, and Persaud (1993). The outstanding balance, Q, increases as the CPI increases, while the repayment rate, i, increases if a wage index increases.

Hungary and Poland, are indexed to both (1) a consumer price index (via Q), as PLAMs, and to (2) a wage index (via i).[15]

PLAMs have increased risk of default and DIMs try to counterbalance this effect if the revenues or, in the case of a residential mortgage, future wages go down. In such case DIMs then lower the monthly payment but increase the outstanding balance. This makes negative amortization possible increasing foreclosure risk. Consequently, maturity T becomes random and can increase as well. Such feature makes DIMs somehow similar to Amortizing Shared Appreciation Mortgages (ASAMs) we discuss in the present paper. The problem with DIMs is that they have maturity risk, i.e. maturity can become infinite and the loan will never be repaid. In contrast we control for this risk in ASAMs.

2.5 Summary of key results for non-amortizing SIMs

SAMs and non amortizing SIMs have extensively been discussed in Ebrahim et al. (2011). We therefore only succinctly summarise here the key result which matters for the analysis of ASIMs that follows. If interest i and participation parameters θ_P, θ_H are *constant* in time and if the contract is *non-amortizing*,[16] then the budget constraint (4) gives[17]

$$Q_0 = \underbrace{i\, Q_0 \frac{1}{r}\left(1 - e^{-rT}\right)}_{\text{annuity with } r,T} + \theta_P \underbrace{E\left[\int_0^T e^{-rt}\left(P_t - K\right)^+ dt\right]}_{\text{profit cap } C(P_0,K,T)} + \theta_H \underbrace{E\left[e^{-rT}\left(H_T - H_0\right)^+\right]}_{\text{call option } c(H_0,H_0,T)} + \underbrace{e^{-rT} Q_0}_{PV \text{ balloon}} \tag{8}$$

[15]For creation of occupational income indexes see Shiller (1993).

[16]For *amortizing* contracts appropriate incentives can be created to encourage borrowers to follow optimized, endogenous time-profiles of i_t and/or $\theta_{P,t}$ (see Section 4.4).

[17]For positive income-house price correlation $\rho > 0$ this formula corroborates pro-cyclicality effect of business cycles on both the house prices and incomes. In the perfectly ($\rho = +1$) correlated case, in our two-GBMs setup, both the P and the H would be driven by the same Brownian motion. This indicates that when the profit cap (second term) is small, the call option (third term) must decrease too. Therefore (for positive correlations), at the peak of house prices H there is less need to substitute FRM with SIMs or ASIMs. Conversely, at the through, while there is need to save underwater FRMs, there is less resources to share via conversion to SIMs or ASIMs. We leave for further research the estimation of these business cycle effects.

where $(P_t - K)^+$ is the payoff of an infinitesimal caplet. We call the function C the *profit cap* and c is the call option function. Both functions are discussed in the Appendix A and can be computed explicitly (see Ebrahim et al., 2011).[18] Condition (8) tells us that the present value of the loan must be equal to the sum of:

- Interest fraction i of annuity with interest r and maturity T
- Participation fraction θ_P in the income from operations (profit cap struck at K)
- Participation fraction θ_H in the appreciation of property value (call option struck at H_0)
- Present value of the balloon payment (equal to initial principal Q_0) at maturity.

It is easy to express i as a function of the other two parameters θ_P, θ_H of the model to see that for positive participation ratios $\theta_P > 0$ and $\theta_H > 0$ the interest i must be *reduced*[19] below r

$$i = r \left[1 - \frac{\theta_P C(P_0, K, T) + \theta_H c(H_0, H_0, T)}{Q_0 \left(1 - e^{-rT}\right)} \right] < r, \tag{9}$$

as both the cap C and the call c take positive values. In the next sections we introduce the amortization feature and establish its applicability to a SIM.

3 Integrating the amortization feature into PMs

The amortization feature has been successfully implemented in FRMs which were introduced in the US in the 1930's to heal the mortgage market post Great Depression. A standard, fully amortizing loan allows the whole balance to be fully repaid at maturity, so that $Q_T = 0$. Absence of balloon payments reduces the risk of default inherent in

[18] Note that C also depends on r, δ and σ which we omitted here for simplicity.

[19] See Ebrahim et al. (2011).

non-amortizing loans discussed in the previous section. However, we will show that implementing the amortization feature in a SIM is more challenging. This is because the "sharing" feature introduces additional degrees of complexity which must be tackled.

In the simplest case of absence of default or prepayment risk it is possible to set a *constant* coupon rate $x_t \to x$ in a FRM, so that the loan is fully repaid at T

$$x = \frac{rQ_0}{1 - e^{-rT}} = \frac{rQ_t}{1 - e^{-r(T-t)}} \; . \tag{10}$$

It is then easily seen that if we increase the maturity T, the annual payment x will decrease but will always be greater than the accrued interest rQ_t

$$x \geq x\left(1 - e^{-r(T-t)}\right) = rQ_t \; , \tag{11}$$

For $T \to \infty$ the initial balance Q_0 is never repaid and, as expected, must be equal to the present value of the perpetuity $\frac{x}{r}$. For finite maturities the amount owed to the lender

$$Q_t = \frac{x}{r}\left(1 - e^{-r(T-t)}\right) \tag{12}$$

is the solution to a boundary value problem involving the ordinary differential equation

$$\frac{dQ_t}{dt} = rQ_t - x \tag{13}$$

with terminal condition $Q_T = 0$. Equation (13) tells us that in order to progressively repay the principal Q_0, infinitesimal changes to intermediate balances Q_t must be negative. That is, because $x \geq rQ_t$ for $dt > 0$ (see (11) above), we must have $dQ_t < 0$. This ensures convergent solutions and requires both sides of the equality (13) to be negative. This also results from (11). This is to say, again, that repayment is possible if the coupon flow

x is *greater* than the interest flow on principal rQ_t. As $t \to T$, equation (12) implies the remaining balance Q_t and interest flow rQ_t both converge to zero. At the same time, the annual amortization flow, i.e. the annual payment net of the annual interest $x - rQ_t$, exponentially increases to reach x at maturity

$$x - rQ_t = -\frac{dQ_t}{dt} = xe^{-r(T-t)} = xe^{-rT}e^{rt} \tag{14}$$

$$\lim_{t \to T} (x - rQ_t) = x \tag{15}$$

4 Interaction of amortization and shared income features

4.1 Debt restructuring: The case of salvaging an underwater FRM

Perhaps the best way to introduce a SIM in an amortization context is to understand how a SIM can be employed to restructure an underwater (negative equity), *amortizing* FRM. Assume such FRM is currently underwater i.e. at time $t = 0$ the outstanding balance is Q_0 but such that $Q_0 > H_0$, where H_0 is the value of the property. This FRM currently requires an annual outflow of funds equal to x. The lender offers to reduce the annual payments to y in such a way that $y < x$. The outflow y would correspond to a second fixed rate mortgage FRM*, with the same maturity T as the original FRM but commanding a lower outstanding balance $Q_0^\star$, lower than the property value H_0

$$Q_0^\star = Q_0 - C < H_0 < Q_0 \, , \tag{16}$$

where the borrower traded in a cap C with fixed maturity T. The cap C is the participating component and is based on the forthcoming income flow of the household, net of a cash flow safety threshold K. The latter can be interpreted as e.g. a minimal subsistence level for a household.

To summarize, after restructuring the debt, the lender now fully ($\theta_P = 1$) participates in the income cap C. Thanks to this transformation the loan has a reduced balance outstanding and is no longer underwater. This new, lower balance is now again collateralized by the real estate asset ($Q_0^\star < H_0$). The borrower pays less in regular interest/repayments. However, the future income prospects are now reduced. The following expression summarizes how the repayment of original balance Q_0 has been rescheduled

$$\text{FRM:} \qquad Q_0 = \underbrace{\frac{x}{r}\left(1 - e^{-rT}\right)}_{\text{original annuity with } r, T, Q_0} \tag{17}$$

$$\Downarrow$$

$$\text{APM:} \qquad Q_0 = \underbrace{\frac{y}{r}\left(1 - e^{-rT}\right)}_{\text{modified annuity with } r, T, Q_0^\star} + \underbrace{\theta_P E\left[\int_0^T e^{-rt}\left(P_t - K\right)^+ dt\right]}_{\text{profit cap } C} . \tag{18}$$

where $y < x$ so that reduction in annual payment is achieved. The new annual payment, y, is based on the new (renegotiated) balance

$$Q_0^\star = Q_0 - C\,, \tag{19}$$

where C is the cap function discussed earlier.

A numerical example for full participation $\theta_P = 1$ has been constructed in Table 1 and illustrated on Figure 1. The profit cap $C = C\left(P_0, K, T, r, \delta, \sigma\right)$ has been computed using a formula discussed in the Appendix A. We notice in particular that, alternatively to reducing annual payments from x to y, the repayment velocity can be kept at x and the maturity *of the modified annuity only* shortened to $T_x^\star < T$ as follows

$$Q_0 = \frac{x}{r}\left(1 - e^{-rT_x^\star}\right) + C \tag{20}$$

As a result, the modified loan eliminates default and allows mortgage to be considered as fully amortized at maturity T, upon expiry of the profit cap portion C.

