Classifying the Clique-Width of *H*-Free Bipartite Graphs*

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Abstract. Let G be a bipartite graph, and let H be a bipartite graph with a fixed bipartition (B_H, W_H). We consider three different, natural ways of forbidding H as an induced subgraph in G. First, G is H-free if it does not contain H as an induced subgraph. Second, G is strongly H-free if no bipartition of G contains an induced copy of H in a way that respects the bipartition of H. Third, G is weakly H-free if G has at least one bipartition that does not contain an induced copy of H in a way that respects the bipartition of H. Lozin and Volz characterized all bipartite graphs H for which the class of strongly H-free bipartite graphs has bounded clique-width. We extend their result by giving complete classifications for the other two variants of H-freeness.

Keywords: clique-width; bipartite graph; graph class

1 Introduction

The *clique-width* of a graph G is a well-known graph parameter that has been studied both in a structural and in an algorithmic context. It is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i;
(ii) taking the disjoint union of two labelled graphs G_1 and G_2;
(iii) joining each vertex with label i to each vertex with label j ($i \neq j$);
(iv) renaming label i to j.

We refer to the surveys of Gurski [19] and Kamiński, Lozin and Milanič [21] for an in-depth study of the properties of clique-width.

We say that a class of graphs has *bounded* clique-width if every graph from the class has clique-width at most c for some constant c. As many NP-hard graph problems can be solved in polynomial time on graph classes of bounded clique-width [13,22,27,28], it is natural to determine whether a certain graph class has

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bounded clique-width and to find new graph classes of bounded clique-width. In particular, many papers have determined the clique-width of graph classes characterized by one or more forbidden induced subgraphs [1,2,5–12,15,20,23–26].

In this paper we focus on classes of bipartite graphs characterized by a forbidden induced subgraph H. A graph G is H-free if it does not contain H as an induced subgraph. If G is bipartite, then when considering notions for H-freeness, we may assume without loss of generality that H is bipartite as well. For bipartite graphs, the situation is more subtle as one can define the notion of freeness with respect to a fixed ordered bipartition (B_H, W_H) of H. This leads to two other notions (see also Section 2 for formal definitions). We say that a bipartite graph G is strongly H-free if no bipartition of G contains an induced copy of H in a way that respects the bipartition of H. Strongly H-free graphs have been studied with respect to their clique-width, although under less explicit terminology (see e.g. [21,24,25]). In particular, Lozin and Volz [25] completely determined those bipartite graphs H, for which the class of strongly H-free graphs has bounded clique-width (we give an exact statement of their result in Section 3). If G has at least one bipartition that does not contain an induced copy of H in a way that respects the bipartition of H then G is said to be weakly H-free. As we shall see, any H-free graph is strongly H-free, and any strongly H-free graph is weakly H-free, whereas the two reverse statements do not always hold. Moreover, as far as we are aware, the notion of being weakly H-free has not been studied with respect to the clique-width of bipartite graphs.

Our Results: We completely classify the classes of H-free bipartite and weakly H-free bipartite graphs of bounded clique-width. In this way, we have identified a number of new graph classes of bounded clique-width. Before stating our classification results precisely in Section 3, we first give some terminology and examples in Section 2. In Section 4 we give the proofs of our results.

2 Preliminaries

We first give some terminology on general graphs and notation to denote various well-known graphs. In Section 2.1 we introduce labelled bipartite graphs. We illustrate the definitions of H-freeness, strong H-freeness and weak H-freeness of bipartite graphs with some examples. As we will explain, these examples also make clear that all three notions are different from each other.

General graphs: Let G and H be graphs. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. A bijection $f : V_G \to V_H$ is called a (*graph*) *isomorphism* when $uv \in E_G$ if and only if $f(u)f(v) \in E_H$. If such a bijection exists then G and H are *isomorphic*. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. A graph G is $(H_1, \ldots, H_p)$*-free* if no H_i is an induced subgraph of G. If $p = 1$ we may write H_1-free instead of (H_1)-free. The *disjoint union* $G + H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. We denote the disjoint union of r vertex-disjoint copies of G by rG.

Special Graphs: For $r \geq 1$, the graphs C_r, K_r, P_r denote the cycle, complete graph and path on r vertices, respectively, and the graph $K_{1,r}$ denotes the star on $r+1$ vertices. If $r = 3$, the graph $K_{1,r}$ is also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{h,i,j}$ denote the tree that has only one vertex x of degree 3 and that has exactly three leaves, which are of distance h, i and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. A graph $S_{h,i,j}$ is said to be a *subdivided claw*. A graph G is a *linear forest* if every connected component of G is a path.

2.1 Labelled Bipartite Graphs

A graph G is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. Let H be a bipartite graph. We say that H is a *labelled* bipartite graph if we are also given a *black-and-white labelling* ℓ, which is a labelling that assigns either the colour "black" or the colour "white" to each vertex of H in such a way that the two resulting monochromatic colour classes B_H^ℓ and W_H^ℓ form a *bipartition* of V_H into two (possibly empty) independent sets. From now on we denote a graph H with such a labelling ℓ by $H^\ell = (B_H^\ell, W_H^\ell, E_H)$. Here the pair (B_H^ℓ, W_H^ℓ) is *ordered*, that is, $(B_H^\ell, W_H^\ell, E_H)$ and $(W_H^\ell, B_H^\ell, E_H)$ are different labelled bipartite graphs.

