AVERAGING METHOD APPLIED TO THE THREE-DIMENSIONAL PRIMITIVE EQUATIONS

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ABSTRACT. In this article we study the small Rossby number asymptotics for the three-dimensional primitive equations of the oceans and of the atmosphere. The fast oscillations present in the exact solution are eliminated using an averaging method, the so-called renormalisation group method.

1. **Introduction.** The geophysical fluids are influenced by rotational and stratification effects. The study of the limit of the equations describing these flows, as the rotation and the stratification are very important, is a problem of major interest from the theoretical and computational points of view.

In this article we study the small Rossby number asymptotics for the three-dimensional primitive equations of the ocean and the atmosphere. When a small parameter, related to the Rossby number, goes to zero, the solution undergoes fast oscillations which we would like to eliminate by an averaging method. In order to average the exact solution, we use the so-called renormalisation group method, which was introduced by Schochet in [22, 23]. The form of the method that we use here is due to Ziane [32]. The method was introduced in a physical context by Chen, Goldenfeld and Oono [6] and used in a mathematical context, for rotation fluids and geophysical flows by Chemin [7], Embid-Majda [9], Genier [12] and many others. Many more articles on the subject of the renormalisation group method are available in the physics and mathematical literatures; we mention here the works of Gallagher [10, 11], of Babin, Mahalov, Nicolaenko [3]-[5], of Moise, Temam, Ziane [16]. In the context of ODEs Temam and Wirosoetisno applied the method to higher orders [24].

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This article is organized as follows: in Section 2 we introduce the three-dimensional primitive equations and recall some important results on the global-in-time existence of weak and very strong solutions. In Section 3 we recall the idea of the renormalisation group method, given in an abstract context. In Section 4 we apply the renormalised group method to the primitive equations, we construct an approximate solution and we study the error between the exact solution and the approximate solution. The Appendix contains a technical result regarding the way we can bound some small denominators, result necessary for the error estimates.

2. **The three-dimensional primitive equations.** In this section we introduce the three dimensional primitive equations written in a non-dimensional form and we recall the available results on the global in time existence and regularity of the solutions. The equations are considered on the domain

$$\mathcal{M} = [0, L_1] \times [0, L_2] \times \left[-\frac{L_3}{2}, \frac{L_3}{2} \right].$$

The primitive equations read:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + v \frac{\partial u}{\partial x_2} + w \frac{\partial u}{\partial x_3} - \frac{1}{\varepsilon} v + \frac{1}{\varepsilon} \frac{\partial p}{\partial x_1} = \nu_v \Delta u + S_u, \\
&\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} + v \frac{\partial v}{\partial x_2} + w \frac{\partial v}{\partial x_3} + \frac{1}{\varepsilon} u + \frac{1}{\varepsilon} \frac{\partial p}{\partial x_2} = \nu_v \Delta v + S_v, \\
&\frac{\partial p}{\partial x_3} = -N\rho, \\
&\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_3} = 0, \\
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1} + v \frac{\partial \rho}{\partial x_2} + w \frac{\partial \rho}{\partial x_3} - \frac{N}{\varepsilon} w = \nu_\rho \Delta \rho + S_\rho,
\end{aligned}
\tag{2.1}
$$

where (u, v, w) is the three-dimensional velocity, p is the pressure, ρ is the density and ε is the Rossby number. Here ν_v and ν_ρ are the non-dimensional eddy viscosities, N is the Burgers number and (S_u, S_v, S_ρ) is a forcing term.

The variable ρ is the perturbation of the density from a stably-stratified profile, the full density of the fluid being given by

$$\rho_{full} = \rho_0 + \bar{\rho} + \rho, \tag{2.2}$$

where $\bar{\rho}$ is the density stratification profile which is assumed to be linear since the Brunt-Väisälä frequency

$$N^2 = \frac{g}{\rho_0} \frac{d\bar{\rho}}{dx_3} \tag{2.3}$$

is assumed to be constant. The total pressure is given by

$$p_{full} = p_0 + \bar{p} + p, \tag{2.4}$$

where p_0, $\bar{p}$, p are respectively in hydrostatic equilibrium with ρ_0, $\bar{\rho}$ and ρ. For simplicity, we assume periodic boundary conditions for the perturbation variables.

The variables of the system are of two types: u, v, ρ are the diagnostic variables for which we prescribe an initial condition while p and w are the prognostic variables that can, at each instant of time, be determined in terms of the prognostic variables.

The vertical velocity w is determined in terms of $U = (u, v, \rho)$ from the incompressibility condition $(2.1)_4$ and the periodicity and antisymmetry (below) conditions in x_3,

$$w = w(U) = -\int_0^{x_3} (u_{x_1} + v_{x_2})(x_1, x_2, z, t) dz.$$

The pressure is determined from the hydrostatic relation $(2.1)_3$, up to the surface pressure p_s,

$$p(x_1, x_2, x_3, t) = p_s(x_1, x_2, t) - N \int_0^{x_3} \rho(x_1, x_2, z, t) dz.$$

Moreover, we assume the following symmetry properties on the variables:

$$
\begin{aligned}
&u(x_1, x_2, -x_3) = u(x_1, x_2, x_3) \qquad v(x_1, x_2, -x_3) = v(x_1, x_2, x_3), \\
&p(x_1, x_2 - x_3) = p(x_1, x_2, x_3), \qquad w(x_1, x_2, -x_3) = -w(x_1, x_2, x_3), \\
&\rho(x_1, x_2, -x_3) = -\rho(x_1, x_2, x_3);
\end{aligned}
$$

these assumptions are typical in numerical simulations of stratified turbulence (see e.g. [1]). In order for these symmetries to persist, we need S_u, S_v to be even in x_3 and S_ρ to be odd in x_3.

We introduce the following function spaces:

$$\mathbf{V} = \left\{ (u,v,\rho) \in \dot{H}_{per}^1(\mathcal{M})^3; u,v \text{ even in } x_3, \rho \text{ odd in } x_3, \int_{-L_3/2}^{L_3/2} (u_{x_1} + v_{x_2})\, dx_3 = 0 \right\},$$

$$\mathbf{H} = \text{ the closure of } \mathbf{V} \text{ in } \dot{L}^2(\mathcal{M})^3,$$

$$\mathbf{V}_2 = \text{ the closure of } \mathbf{V} \cap \dot{H}_{per}^2(\mathcal{M})^3 \text{ in } \dot{H}_{per}^2(\mathcal{M})^3. \tag{2.5}$$

In (2.5) and elsewhere, we denote by $\dot{H}_{per}^m(\mathcal{M})$, with $m \geq 0$ integer, the functions of $H_{per}^m(\mathcal{M})$ with zero average on $\mathcal{M}$.

2.1. **Variational formulation of the problem.** The variational formulation is the following: Given $t_* > 0$ arbitrary, $U_0 = (u_0, v_0, \rho_0) \in \mathbf{H}$ and $S = (S_u, S_v, S_\rho) \in L^2(0, t_*; \mathbf{H})$, find $U : (0, t_*) \to \mathbf{V}$ such that

$$\frac{d}{dt}(U, \tilde{U})_{L^2} + a(U, \tilde{U}) + b(U, U, \tilde{U}) + \frac{1}{\varepsilon} e(U, \tilde{U}) = (S, \tilde{U})_{L^2}, \quad \forall \tilde{U} \in \mathbf{V}_2, \tag{2.6}$$

and

$$U(0) = U_0. \tag{2.7}$$

In (2.6) we introduced the following forms:

$a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ bilinear, continuous, coercive:

$$a(U, \tilde{U}) = \nu_v\big((u, \tilde{u})\big) + \nu_v\big((v, \tilde{v})\big) + \nu_\rho\big((\rho, \tilde{\rho})\big), \quad a(U, U) \geq c_1 \|U\|^2 \quad \forall U, \tilde{U} \in \mathbf{V}, \tag{2.8}$$

where $((\phi, \tilde{\phi})) = (\nabla \phi, \nabla \tilde{\phi})_{L^2}$,

$b : \mathbf{V} \times \mathbf{V} \times \mathbf{V}_2 \to \mathbb{R}$ bilinear, continuous:

$$b(U, U^\#, \tilde{U}) = \int_\mathcal{M} \left(u \frac{\partial u^\#}{\partial x_1} \tilde{u} + u \frac{\partial u^\#}{\partial x_2} \tilde{u} + w(U) \frac{\partial u^\#}{\partial x_3} \tilde{u} \right) d\mathcal{M}$$

$$+ \int_\mathcal{M} \left(u \frac{\partial v^\#}{\partial x_1} \tilde{v} + u \frac{\partial v^\#}{\partial x_2} \tilde{v} + w(U) \frac{\partial v^\#}{\partial x_3} \tilde{v} \right) d\mathcal{M} \tag{2.9}$$

$$+ \int_\mathcal{M} \left(u \frac{\partial \rho^\#}{\partial x_1} \tilde{\rho} + u \frac{\partial \rho^\#}{\partial x_2} \tilde{\rho} + w(U) \frac{\partial \rho^\#}{\partial x_3} \tilde{\rho} \right) d\mathcal{M},$$

$e : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ bilinear, continuous:

$$e(U, \tilde{U}) = \int_\mathcal{M} (-v\tilde{u} + u\tilde{v})\, d\mathcal{M} + N \int_\mathcal{M} (\rho \tilde{w} - w \tilde{\rho})\, d\mathcal{M}. \tag{2.10}$$

We also define the linear operators

$$A : \mathbf{V} \to \mathbf{V}', \qquad \langle AU, \tilde{U} \rangle_{\mathbf{V}', \mathbf{V}} = a(U, \tilde{U}), \qquad \forall U, \tilde{U} \in \mathbf{V}, \tag{2.11}$$

$$L : \mathbf{V} \to \mathbf{V}', \qquad \langle LU, \tilde{U} \rangle_{\mathbf{V}', \mathbf{V}} = e(U, \tilde{U}), \qquad \forall U, \tilde{U} \in \mathbf{V}, \tag{2.12}$$

and the bilinear operator

$$B : \mathbf{V} \times \mathbf{V} \to \mathbf{V}_2', \quad \langle B(U, U^\#), \tilde{U} \rangle_{\mathbf{V}', \mathbf{V}} = b(U, U^\#, \tilde{U}), \quad \forall U, U^\# \in \mathbf{V}, \tilde{U} \in \mathbf{V}_2, \tag{2.13}$$

where $\mathbf{V}'$ denotes the dual space of $\mathbf{V}$.