Alternatively to committing the entire portion ($\theta_P = 100\%$) of the cap C, the participation ratio θ_P can be negotiated to reflect the actual proportion of negative equity to be compensated by a portion of C. This has been illustrated in Figure 2. The new income sharing fraction is

$$\theta_P^\star = \frac{Q_0 - H_0}{C} = \frac{500\,000 - 300\,000}{332\,251} \approx 60.2\% \,. \tag{21}$$

As is now very clear from this example the possibility of quickly and precisely estimating the value of the cap $C = \$\ 332\,251$ via the closed-form formula is very handy for such a debt restructuring task. The moderator can instantly assess the exact portion $\theta_P^\star$ of the income flow P to be committed in order to completely eliminate the negative equity. It is then straightforward to compute the new coupon rate $y^\star$ or new maturity $T_x^\star$ as

$$y^\star = \frac{rH_0}{1 - e^{-rT}} = \frac{0.05 \times 300000}{1 - \exp\left(-0.05 \times 25\right)} \approx \$\ p.a.\ 21023 \tag{22}$$

$$T_x^\star = \frac{\ln\left(\frac{x}{x - H_0 r}\right)}{r} = \frac{\ln\left(\frac{35\,039}{35\,039 - 300000 \times 0.05}\right)}{0.05} \approx 11.18\ years \tag{23}$$

Numerical values for both examples have been collected in Table 1.

The base case (y_1) corresponds to $\{y, 1\}$, i.e. the annual payment x is reduced to $y < x$ while participation θ_P is left at 100%. The optimal case ($y_\theta^\star$) corresponds to $\{y^\star, \theta_P^\star\}$, i.e. the optimal participation $\theta_P^\star < 100\%$ is first solved for and then the annual payment is adjusted to $y^\star$ such that $y < y^\star < x$.

Description of variable	Notation	Base case (y_1)	Optimal case ($y^\star_\theta$)
Balance	Q_0	\$ 500 000	
House value	H_0	\$ 300 000	
Initial wage per year	P_0	\$ $p.a.$ 100 000	
Riskless rate	r	5% $p.a.$	
Cash yield	δ	3% $p.a.$	
Expected income growth	$r-\delta$	2% $p.a.$	
Volatility of income	σ	2% $p.a.$	
Maturity remaining	T	25 *years*	
Initial annual payment	x	\$ $p.a.$ 35 039	
Modified annual payment	$y, y^\star$	\$ $p.a.$ 11 755	\$ $p.a.$ 21023
Annual payment reduction	$y-x, y^\star-x$	−\$ $p.a.$ 23 284	−\$ $p.a.$ 14016.
Alternative maturity	$T^\star_x < T$	5.4723 *years*	11.1757 *years*
Safety threshold	K	\$ $p.a.$ 100 000	
Value of the cap	C	\$332251	
Income participation	$\theta_P, \theta^\star_P = \frac{H_0-Q_0}{C}$	100%	60.2%
New (reduced) balance	$Q^\star_0 = Q_0 - C$	\$ 167 749	

Table 1: **Numerical Example:** The base case (y_1) corresponds to $\{y, 1\}$, i.e. the annual payment x is reduced to $y < x$ while participation θ_P is left at 100%. The optimal case (y^*_θ) corresponds to $\{y^*, \theta^*_P\}$, i.e. the optimal participation $\theta^*_P < 100\%$ is first solved for and then the annual payment is adjusted to y^* such that $y < y^\star < x$.

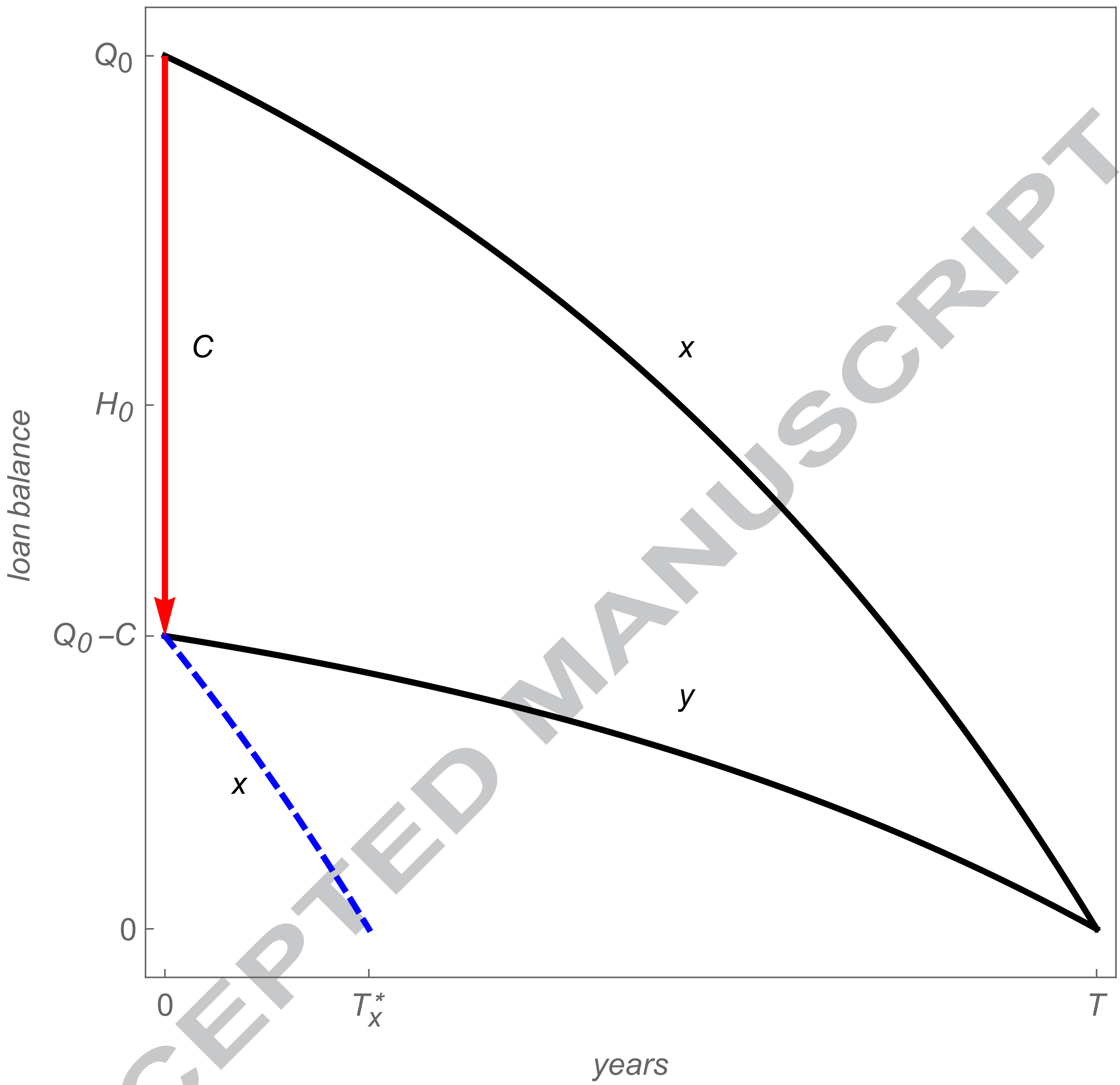

Figure 1: **Home loan modification with a SIM:** ***Full participation and over-adjustment.*** This figure presents home loan modification with full income participation $\theta_P = 100\%$. The value of entire income cap C is committed to eliminate negative equity $Q_0 - H_0$. This means that any accruing income surplus above threshold K will immediately be used to repay the mortgage balance and interest. The speed of repayment is adjusted down from x to y to maintain maturity at T. Without such adjustment the maturity is shortened to T_a^*.