We say that two labelled bipartite graphs H_1^ℓ and $H_2^{\ell^*}$ are *isomorphic* if the (unlabelled) graphs H_1 and H_2 are isomorphic, and if in addition there exists an isomorphism $f : V_{H_1} \to V_{H_2}$ such that for all $u \in V_{H_1}$, $u \in W_{H_1}^\ell$ if and only if $f(u) \in W_{H_2}^{\ell^*}$. Moreover, if $H_1 = H_2$, then ℓ and ℓ^* are said to be *isomorphic* labellings. For example, the bipartite graphs $(\{u, v\}, \emptyset)$ and $(\{x, y\}, \emptyset)$ are isomorphic, and the labelled bipartite graph $(\{u, v\}, \emptyset, \emptyset)$ is isomorphic to the labelled bipartite graph $(\{x, y\}, \emptyset, \emptyset)$. However, $(\{x, y\}, \emptyset, \emptyset)$ is neither isomorphic to $(\emptyset, \{x, y\}, \emptyset)$ nor to $(\{x\}, \{y\}, \emptyset)$ (also see Fig. 1).

We write $H_1^\ell \subseteq_{li} H_2^{\ell^*}$ if $H_1 \subseteq_i H_2$, $B_{H_1}^\ell \subseteq B_{H_2}^{\ell^*}$ and $W_{H_1}^\ell \subseteq W_{H_2}^{\ell^*}$. In this case we say that H_1^ℓ is a *labelled* induced subgraph of $H_2^{\ell^*}$. Note that the two labelled bipartite graphs $H_1^{\ell_1}$ and $H_2^{\ell_2}$ are isomorphic if and only if $H_1^{\ell_1}$ is a labelled induced subgraph of $H_2^{\ell_2}$, and vice versa.

(a) $(2P_1)^b$ (b) $(2P_1)^{\overline{b}}$ (c) $(P_1)^b + (P_1)^{\overline{b}}$

Fig. 1: The three pairwise non-isomorphic labellings of $2P_1$. The labellings b and $\overline{b}$ will be formally defined later.

Let G be an (unlabelled) bipartite graph, and let H^ℓ be a labelled bipartite graph. The graph G is *strongly* H^ℓ-free if for every labelling ℓ^* of G, G^{ℓ^*} does not contain H^ℓ as a labelled induced subgraph. The graph G is *weakly* H^ℓ-free if there is a labelling ℓ^* of G such that G^{ℓ^*} does not contain H^ℓ as a labelled induced subgraph. Note that these two notions of freeness are only defined for (unlabelled) bipartite graphs. Let $\{H_1^{\ell_1}, \dots, H_p^{\ell_p}\}$ be a set of labelled bipartite graphs. Then a graph G is *strongly (weakly)* $(H_1^{\ell_1}, \dots, H_p^{\ell_p})$*-free* if G is strongly (weakly) $H_i^{\ell_i}$-free for $i = 1, \dots, p$.

The following lemma shows that for all labelled bipartite graphs H^ℓ, the class of H-free graphs is a (possibly proper) subclass of the class of strongly H^ℓ-free bipartite graphs and that the latter graph class is a (possibly proper) subclass of the class of weakly H^ℓ-free bipartite graphs.

Lemma 1. *Let G be a bipartite graph and H^ℓ be a labelled bipartite graph. The following two statements hold:*

(i) If G is H-free, then G is strongly H^ℓ-free.
(ii) If G is strongly H^ℓ-free, then G is weakly H^ℓ-free.

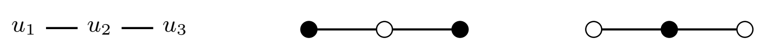
Moreover, the two reverse statements are not necessarily true.

Proof. Statements (i) and (ii) follow by definition.

The following examples, which are also depicted in Fig. 2, show that the reverse statements may not necessarily be true. Let G be isomorphic to P_3 with $V_G = \{u_1, u_2, u_3\}$ and $E_G = \{u_1u_2, u_2u_3\}$. Let $H = 2P_1$. We denote the vertex set and edge set of H by $V_H = \{x_1, x_2\}$ and $E_H = \emptyset$, respectively.

Let $H^\ell = (P_1)^b + (P_1)^{\bar{b}} = (\{x_1\}, \{x_2\}, \emptyset)$ (see also Fig. 1). We first notice that G is not H-free, because $G[\{u_1, u_3\}]$ is isomorphic to $2P_1$. However, we do have that G is strongly H^ℓ-free, because H^ℓ is neither a labelled induced subgraph of $G^b = (\{u_1, u_3\}, \{u_2\}, E_G)$ nor of $G^{\bar{b}} = (\{u_2\}, \{u_1, u_3\}, E_G)$.

Let $H^{\ell^*} = (2P_1)^b = (\{x_1, x_2\}, \emptyset, E_H)$ (see also Fig. 1). Then G is not strongly H^{ℓ^*}-free, because $(\{u_1, u_3\}, \emptyset, \emptyset)$ is isomorphic to H^{ℓ^*}. However, G is weakly H^{ℓ^*}-free, because H^{ℓ^*} is not a labelled induced subgraph of $G^{\bar{b}} = (\{u_2\}, \{u_1, u_3\}, E_G)$. □



(a) G (b) G^b (c) $G^{\bar{b}}$

Fig. 2: The graph $G = P_3$ and its two labellings.