Problem (2.6) can be thus written as an abstract evolution equation:

$$\frac{dU}{dt} + AU + B(U, U) + \frac{1}{\varepsilon} LU = S \quad \text{in } \mathbf{V}_2', \tag{2.14}$$
$$U(0) = U_0.$$

The existence of weak solutions for (2.6) was proved in [14], while the existence and uniqueness, globally in time, of strong solutions was proved in [8] and [13]. The high order regularity of the solution of (2.6) was proved in [18]. All these results are collected in the following theorem:

Theorem 2.1. *Given $U_0 \in \mathbf{H}$ and $S \in L^\infty(\mathbb{R}_+, \mathbf{H})$, there exists at least one solution U of (2.6) with initial condition (2.7) such that*

$$U \in L^\infty(\mathbb{R}_+, \mathbf{H}) \cap L^2(0, t_*, \mathbf{V}) \text{ for all } t_* > 0. \tag{2.15}$$

If $U_0 \in \mathbf{V}$ and $S \in L^\infty(\mathbb{R}_+, \mathbf{H})$, there exists a unique solution of (2.6)–(2.7) such that

$$U \in L^\infty(\mathbb{R}_+, \mathbf{H}) \cap L^2(0, t_*, \mathbf{V}_2) \text{ for all } t_* > 0. \tag{2.16}$$

Moreover, given $m \in \mathbb{N}, m \geq 2$, *if* $U_0 \in (\dot{H}_{per}^m(\mathcal{M}))^3 \cap \mathbf{V}$ *and* $S \in L^\infty(\mathbb{R}_+, (\dot{H}_{per}^m(\mathcal{M}))^3 \cap \mathbf{V})$, *the solution satisfies*

$$U \in L^\infty(\mathbb{R}_+, (\dot{H}_{per}^m(\mathcal{M}))^3) \cap L^2(0, t_*, (\dot{H}_{per}^{m+1}(\mathcal{M}))^3) \text{ for all } t_* > 0. \tag{2.17}$$

3. **The renormalisation group method.** The averaging method we will use in what follows is known as the renormalisation group method. This allows us to study the asymptotic solutions of an equation which can be written in the following general form:

$$\frac{dU}{dt} + \frac{1}{\varepsilon}LU = \mathcal{F}(U), \tag{3.1}$$
$$U(0) = U_0,$$

where ε is a small parameter, L is a diagonalizable, antisymmetric linear operator and $\mathcal{F}$ is a non-linear operator. The fact that L is antisymmetric explains why the solutions display large oscillations when ε is small. This problem has two natural time scales: the slow time t and the fast time $s = t/\varepsilon$. Problem (3.1), written in the fast time variable, becomes

$$\frac{dV}{ds} + LV = \varepsilon\mathcal{F}(V), \tag{3.2}$$
$$V(0) = U_0,$$

where we denoted $V(s) = U(\varepsilon s)$. We start by writing a naive perturbation expansion for V,

$$V = V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \dots \tag{3.3}$$

We substitute (3.3) into (3.2) and we finally derive

$$\frac{dV^0}{ds} + LV^0 = 0, \tag{3.4}$$

$$\frac{dV^1}{ds} + LV^1 = \mathcal{F}(V^0), \tag{3.5}$$

$$\frac{dV^2}{ds} + LV^2 = \nabla_V\mathcal{F}(V^0)V^1, \tag{3.6}$$

and so on.

From (3.4) we find $V^0(s) = e^{-Ls}U_0$. Using the variation of constants formula to (3.5), we obtain

$$V^1(s) = e^{-Ls}\int_0^s e^{L\tau}\mathcal{F}(e^{-L\tau}U_0)d\tau.$$

We decompose

$$e^{L\tau}\mathcal{F}(e^{-L\tau}U_0) = \mathcal{F}_r(U_0) + \mathcal{F}_n(\tau, V_0), \tag{3.7}$$

where the term $\mathcal{F}_r$ which is independent of time is called resonant and the remaining, time-dependent term $\mathcal{F}_n$ is called non-resonant. We define

$$\mathcal{F}_{np}(\tau, U_0) = \int_0^\tau \mathcal{F}_n(\tau', U_0)d\tau',$$

and we can write

$$V^1(s) = e^{-Ls}\{s\mathcal{F}_r(U_0) + \varepsilon\mathcal{F}_{np}(s, U_0)\}. \tag{3.8}$$

We thus find our leading-order approximate solution,

$$V^\varepsilon(s) = V^0 + \varepsilon V^1 = e^{-Ls}\{U_0 + \varepsilon s\mathcal{F}_r(U_0) + \varepsilon\mathcal{F}_{np}(s, U_0)\}. \tag{3.9}$$

In (3.9) we remove the term εs by searching for a function $\bar{U}$ having $U_0 + \varepsilon s\mathcal{F}_r(U_0)$ as first order Taylor expansion. This justifies to introduce the first order renormalised group equation

$$\frac{d\bar{U}}{ds} = \varepsilon\mathcal{F}_r(\bar{U}), \tag{3.10}$$
$$\bar{U}(0) = U_0,$$

and to consider the first-order approximate solution

$$\tilde{U}^1(s) = e^{-Ls}\{\bar{U}(s) + \varepsilon\mathcal{F}_{np}(s, \bar{U}(s))\}. \tag{3.11}$$

The main issue now is to solve equation (3.10) and to compare the approximate solution (3.11) to the exact solution of (3.2) and to prove that the error is of order ε in an interval of time s of order $O(1/\varepsilon)$. For more details on this method, see e.g. [15],[17], [24], [19].

4. **Averaging the three-dimensional primitive equations.** As announced before, in this section we are interested in applying the renormalization group method described in Section 3 to the three-dimensional primitive equations. The first step is to deduce the renormalised group system (3.10) that corresponds to the primitive equations and to study the well-posedness of this system. Thus, we first introduce the fast time $s = t/\varepsilon$ in system (2.1). Since all the functions we are working with are (space) periodic, they admit Fourier series expansions. Thus, we write

$$f(x,t) = \sum_k f_k(t)e^{ik \cdot x},$$

where $x = (x_1, x_2, x_3)$ and $k = (k_1, k_2, k_3) \in \mathbf{Z}_{\mathcal{M}}$, where

$$\mathbf{Z}_{\mathcal{M}} = (2\pi\mathbb{Z})^3/\mathcal{M} = \left\{(2\pi l_1/L_1, 2\pi l_2/L_2, 2\pi l_3/L_3)\big|(l_1, l_2, l_3) \in \mathbb{Z}^3\right\}; \tag{4.1}$$

any wavevector k is henceforth understood to live in $\mathbf{Z}_{\mathcal{M}}$.

Thus the primitive equations written in the fast time variable $s = t/\varepsilon$ and in Fourier modes, read

$$\begin{aligned}
&u_k' + \varepsilon \sum_{j+l=k} i(l_1 u_j u_l + l_2 v_j u_l + l_3 w_j u_l) - v_k + ik_1 p_k = -\varepsilon\nu_v |k|^2 u_k + \varepsilon S_{u,k}, \\
&v_k' + \varepsilon \sum_{j+l=k} i(l_1 u_j v_l + l_2 v_j v_l + l_3 w_j v_l) + u_k + ik_2 p_k = -\varepsilon\nu_v |k|^2 v_k + \varepsilon S_{v,k}, \\
&ik_3 p_k = -N\rho_k, \\
&k_1 u_k + k_2 v_k + k_3 w_k = 0, \\
&\rho_k' + \varepsilon \sum_{j+l=k} i(l_1 u_j \rho_l + l_2 v_j \rho_l + l_3 w_j \rho_l) - N w_k = -\varepsilon\nu_\rho |k|^2 \rho_k + \varepsilon S_{\rho,k}.
\end{aligned} \tag{4.2}$$

For $k_3 \neq 0$ we obtain the k-component of the diagnostic variables p and w in terms of the prognostic variables,

$$\begin{aligned}
p_k &= -\frac{N}{ik_3}\rho_k, \\
w_k &= -\frac{k_1 u_k + k_2 v_k}{k_3}.
\end{aligned} \tag{4.3}$$