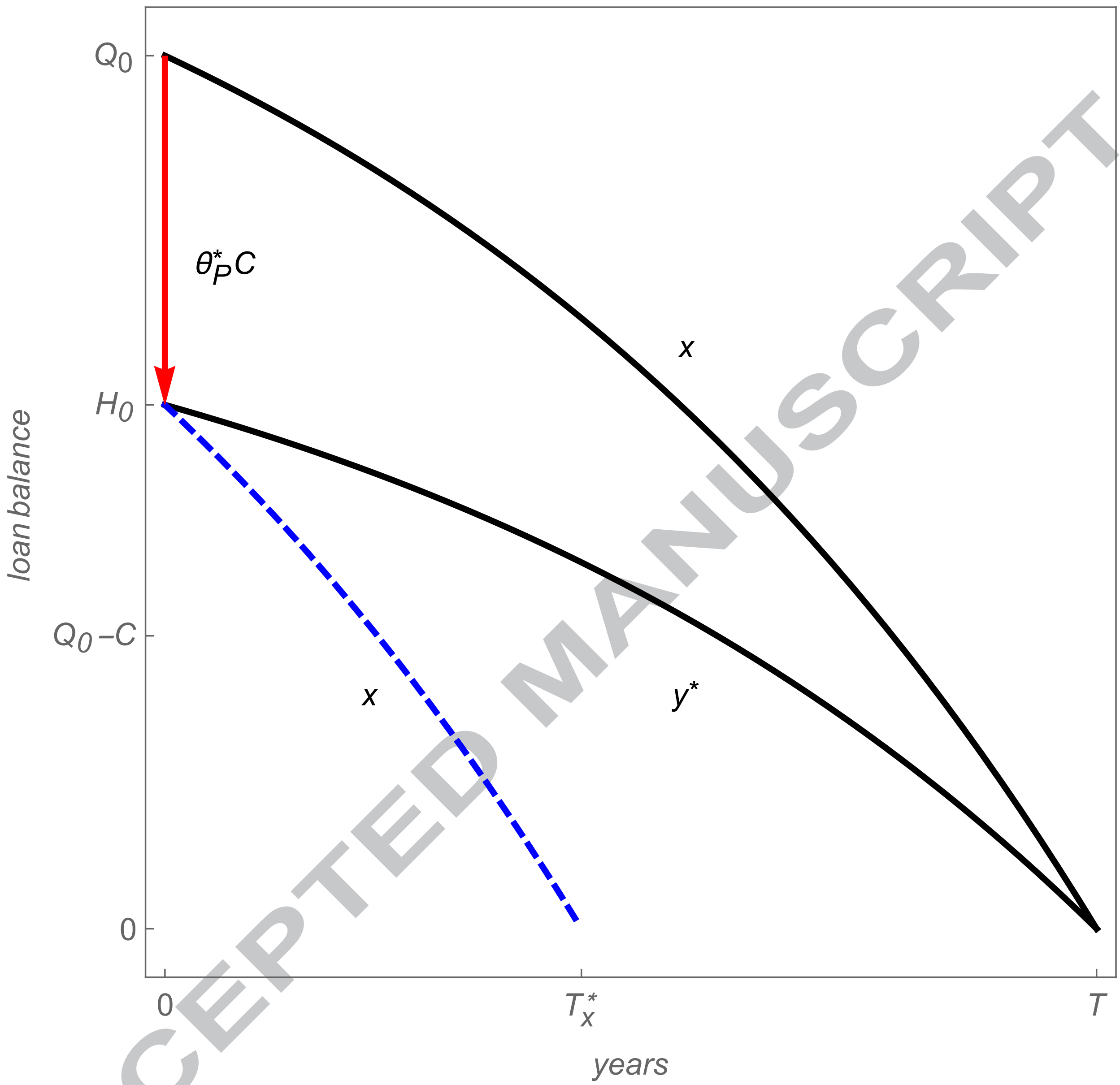

Figure 2: **Home loan modification with a SIM: *Optimal adjustment*.**
This figure presents home loan modification income participation θ_P optimally adjusted to match the negative equity $\theta_P = (Q_0 - H_0)/C$. The value of a *fraction* $\theta_P < 100\%$ of the income cap C is committed to eliminate negative equity $Q_0 - H_0$. The speed of repayment can be adjusted down from x to y^* to maintain maturity at T. Without such adjustment the maturity is shortened to T_x^*.

4.2 Debt restructuring when the underlying income flow is discontinuous

Large, sudden declines in income flow are consistent with the majority of (residential) mortgage defaults post 2008 crisis, at least in the U.S. Whether the income level will take a relatively long time to be restored will primarily depend on the parameters of the income process which are already present in our analysis, i.e. the growth rate (equal to the riskless rate r under the pricing measure) and the income volatility σ.

However, our approach can be augmented to incorporate another source of uncertainty to capture income discontinuity, i.e. jumps, to represent the large, sudden decline in income flow. One possible approach is to use the sum of a Brownian motion diffusion process (2) and a Poisson jump process, as in Merton (1976) (with a risk-neutral jump compensator λ).

Default can be modelled using the reduced form (hazard rate) approach of Jarrow and Turnbull (1995) to credit risk who extend Merton (1976) results to include unpredictable and undiversifiable shocks. Such an approach seems more appropriate in our case as it allows separating the occupational income fluctuations from individual circumstances. For a specific "class" of occupational income such individual shocks can be diversified away. This corresponds to a situation where the occupational risk is made tradable on "macro markets," as in Shiller (1993), with the moral hazard component removed.

We make no assumptions concerning why negative shocks occur. Rather, we specify the dynamics of default via the default rate (or intensity) λ of the Poisson event triggering personal ruin. The advantage in practice would be that the lender will be able to price the credit sensitive, modified loan as if it were default-free, using the risk free rate adjusted by the level of intensity estimated from historical data. In the simplest case this will only require replacing the riskless interest rate parameter r by $r + \lambda$.

In the most simple case there would be just one such adverse event arriving with intensity λ over the life span $[0, T]$. The random default time τ is then a Poisson event independent of state variables governing the diffusion processes for income and house prices. To incorporate the risk of default into debt restructuring process we have to adjust both the modified annuity and the profit cap components present in (18). This is to reflect the fact that upon occurrence of a sudden drop of personal income to zero, both the fixed as well as the income-contingent flows will cease to be paid to the lender. Interest payment interruption also triggers a default on the remaining balance. Consequently, the valuation condition (18) becomes

$$Q_0 = \underbrace{\int_0^T e^{-\lambda t} y_\lambda e^{-r(T-t)} dt}_{\text{modified annuity with } r, T, Q^\star_{0\lambda}} + \underbrace{\theta_P E\left[\int_0^T e^{-\lambda t} e^{-rt} (P_t - K)^+ dt\right]}_{\text{profit cap } C_\lambda}, \tag{24}$$

where $e^{-\lambda t} = \Pr(t > \tau)$ is the cumulative probability of the loan surviving beyond default time τ. As long as the default event does not occur early ($t < \tau$), the lender receives the contractual repayment y_λ (incorporating both interest and partial principal repayment) plus the participation fraction θ_P based on current income P_t above the threshold K. When default hits before maturity, i.e. $\tau < T$, both the fixed and contingent flows cease immediately and the lender receives nothing.

In practice, the lender will follow the same procedure as before. In addition, the lender will now estimate and incorporate the information about likelihood of default. First, the portion corresponding to the profit cap can be computed as

$$C_\lambda = C(P_0, K, T, r + \lambda, \delta + \lambda, \sigma) \ , \tag{25}$$

where C is the profit cap function (see Appendix A) with parameters r and δ adjusted for jumps i.e. replaced by $r + \lambda$ and $\delta + \lambda$, respectively.[20] This quantity then serves to com-

[20]Profit cap C_λ sums a continuum of infinitesimal caplets on revenue P struck at K over a continuum of ma-

pute the reduced modified principal $Q^{\star}_{0\lambda} = Q_0 - \theta_P C_\lambda$ (if θ_P is imposed) or the sharing fraction $\theta_{P\lambda}$ (if reduction of outstanding balance from Q_0 to $Q^{\star}_0 = Q_0 - \theta_{P\lambda} C_\lambda$ is required). Note that for $\lambda > 0$

$$Q^{\star}_{0\lambda} > Q^{\star}_0 \text{ or } \theta_{P\lambda} > \theta_P \tag{26}$$

because, with the possibility of default, the cap C_λ will in (25) be worth *less* than C in (18). The "power" to achieve reduction in principal is lower here due to presence of additional risk.[21] Consequently, if the maturity is kept the same, a lower effective principal reduction will have the effect of increasing the fixed, regular portion payment, y_λ or $y^{\star}_\lambda$, to be determined in the next step from

$$y_\lambda = Q^{\star}_{0\lambda}\,(r+\lambda)\left[1 - e^{-(r+\lambda)T}\right]^{-1} \text{ or } y^{\star}_\lambda = Q^{\star}_0\,(r+\lambda)\left[1 - e^{-(r+\lambda)T}\right]^{-1} \tag{27}$$

In our numerical example from the previous section,[22] we introduce a small income loss risk $\lambda = 1/100 = 0.01$ (occurring once in 100 years). This reduces the income cap capacity from $C = \$\ 332\ 251$ to $C_\lambda = \$\ 285\ 973$ i.e. by about 14%. As expected, the debt restructuring capacity of the cap is very sensitive to the risk of sudden income loss. This drawback should be accounted for by the lender when restructuring the loan. Consequently, the borrower should participate more and $\theta_{P\lambda} = 69.9\%$, an increase from $\theta_P = 60.2\%$. Finally, to further compensate for the risk of default, the repayment rate on the fixed tranche increases to $y^{\star}_\lambda = \$\ 23\ 170$ from $y^{\star} = \$\ 21\ 023$ per annum.

turities $t \in [0, T]$. Each caplet is weighted by the probability $e^{-\lambda t}$ of borrower's income source "surviving" at least until t.

[21] However, in our contract valuation model, the reduction would always be *positive* no matter how large the jump risk, as measured by λ. Therefore, a rational borrower would always prefer such a reduction.

[22] See Table 1 for the base case parameter values.

4.3 Improving affordability with Amortizing Shared Income Mortgages

Our analysis and numerical examples in the two previous sections can be directly reinterpreted as employing Amortizing Shared Income Mortgage to improve affordability. In our setup by improving affordability we intend lowering cash flows in the early, hard to repay years, from x per year to y per year, where $y < x$ in our examples.[23] Overall the product has the same present value but lower future cash flows, thanks to the income cap.

Alternatively, improving affordability would be characterized by a relatively lower interest rate i. See Ebrahim et al. (2011) who perform such analysis for non-amortizing Shared Income Mortgages. It is straightforward to extend their analysis to Amortizing SIMs. However it is beyond the scope of the present paper and we leave this avenue for future research.