Next, we prove a lemma which is related to Lemma 1 and which follows immediately from the corresponding definitions.

Lemma 2. *Let G and H be bipartite graphs. Then G is H-free if and only if G is strongly H^ℓ-free for all black-and-white labellings ℓ of H.*

A graph G that contains a graph H as an induced subgraph may be weakly H^ℓ-free for all black-and-white labellings ℓ of H; take for instance the graphs

$G = P_3$ and $H = 2P_1$ as in the proof of Lemma 1. However, we can make the following observation, which also follows directly from the corresponding definitions.

Lemma 3. *Let H be a bipartite graph with a unique black-and-white labelling ℓ (up to isomorphism). Then every bipartite graph G is H-free if and only if it is weakly H^ℓ-free.*

Note that there exist both connected bipartite graphs (for example $H = P_6$) and disconnected bipartite graphs (for example $H = 2P_2$) that satisfy the condition of Lemma 3.

Let $H^\ell = (B_H^\ell, W_H^\ell, E_H)$ be a labelled bipartite graph. The *opposite* of H^ℓ is defined as the labelled bipartite graph $H^{\overline{\ell}} = (W_H^\ell, B_H^\ell, E_H)$; in other words it is the labelled bipartite graph obtained from H^ℓ by recolouring the black vertices to be white and vice versa. We say that $\overline{\ell}$ is the *opposite* black-and-white labelling of ℓ. Suppose that H is a bipartite graph such that among all its black-and-white labellings, all those that maximize the number of black vertices are isomorphic. In this case we pick one of such labelling and call it b (see also Fig. 1). Note that there are graphs for which such a labelling does not exist. For example, the graph $S_{1,2,2}$ has two non-isomorphic labellings and both of them have the same number of black vertices (see also Fig. 3). If such a unique labelling b does exist, we let $\overline{b}$ denote the opposite labelling to b.

Fig. 3: The two labellings of $S_{1,2,2}$.

Two black-and-white labellings of a bipartite graph H are said to be *equivalent* if they are isomorphic or opposite to each other; otherwise they are said to be *non-equivalent*. Note that if a linear forest has two non-equivalent labellings then it must contain at least two components with an odd number of vertices. The following lemma follows directly from the definitions.

Lemma 4. *Let ℓ and ℓ^* be two equivalent black-and-white labellings of a bipartite graph H. Then the class of strongly (weakly) H^ℓ-free graphs is equal to the class of strongly (weakly) H^{ℓ^*}-free graphs.*

We will also need the following two lemmas.

Lemma 5. *Let H^ℓ be a labelled bipartite graph. Then $H \subseteq_i P_2 + P_4$ or $H \subseteq_i P_6$ if and only if $H^\ell \subseteq_{li} (P_2 + P_4)^b$ or $H^\ell \subseteq_{li} (P_6)^b$.*

Proof. Clearly, if $H^\ell \subseteq_{li} (P_2+P_4)^b$ or $H^\ell \subseteq_{li} (P_6)^b$ then $H \subseteq_i P_2+P_4$ or $H \subseteq_i P_6$.

Now suppose $H \subseteq_i P_2+P_4$ or $H \subseteq_i P_6$ and let ℓ be a labelling of H. We will show that $H^\ell \subseteq_{li} (P_2+P_4)^b$ or $H^\ell \subseteq_{li} (P_6)^b$. Note that P_2+P_4 and P_6 have a unique labelling b (up to isomorphism). We may therefore assume that $H \notin \{P_2+P_4, P_6\}$. If H^ℓ is not a labelled induced subgraph of one of $\{(P_2+P_4)^b, (P_6)^b\}$ then H must have two non-equivalent black-and-white labellings. Since H is a linear forest, it must have at least two components with an odd number of vertices. Therefore $H \in \{2P_1, 3P_1, P_1+P_3, 2P_1+P_2\}$. However, in all these cases, for every labelling ℓ of H, $H^\ell \subseteq_{li} (P_6)^b$ or $H^\ell \subseteq_{li} (P_2+P_4)^b$. This completes the proof. □

Lemma 6. *Let $H \in \mathcal{S}$. Then H is $(2P_1+2P_2, 2P_1+P_4, 4P_1+P_2, 3P_2, 2P_3)$-free if and only if $H = sP_1$ for some integer $s \geq 1$ or H is an induced subgraph of one of the graphs in $\{K_{1,3}+3P_1, K_{1,3}+P_2, P_1+S_{1,1,3}, S_{1,2,3}\}$.*

Proof. Let $H \in \mathcal{S}$. First suppose that $H = sP_1$ for some integer $s \geq 1$ or that H is an induced subgraph of one of the graphs in $\{K_{1,3}+3P_1, K_{1,3}+P_2, P_1+S_{1,1,3}, S_{1,2,3}\}$. It is readily seen that H is $(2P_1+2P_2, 2P_1+P_4, 4P_1+P_2, 3P_2, 2P_3)$-free. We may therefore assume that H contains at least one edge.

Now suppose that H is $(2P_1+2P_2, 2P_1+P_4, 4P_1+P_2, 3P_2, 2P_3)$-free. Let $D_1, \ldots D_r$ be the connected components of H, where $|V_{D_1}| \leq \cdots \leq |V_{D_r}|$. Since H contains an edge, we find that $|V_{D_r}| \geq 2$. Since H is $(4P_1+P_2)$-free, it follows that $r \leq 4$.