Putting (4.3) into (4.2), we obtain

$$\begin{aligned}
&u_k' + \varepsilon \sum_{j+l=k} i(l_1 u_j + l_2 v_j + l_3 w_j)u_l - v_k - N\frac{k_1}{k_3}\rho_k = -\varepsilon\nu_v |k|^2 u_k + \varepsilon S_{u,k}, \\
&v_k' + \varepsilon \sum_{j+l=k} i(l_1 u_j + l_2 v_j + l_3 w_j)v_l - u_k - N\frac{k_2}{k_3}\rho_k = -\varepsilon\nu_v |k|^2 u_k + \varepsilon S_{v,k}, \\
&\rho_k' + \varepsilon \sum_{j+l=k} i(l_1 u_j + l_2 v_j + l_3 w_j)\rho_l + N\frac{k_1}{k_3}u_k + N\frac{k_2}{k_3}v_k = -\varepsilon\nu_\rho |k|^2 \rho_k + \varepsilon S_{\rho,k}.
\end{aligned} \tag{4.4}$$

The k-components of the operators A, L and B are thus

$$A_k = \text{diag}(\nu_v |k|^2, \nu_v |k|^2, \nu_\rho |k|^2), \tag{4.5}$$

$$L_k = \begin{pmatrix} 0 & -1 & -Nk_1/k_3 \\ 1 & 0 & -Nk_2/k_3 \\ Nk_1/k_3 & Nk_2/k_3 & 0 \end{pmatrix}, \tag{4.6}$$

$$B_k = \begin{pmatrix} \sum_{j+l=k} i(l_1 - \delta_j^1 l_3)u_j u_l + \sum_{j+l=k} i(l_2 - \delta_j^2 l_3)u_j u_l \\ \sum_{j+l=k} i(l_1 - \delta_j^1 l_3)u_j v_l + \sum_{j+l=k} i(l_2 - \delta_j^2 l_3)v_j v_l \\ \sum_{j+l=k} i(l_1 - \delta_j^1 l_3)u_j \rho_l + \sum_{j+l=k} i(l_2 - \delta_j^2 l_3)v_j \rho_l \end{pmatrix}, \tag{4.7}$$

where $\delta_j^i = \begin{cases} j_i/j_3, & \text{if } j_3 \neq 0, \\ 0, & \text{if } j_3 = 0, \end{cases}$

for $i = 1, 2$.

For $k_3 = 0$, we know that $k_1 u_k + k_2 v_k = 0$, which corresponds to

$$\int_{-L_3/2}^{L_3/2} (u_{x_1} + v_{x_2})\, dx_3 = 0. \tag{4.8}$$

We also know that $\rho_k = 0$. We introduce the following notations: $\mathbf{v} = (u,v)$, $\mathbf{v}^\perp = (-v,u)$, $S_\mathbf{v} = (S_u, S_v)$ and for all $k \in \mathbb{Z}_\mathcal{M}$ we write $\tilde{k} = (k_1, k_2)$ and denote $\tilde{k} \wedge \tilde{l} = k_1 k_2 - k_2 l_1$.

From (4.8) we find the k-component of the pressure,

$$|\tilde{k}|^2 p_k = -\varepsilon \sum_{j+l=k}(l_1 - \delta_j^1 l_3)u_j\tilde{k}\cdot\mathbf{v}_l - \varepsilon \sum_{j+l=k}(l_2 - \delta_j^2 l_3)v_j\tilde{k}\cdot\mathbf{v}_l + i\tilde{k}\wedge\mathbf{v}_k - \varepsilon i S_{\mathbf{v},k}\cdot\tilde{k}. \quad (4.9)$$

Writing $P_k(\mathbf{v}_l) = \mathbf{v}_l - \tilde{k}(\tilde{k}\cdot\mathbf{v}_l)/|\tilde{k}|^2$ and introducing the pressure p_k given by (4.8) into (4.2)$_1$, (4.2)$_2$, we find

$$\mathbf{v}_k' + \varepsilon \sum_{j+l=k} i(l_1 - \delta_j^1 l_3)u_j P_k(\mathbf{v}_l) + \varepsilon \sum_{i+j=k} i(l_2 - \delta_j^2 l_3)v_j P_k(\mathbf{v}_l) + \mathbf{v}_k^\perp + \tilde{k}\frac{\tilde{k}\wedge\mathbf{v}_k}{|\tilde{k}|^2}$$
$$= -\epsilon\nu_v|k|^2\mathbf{v}_k + \varepsilon P_k(S_{\mathbf{v},k}), \quad (4.10)$$

We note that the unknowns u_k, v_k are not independent due to the constraint $k_1 u_k + k_2 v_k = 0$. We also remark here that $\mathbf{v}_{\tilde{k}}^\perp + \tilde{k}(\tilde{k}\wedge\mathbf{v}_k)/|\tilde{k}|^2 = 0$. In this notation, the operators read

$$A_k = \mathrm{diag}(\nu_v|\tilde{k}|^2, \nu_v|\tilde{k}|^2, 0), \quad (4.11)$$

$$L_k = 0_3, \quad (4.12)$$

$$B_k = \begin{pmatrix} \sum_{j+l=k} i(l_1 - \delta_j^1 l_3)u_j P_k(\mathbf{v}_l) & +\sum_{i+j=k} i(l_2 - \delta_j^2 l_3)v_j P_k(\mathbf{v}_l) \\ 0 \end{pmatrix}. \quad (4.13)$$

In order to deduce the renormalized group system, we need to compute, as in (3.7), $e^{L\tau}\mathcal{F}(e^{-L\tau}U_0)$ mode by mode to find the resonant $\mathcal{F}_r$ and the non-resonant part $\mathcal{F}_n$ of $\mathcal{F}$. We recall that in our case $\mathcal{F}(U) = S - AU - B(U,U)$, so we need to compute

$$e^{L\tau}S, \quad e^{L\tau}A(e^{-L\tau}U_0), \quad e^{L\tau}B(e^{-L\tau}U_0 \text{ and } e^{-L\tau}U_0). \quad (4.14)$$

We need the eigenvalues and the eigenvectors of L to compute the terms in (4.14).

For $k_3 \neq 0$, the eigenvalues of L_k are $\omega_k^0 = 0$ and $i\omega_k^\pm$, where $\omega_k^\pm = \pm\dfrac{|k|_N}{k_3}$ and $|k|_N := (N^2 k_1^2 + N^2 k_2^2 + k_3^2)^{1/2}$. If $|\tilde{k}| \neq 0$, the corresponding eigenvectors are

$$X_k^0 = \frac{1}{|k|_N}\begin{pmatrix} Nk_2 \\ -Nk_1 \\ k_3 \end{pmatrix} \text{ and } X_k^\pm = \frac{1}{\sqrt{2}|\tilde{k}||k|_N}\begin{pmatrix} -k_2 k_3 \pm ik_1|k|_N \\ k_1 k_3 \pm ik_2|k|_N \\ N|\tilde{k}|^2 \end{pmatrix}. \quad (4.15)$$

If $|\tilde{k}| = 0$, we have $i\omega_k^\pm = \pm\mathrm{sgn}(k_3)i$ and the corresponding eigenvectors are

$$X_k^0 = \begin{pmatrix} 0 \\ 0 \\ \mathrm{sgn}\,k_3 \end{pmatrix} \text{ and } X_k^\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i\,\mathrm{sgn}\,(k_3) \\ 0 \end{pmatrix}. \quad (4.16)$$

For $k_3 = 0$, we introduce the following vectors in order to have unified notations for all cases:

$$X_k^0 = \frac{1}{|\tilde{k}|}\begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix} \text{ and } X_k^\pm = \frac{1}{\sqrt{2}|\tilde{k}|}\begin{pmatrix} \pm ik_1 \\ \pm ik_2 \\ |\tilde{k}| \end{pmatrix}. \quad (4.17)$$

Introducing the matrix $Q_k = \begin{pmatrix} (X_k^+)^t \\ (X_k^-)^t \\ (X_k^0)^t \end{pmatrix}$ for all k, we obtain that

$$L_k = Q_k^{-1}\tilde{L}_k Q_k, \text{ with } L_k = \mathrm{diag}(iw_k^+, iw_k^-, 0) \text{ and } Q_k = (X_k^-, X_k^+, X_k^0)$$

We introduce the new unknown $V = (v^+, v^-, v^0)$ given by $V = QU$:

$$v_k^+ = X_k^+ \cdot U_k, \quad v_k^- = X_k^- \cdot U_k, \quad v_k^0 = X_k^0 \cdot U_k. \quad (4.18)$$

We remark that $v_k^0|k|_N = Nk_2 u_k - Nk_1 v_k + k_3\rho_k$ is the quasigeostrophic potential vorticity.

Equation (2.14), written in terms of the new variable V, becomes

$$\frac{dV}{dt} + \frac{1}{\varepsilon}\tilde{L}V + \tilde{A}V + \tilde{B}(V,V) = \tilde{S}, \quad (4.19)$$

with $\tilde{S} = QS$, $\tilde{A} = QAQ^{-1}$ and $\tilde{B}(\cdot,\cdot) = QB(Q^{-1}\cdot, Q^{-1}\cdot)$. Equation (4.19) is now equivalent to

$$\frac{dV}{ds} + \tilde{L}V = \varepsilon\tilde{\mathcal{F}}(V), \quad (4.20)$$

with $\tilde{\mathcal{F}}(V) = \tilde{S} - \tilde{A}V - \tilde{B}(V,V)$.

We now compute, mode by mode, the resonant and the non-resonant parts of $\tilde{\mathcal{F}}$ and write the renormalised group system in terms of the new variable.