Finally, we note that the income stream volatility σ enters our formulas explicitly. So does the probability of job loss, via a default intensity parameter λ. We anticipate that those borrowers with more volatile income streams and higher risk of job loss will prefer the APM. They should be required to pay a higher annual repayment amount and the reduction of their principal should be lower. Such a built-in, negative feedback loop (provided via parameters σ and λ, assuming both are observable by lenders and correctly estimated ex ante), should provide appropriate prevention mechanisms against adverse selection issues. Conversely, if either σ or λ is not measurable or not measured (e.g. deliberately, complicitly or to reduce costs, etc.), lenders will be dealing with a pooled equilibrium (*volatile-and-risky* mixed with *smooth-and-safe* income streams). To achieve separation globally, regulation may therefore require lenders to offer certain products to certain groups.

While the historical data (which a lender can gather from past borrower's choices) will

[23]On the contrary, if these products were launched and were expensive, they would provide information about (i.e. the market price of) the ex-ante costs of foreclosure.

provide an estimate of the extra costs generated by moral hazard and adverse selection, any lack of information ex-ante will always be a problem. As a second step, lenders can structure a *range of products* which borrowers will self-select and in that way reveal their "type" (their specific risk level of defaulting on mortgage payments, their likelihood of moving house, their probability to prepay early, etc.), thus achieving the separating equilibrium and reducing information asymmetry ex-ante. In the context of mortgage loans the real estate literature hints how this can be achieved. Building on the seminal insights of Rothschild and Stiglitz (1976) and Stiglitz and Weiss (1981) various *screening instruments* have been advocated, in particular: the initial loan-to-value ratio (Brueckner, 2000), points (Brueckner, 1994; Chari and Jagannathan, 1989; LeRoy, 1996; Stanton and Wallace, 1998), prepayment penalties (Chari and Jagannathan, 1989), the type (fixed or variable) of the contract rate (Posey and Yavas, 2001), etc.[24]

4.4 Controlling amortization in an Amortizing Shared Income Mortgage (ASIM) optimally

In a SIM the borrower benefits from a lower amortization flow y such that $y < x$. This reduction is financed via sharing with the lender a fraction θ_P of the income flow P_t net of a threshold K. In our example in the previous section the arrangement had European options only. Both the fixed rate amortized portion and the shared income portion had to last until maturity T. However, this may not be always desirable.

For example, the customer may prefer committing a higher fraction θ_P to compensate for the *agreement to automatically terminate all payments exactly at the random instant*[25] τ *when the shared income portion repays the then outstanding balance* Q_τ. If the strength of the

[24] In contrast to these analyses which are set up in static frameworks, dynamic moral hazard environment has been considered only recently, in a study of optimal mortgage design (see Piskorski and Tchistyi, 2010).

[25] In this section we assume $\lambda = 0$ i.e. there are no income terminating "Poisson events" here. Our analysis can easily be extended to the case $\lambda > 0$.

income flow P is high enough there is high probability for such early termination instant τ to happen before maturity T. Because τ is random, this may or may not occur *before* maturity T. This resembles an early exercise of an American option. However, this is where the similarity ends. Termination is not at callers choice but imposed by a random event.

In our case it is never too late to terminate the agreement once the cumulated income

$$\pi_\tau = \int_0^\tau e^{r(\tau-t)} \left(P_t - K\right)^+ dt \tag{28}$$

meets the remaining balance Q_τ, as given by expression (12) for $t = \tau$. This is because the balance always decreases while the cumulated income is a *non decreasing* process. The latter is a sum of call options (caplets) which provide a random but non-negative payout flow. This is illustrated in Figure 3 representing the *present values* of accumulated revenues. Three random possible shared income paths are drawn. Path[26] π_3 is the lowest and has a long plateau so that it only meets the (present value of) remaining balance $e^{-rt}Q_t$ very late, at about 21.5 years. Path π_1 accumulates income very quickly while π_2 has a "typical" behaviour. In our sample most of the paths (not represented on the graph for the sake of esthetics) are like path π_2 i.e. they lie close to the bold line C_t representing the present value of the income cap. Line C_t represents in a sense the expected behaviour of income to come. Most importantly, C_t can be calculated analytically using the cap formula in Appendix A. It is assumed that $P_0 = K = \$\,100\,000$, $\sigma = 2\%$, $r = 5\%$, $T = 25$ years and the initial balance is $Q_0 = \$\,500\,000$. We can solve numerically for the expected time $\bar{\tau}$ (about 14.31 years) when the income is expected to meet the remaining balance (about \$ 141 837) at point marked A on the Figure 3.

We also proceed with some comparative statics. Figure 3 also illustrates what will

[26]Note that in what follows we use subscripts to π's to differentiate between three different time-dependent evolutions of cumulated income $\pi_\tau : \tau \in [0, T]$ i.e. τ does *not* indicate a termination moment.

happen if volatility of income increases from 2% to 50%. Then the point A will move along the present value of the balance curve up to the point labelled A_σ. This is to be expected because the income cap is a sum of call options and these appreciate with rising volatility. In contrast, when the interest rate increases to $r = 10\%$ both curves move to cross at the point labelled A_r. It is quite straightforward to understand why the present value of the remaining balance has to become lower and has to remain anchored at initial and terminal points $\{0, Q_0\}, \{T, 0\}$. It thus becomes a more convex curve if interest rate r rises. It is less intuitive to see why the location of the cap C_t rises with r. In fact, this is because the Brownian motion which drives C_t appreciates more rapidly at the rate $r - \delta$ and the income cap accumulates faster.

As a general rule, the expected termination time $\bar{\tau}$ decreases with increasing income volatility and interest rate r. This is illustrated on Figure 4.

Figure 5 and Table 2 further illustrate the dependence of expected terminal balance $Q_{\bar{\tau}}$ and expected termination time $\bar{\tau}$ on income volatility σ and discount rate r.

4.5 Incentivising borrowers to increase income participation

In our two introductory examples in the previous sections we established that by significantly increasing the participation ratio, θ_P, various features can be achieved. In particular, negative equity can be reduced or completely eliminated, the termination time τ can be shortened (in expectation), the coupon rate of repayment can be reduced or the maturity shortened as a result.

However, having restructured the mortgage with the help of the cap C (see previous sections), the lender may wonder whether C will actually be repaid over time. This section dealt with the problem of incentivising the borrower to repay the debt in such timely manner. With given participation rate and annual payments $\{\theta_P, y\}$ timely repayment of the principal amount $Q_0 > 0$ in full cannot be guaranteed *ex ante*, because repayment

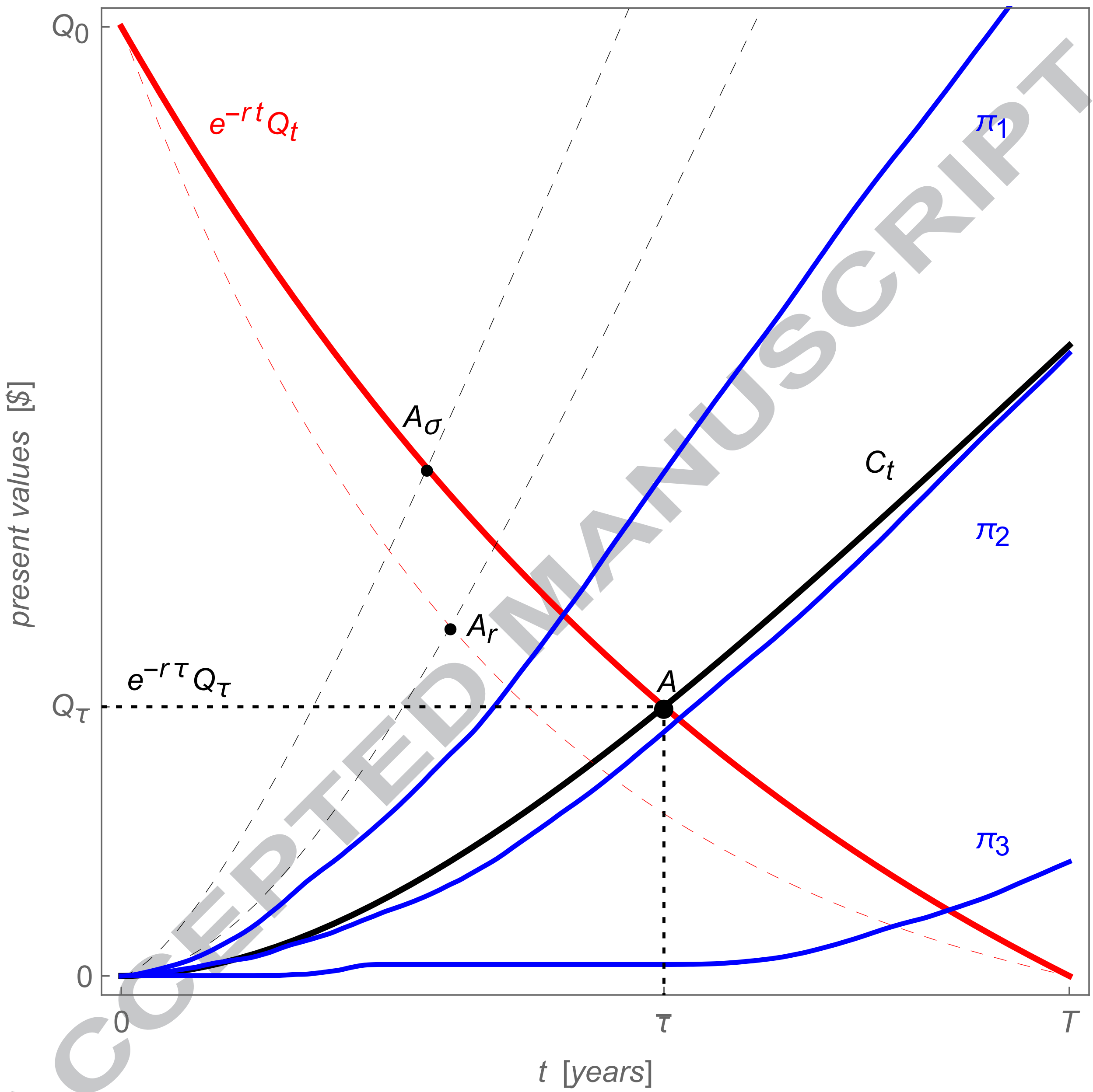

Figure 3: **Automatically terminating agreement:** All payments cease exactly at the random instant τ when the shared income portion repays the then outstanding balance Q_τ.