Suppose that $r=4$. Because H is $(2P_1+2P_2)$-free, it follows that $D_3 = P_1$, so $H = 3P_1 + D_3$. Because H is $(2P_1+P_4)$-free, D_3 must be P_4-free. As $H \in \mathcal{S}$, this means that D_3 is isomorphic to one of $\{K_{1,3}, P_2, P_3\}$. Hence, H is an induced subgraph of $K_{1,3}+3P_1$.

Now suppose that $r=3$ Because H is $3P_2$-free, it follows that $D_1 = P_1$. Since H is $2P_3$-free and $H \in \mathcal{S}$, it follows that D_2 is P_3-free, so $D_2 \in \{P_1, P_2\}$. Because H is $(2P_1+P_4)$-free, D_3 must be P_4-free. As $H \in \mathcal{S}$, this means that $D_3 \in \{K_{1,3}, P_2, P_3\}$. Because H is $(4P_1+P_2)$-free, the combination $D_2 = P_2$ and $D_3 = K_{1,3}$ is not possible. Hence, if $D_2 = P_1$ then H is an induced subgraph of $K_{1,3}+2P_1$ and if $D_2 = P_2$ then H is an induced subgraph of $P_1+P_2+P_3$. This means that H is an induced subgraph of $K_{1,3}+3P_1$ or of $S_{1,2,3}$, respectively.

Now suppose that $r=2$. Because H is $2P_3$-free and $H \in \mathcal{S}$, we find that $D_1 \in \{P_1, P_2\}$ and that D_2 is either a path or a subdivided claw. Because H is $(2P_1+P_4)$-free, D_2 is P_6-free. Suppose that D_2 is a path. Then $D_2 \subseteq_i P_5$. If $D_2 = P_5$ then $D_1 = P_1$, as H is $3P_2$-free. Hence we find that H is an induced subgraph of P_1+P_5 or P_2+P_4, which are induced subgraphs of $P_1+S_{1,1,3}$ and $S_{1,2,3}$, respectively. Suppose that D_2 is a subdivided claw, say $D_2 = S_{a,b,c}$ for some $1 \leq a \leq b \leq c$. Then, because H is $(2P_1+2P_2)$-free, $a=b=1$. Because H is $(2P_1+P_4)$-free, $c \leq 3$. Moreover, if $2 \leq c \leq 3$ then $D_1 = P_1$, as $H = (2P_1+2P_2)$-free. Hence, we find that H is an induced subgraph of $K_{1,3}+P_2$ or $P_1+S_{1,1,3}$.

Now suppose that $r=1$, in which case H is connected. As $H \in \mathcal{S}$, we find that H is either a path or a subdivided claw. If H is a path then, as H is $2P_3$-free,

H is an induced subgraph of P_6, which means that $H \subseteq_i S_{1,2,3}$. Suppose that H is a subdivided claw, say $H = S_{a,b,c}$ for some $1 \leq a \leq b \leq c$. Because H is $3P_2$-free, we find that $a = 1$. Because H is $2P_3$-free, we find that $b \leq 2$ and that $c \leq 3$. Hence, H is an induced subgraph of $S_{1,2,3}$. This completes the proof. □

3 The Classifications

A full classification of the boundedness of the clique-width of strongly H^ℓ-free bipartite graphs was given by Lozin and Volz [25], except that in their result the trivial case when H^ℓ or $H^{\overline{\ell}} = (sP_1)^b$ for some $s \geq 1$ was missing. Their proof is correct except that it overlooked this case, which occurs when one of the colour classes of the labelled graph H^ℓ is empty. However, strongly $(sP_1)^b$-free bipartite graphs can have at most $2s - 2$ vertices, and as such form a class of bounded clique-width. Below we state their result after incorporating this small correction, followed by our results for the other two variants of freeness. We refer to Fig. 4 for pictures of the labelled bipartite graphs used in Theorems 1 and 3.

Theorem 1 ([25]). *Let H^ℓ be a labelled bipartite graph. The class of strongly H^ℓ-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- H^ℓ *or* $H^{\overline{\ell}} = (sP_1)^b$ *for some* $s \geq 1$
- H^ℓ *or* $H^{\overline{\ell}} \subseteq_{li} (K_{1,3} + 3P_1)^b$
- H^ℓ *or* $H^{\overline{\ell}} \subseteq_{li} (K_{1,3} + P_2)^b$
- H^ℓ *or* $H^{\overline{\ell}} \subseteq_{li} (P_1 + S_{1,1,3})^b$
- H^ℓ *or* $H^{\overline{\ell}} \subseteq_{li} (S_{1,2,3})^b$

Theorem 2. *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H = sP_1$ *for some* $s \geq 1$
- $H \subseteq_i K_{1,3} + 3P_1$
- $H \subseteq_i K_{1,3} + P_2$
- $H \subseteq_i P_1 + S_{1,1,3}$
- $H \subseteq_i S_{1,2,3}$.