We compute $e^{\tilde{L}\tau}\tilde{\mathcal{F}}(e^{-\tilde{L}\tau}V)$ for $k_3 = 0$. Since $e^{\tilde{L}\tau}\tilde{S}$ and $e^{\tilde{L}\tau}\tilde{A}(e^{-\tilde{L}\tau}V_0)$ are fully resonant, the only term we need to compute is $e^{\tilde{L}\tau}\tilde{B}(e^{-\tilde{L}\tau}V, e^{-\tilde{L}\tau}V)$. The k-mode of equation (4.20) reduces to an equation for v_k^0:

$$(v_k^0)' + i \sum_{\substack{j+l=k \\ j_3=l_3=0}} \frac{1}{|\tilde{k}|}(l_1 u_j + l_2 v_j)(k_2 u_l - k_1 v_l) + i \sum_{\substack{j+l=k \\ j_3 \neq 0}} \frac{1}{|\tilde{k}|}(k_1 u_j + k_2 v_j)$$

$$(k_2 u_l - k_1 v_l) = \nu_v |\tilde{k}|^2 v_k^0 + S_k^0, \tag{4.21}$$

with $U = (u, v, \rho) = Q^{-1}V$ and $S_k^0 = X_k^0 \cdot S_k$.

When we compute $\tilde{B}_k(e^{-\tilde{L}\tau}V, e^{-\tilde{L}\tau}V)$, the term corresponding to

$$i \sum_{\substack{j+l=k \\ j_3=l_3=0}} \frac{1}{|\tilde{k}|}(l_1 u_j + l_2 v_j)(k_2 u_l - k_1 v_l)$$

is fully resonant. The only term that could have a nonresonant part is

$$i \sum_{\substack{j+l=k \\ j_3 \neq 0}} \frac{1}{|\tilde{k}|}(k_1 u_j + k_2 v_j)(k_2 u_l - k_1 v_l). \tag{4.22}$$

Thus, we replace U by $Q^{-1}e^{\tilde{L}\tau}V$ in (4.22) and we notice that the only possible resonant terms come from the interactions $\omega_j^+ + \omega_l^+ = 0$ and $\omega_j^- + \omega_l^- = 0$ since $\omega_j^+ + \omega_l^- = 0$ would imply that j_3 and l_3 have the same sign, which contradicts the fact that $j_3 + l_3 = 0$.

The coefficient corresponding to $\omega_l^+ + \omega_j^+ = 0$ with $j_3 + l_3 = 0$ is denoted by B_{ljk}^{++0} and similar notations are used for all the coefficients. With these notations, the resonant part of (4.22) is

$$i \sum_{\substack{j+l=k \\ \omega_j^+ + \omega_l^+ = 0}} B_{ljk}^{++0} v_l^+ v_j^+ + i \sum_{\substack{j+l=k \\ \omega_j^- + \omega_l^- = 0}} B_{ljk}^{--0} v_l^- v_j^- + i \sum_{j+l=k} B_{ljk}^{000} v_l^0 v_j^0, \tag{4.23}$$

where

$$\begin{aligned} B_{ljk}^{++0} &= \frac{1}{|\tilde{k}|} \frac{j_3(\tilde{j} \wedge \tilde{l}) - i(\tilde{k} \cdot \tilde{j})|j|_N}{\sqrt{2}|\tilde{j}||j|_N} \frac{-l_3(\tilde{l} \cdot \tilde{k}) + i(\tilde{k} \wedge \tilde{l})|l|_N}{\sqrt{2}|\tilde{l}||l|_N}, \\ B_{ljk}^{--0} &= \overline{B_{ljk}^{++0}}, \\ B_{ljk}^{000} &= \frac{N^2}{|\tilde{k}|} \frac{(\tilde{k} \wedge \tilde{j})(\tilde{k} \cdot \tilde{l})}{|j|_N |l|_N}. \end{aligned} \tag{4.24}$$

Proposition 4.1. *The coefficients B_{ljk}^{++0} and B_{ljk}^{--0} have the following symmetry properties:*

$$B_{ljk}^{++0} + B_{ljk}^{++0} = 0, \quad B_{ljk}^{--0} + B_{ljk}^{--0} = 0 \quad \forall l, j, k, \in \mathbb{Z}_{\mathcal{M}} \text{ with } j + l = k. \tag{4.25}$$

Proof. Relation (4.25) follows from direct computations. $\qquad\qquad\square$

Proposition 4.1 implies that the renormalised group equation will not contain terms in v^+ and v^-. Writing (4.21) with U replaced by $Q^{-1}V = Q^{-1}(v^+, v^-, v^0)^t$, we find the following renormalised group equation for the case $k_3 = 0$,

$$(v_k^0)' + i \sum_{\substack{l+j=k \\ l_3=j_3=0}} \frac{(\tilde{l} \wedge \tilde{j})(\tilde{l} \cdot \tilde{k})}{|\tilde{j}||\tilde{l}||\tilde{k}|} v_l^0 v_j^0 + iN^2 \sum_{\substack{l+j=k \\ j_3 \neq 0}} \frac{(\tilde{l} \wedge \tilde{j})(\tilde{l} \cdot \tilde{k})}{|j|_N |l|_N |\tilde{k}|} v_l^0 v_j^0$$

$$- \nu_v |\tilde{k}|^2 v_k^0 = S_k^0. \tag{4.26}$$

Collecting the nonlinear terms in (4.26), we find the following relation for the case $k_3 = 0$,

$$(v_k^0)' + iN^2 \sum_{\substack{l+j=k \\ j_3 \neq 0}} \frac{(\tilde{l} \wedge \tilde{j})(\tilde{l} \cdot \tilde{k})}{|j|_N |l|_N |\tilde{k}|} v_l^0 v_j^0 - \nu_v |k|^2 v_k^0 = S_k^0. \tag{4.27}$$

We also need to do the same kind of computations for the case $k_3 \neq 0$. In order to simplify the writings, we denote

$$Q_k^{-1} = \begin{pmatrix} \overrightarrow{\xi}_{1,k} \\ \overrightarrow{\xi}_{2,k} \\ \overrightarrow{\xi}_{3,k} \end{pmatrix}.$$

We need to compute the resonant parts of the terms $e^{\tilde{L}\tau}\tilde{A}(e^{-\tilde{L}\tau}V)$ and $e^{\tilde{L}\tau}\tilde{B}(e^{-\tilde{L}\tau}V, e^{-\tilde{L}\tau}V)$. For the linear term we compute

$$
\begin{aligned}
e^{\tilde{L}_k\tau}\tilde{A}_k(e^{-\tilde{L}_k\tau}V_k) &= \nu_v|k|^2 I_3 \\
&+ (\nu_\rho + \nu_{\mathbf{v}}) \begin{pmatrix} k_2^2 & -k_1 k_2 e^{i(\omega_k^- - \omega_k^+)\tau} & k_2 k_3 e^{i\omega_k^+\tau} \\ -k_1 k_2 e^{i(\omega_k^+ - \omega_k^-)\tau} & k_1^2 & -k_1 k_2 e^{i\omega_k^-\tau} \\ N|\tilde{k}|k_2 e^{i\omega_k^+\tau} & -N|\tilde{k}|k_1 e^{i\omega_k^-\tau} & k_3^2 \end{pmatrix}
\end{aligned}
\tag{4.28}
$$

which implies that the resonant part is

$$
\begin{aligned}
\tilde{A}_{r,k} &= \mathrm{diag}\Big(\nu_\rho k_2^2 + \nu_{\mathbf{v}}(k_1^2 + k_3^2), \nu_\rho k_1^2 + \nu_{\mathbf{v}}(k_2^2 + k_3^2), \nu_\rho k_3^2 + \nu_{\mathbf{v}}(k_1^2 + k_2^2)\Big) \\
&:= \mathrm{diag}(\nu_k^+, \nu_k^-, \nu_k^0)|k|^2,
\end{aligned}
\tag{4.29}
$$

with $\nu^+ = \dfrac{\nu_\rho k_2^2 + \nu_{\mathbf{v}}(k_1^2 + k_3^2)}{|k|^2}, \nu^- = \dfrac{\nu_\rho k_1^2 + \nu_{\mathbf{v}}(k_2^2 + k_3^2)}{|k|^2}, \nu^0 = \dfrac{\nu_\rho k_3^2 + \nu_{\mathbf{v}}(k_1^2 + k_2^2)}{|k|^2}.$

Remark 4.1. From formula (4.29) we easily notice that the operator $\tilde{A}_r$ is still coercive.