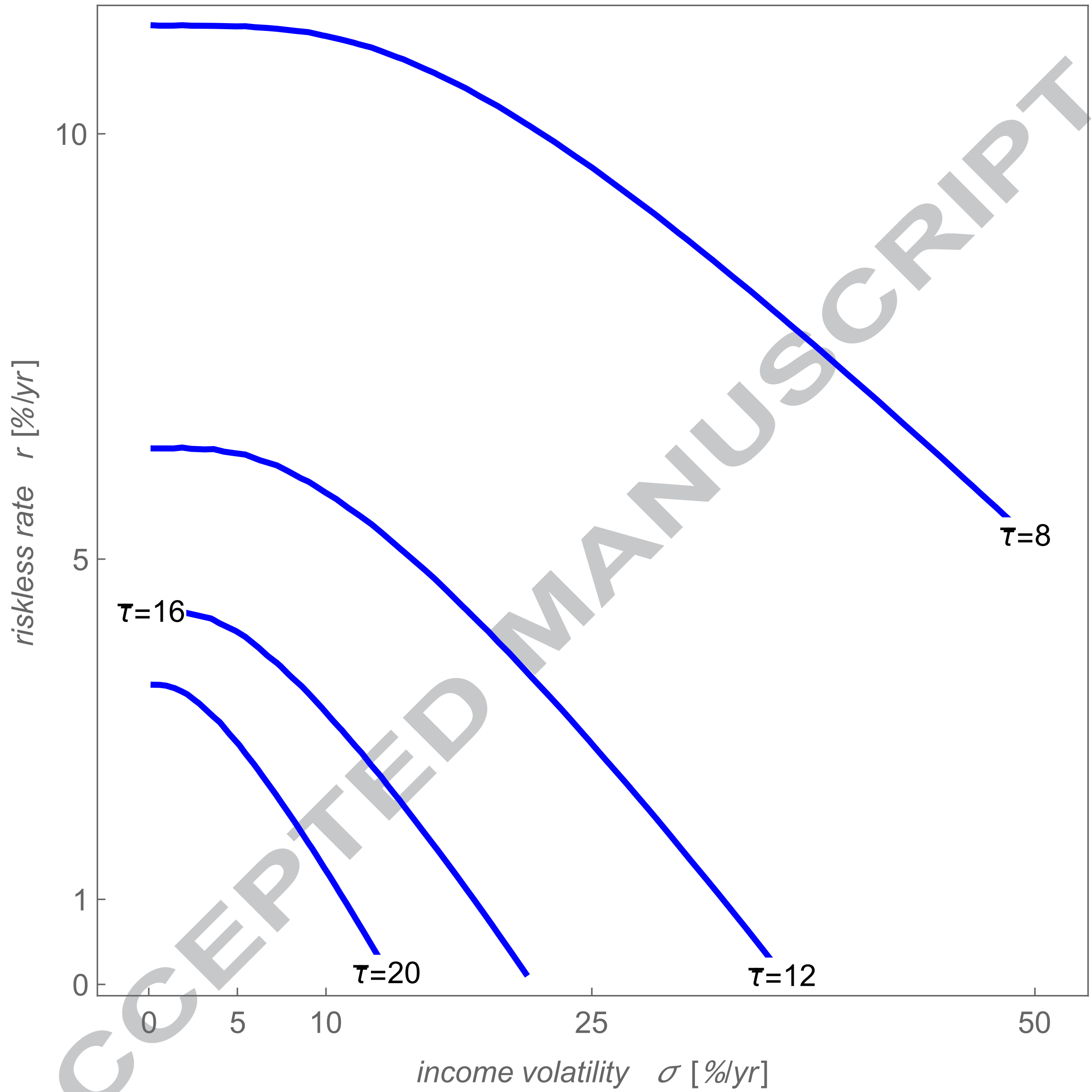

Figure 4: **Termination:** Expected termination time $\bar{\tau}$ as a function of the income volatility σ and the riskless interest rate r. The higher the volatility σ and/or the higher the riskless rate r, the lower the expected termination time $\bar{\tau}$.

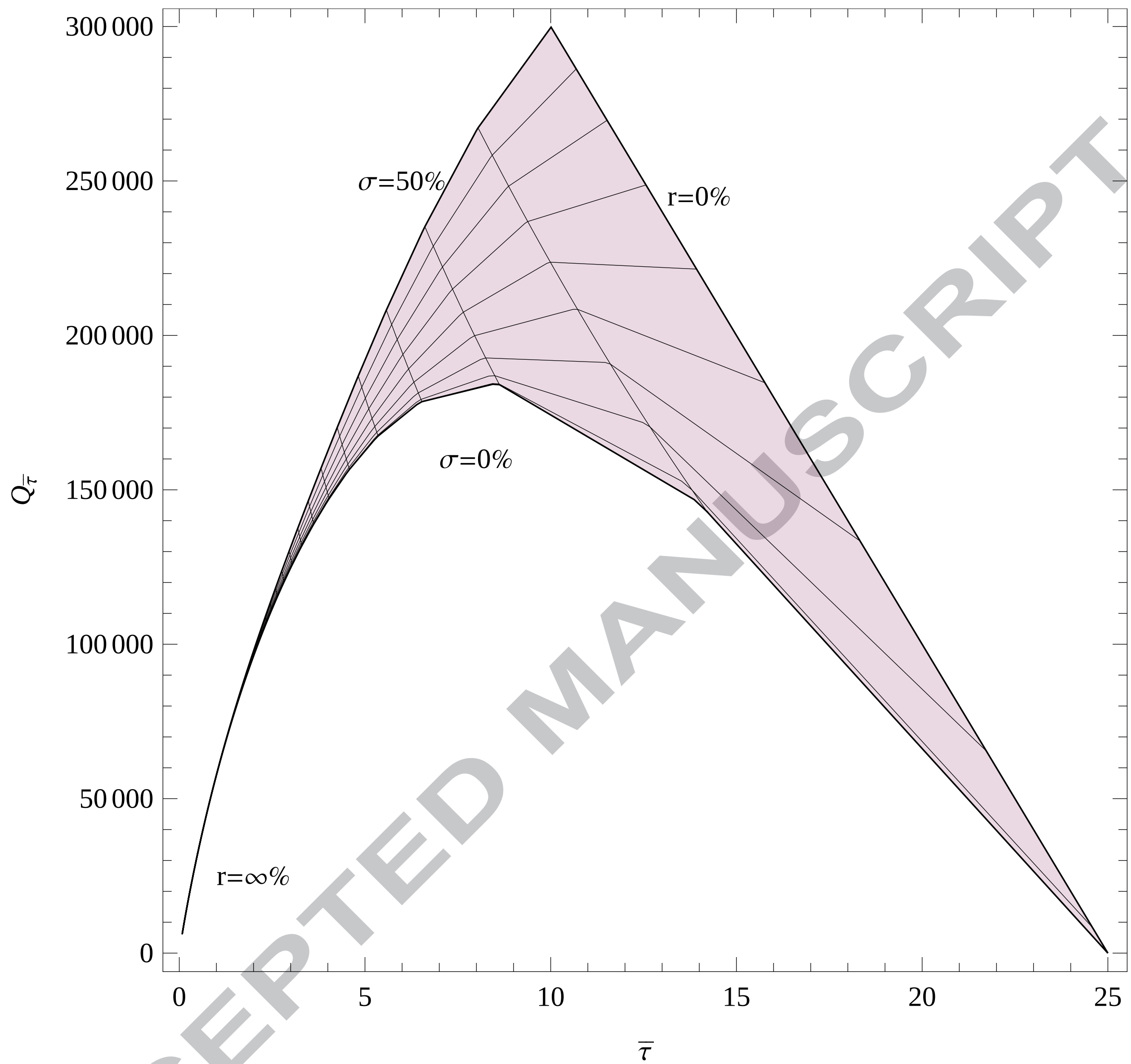

Figure 5: **Expected terminal balance $Q_{\bar{\tau}}$ and termination time $\bar{\tau}$:** Dependence of the present value of the expected termination balance $Q_{\bar{\tau}}$ and the expected termination time $\bar{\tau}$ on income volatility σ and discount rate r. When the volatility σ increases: 1) the expected termination balance $Q_{\bar{\tau}}$ increases; and 2) the expected termination time $\bar{\tau}$ decreases. When the discount rate r increases: 1) the expected termination balance $Q_{\bar{\tau}}$ increases initially (wealth effect, C accumulates faster) and then decreases (discounting effect) while 2) the expected termination time $\bar{\tau}$ decreases monotonically. The corresponding values are computed in Table 2.