Theorem 3. *Let H^ℓ be a labelled bipartite graph. The class of weakly H^ℓ-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

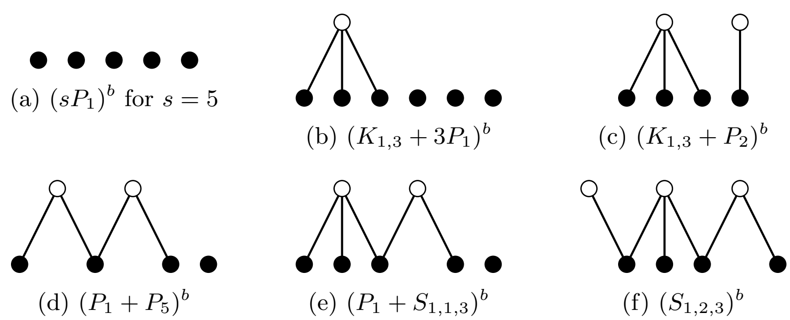
- H^ℓ *or* $H^{\overline{\ell}} = (sP_1)^b$ *for some* $s \geq 1$
- H^ℓ *or* $H^{\overline{\ell}} \subseteq_{li} (P_1 + P_5)^b$
- $H^\ell \subseteq_i (P_2 + P_4)^b$
- $H^\ell \subseteq_i (P_6)^b$.

Note that Theorem 2 is exactly the unlabelled variant of Theorem 1. Indeed, if H^ℓ is a labelled bipartite graph then the class of H-free bipartite graphs is contained in the class of strongly H^ℓ-free bipartite graphs (by Lemma 1), so all the bounded cases carry over. However, we need to do some more work to deal with the unbounded cases, so Theorem 2 does not follow from Theorem 1 as a direct corollary.

Also note that by Lemma 5, we can also state Theorem 3 as follows. (We originally stated the theorem in this form in the extended abstract of this paper [17].)

Theorem 3 (equivalent formulation). *Let H^ℓ be a labelled bipartite graph. The class of weakly H^ℓ-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- H^ℓ *or* $H^{\overline{\ell}} = (sP_1)^b$ *for some* $s \geq 1$
- H^ℓ *or* $H^{\overline{\ell}} \subseteq_{li} (P_1 + P_5)^b$
- $H \subseteq_i P_2 + P_4$
- $H \subseteq_i P_6$.



(a) $(sP_1)^b$ for $s = 5$ (b) $(K_{1,3} + 3P_1)^b$ (c) $(K_{1,3} + P_2)^b$

(d) $(P_1 + P_5)^b$ (e) $(P_1 + S_{1,1,3})^b$ (f) $(S_{1,2,3})^b$

Fig. 4: The labelled bipartite graphs used in Theorems 1 and 3.

4 The Proofs of Our Results

We first recall a number of basic facts on clique-width known from the literature. We then state a number of other lemmas which we use to prove Theorems 2 and 3.

4.1 Facts about Clique-width

For two disjoint vertex subsets X and Y in a (not necessarily bipartite) graph G, the *bipartite complementation* operation with respect to X and Y acts on G by replacing every edge with one end-vertex in X and the other one in Y by a non-edge and vice versa. The *bipartite complement* of a bipartite graph *with respect* *to* a bipartition (B, W) is the bipartite graph with bipartition (B, W) obtained from G by applying a bipartite complementation between B and W. For instance, the graph $2P_2$ has a unique bipartition (up to isomorphism) and it therefore has only one bipartite complement, namely $2P_2$. On the other hand $2P_1$ does not have a unique bipartition, and both $2P_1$ and P_2 can be obtained as bipartite complements of it, depending on the choice of partition. The *edge subdivision* operation replaces an edge vw in a graph by a new vertex u with edges uv and uw.

Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k, γ)-*obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

If we allow arbitrarily many applications of γ then we write that $\mathcal{G}'$ is (∞, γ)-obtained from $\mathcal{G}$.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [14,23].

Fact 2. Bipartite complementation preserves boundedness of clique-width [21].

Fact 3. For a class of graphs $\mathcal{G}$ of *bounded* maximum degree, let $\mathcal{G}'$ be a class of graphs that is $(\infty, \texttt{es})$-obtained from $\mathcal{G}$, where $\texttt{es}$ is the edge subdivision operation. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}'$ has bounded clique-width [21].

We also use some other elementary results on the clique-width of graphs. In order to do so we need the notion of a *wall*. We do not formally define this notion, but instead refer to Fig. 5, in which three examples of walls of different height are depicted. A k*-subdivided wall* is a graph obtained from a wall after subdividing each edge exactly k times for some constant $k \geq 0$. The next well-known lemma follows from combining Fact 3 with the fact that walls have maximum degree 3 and unbounded clique-width (see e.g. [21]).

Lemma 7. *For every constant k, the class of k-subdivided walls has unbounded clique-width.*

We let $\mathcal{S}$ be the class of graphs each connected component of which is either a subdivided claw $S_{h,i,j}$ for some $1 \leq h \leq i \leq j$ or a path P_r for some $r \geq 1$. Note that every graph in $\mathcal{S}$ is of maximum degree at most 3 and every connected component of a graph in $\mathcal{S}$ has at most one vertex of degree 3. This leads to the following lemma, which is well known and follows from the fact that walls have maximum degree at most 3 and from Lemma 7 by choosing an appropriate value for k (also note that k-subdivided walls are bipartite for all $k \geq 0$).