We also need to compute $(e^{\tilde{L}\tau}\tilde{B}(e^{-\tilde{L}\tau}V, e^{-\tilde{L}\tau}V))_k$ for $k_3 \neq 0$. We write

$$B_k = B_k^1 + B_k^2,$$

with

$$B_k^1 = \sum_{j+l=k} i(l_1 - \delta_j^1 l_3)u_j U_l, \quad B_k^2 = \sum_{j+l=k} i(l_2 - \delta_j^2 l_3)v_j U_l,$$

and we compute the resonant part for each term B_k^i with $i = 1, 2$.
We find

$$e^{\tilde{L}_k\tau}\tilde{B}_k^1(e^{-\tilde{L}\tau}V, e^{-\tilde{L}\tau}V) \tag{4.30}$$

$$
\begin{aligned}
&= e^{\tilde{L}_k\tau}\sum^n i(l_1 - \delta_j^1 l_3)\overrightarrow{\xi}_{1,j} \cdot \begin{pmatrix} e^{-i\tau\omega_j^+}v_j^+ \\ e^{-i\tau\omega_j^-}v_j^- \\ v_j^0 \end{pmatrix} Q_l^{-1} \begin{pmatrix} e^{-i\tau\omega_l^+}v_l^+ \\ e^{-i\tau\omega_l^-}v_l^- \\ v_l^0 \end{pmatrix} \\
&+ e^{\tilde{L}_k\tau}\sum_{j_3=0}^c i(l_1 - \delta_j^1 l_3)\overrightarrow{\xi}_{1,j} \cdot \begin{pmatrix} 0 \\ 0 \\ v_l^0 \end{pmatrix} Q_l^{-1} \begin{pmatrix} e^{-i\tau\omega_l^+}v_l^+ \\ e^{-i\tau\omega_l^-}v_l^- \\ v_l^0 \end{pmatrix} \\
&+ e^{\tilde{L}_k\tau}\sum_{l_3=0}^c il_1 \overrightarrow{\xi}_{1,j} \cdot \begin{pmatrix} e^{-i\tau\omega_j^+}v_j^+ \\ e^{-i\tau\omega_j^-}v_j^- \\ v_j^0 \end{pmatrix} Q_l^{-1} \begin{pmatrix} 0 \\ 0 \\ v_l^0 \end{pmatrix},
\end{aligned}
$$

$$\tag{4.31}$$

where $\sum^c$ is a notation for the cyclic sum $\sum_{j+l=k}$ and $\sum^n$ indicates the cyclic sums for which $j_3 \neq 0, l_3 \neq 0, k_3 \neq 0$. After similar computations for $\tilde{B}_k^2$, the resonant part of the nonlinear term

is a three-dimensional vector having the following components:

$$
\begin{aligned}
\tilde{B}_{r,1} = & \sum_{\omega_l^+ + \omega_k^+ = \omega_k^+}^{n} B_{jlk}^{+++} X_k^+ \cdot X_l^- v_j^+ v_l^+ + \sum_{\omega_j^+ + \omega_l^- = \omega_k^+}^{n} B_{jlk}^{+-+} X_k^+ \cdot X_l^+ v_l^- v_j^+ \\
& + \sum_{\omega_j^+ = \omega_k^+}^{n} B_{jlk}^{+0+} X_k^+ \cdot X_l^0 v_j^+ v_l^- + \sum_{\omega_j^- + \omega_l^+ = \omega_k^+}^{n} B_{jlk}^{-++} X_k^+ \cdot X_l^- v_j^- v_l^+ \\
& + \sum_{\omega_l^- + \omega_j^- = \omega_k^+}^{n} B_{jlk}^{--+} X_k^+ \cdot X_l^+ v_j^- v_l^- + \sum_{\omega_j^- = \omega_k^+}^{n} B_{jlk}^{-0+} X_k^+ \cdot X_l^0 v_j^- v_l^0 \\
& + \sum_{\omega_l^+ = \omega_k^+}^{n} B_{jlk}^{0++} X_k^+ \cdot X_l^- v_j^0 v_l^+ + \sum_{\omega_l^- = \omega_k^+}^{n} B_{jlk}^{0-+} X_k^+ \cdot X_l^+ v_j^0 v_l^- \\
& + \sum_{\substack{j_3 = 0 \\ \omega_l^+ = \omega_k^+}}^{c} B_{jlk}^{0++} X_k^+ \cdot X_l^- v_j^0 v_l^+ + \sum_{\substack{j_3 = 0 \\ \omega_l^- = \omega_k^+}}^{c} B_{jlk}^{0-+} X_k^+ \cdot X_l^+ v_j^0 v_l^- \\
& + \sum_{\substack{l_3 = 0 \\ \omega_j^+ = \omega_k^+}}^{c} B_{jlk}^{+0+} X_k^- \cdot X_l^0 v_j^+ v_l^0 + \sum_{\substack{l_3 = 0 \\ \omega_j^- = \omega_k^+}}^{c} B_{jlk}^{-0+} X_k^- \cdot X_l^0 v_j^- v_l^0,
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
\tilde{B}_{r,2} = & \sum_{\omega_l^+ + \omega_j^+ = \omega_k^-}^{n} B_{jlk}^{++-} X_k^- \cdot X_l^- v_j^+ v_l^+ + \sum_{\omega_j^+ + \omega_l^- = \omega_k^-}^{n} B_{jlk}^{+--} X_k^- \cdot X_l^+ v_j^+ v_l^- \\
& + \sum_{\omega_j^+ = \omega_k^-}^{n} B_{jlk}^{+0-} X_k^- \cdot X_l^0 v_j^- v_l^0 + \sum_{\omega_j^- + \omega_l^+ = \omega_k^-}^{n} B_{jlk}^{-+-} X_k^- \cdot X_l^- v_j^- v_l^+ \\
& + \sum_{\omega_j^- + \omega_l^- = \omega_k^-}^{n} B_{jlk}^{---} X_k^- \cdot X_l^+ v_j^- v_l^- + \sum_{\omega_j^- = \omega_k^-}^{n} B_{jlk}^{-0-} X_k^- \cdot X_l^0 v_j^- v_l^0 \\
& + \sum_{\omega_l^+ = \omega_k^-}^{n} B_{jlk}^{0+-} X_k^- \cdot X_l^- v_j^0 v_l^+ + \sum_{\omega_j^- = \omega_k^-}^{n} B_{jlk}^{0--} X_k^- \cdot X_l^+ v_j^0 v_l^- \\
& + \sum_{j_3 = 0 \, \omega_l^+ = \omega_k^-}^{c} B_{jlk}^{0+-} X_k^- \cdot X_l^- v_j^0 v_l^- + \sum_{j_3 = 0 \, \omega_l^- = \omega_k^-}^{c} B_{jlk}^{0--} X_k^- \cdot X_l^+ v_j^0 v_l^- \\
& + \sum_{\substack{l_3 = 0 \\ \omega_j^+ = \omega_k^-}}^{c} B_{jlk}^{+0-} X_k^- \cdot X_l^0 v_j^- v_l^0 + \sum_{l_3 = 0 \, \omega_j^- = \omega_k^-}^{c} B_{jlk}^{-0-} X_k^- \cdot X_l^0 v_j^- v_l^0,
\end{aligned}
\tag{4.33}
$$

and

$$
\begin{aligned}
\tilde{B}_{r,3} = & \sum_{\omega_j^+ + \omega_l^+ = 0}^{n} B_{jlk}^{++0} X_k^0 \cdot X_l^- v_j^+ v_l^+ + \sum_{\omega_j^+ + \omega_l^- = 0}^{n} B_{jlk}^{+-0} X_k^0 \cdot X_l^+ v_j^+ v_l^- \\
& + \sum_{\omega_j^- + \omega_l^+ = 0}^{n} B_{jlk}^{-+0} X_k^0 \cdot X_l^- v_j^- v_l^+ + \sum_{\omega_j^- + \omega_l^- = 0}^{n} B_{jlk}^{--0} X_k^0 \cdot X_l^+ v_j^- v_l^- \\
& + \sum_{}^{n} B_{jlk}^{000} X_k^0 \cdot X_l^0 v_j^0 v_l^0 + \sum_{j_3 = 0}^{c} B_{jlk}^{000} X_k^0 \cdot X_l^0 v_j^0 v_l^0 \\
& + \sum_{l_3 = 0}^{c} B_{jlk}^{000} X_k^0 \cdot X_l^0 v_j^0 v_l^0.
\end{aligned}
\tag{4.34}
$$

Proposition 4.2. *The coefficients $B_{jlk}^{s_1 s_2 s_3}$ with $s_1 = +, -, 0$, satisfy the following properties:*

$$B_{jlk}^{+++} = \frac{iN}{\sqrt{2}|\tilde{j}||j|_N} \left\{ j_3(\tilde{j} \wedge \tilde{l}) - i(\tilde{j} \cdot \tilde{l})|j|_N + i|\tilde{j}|^2|j|_N \frac{l_3}{j_3} \right\},$$

$$B_{jlk}^{-++} = \frac{iN}{\sqrt{2}|\tilde{j}||j|_N} \left\{ j_3(\tilde{j} \wedge \tilde{l}) + i(\tilde{j} \cdot \tilde{l})|j|_N - i|\tilde{j}|^2|j|_N \frac{l_3}{j_3} \right\},$$

$$B_{jlk}^{0++} = \frac{iN^2}{|j|_N}\tilde{l} \wedge \tilde{j},$$

$$B_{jlk}^{+++} = B_{jlk}^{+-+} = B_{jlk}^{+0+} = B_{jlk}^{++0} = B_{jlk}^{+-0} = B_{jlk}^{++-} = B_{jlk}^{+--} = B_{jlk}^{+0-}, \qquad (4.35)$$

$$\overline{B_{jlk}^{+++}} = B_{jlk}^{-++} = B_{jlk}^{--+} = B_{jlk}^{-0+} = B_{jlk}^{-+0} = B_{jlk}^{--0} = B_{jlk}^{-+-}$$

$$= B_{jlk}^{---} = B_{jlk}^{-0-},$$

$$B_{jlk}^{0++} = B_{jlk}^{0-+} = B_{jlk}^{000} = B_{jlk}^{0+-} = B_{jlk}^{0--}.$$

These coefficients also have the following properties:

$$B_{jlk}^{+-0}X_k^0 \cdot X_l^+ + B_{ljk}^{-+0}X_k^0 \cdot X_j^- = 0, \qquad (4.36)$$

$$B_{jlk}^{++0}X_k^0 \cdot X_l^- + B_{ljk}^{++0}X_k^0 \cdot X_l^- = 0, \qquad (4.37)$$

$$B_{jlk}^{--0}X_k^0 \cdot X_l^+ + B_{ljk}^{--0}X_k^0 \cdot X_j^+ = 0. \qquad (4.38)$$