Panel A: Dependence of $\bar{\tau}$ on r and σ

$r \downarrow \quad \sigma \rightarrow$	**0.001**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
$\mathbf{1 \times 10^{-8}}$	25.0	22.5	16.7	13.3	11.3	10.0
0.05	14.3	13.0	11.1	9.8	8.8	8.1
0.1	8.7	8.6	8.1	7.6	7.1	6.6
0.15	6.6	6.6	6.4	6.1	5.9	5.6
0.2	5.4	5.4	5.3	5.2	5.0	4.8
0.25	4.6	4.6	4.6	4.5	4.4	4.3
0.3	4.1	4.1	4.0	4.0	3.9	3.8
0.35	3.6	3.6	3.6	3.6	3.6	3.5
0.4	3.3	3.3	3.3	3.3	3.3	3.2
0.45	3.1	3.1	3.1	3.0	3.0	3.0
0.5	2.8	2.8	2.8	2.8	2.8	2.8

Panel B: Dependence of $Q_{\bar{\tau}}$ on r and σ

$r \downarrow \quad \sigma \rightarrow$	**0.001**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
$\mathbf{1 \times 10^{-8}}$	0.	50181.	165670.	233080.	273200.	299790.
0.05	141600.	165230.	201390.	229240.	250640.	267530.
0.1	183960.	186080.	197080.	210890.	224030.	235790.
0.15	178840.	179340.	183720.	191520.	200290.	208930.
0.2	167870.	168060.	170150.	174910.	181060.	187610.
0.25	157100.	157200.	158330.	161400.	165850.	170940.
0.3	147630.	147680.	148360.	150420.	153730.	157750.
0.35	139460.	139490.	139930.	141360.	143870.	147080.
0.4	132380.	132400.	132700.	133730.	135660.	138260.
0.45	126190.	126210.	126420.	127180.	128690.	130820.
0.5	120730.	120740.	120890.	121470.	122660.	124430.

Table 2: **The present value of the expected terminal balance $Q_{\bar{\tau}}$ and the expected termination time $\bar{\tau}$:** Dependence of the expected terminal balance $Q_{\bar{\tau}}$ and the expected termination time $\bar{\tau}$ on income volatility σ and discount rate r. See caption in Figure 5 for more explanations.

cash flows are partly random by design. Such a loan cannot be made fully amortizing for all scenarios. This is because of how the cap C is constructed. If the contract is not adjusted *en route*, we shall expect the actual terminal amount Q_T to end up away from its full repayment target $\hat{Q}_T = 0$, being underpaid or overpaid, with probability greater than zero.

The lender and the borrower can, however, adjust at least one of the contract parameters, y or θ_P or both, with repayment target in mind. For example, the lender will initially offer a constant annual rate y such that $0 \leq y < x$, where x would guarantee full repayment without participation. In order to collect the remaining portion of the debt debt, the lender will then command a participation policy[27] $\theta_t \in [0, 1]$, which[28] minimizes the squared[29] distance from a set terminal target amount $\hat{Q}_T$

$$J = \min_{\{\theta_t : t \in [0,T]\}} E\left[\left(Q_T - \hat{Q}_T\right)^2\right] . \tag{29}$$

Note that J will be minimized by controlling participation θ_t over time $t \in [0, T]$.

In what follows we will assume *full repayment target*, so we set $\hat{Q}_T = 0$. Note that then the minimization program (29) will penalize terminal underpayments ($Q_T > 0$) as well as overpayments ($Q_T < 0$). Furthermore, the minimization programme (29) is subject to the following budget condition, derived from (4)

$$Q_0 = e^{-rT} E\left[Q_T\right] + i \int_0^T e^{-rt} E\left[Q_t\right] dt + \int_0^T e^{-rt} E\left[\theta_t \left(P_t - K\right)^+\right] dt \tag{30}$$

For this Shared Income Mortgage the required coupon rate i is reduced (below r) to

[27] To simplify notation, we dropped the subscript P form time-dependent participation ratio $\theta_{P,t}$.

[28] By setting the upper admissible bound on θ_t equal to one, we explicitly assumed that the maximum the lender can require from the borrower is full participation. Similarly, we constrained the lender from further investing into the project (no negative participation ratios).

[29] Our choice is arbitrary. Other loss functions are possible.

reward lender's participation. The lender is compensated proportionally to the available excess intermediate profit flow $(P_t - K)^+$ i.e. via a cap on income in the good but not in the bad states.

Note that $E[Q_T]$ is not necessarily equal to $\hat{Q}_T$ (zero if full repayment is sought) in situations when the maturity repayment target amount is expected to be met with probability less than one. Such situations can occur if e.g. the lender offers a low repayment rate i, and when the available earnings, P_t, are also low. This budget condition relies on the full time-profile of the future participation policy θ_t, where $t \in [0, T]$. Furthermore, it is straightforward to show that Q_t is given by

$$\begin{aligned} Q_t &= e^{-r(T-t)} E_t[Q_T] + i \int_t^T e^{-r(s-t)} E_t[Q_s]\, ds \\ &\quad + \int_t^T e^{-r(s-t)} E_t\left[\theta_s (P_s - K)^+\right] ds\,, \end{aligned} \tag{31}$$

(see Appendix B) and define, as τ, the time at which the loan will be repaid

$$\tau = \inf\{t : Q_t = 0\}\ . \tag{32}$$

If $\tau < T$ the loan has been repaid before maturity and the optimal policy is to set θ_t to zero for the remaining time segment $t \in [\tau, T]$. Indeed, once Q hits zero at a stopping time τ, only zero-participation policy ($\theta_t = 0$) precludes the generation of subsequent overpayment or underpayment at maturity (with probability one). This instantly minimizes variance-like target (29) for all paths, as opposed to maintaining, after time τ, any stochastic cash flows linked to debt and participation.

It is clear that full description of the state of the system at time t requires knowledge of the current balance Q_t as well as current earnings flow P_t, the two state variables. We define the value function

$$J_t = J\left(Q_t, P_t, t\right) = \min_{\{\theta_s : s \in [t,T]\}} E_t\left[Q_T^2\right] . \tag{33}$$

Standard steps (see Appendix C) give the Hamilton-Jacobi-Bellman (HJB) equation for the value function J

$$\min_{\theta}\left\{\frac{\partial J}{\partial t} + \frac{\partial J}{\partial Q}\left[\left(r-i\right)Q - \theta\left(P-K\right)^{+}\right] + \frac{\partial J}{\partial P}\left(r-\delta\right)P + \frac{\partial^2 J}{\partial P^2}\frac{\sigma^2 P^2}{2}\right\} = 0 , \tag{34}$$

where we dropped explicit time dependence (subscript t) from Q_t, P_t and θ_t. Because the control θ is constrained, the first-order condition for optimality need not be satisfied, i.e. there may not be interior minimum for $\theta \in [0, 1]$. However, the control θ appears only as a coefficient in the second term. Thus, the Hamilton-Jacobi-Bellman equation is minimized by choosing θ to minimize the second term. This term is clearly minimized by selecting

$$\theta^{\star} = \begin{cases} 1 & \text{if} \quad \frac{\partial J}{\partial Q}\left(P-K\right)^{+} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

whenever this strategy is admissible. The choice of $\theta^{\star} = 0$ is arbitrary because when e.g. $P \leq K$ there is no excess flow to share and so any admissible control θ is allowed.

Typically, we expect $\frac{\partial J}{\partial Q}$ to be positive, because in the case of mortgage amortization the starting balance Q_0 is positive. Consequently, increasing the loan value Q_t up from zero necessarily makes the full repayment goal ($\hat{Q}_T = 0$) harder to reach. This in turn increases the variance-like terminal objective function (33). As a result, the optimal repayment policy is *full participation* ($\theta_t^{\star} = 1$), whenever the loan has not been repaid yet ($Q_t > 0$) and if there is positive excess cash flow available to share ($P_t > K$). We have thus shown that the optimal policy is to set participation to maximum, so that the mortgage will be repaid as soon as possible. If Q attains zero before maturity, at some τ such that $\tau < T$, participation should simply be "switched off" for the remaining time until maturity T, because the loan has just been repaid in full early.

4.6 Commercial and partially amortizing mortgages

Commercial mortgages typically assume an amortization schedule of length T_a, which is typically much longer than the term $T : T < T_a$ of the loan. At maturity T a remaining *balloon payment* Q_a is then required to cover the unpaid balance. For a standard repayment fixed rate mortgages (FRM) the annual payment rate is therefore calculated, using identity (10), *as if* the mortgage had the maturity equal to T_a

$$x_a = \frac{rQ_0}{1 - e^{-rT_a}} . \tag{36}$$

Similarly, the balloon payment Q_a is computed as the remaining balance, at time T, for a mortgage with annual payment x_a and "maturity" T_a

$$Q_a = \frac{x_a}{r}\left(1 - e^{-r(T_a - T)}\right) = Q_0\frac{1 - e^{-r(T_a - T)}}{1 - e^{-rT_a}} \qquad : \qquad 0 < T < T_a . \tag{37}$$

The balloon payment is a quantity which can be computed at time $t = 0$ when T_a and T are known.