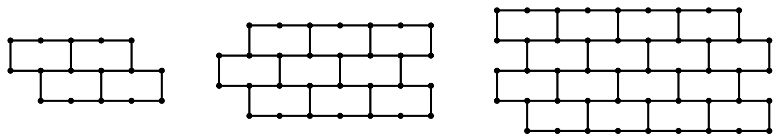


Fig. 5: Walls of height 2, 3, and 4, respectively.

Lemma 8. *Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. If $H_i \notin \mathcal{S}$ for $i = 1, \ldots, p$ then the class of $(H_1, \ldots, H_p)$-free bipartite graphs contains all $\big(\max\{|V_{H_1}|, \ldots, |V_{H_p}|\}\big)$-subdivided walls, and thus has unbounded clique-width.*

The following lemma is due to Lozin and Rautenbach [24].

Lemma 9 ([24]). *Let $\{H_1^{\ell_1}, \ldots, H_p^{\ell_p}\}$ be a finite set of labelled bipartite graphs. For $i = 1, \ldots, p$, let F_i denote the bipartite complement of H_i with respect to $(B_{H_i}^{\ell_i}, W_{H_i}^{\ell_i})$. If $H_i \notin \mathcal{S}$ for all $1 \leq i \leq p$ or $F_i \notin \mathcal{S}$ for all $1 \leq i \leq p$, then the class of strongly $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-free bipartite graphs has unbounded clique-width.*

In the next two lemmas we list a number of classes of H-free bipartite graphs that have unbounded clique-width. The first of these is due to Lozin and Volz.

Lemma 10 ([25]). *The class of $2P_3$-free graphs is unbounded.*

Lemma 11. *The class of H-free bipartite graphs has unbounded clique-width if $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2\}$.*

Proof. Let $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2\}$, and let $\{H^{\ell_1}, \ldots, H^{\ell_p}\}$ be the set of all non-equivalent labelled bipartite graphs isomorphic to H. For $i = 1, \ldots, p$, let F_i denote the bipartite complement of H with respect to $(B_H^{\ell_i}, W_H^{\ell_i})$. We will show that every F_i does not belong to $\mathcal{S}$. Then, by Lemma 9 the class of strongly $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-free bipartite graphs has unbounded clique-width. Because a bipartite graph is H-free if and only if it is strongly $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-free (by Lemmas 2 and 4), this means that the class of H-free bipartite graphs has unbounded clique-width.

Suppose $H \in \{2P_1 + 2P_2, 2P_1 + P_4\}$. Let $V_H = \{x_1, \ldots, x_6\}$ with $E_H = \{x_1x_2, x_3x_4\}$ if $H = 2P_1 + 2P_2$ and $E_H = \{x_1x_2, x_2x_3, x_3x_4\}$ if $H = 2P_1 + P_4$. Then H has only two non-equivalent black-and-white labellings. We may assume without loss of generality that one of these two labellings colours x_1, x_3, x_5, x_6 black and x_2, x_4 white, whereas the other one colours x_1, x_3, x_5 black and x_2, x_4, x_6 white. Let F_1 and F_2 be the bipartite complements corresponding to the first and second labellings, respectively. The vertices x_2, x_4, x_5, x_6 induce a C_4 in F_1, whereas the vertices x_1, x_4, x_5, x_6 induce a C_4 in F_2. Hence, F_1 and F_2 do not belong to $\mathcal{S}$.

Suppose $H = 4P_1 + P_2$. Let $V_H = \{x_1, \ldots, x_6\}$ and $E_H = \{x_1x_2\}$. Then H has three non-equivalent black-and-white labellings. We may assume without loss of generality that the first one colours x_1, x_3, x_4, x_5, x_6 black and x_2 white, the second one colours x_1, x_3, x_4, x_5 black and x_2, x_6 white, and the third one colours x_1, x_3, x_4 black and x_2, x_5, x_6 white. Let F_1, F_2, F_3 denote the corresponding bipartite complements. The vertices $x_2, \ldots, x_6$ induce a $K_{1,4}$ in F_1. The vertices x_2, x_3, x_4, x_6 induce a C_4 in F_2 and F_3. Hence, none of F_1, F_2, F_3 belongs to $\mathcal{S}$.

Suppose $H = 3P_2$. Let $V_H = \{x_1, \ldots, x_6\}$ and $E_H = \{x_1x_2, x_3x_4, x_5x_6\}$. Let ℓ be a black-and-white labelling of H that colours x_1, x_3, x_5 black and x_2, x_4, x_6 white. Then every other labelling ℓ^* of H is isomorphic to ℓ. The bipartite complement of H with respect to (B_H^ℓ, W_H^ℓ) is isomorphic to C_6, which does not belong to $\mathcal{S}$. □

The last lemma we need before proving the main results of this paper is the following one (we use it several times in the proof of Theorem 3).

Lemma 12. *Let H^ℓ be a labelled bipartite graph. The class of weakly H^ℓ-free bipartite graphs has unbounded clique-width in both of the following cases:*

(i) H^ℓ contains a vertex of degree at least 3, or
(ii) H^ℓ contains four independent vertices, not all of the same colour.

Proof. Let b_1 be a black-and-white labelling of $4P_1$ that colours three vertices black and one vertex white. Let b_2 be a black-and-white labelling of $4P_1$ that colours two vertices black and two vertices white. We show below that the class of weakly H^ℓ-free bipartite graphs has unbounded clique-width if $H^\ell \in \{(K_{1,3})^b, (4P_1)^{b_1}, (4P_1)^{b_2}\}$. Then we are done by Lemma 4.