Interchanging l and j in (4.32)–(4.34) and using properties (4.36)–(4.38), the renormalised group system finally reads

$$\frac{dv_k^+}{dt} + \nu_k^+|k|^2 v_k^+ + \sum_{\substack{\omega_j^{s_1} + \omega_l^{s_2} = \omega_k^+ \\ s_1, s_2 = +, -}}^{n} B^{s_1 s_2 +}X_k^+ = X_l^{-s_2}v_j^{s_1}v_l^{s_2} + \sum_{\substack{\omega_j^{s_1} = \omega_k^+ \\ s_1 = +, -}}^{c} B_{jlk}^{s_1 0 +}X_k^+ \cdot X_l^0 v_j^{s_1} v_l^0$$

$$+ \sum_{\substack{\omega_l^{s_1} = \omega_k^+ \\ s_1 = +, -}}^{c} B_{jlk}^{0s_1 +}X_k^+ \cdot X_l^{-s_1}v_j^0 \cdot v_l^{s_1} = S_k^+,$$

$$\frac{dv_k^-}{dt} + \nu_k^-|k|^2 v_k^- + \sum_{\substack{\omega_j^{s_1} + \omega_l^{s_2} = \omega_k^- \\ s_1, s_2 = +, -}}^{n} B_{jlk}^{s_1 s_2 -}X_k^- \cdot X_l^{-s_2}v_j^{s_1}v_l^{s_2} + \sum_{\substack{\omega_j^{s_1} = \omega_k^- }}^{c} B_{jlk}^{s_1 0 -}X_k^- \cdot X_l^0 v_j^{s_1} v_l^0$$

$$+ \sum_{\substack{\omega_l^{s_1} = \omega_k^- \\ s_1 = +, -}}^{c} B_{jlk}^{0s_1 -}X_k^- \cdot X_l^{-s_1}v_j^0 v_l^{s_1} = S_k^-,$$

$$\frac{dv_k^0}{dt} + \nu_k^0|k|^2 v_k^0 + iN^2 \sum_{j+l=k} \frac{\tilde{l} \wedge \tilde{j}}{|j|_N|k|_N}|l|_N v_j^0 v_l^0 = S_k^0. \qquad (4.39)$$

4.1. Study of the renormalised system. In what follows we consider the case where three-wave resonances are not possible, i.e. $\omega_j^{\pm} + \omega_l^{\pm} \neq \omega_k^{\pm}$ for $j + l = k$. In the Appendix we see that this scenario cannot happen when the Burgers number N lies outside a certain quasi-resonant set. In this case the renormalised equation is

$$\frac{dv_k^+}{dt} + \nu_k^+|k|^2 v_k^+ + \sum_{\substack{\omega_j^{s_1} = \omega_k^+ \\ s_1 = +, -}}^{c} B_{jlk}^{s_1 0 +}X_k^+ \cdot X_l^0 v_j^{s_1} v_l^0 + \sum_{\substack{\omega_l^{s_1} = \omega_k^+ \\ s_1 = +, -}}^{c} B_{jlk}^{0s_1 +}X_k^+ \cdot X_l^{-s_1}v_j^0 v_l^{s_1} = S_k^+,$$

$$\frac{dv_k^-}{dt} + \nu_k^-|k|^2 v_k^- + \sum_{\substack{\omega_j^{s_1} = \omega_k^- \\ s_1 = +, -}}^{c} B_{jlk}^{s_1 0 -}X_k^- \cdot X_l^0 v_j^{s_1} v_l^0 + \sum_{\substack{\omega_l^{s_1} = \omega_k^- \\ s_1 = +, -}}^{c} B_{jlk}^{0s_1 -}X_k^- \cdot X_l^{-s_1}v_j^0 v_l^{s_1} = S_k^-, \quad (4.40)$$

$$\frac{dv_k^0}{dt} + \nu_k^0|k|^2 v_k^0 + iN^2 \sum^{c} \frac{\tilde{l} \wedge \tilde{j}}{|j|_N|k|_N}|l|_N v_j^0 v_l^0 = S_k^0.$$

In system (4.40) we notice that the last equation decouples completely from the first two equations so we can start by studying the well-posedness of this equation. We also notice that the first two

equations on (v^+, v^-) are bilinear and can be written as

$$\frac{d}{dt}\begin{pmatrix} v^+ \\ v^- \end{pmatrix} + A_\pm \begin{pmatrix} v^+ \\ v^- \end{pmatrix} + B_\pm(v^0)\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix}, \tag{4.41}$$

with $A_\pm, B_\pm(v^0), S^+, S^-$ given mode-by-mode by equations $(4.40)_1$–$(4.40)_2$.

As announced earlier, we start by studying equation $(4.40)_3$ for v_k^0. Introducing the quasi-geostrophic potential vorticity $q_k = |k|_N v_k^0$ as new unknown, equation $(4.40)_3$ becomes

$$\frac{dq_k}{dt} + \nu_k^0 |k|^2 q_k + i \sum{}^c \frac{\tilde{l} \wedge \tilde{j}}{|j|_N^2} q_l q_j = S_k^0 |k|_N. \tag{4.42}$$

This equation is known as the three-dimensional quasi-geostrophic equation and it was studied by Babin, Mahalov and Nicolaenko in [2]. We thus have:

Theorem 4.1. *[BMN99] Let $m > 0$ be fixed, arbitrary. Let S^0 be a forcing term belonging to $\dot{H}^{m-1}(\mathcal{M})$. Then a solution $q(t)$ of the quasi-geostrophic equations belonging to $\dot{H}^{m-1}(\mathcal{M})$ exists for all $t > 0$ and is unique. More precisely*

$$q \in L^\infty(\mathbb{R}_+, \dot{H}_{per}^m(\mathcal{M})) \tag{4.43}$$

and taking $r > 0$ arbitrary and fixed, we have

$$\int_t^{t+r} |q(t')|_{H^{m+1}}^2 \, dt' \leq K_m, \ \forall t > t_m(q(0)), \tag{4.44}$$

where by K_m we denote a constant depending on ν^0 but independent of the initial condition and $t_m(q(0))$ is a time depending on the H^m-norm of the initial data $q(0)$.

Knowing these results on the regularity of v^0, we can obtain the existence and uniqueness of a solution (v^+, v^-) in $L^\infty(\mathbb{R}_+, (\dot{H}^m(\mathcal{M}))^2)$, for all $m > 0$, provided the initial condition $(v^+(0), v^-(0))$ and the forcing term (S^+, S^-) are in $\dot{H}^m(\mathcal{M})^2$. Gathering the informations on v^0 with the results on (v^+, v^-), we are actually able to prove the following result:

Theorem 4.2. *Let us consider $m \in \mathbb{N}$, $V_0 = (v_0^+, v_0^-, v_0^0) \in \dot{H}_{per}^m(\mathcal{M})^3$ and $S = (S^+, S^-, S^0) \in \dot{H}_{per}^m(\mathcal{M})^3$. Then there exists a unique solution V of system (4.40) such that*

$$V = (v^+, v^-, v^0) \in L^\infty\big(\mathbb{R}_+(\dot{H}_{per}^m(\mathcal{M}))^3\big) \ and \ V(0) = V_0. \tag{4.45}$$

Moreover, if $r > 0$ is a fixed arbitrary constant, then

$$\int_t^{t+r} |V(t')|_{H^{m+1}}^2 \, dt' \leq b_m, \quad \forall t \geq t_m(V_0), \tag{4.46}$$

where b_m is a constant independent of the initial data.

Proof. Taking into account the results on v^0, it remains to prove the regularity results on (v^+, v^-). By direct computations we can prove that

$$\left(B_\pm(v^0)\begin{pmatrix} v^+ \\ v^- \end{pmatrix}, \begin{pmatrix} v^+ \\ v^- \end{pmatrix} \right) = 0,$$

which together with the coercivity of the operator $A_\pm$ implies the results (4.45)-(4.46) for $m = 0$.