A commercial mortgage can also be restructured to incorporate a participation cap C and reduce the annual repayment amount x_a. Incentivising of the borrower can then be achieved by setting the repayment target in equation (29) equal to the balloon amount Q_a, that is

$$\hat{Q}_T = Q_a , \tag{38}$$

where Q_a can be computed using (37).

The above observations are very important because they show we can easily generalize[30] our earlier results to the *intermediate* situation (where the mortgage is neither fully amortizing or non-amortizing) of partial amortization, as is the case of commercial mortgages (with balloon payments).

5 Suggestions for future research

Our results are particularly significant for properties such as pre-fabricated homes, industrial properties, shop houses, etc. with little appreciation potential (i.e., where the underlying land is held on a leasehold basis). Properties such as mines, nuclear power plants, etc. with depreciation potential are also good candidates for an ASIM.

Future research should also encompass other variants of APMs. Here, we only briefly enumerate these possible variants, together with their corresponding property types. Single family homes, which combine residence with investment (where the underlying land is owned on a freehold basis) are a good candidate for a ASAM (Amortizing Shared Appreciation Mortgages). Undeveloped land or underdeveloped property with endemic growth options are good candidates for an ASEM (Amortizing Shared Equity Mortgages). The last case constitutes a special case and has been the focus of Titman (1985).

Finally, experts in different fields of finance can learn from the example of APMs to structure equivalent Amortizing Participation Bonds (APBs) for corporations where there is no informational advantage to borrowers or managers of firms. Firms, like different classes of real estate, operate in different sectors of the economy. This endows them with complementary and diversifiable characteristics (see Fama and French (1998) and Daniel and Titman, 1997). For instance, a pure value firm operates in a mature (or declining) sector of the economy. In contrast, a pure growth firm is endowed with growth options. In between these two extremes, we find firms with mixed value-growth or growth-value

[30] We thank one anonymous referee for pointing us towards this direction.

characteristics. An APB can be employed for a Value-Growth (or a Growth-Value) firm. In contrast, a pure value firm is a prime candidate for an Amortizing Shared Income Bond (ASIB) while a pure growth firm is a prime candidate for an Amortizing Shared Equity Bond (ASEB). Unfortunately, there is no equivalent case of a firm, which is both a consumption good as well as an investment like residential real estate. Therefore, in the corporate sector, we may not be able to structure an equivalent Amortizing Shared Appreciation Bond. Further study is also needed to adapt an APM to an already default conventional mortgage (or loan) in a workout situation.

6 Policy implications

First, we recommend that policy makers should aim at developing Macro Markets (see Shiller, 1993) for sectorial salary/income indices to facilitate monitoring, estimation of earning capacity, etc. This helps provide the necessary information to estimating the parameters of the Shared Income mortgage and thus lowering the monitoring costs.

The second recommendation stemming from our study is to shift policymakers's focus *from initial stages to the final stages* when designing and introducing mortgage contracts whose explicit payoff depends on borrower's income profile over time. This is because, unlike the fixed rate mortgages, which policymakers are typically familiar with, the Shared Income products we study in our paper need to have *penalties imposed at maturity* in order to incentivize borrowers to perform repayment in timely manner.[31]

Finally, as an additional policy implication, we conclude that in the particular case of Shared Income loans, the candidate screening instrument of choice (which our present theoretical work uncovered) is the size of the repayment penalty to be applied at maturity. This feature is novel and distinct in its scope from the well studied and understood

[31] In contrast, fixed rate mortgages have prepayment penalties imposed at the initial stages of their repayment schedule.

prepayment penalty discussed in the literature. Lenders will thus be able to offer a range of contracts with various combinations of contract rates and more or less onerous end of contract terminal penalty profiles. As a result they will be able to separate and possibly ration credit to potentially problematic customers e.g. when observing preference for the lowest contract rates combined with very high end term penalties.

7 Concluding remarks

This paper aims to mitigate financial fragility by focusing on Amortizing Participating Mortgages. APMs belong to the same class of home loan credit facilities as advocated in the Dodd-Frank Wall Street Reform and Consumer Protection Act 2010.

We first illustrate that unlike interest bearing fragile facilities, optimal structure of participating mortgages necessitates a meticulous evaluation of their stochastic amortization schedule. Since participating mortgages repayment flow comprises two components, i.e., deterministic and stochastic, flows, optimal participation ratios are a solution to a class of optimal stochastic control problems. We illustrate this specificity of participating mortgages by formulating and solving such a problem. We then provide the corresponding Hamilton-Jacobi-Bellman equation and discuss the implications of the optimal control, which we obtain in closed form.

Second, our approach offers a method to obtain closed-form solutions to APMs. We provide detailed numerical examples of employing these facilities as workout loans. We argue that APMs reduce the endemic agency costs of debt by reducing deadweight costs. These benefits increase the demand for real estate and reduce the fragility of the financial system thereby forestalling foreclosures. This, in effect, improves the value of real assets in conjunction with enhancing the resilience of the financial architecture and invigorating economic growth. In this respect our approach offers a novel *ex-ante* renegotiation to mitigate the foreclosure problem.

References

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–654.

Brueckner, J., 1994. Borrower mobility, adverse selection, and mortgage points. Journal of Financial Intermediation 3, 416–441.

Brueckner, J. K., 2000. Mortgage default with asymmetric information. The Journal of Real Estate Finance and Economics 20, 251–274.

Buckley, R., Lipman, B., Persaud, T., 1989. Mortgage design under inflation and real wage uncertainty : the use of a dual indexed instrument. Infrastructure and Urban Development Department discussion paper No. INU-62, Washington, DC, World Bank.

Buckley, R., Lipman, B., Persaud, T., 1993. Mortgage design under inflation and real wage uncertainty: The use of a dual index instrument. World Development 21, 455–464.

Chari, V., Jagannathan, R., 1989. Adverse selection in a model of real estate lending. The Journal of Finance 44, 499–508.

Chiquier, L., 1998. Dual index mortgages: lessons from international practice and conditions of development in Poland. Housing Finance International 13, 8–23, published by International Union for Housing Finance.

Chiquier, L., Lea, M. J., 2009. Housing Finance Policy in Emerging Markets. World Bank e-Library, World Bank.

CoreLogic, 2014. Equity report: Second quarter 2014. Irvine, California.

Daniel, K., Titman, S., 1997. Evidence on the characteristics of cross sectional variation in stock returns. Journal of Finance 52, 1–33.

Dodd-Frank, 2010. Wall Street reform and consumer protection act. Title XIV: Mortgage Reform and Anti-Predatory Lending Act, Sec. 1406 Study of shared appreciation mortgages.

Ebrahim, M., Hussain, S., 2010. Financial development and asset valuation: The special case of real estate. Journal of Banking and Finance 34, 150–162.

Ebrahim, M. S., 1996. On the design and pareto-optimality of participating mortgages. Real Estate Economics 24, 407–419.

Ebrahim, M. S., Shackleton, M. B., Wojakowski, R. M., 2011. Participating mortgages and the efficiency of financial intermediation. Journal of Banking & Finance 35, 3042 – 3054.

Fama, E. F., French, K. R., 1998. Value versus growth: The international evidence. Journal of Finance 53, 1975–1999.

He, Z., 2009. Optimal executive compensation when firm size follows geometric Brownian motion. Review of Financial Studies 22, 859–892.

Hosios, A., Peters, M., 1989. Repeated insurance contracts with adverse selection and limited commitment. The Quarterly Journal of Economics 104, 229–53.

Jarrow, R. A., Turnbull, S. M., 1995. Pricing derivatives on financial securities subject to credit risk. The Journal of Finance 50, 53–85.

Kau, J. B., Keenan, D. C., Muller, W. J., Epperson, J. F., 1992. A generalized valuation model for fixed-rate residential mortgages. Journal of Money, Credit and Banking 24, 279–299.

Lee, P.-T., Rosenfield, D. B., 2005. When to refinance a mortgage: A dynamic programming approach. European Journal of Operational Research 166, 266–277.

LeRoy, S. F., 1996. Mortgage valuation under optimal prepayment. The Review of Financial Studies 9, 817–844.

Lessard, D. R., Modigliani, F., 1975. Inflation and the housing market: Problems and potential solutions. In: *New Mortgage Designs for an Inflationary Environment*, Federal Reserve Bank of Boston, Conference Series No. 14, pp. 13–45.

Levine, R., Loayza, N., Beck, T., 2000. Financial intermediation and growth: Causality and causes. Journal of Monetary Economics 46, 31–77.

Merton, R. C., 1969. Lifetime portfolio selection under uncertainty: The continuous-time case. The Review of Economics and Statistics 51, 247–257.

Merton, R. C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 373–413.

Merton, R. C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3, 125–144.

Modigliani, F., 1974. Some economic implications of the indexing of financial assets with special reference to mortgages, paper Presented at the Conference on the New Inflation, Milano, Italy, June 24–26.

Piskorski, T., Tchistyi, A., 2010. Optimal mortgage design. Review of Financial Studies 23, 3098–3140.