Consider a 1-subdivided wall G' obtained from a wall G. Recall that 1-subdivided walls are bipartite. Moreover, the vertices that were introduced when subdividing every edge of G all have degree 2 and the set of these vertices forms one class of a bipartition (B, W) of G'. Let this class be B. Then $(K_{1,3})^b$ is not a labelled induced subgraph of $(B, W, E_{G'})$. Hence, G' is weakly $(K_{1,3})^b$-free. This means that the class of weakly $(K_{1,3})^b$-free graphs contains the class of 1-subdivided walls. As such, it has unbounded clique-width by Lemma 7. The bipartite complement G'' of G' with respect to (B, W) is weakly $(4P_1)^{b_1}$-free, as $(K_{1,3})^b$ is the bipartite complement of $(4P_1)^{b_1}$ and $(K_{1,3})^b$ is not a labelled induced subgraph of $(B, W, E_{G'})$. Hence, the class of weakly $(4P_1)^{b_1}$-free graphs has unbounded clique-width by Fact 2. The class of weakly $(4P_1)^{b_2}$-free bipartite graphs has unbounded clique-width by Lemma 1 and Theorem 1. □

4.2 The Proof of Theorem 2

Proof. First suppose that H does not contain an edge, so $H = sP_1$ for some $s \geq 1$. Then every H-free bipartite graph G has at most $s-1$ vertices in each partition class for every bipartition. This means that the clique-width of G is at most $2s-2$. For the remainder of the proof we therefore assume that H contains at least one edge.

If $H \in \{K_{1,3}+3P_1, K_{1,3}+P_2, P_1+S_{1,1,3}, S_{1,2,3}\}$ then the claim follows from combining Lemma 1 with Theorem 1. Now suppose that H is not an induced subgraph of one of the graphs in $\{K_{1,3}+3P_1, K_{1,3}+P_2, P_1+S_{1,1,3}, S_{1,2,3}\}$. Then by Lemma 6, either $H \notin \mathcal{S}$ or, H is not $(2P_1+2P_2, 2P_1+P_4, 4P_1+P_2, 3P_2, 2P_3)$-free. Hence, the clique-width of the class of H-free bipartite graphs is unbounded by Lemmas 8, 10 and 11. $\square$

4.3 The Proof of Theorem 3

Proof. We first consider the bounded cases. First suppose $H^\ell = (sP_1)^b$ for some $s \geq 1$ (the $H^{\bar{\ell}} = (sP_1)^b$ case is equivalent). Then every weakly H^ℓ-free bipartite graph has a bipartition (B, W) with $|B| \leq s-1$. The clique-width of such graphs is at most $s+1$: first introduce the vertices of B using distinct labels and then use two more labels for the vertices of W, introducing them one-by-one.

Suppose $H^\ell = (P_2+P_4)^b$ or $(P_6)^b$. Then $H \subseteq_i S_{1,2,3}$, which implies that that the class of H-free bipartite graphs has bounded clique-width by Theorem 2. All black-and-white labellings of P_2+P_4 are isomorphic. Similarly, all black-and-white labellings of P_6 are isomorphic. Hence, the class of H-free bipartite graphs coincides with the class of weakly H^ℓ-free graphs by Lemma 3. We therefore conclude that the latter class also has bounded clique-width.

Finally, let $H^\ell = (P_1+P_5)^b$. Note that in a $(P_1+P_5)^b$, the set of four black vertices have every possible neighbourhood among the two white vertices. Therefore, if we apply a bipartite complementation in a labelled bipartite graph between a subset of the white vertices and the set of *all* black vertices, then any six vertices that form a $(P_1+P_5)^b$ in the original graph will still form a $(P_1+P_5)^b$ in the obtained graph and vice versa. Suppose G is a weakly $(P_1+P_5)^b$-free bipartite graph. Then G has a labelling ℓ^* such that $(P_1+P_5)^b$ is not a labelled induced subgraph of $(B_G^{\ell^*}, W_G^{\ell^*}, E_G)$. If $|B_G^{\ell^*}|$ is even, then we delete a vertex of $B_G^{\ell^*}$. We may do this by Fact 1. Hence $|B_G^{\ell^*}|$ may be assumed to be odd. Let X be the subset of $W_G^{\ell^*}$ that consists of all vertices that are adjacent to less than half of the vertices of $B_G^{\ell^*}$. We apply a bipartite complementation between X and $B_G^{\ell^*}$. We may do this by Fact 2. Let G_1 be the resulting bipartite graph, with bipartition classes $B_{G_1}^{\ell^*} = B_G^{\ell^*}$ and $W_{G_1}^{\ell^*} = W_G^{\ell^*}$. Since $|B_{G_1}^{\ell^*}| = |B_G^{\ell^*}|$ is odd, in the graph G_1 every vertex of $W_{G_1}^{\ell^*}$ is adjacent to more than half of the vertices in $B_{G_1}^{\ell^*}$.