The results (4.45)–(4.46) for an arbitrary $m \in \mathbb{N}, m \geq 1$ follow from a recursive argument. Thus, we suppose that we have

$$(v^+, v^-) \in L^\infty(\mathbb{R}_+, (\dot{H}_{per}^{m-1}(\mathcal{M}))^2),$$
$$\int_t^{t+r} |(v^+, v^-)(t')|_{H^m}^2 \, dt' \leq b_{m-1}, \quad \forall t > t_{m-1}(v_0^+, v_0^-), \tag{4.47}$$

where by b_{m-1} we denote a constant independent of the initial condition (v^+, v^-). We seek to prove that

$$(v^+, v^-) \in L^\infty(\mathbb{R}_+, (\dot{H}_{per}^m(\mathcal{M}))^2),$$
$$\int_t^{t+r} |(v^+, v^-)(t')|_{H^{m+1}}^2 \, dt' \leq b_m, \quad \forall t > t_m(v_0^+, v_0^-). \tag{4.48}$$

In order to derive a priori estimates for (v^+, v^-), we multiply $(4.40)_1$ by $|k|^{2m}v_k^+$, $(4.40)_2$ by $|k|^{2m}v_k^-$, sum in $k \in \mathbb{Z}_\mathcal{M}$ and add the resulting equations. We obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|v^+|_{H^m}^2 + |v^-|_{H^m}^2\right\} + \min(\nu_\mathbf{v}, \nu_\rho)\left\{|v^+|_{H^{m+1}}^2 + |v^+|_{H^{m+1}}^2\right\}$$
$$\leq |(B_\pm(v^0)(v^+, v^-), (-\Delta)^m(v^+, v^-))| + |((S^+, S^-), (-\Delta)^m(v^+, v^-))|. \tag{4.49}$$

We need to estimate the terms from the right hand side of (4.49). The last term can be easily estimated as

$$|((S^+, S^-), (-\Delta)^m(v^+, v^-))| \leq |(S^+, S^-)|_{H^m}|(v^+, v^-)|_{H^m}$$
$$\leq c_1\left\{|S^+|_{H^m}^2 + |S^-|_{H^m}^2\right\} + \frac{1}{2}\min(\nu_\mathbf{v}, \nu_\rho)\left\{|v^+|_{H^{m+1}}^2 + |v^-|_{H^{m+1}}^2\right\}. \tag{4.50}$$

We also need to estimate the first term from the right hand side of (4.49). Since

$$|B_{jlk}^{s_1 0 s_2}| \leq \frac{N}{\sqrt{2}|\tilde{j}||j|_N}\left\{|j_3||\tilde{j}||\tilde{l}| + |\tilde{j}||\tilde{l}||j|_N + |\tilde{j}|^2|j|_N\frac{|l_3|}{|j_3|}\right\}$$
$$\leq \frac{N}{\sqrt{2}}\left\{2|\tilde{l}| + |\tilde{j}|\frac{|l_3|}{|j_3|}\right\}.$$

and $|B_{jlk}^{0 s_1 s_2}| \leq N^2|\tilde{l}|$, the most difficult terms to estimate will be the terms involving $|\tilde{j}||l_3|/|j_3|$. An important aspect here is the fact that these terms appear only in some particular cases, meaning when $\omega_j^{s_1} = \omega_k^{s_2}$ or $\omega_l^{s_1} = \omega_k^{s_2}$, for $s_1, s_2 = +, -$ and $j + l = k$. Taking for example the case when $\omega_j^{s_1} = \omega_k^{s_2}$, this implies that $|j_3|/|k_3| = |j|_N/|k|_N = |\tilde{j}|/|\tilde{k}| = \alpha$, with $\alpha > 0$. Since $j_3 + l_3 = k_3$, it means that when $j_3 = \alpha k_3$ we get $l_3 = (1 - \alpha)k_3$ and when $j_3 = -\alpha k_3$ we have $l_3 = (1 + \alpha)k_3$. Thus, we can estimate

$$\frac{|l_3|}{|j_3|}|\tilde{j}| \leq (1 + \alpha)|\tilde{k}| = |\tilde{k}| + |\tilde{j}| \leq 2(|\tilde{j}| + |\tilde{l}|).$$

The same kind of estimate is available for the case $\omega_l^{s_1} = \omega_k^{s_2}$. We can thus bound the most difficult terms from $(B_\pm(v^0)(v^+, v^-), (-\Delta)^m(v^+, v^-))$ as

$$\left|\sum_k \sum_{\substack{\omega_j^{s_1} = \omega_k^{s_2} \\ s_1, s_2 = +, -}}^c B_{jlk}^{s_1 0 s_2} X_k^{s_2} \cdot X_l^0 v_j^{s_1} v_l^0 v_k^{s_2}|k|^{2m}\right|$$

$$\leq \sum_k \sum_{s_1, s_2}^c (|\tilde{j}| + |\tilde{l}|)|k|^{2m}|v_j^{s_1}||v_k^{s_2}||v_l^0|$$

$$\leq c\sum_k \sum_{s_1, s_2}^c (|j|^{m+1} + |l|^{m+1})|k|^m|v_j^{s_1}||v_k^{s_2}||v_l^0|$$

$$= c\sum_{\substack{j+l+k=0 \\ s_1, s_2}} |j|^{m+1}|v_j^{s_1}||k|^m|v_k^{s_2}||v_l^0| + c\sum_{\substack{j+l+k=0 \\ s_1, s_2}} |k|^m|v_k^{s_2}||v_j^{s_1}||l|^{m+1}|v_l^0| \tag{4.51}$$

$$\leq c\sum_{s_1, s_2} |(-\Delta)^{\frac{m+1}{2}}v^{s_1}|_{L^2}|(-\Delta)^{\frac{m}{2}}v^{s_2}|_{L^3}|v^0|_{L^6}$$

$$+ c\sum_{s_1, s_2} |(-\Delta)^{\frac{m}{2}}v^{s_2}|_{L^3}|(-\Delta)^{\frac{m+1}{2}}v^0|_{L^2}|v^{s_1}|_{L^6}$$

$$\leq c\sum_{s_1, s_2} |v^{s_1}|_{H^{m+1}}|v^{s_2}|_{H^m}^{1/2}|v^{s_2}|_{H^{m+1}}^{1/2}|v^0|_{H^1}$$

$$+ c\sum_{s_1, s_2} |v^{s_2}|_{H^m}^{1/2}|v^{s_2}|_{H^{m+1}}^{1/2}|v^{s_1}|_{H^1}|v^0|_{H^{m+1}}$$

$$\leq \frac{1}{4}\min(\nu_\mathbf{v}, \nu_\rho)(|v^+|_{H^{m+1}}^2 + |v^-|_{H^{m+1}}^2) + f(t)(|v^+|_{H^m}^2 + |v^-|_{H^m}^2),$$

where $f(t) = c(1 + |v^0|_{H^1}^4 + |v^0|_{H^{m+1}}^2)$.

Introducing (4.50) and (4.51) into (4.49), we find

$$\frac{d}{dt}\{|v^+|_{H^m}^2 + |v^-|_{H^m}^2\} + \min(\nu_\mathbf{v}, \nu_\rho)\{|v^+|_{H^{m+1}}^2 + |v^-|_{H^{m+1}}^2\}$$
$$\leq f(t)\{|v^+|_{H^m}^2 + |v^-|_{H^m}^2\} + c_1\{|S^+|_{H^m}^2 + |S^-|_{H^m}^2\}. \tag{4.52}$$

Applying the classical Gronwall lemma to (4.52), we obtain estimates in $L^\infty(0, t_*, H^m)$ for all $t_* > 0$, with the bounds depending on the initial data. The bounds uniform in time are obtained applying the uniform Gronwall lemma and using the fact that uniform bounds in time for v^0 are already proved. This proved (4.48) and thus Theorem 4.2. □

4.2. **First-order error estimates.** Having proved that the renormalized group system is well-posed in all Sobolev spaces, we can now use the solution to construct the first order approximate solution, as in (3.11),

$$\tilde{U}^1(s) = e^{-Ls} \left\{ \bar{U}(s) + \varepsilon \mathcal{F}_{np}(s, \bar{U}(s)) \right\}, \tag{4.53}$$

where $\bar{U}(s)$ is the solution of the renormalized equation,

$$\frac{d\bar{U}}{ds} = \varepsilon \mathcal{F}_r(\bar{U}),$$
$$\bar{U}(0) = U_0.$$

We need to compare $\tilde{U}^1$ to the exact solution V, which satisfies (3.2), meaning that we need to evaluate the error $W(s) = \tilde{U}^1(s) - V(s)$. The error satisfies

$$\frac{dW}{ds} + LW + \varepsilon AW + \varepsilon B(W, W) + \varepsilon B(\tilde{U}^1, W) + \varepsilon B(W, \tilde{U}^1) = \varepsilon^2 R_\varepsilon, \tag{4.54}$$
$$W(0) = 0,$$

where

$$R_\varepsilon = - A e^{-sL} \mathcal{F}_{np}(s, \bar{U}) - B(e^{-sL}\bar{U}, e^{-sL}\mathcal{F}_{np}(s, \bar{U})) - B(e^{-sL}\mathcal{F}_{np}(s, \bar{U}), e^{-sL}\bar{U})$$
$$- \varepsilon B(e^{-sL}\mathcal{F}_{np}(s, \bar{U}), e^{-sL}\mathcal{F}_{np}(s, \bar{U})) - e^{-sL}\nabla_{\bar{U}}\mathcal{F}_{np}(s, \bar{U}) \cdot \mathcal{F}_r(\bar{U}). \tag{4.55}$$

4.2.1. L^2 *error estimates.* In order to obtain the L^2 error estimates we take the scalar product in $(L^2(\mathcal{M}))^3$ of (4.54) with W and using the anti-symmetry property of L and the coercivity property of A, we find

$$\frac{1}{2}\frac{d}{ds}|W|^2_{L^2} + \varepsilon c_1\|W\|^2 \leq \varepsilon|b(W, W, W)| + \varepsilon|b(\tilde{U}^1, W, W)| + \varepsilon|b(W, \tilde{U}^1, W)|$$
$$+ \varepsilon^2 c_0 |R_\varepsilon|_{\mathbf{V}'}\|W\|, \tag{4.56}$$

where $\mathbf{V}'$ is the dual space of $\mathbf{V}$.