Posey, L. L., Yavas, A., 2001. Adjustable and fixed rate mortgages as a screening mechanism for default risk. Journal of Urban Economics 49, 54 – 79.

Real Capital Analytics, 2014. US economic activity report. http://www.rcanalytics.com.

Rothschild, M., Stiglitz, J., 1976. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. The Quarterly Journal of Economics 90, 629–649.

Shackleton, M. B., Wojakowski, R. M., 2007. Finite maturity caps and floors on continuous flows. Journal of Economic Dynamics and Control 31, 3843 – 3859.

Shiller, R. J., 1993. Macro Markets: Creating Institutions for Managing Society's Largest Economic Risks. Oxford University Press Inc., New York.

Shiller, R. J., 2014. Why is housing finance still stuck in such a primitive stage? American Economic Review 104, 73–76, paper presented at the American Economic Association annual meeting, Philadelphia, January 3-5, Session: Is Neglect Benign? The Case of United States Housing Finance Policy.

Shiller, R. J., Wojakowski, R. M., Ebrahim, M. S., Shackleton, M. B., 2013. Mitigating financial fragility with continuous workout mortgages. Journal of Economic Behavior & Organization 85, 269–285.

Smith, Jr., C. W., Warner, J. B., 1979. On financial contracting: An analysis of bond covenants. Journal of Financial Economics 7, 117–161.

Stanton, R., Wallace, N., 1998. Mortgage choice: What's the point? Real Estate Economics 26, 173–205.

Stiglitz, J. E., Weiss, A., 1981. Credit Rationing in Markets with Imperfect Information. American Economic Review 71, 393–410.

Titman, S., 1985. Urban land prices under uncertainty. The American Economic Review 75, 505–514.

A Appendix: Cap and call formulae

Caps C on flow s with strike flow level k for finite horizon T can be computed using the following closed-form formula (see Shackleton and Wojakowski, 2007):

$$\begin{aligned} C(s_0,k,T,r,\delta,\sigma) &= -As_0^a\left(\mathbf{1}_{s_0>k} - N(d_a)\right) + \frac{s_0}{\delta}\left(\mathbf{1}_{s_0>k} - e^{-\delta T}N(d_1)\right) \\ & \quad -\frac{k}{r}\left(\mathbf{1}_{s_0>k} - e^{-rT}N(d_0)\right) + Bs_0^b\left(\mathbf{1}_{s_0>k} - N(d_b)\right). \end{aligned} \tag{39}$$

where

$$\mathbf{1}_z = \begin{cases} 1 & \text{if} \quad z \text{ is true} \\ 0 & \text{if} \quad z \text{ is false} \end{cases} \tag{40}$$

and

$$\begin{aligned} A &= \frac{k^{1-a}}{a-b}\left(\frac{b}{r} - \frac{b-1}{\delta}\right), \\ B &= \frac{k^{1-b}}{a-b}\left(\frac{a}{r} - \frac{a-1}{\delta}\right), \end{aligned} \tag{41}$$

and

$$a,b = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}, \tag{42}$$

whereas the cumulative normal integrals $N(\cdot)$ are labelled with parameters d_β

$$d_\beta = \frac{\ln s_0 - \ln k + \left(r - \delta + \left(\beta - \frac{1}{2}\right)\sigma^2\right)T}{\sigma\sqrt{T}} \tag{43}$$

(different to the standard textbook notation) for elasticity β which takes one of four values $\beta \in \{a, b, 0, 1\}$.

Standard Black-Scholes Black and Scholes (1973) call on S with strike value of K can be computed using

$$c\left(S_0, K, r, \delta, \sigma, T\right) = S_0 e^{-\delta T} N\left(d_1\right) - K e^{-rT} N\left(d_0\right) \tag{44}$$

where d_0 and d_1 can be computed using formula (43) in which values S_0 and K can (formally) be used in place of flows s_0 and k.

Both floor (39) and put (44) formulae assume that the underlying flow s or asset S follows the stochastic differential equation

$$\frac{ds_t}{s_t} = \frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma dZ_t \tag{45}$$

with initial values s_0 and S_0, respectively. Clearly, (45) describes a geometric Brownian motion under risk-neutral measure where Z_t is the standard Brownian motion, σ is the volatility, r is the riskless rate and δ is the payout flow rate.

B Appendix: Derivation of the expression for remaining balance

Consider

$$d\left(e^{-rs} Q_s\right) = e^{-rs} dQ_s - r e^{-rs} Q_s ds \tag{46}$$

where (see (59))

$$dQ_s = \left[(r - i)\, Q_s - \theta_s \left(P_s - K\right)^+\right] ds \tag{47}$$

Therefore

$$\begin{aligned} d\left(e^{-rs} Q_s\right) &= e^{-rs}\left[(r - i)\, Q_s - \theta_s \left(P_s - K\right)^+\right] ds - r e^{-rs} Q_s ds && (48) \\ &= e^{-rs}\left[-i Q_s - \theta_s \left(P_s - K\right)^+\right] ds && (49) \end{aligned}$$

Integrating from t to T gives

$$\int_t^T d\left(e^{-rs}Q_s\right) = \int_t^T e^{-rs}\left[-iQ_s - \theta_s\left(P_s - K\right)^+\right] ds \tag{50}$$

$$e^{-rT}Q_T - e^{-rt}Q_t = -\int_t^T e^{-rs}\left[iQ_s + \theta_s\left(P_s - K\right)^+\right] ds \tag{51}$$

Rearranging terms gives

$$Q_t = e^{-r(T-t)}Q_T + \int_t^T e^{-r(s-t)}\left[iQ_s + \theta_s\left(P_s - K\right)^+\right] ds \tag{52}$$

$$Q_t = e^{-r(T-t)}Q_T + \int_t^T e^{-r(s-t)} iQ_s ds + \int_t^T e^{-r(s-t)}\theta_s\left(P_s - K\right)^+ ds \tag{53}$$

Finally, taking risk-neutral expectations conditional on information $\mathcal{F}_t$ gives

$$Q_t = e^{-r(T-t)}E_t + \left[Q_T\right] i\int_t^T e^{-r(s-t)}E_t\left[Q_s\right] ds + \int_t^T e^{-r(s-t)}E_t\left[\theta_s\left(P_s - K\right)^+\right] ds \tag{54}$$

For $t = 0$ we recover the budget constraint (30)

$$Q_0 = e^{-rT}E\left[Q_T\right] + i\int_0^T e^{-rt}E\left[Q_t\right] dt + \int_0^T e^{-rt}E\left[\theta_t\left(P_t - K\right)^+\right] dt \tag{55}$$

C Appendix: Derivation of the HJB equation

For small h

$$J_t = \min_{\theta_t} E_t\left[\min_{\{\theta_s: s\in[t+h,T]\}} E_{t+h}\left[Q_T^2\right]\right] = \min_{\theta_t} E_t\left[J_{t+h}\right] \tag{56}$$

Rearranging terms and taking limit $h \to 0$ gives

$$\lim_{h\to 0}\min_{\theta_t} E_t\left[J_{t+h} - J_t\right] = \min_{\theta_t} E_t\left[dJ_t\right] = 0 \tag{57}$$

Itô's lemma gives

$$dJ_t = \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial Q}dQ_t + \frac{\partial J}{\partial P}dP_t + \frac{1}{2}\frac{\partial^2 J}{\partial P^2}(dP_t)^2 \tag{58}$$

Note that this particular form is due to dynamics of Q_t. In fact, it is locally deterministic and is inherited from ODE (13)

$$dQ_t = \{rQ_t - \underbrace{\left[iQ_t + \theta_t (P_t - K)^+\right]}_{x_t}\}dt \tag{59}$$

while P_t follows stochastic dynamics (2). From Itô's Lemma (58), (59) and (2) we obtain

$$\begin{aligned} dJ_t &= \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial Q}\left[(r-i)Q_t - \theta_t (P_t - K)^+\right]dt \\ &\quad + \frac{\partial J}{\partial P}(r-\delta)P_t dt + \frac{\partial J}{\partial P}\sigma P_t dZ_t + \frac{1}{2}\frac{\partial^2 J}{\partial Q^2}\sigma^2 P_t^2 dt \end{aligned} \tag{60}$$

where we used the fact that $(dQ_t)^2 = 0$ and $dQ_t dP_t = 0$, because the evolution of Q_t is deterministic. Inserting dJ_t into the optimality condition (57), taking the expectation and dropping dt gives the HJB equation

$$\min_{\theta_t}\left\{\frac{\partial J}{\partial t} + \frac{\partial J}{\partial Q}\left[(r-i)Q_t - \theta_t (P_t - K)^+\right] + \frac{\partial J}{\partial P}(r-\delta)P_t + \frac{1}{2}\frac{\partial^2 J}{\partial P^2}\sigma^2 P_t^2\right\} = 0 \tag{61}$$

The first order condition for *interior* minimum can, mechanically, be computed

$$-\frac{\partial J}{\partial Q}(P_t - K)^+ = 0 \tag{62}$$

but it is not useful because the HJB equation is *linear* in θ_t.[32] Therefore, candidates for minima should only be looked for at boundaries of the control domain, i.e. either at $\theta_t = 1$ or $\theta_t = 0$.

[32] Note that the second order "condition" (derivative) w.r.t. θ is identically zero implying no local minimum.