Suppose $B_{G_1}^{\ell^*}$ contains three vertices b_1, b_2, b_3 and $W_{G_1}^{\ell^*}$ contains two vertices w_1, w_2 such that $G_1^{\ell^*}[b_1, b_2, b_3, w_1, w_2]$ is isomorphic to $(P_1+2P_2)^b$. Because every vertex of $W_{G_1}^{\ell^*}$ is adjacent to more than half of the vertices in $B_{G_1}^{\ell^*}$, w_1 and w_2 have at least one common neighbour $b_4 \in B_{G_1}^{\ell^*}$. Then $G_1^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$ is isomorphic to $(P_1+P_5)^b$. However, then $G^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$ is also isomorphic to $(P_1+P_5)^b$ (irrespective of whether w_1 or w_2 belong to X), which is a contradiction. We conclude that G_1 is weakly $(P_1+2P_2)^b$-free. As observed above, this means that G_1 has bounded clique-width. Hence G has bounded clique-width.

We now consider the unbounded cases. Let H^ℓ be a labelled bipartite graph that is not isomorphic to one of the (bounded) cases considered already. Suppose that H contains a cycle or an induced subgraph isomorphic to $2P_3$. Then the class of weakly H^ℓ-free graphs has unbounded clique-width by combining Lemma 1 with Theorem 2. Suppose that H contains a vertex of degree at least 3. Then the class of weakly H^ℓ-free bipartite graphs has unbounded clique-width by Lemma 12(i). It remains to consider the case when $H = sP_1 + tP_2 + P_r$ for some constants $1 \leq r \leq 6$, $s \geq 0$ and $t \geq 0$, where $\max\{s,t\} \geq 1$ (as H^ℓ is not a labelled induced subgraph of P_6). We will show that in most of the cases we need to consider, we can find four pairwise disjoint vertices in H^ℓ that are not all of the same colour, in which case we apply Lemma 12(ii).

Suppose $5 \leq r \leq 6$. Assume without loss of generality that three vertices of the copy of P_r in H^ℓ are coloured black. If $r = 6$ or $t \geq 1$ or some copy P_1 in H^ℓ is coloured white, or two copies of P_1 in H^ℓ are coloured black, then we apply Lemma 12(ii). Hence, $H^\ell = (P_1 + P_5)^b$, which is not possible by assumption.

Suppose $r = 4$. If two vertices in the induced subgraph of H^ℓ isomorphic to $sP_1 + tP_2$ have the same colour then we apply Lemma 12(ii). Hence we may assume that $s \leq 2$ and $t \leq 1$, and moreover that $s = 0$ if $t = 1$. Also we would have $H^\ell \subseteq_{li} (P_2 + P_4)^b$ if $s = 0$ and $t = 1$ or if $s = 1$ and $t = 0$. Hence, it remains to consider the case $s = 2$ and $t = 0$, such that one copy of P_1 is coloured black and the other one white. In that case, we apply Lemma 12(ii).

Suppose $r = 3$. Assume without loss of generality that the two vertices of the copy of P_3 in H^ℓ are coloured black. Recall that $s \geq 1$ or $t \geq 1$. If $t \geq 2$, then we apply Lemma 12(ii). Suppose $t = 1$. Then $s = 0$ otherwise H^ℓ would contain an induced $4P_1$ in which not all the vertices are the same colour, in which case we would apply Lemma 12(ii). However, this means that $H^\ell \subseteq_{li} (P_2 + P_4)^b$. Now suppose $t = 0$. Then $s \geq 2$, as otherwise $H^\ell \subseteq_{li} (P_2 + P_4)^b$. If $s \geq 3$ then H^ℓ contains an induced $4P_1$ in which not all the vertices are the same colour, in which case we apply Lemma 12(ii). Hence, $s = 2$ and both copies are coloured black (otherwise we apply Lemma 12(ii)). However, in this case H^ℓ is a labelled induced subgraph of $(P_1 + P_5)^b$, which is not possible by assumption.

Finally suppose that $r \leq 2$. Then we may write $H = sP_1 + tP_2$ instead. We must have $s + t \geq 4$ or $t \geq 3$, otherwise $H^\ell \subseteq_{li} (P_2 + P_4)^b$. If $t = 0$ then since $H^\ell \neq (sP_1)^b$ and $H^\ell \neq (sP_1)^{\overline{b}}$ we can find four copies of P_1 in H that are not all of the same colour and apply Lemma 12(ii). If $t \geq 1, s + t \geq 4$, we can also find four copies of P_1 that are not all of the same colour and apply Lemma 12(ii). Finally, suppose $s = 0, t = 3$. In this case we combine Lemmas 1 and 11. This completes the proof. □

5 Conclusions

We have completely determined those bipartite graphs H for which the class of H-free bipartite graphs has bounded clique-width. We also characterized exactly those labelled bipartite graphs H for which the class of weakly H-free bipartite graphs has bounded clique-width. These results complement the known characterization of Lozin and Volz [25] for strongly H-free bipartite graphs. A natural direction for further research would be to characterize, for each of the three notions of H-freeness, the clique-width of classes of $\mathcal{H}$-free bipartite graphs when $\mathcal{H}$ is an arbitrary set containing at least two graphs. Here, the underlying research question is to determine what kinds of properties ensure that a graph class has bounded clique-width. As mentioned in Section 1, many results exist in the literature. In a series of follow-up papers [3,4,16,18] we have tried to address this question by determining classes of (H_1, H_2)-free (general) graphs, H-free split graphs, H-free chordal graphs and H-free weakly chordal graphs of bounded and unbounded clique-width. In each of these papers, we have applied our results for H-free bipartite graphs as useful lemmas.

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