In order to bound the trilinear terms in the rhs of (4.56), we use the following result (the proof can be found in [20]):

Lemma 4.1. *The form b is trilinear continuous from $\mathbf{V} \times \mathbf{V}_2 \times \mathbf{V}$ into $\mathbb{R}$ and from $\mathbf{V} \times \mathbf{V} \times \mathbf{V}_2$ into $\mathbb{R}$, and the following inequalities hold:*

$$|b(U, U^\#, \tilde{U})| \leq c_2\|U\|\,|U^\#|^{1/2}\|U^\#\|^{1/2}|\tilde{U}|_{\mathbf{V}_2}, \qquad \forall U, U^\# \in \mathbf{V}, \tilde{U} \in \mathbf{V}_2, \tag{4.57}$$

$$|b(U, U^\#, \tilde{U})| \leq c_3\|U\|\,\|U^\#\|^{1/2}|U^\#|^{1/2}_{\mathbf{V}_2}|\tilde{U}|^{1/2}\|\tilde{U}\|^{1/2}, \qquad \forall U, \tilde{U} \in \mathbf{V}, U^\# \in \mathbf{V}_2. \tag{4.58}$$

Furthermore,

$$b(U, \tilde{U}, \tilde{U}) = 0 \qquad \forall U, \tilde{U} \in \mathbf{V},$$
$$b(U, U^\#, \tilde{U}) = -b(U, \tilde{U}, U^\#) \quad \forall U, \tilde{U}, U^\# \in \mathbf{V} \text{ with } \tilde{U} \text{ or } U^\# \in \mathbf{V}_2. \tag{4.59}$$

From (4.59) we find $b(W, W, W) = 0$ and $b(\tilde{U}^1, W, W) = 0$. Inequality (4.58) implies

$$|b(W, \tilde{U}^1, W)| \leq c_3\|\tilde{U}^1\|^{1/2}|\tilde{U}^1|^{1/2}|W|^{1/2}\|W\|^{3/2}. \tag{4.60}$$

Thanks to Young's inequality, (4.56) and (4.60) imply

$$\frac{d}{ds}|W|^2_{L^2} + \varepsilon c_1\|W\|^2 \leq \varepsilon^2 c|R_\varepsilon|^2_{\mathbf{V}'} + \varepsilon c|W|^2_{L^2}\|\tilde{U}^1\|^2\|\tilde{U}^1\|^2_{H^2}. \tag{4.61}$$

It remains to estimate R_ε in $\mathbf{V}'$

$$|R_\varepsilon|_{\mathbf{V}'} \leq c|e^{-sL}\mathcal{F}_{np}(s, \bar{U})|_{H^1} + |B(e^{sL}\bar{U}, e^{-sL}\mathcal{F}_{np}(s, \bar{U}))|_{\mathbf{V}'}$$
$$+ |B(e^{-sL}\mathcal{F}_{np}(s, \bar{U}), e^{-sL}\bar{U})|_{\mathbf{V}'} + |e^{-sL}\nabla_{\bar{U}}\mathcal{F}_{np}(s, \bar{U}) \cdot \mathcal{F}_r(\bar{U})|_{\mathbf{V}'}$$
$$+ \varepsilon|B(e^{-sL}\mathcal{F}_{np}(s, \bar{U}), e^{-sL}\mathcal{F}_{np}(s, \bar{U}))|_{\mathbf{V}'}. \tag{4.62}$$

From (4.58) we find that

$$|B(V, \tilde{V})|_{\mathbf{V}'} \leq c|V|_{H^2}\|\tilde{V}\| \leq c|V|_{H^2}|\tilde{V}|_{H^2}. \tag{4.63}$$

Using (4.63) and the fact that e^{-sL} conserves all Sobolev norms, inequality (4.62) implies

$$|R_\varepsilon|_{\mathbf{V}'} \leq c |\mathcal{F}_{np}(s,\bar{U})|_{H^1} + |\nabla_{\bar{U}} \mathcal{F}_{np}(s,\bar{U}) \cdot \mathcal{F}_r(\bar{U})|_{\mathbf{V}'} + c |\bar{U}|_{H^2} |\mathcal{F}_{np}(s,\bar{U})|_{H^2}$$
$$+ \varepsilon c |\mathcal{F}_{np}(s,\bar{U})|_{H^2}^2. \tag{4.64}$$

We need to bound $|\mathcal{F}_{np}(s,\bar{U})|_{H^2}$ and $|\nabla_{\bar{U}} \mathcal{F}_{np}(s,\bar{U}) \cdot \mathcal{F}_r(\bar{U})|_{\mathbf{V}'}$. We start estimating $\mathcal{F}_{np}(s,\bar{U})$. We recall that $\mathcal{F}_n(s,\bar{U}) = A_n\bar{U} + B_n(\bar{U},\bar{U}) + S_n$, with A_n, B_n respectively given by the time-dependent terms in (4.28) and (4.30).

In order to estimate

$$\mathcal{F}_{np}(s,\bar{U}) = \int_0^s \mathcal{F}_n(\tau,\bar{U}) d\tau,$$

we need to bound the integrals in time of the terms $e^{i\tau\omega_k^\pm}, e^{i\tau(\omega_j^\pm+\omega_l^\pm)}$ and $e^{i\tau(\omega_j^\pm+\omega_l^\pm+\omega_k^\pm)}$. The integral

$$I_1(k) = \frac{e^{i\tau\omega_k^\pm} - 1}{i\omega_k^\pm},$$

is easily estimated as $|I_1(k)| \leq \dfrac{2}{|\omega_k^\pm|} \leq 2$. The integral

$$|I_2(j,l)| = \left| \frac{e^{i\tau(\omega_j^\pm+\omega_l^\pm)} - 1}{i(\omega_j^\pm + \omega_l^\pm)} \right| \leq \frac{2}{|\omega_j^\pm + \omega_l^\pm|}$$

is estimated as $|I_2(j,l)| \leq 2$ when $\omega_j^\pm$ and $\omega_l^\pm$ have the same sign and as follows

$$|I_2(j,l)| \leq \frac{2}{N^2} \frac{|j|_N/|j_3| + |l|_N/|l_3|}{\left| |\tilde{j}|^2/j_3^2 - |\tilde{l}|^2/l_3^2 \right|} \leq \frac{2}{N^2} \left(\frac{|j|_N}{|j_3|} + \frac{|l|_N}{|l_3|} \right) j_3^2 l_3^2$$
$$\leq c(N)|j|^2|l|^2,$$

when $\omega_j^\pm$ and $\omega_l^\pm$ have opposite signs. The integral

$$I_3(j,l,k) = \frac{e^{i\tau(\omega_j^\pm+\omega_l^\pm+\omega_k^\pm)} - 1}{i(\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)},$$

is estimated using Theorem 5.1 proved in the Appendix.

Using the estimates for $I_1(k)$, $I_2(j,l)$ and $I_3(j,l,k)$, we are now able to bound $\mathcal{F}_{np}(s,\bar{U})$ in H^2. Since $A_{np}(s,\bar{U})$ contains only terms of type $I_1(k)$, we find

$$|A_{np}(s,\bar{U})|_{H^m} \leq c|\bar{U}|_{H^{m+2}}, \quad \forall m \in \mathbb{N}. \tag{4.65}$$

In bounding $B_{np}(s,\bar{U})$, the most difficult terms to estimate are those of the type

$$T_1 = \left| \sum_{k\in\mathbb{Z}_\mathcal{M}} {\sum}^n B_{jlk}^{\pm\pm\pm} I_3(j,l,k) X_k^\pm \cdot X_l^\pm v_j^\pm v_l^\pm \right|_{H^m}$$
$$\leq \frac{C(N,L_3,\gamma)}{\mu} \left[\sum_{k\in\mathbb{Z}_\mathcal{M}} \left(\sum_{j+l=k} (|k|+|l|+|j|)^3 |j|^{4+\gamma} |l|^{4+\gamma} |\bar{U}_j||\bar{U}_l| \right)^2 |k|^{2m} \right]^{1/2} \tag{4.66}$$
$$\leq \frac{C(N,L_3,\gamma)}{\mu} |fg|_{H^m},$$

where $f = \sum_{k\in\mathbb{Z}_\mathcal{M}} |\bar{U}_k||k|^8 e^{ik\cdot x}$, $g = \sum_{k\in\mathbb{Z}_\mathcal{M}} |\bar{U}_k||k|^5 e^{ik\cdot x}$ and μ and γ are as in Theorem 5.1.

Since $H^m(\mathcal{M})$ is a multiplicative algebra for $m \geq 2$, we find

$$T_1 \leq \frac{c}{\mu} |\bar{U}|_{m+8} |\bar{U}|_{m+5}, \tag{4.67}$$

and if $m = 0,1$, then

$$T_1 \leq \frac{c}{\mu} |\bar{U}|_{m+9} |\bar{U}|_{m+6}. \tag{4.68}$$

We also remark that

$$|S_{np}|_m \leq |S|_m, \quad \forall m.$$

This allows us to conclude that

$$|\mathcal{F}_{np}(s,\bar{U})|_m \leq c(N,\mu,|S|_m,|\bar{U}|_{m+8}), \quad \forall m \geq 2, \tag{4.69}$$

and thus we obtained the estimate for the particular case $m = 2$.

It remains to bound $|\nabla_{\bar{U}}\mathcal{F}_{np}(s,\bar{U}) \cdot \mathcal{F}_r(\bar{U})|_{\mathbf{V}'}$. Since

$$\mathcal{F}_{np}(s,\bar{U}) = A_{np}(s,\bar{U}) + B_{np}(s,\bar{U}) + S_{np},$$

we find

$$\nabla_{\bar{U}}\mathcal{F}_{np}(s,\bar{U}) \cdot \mathcal{F}_r(\bar{U}) = A_{np}(s,\mathcal{F}_r(\bar{U})) + B_{np}(s,\bar{U},\mathcal{F}_r(\bar{U})) \tag{4.70}$$
$$+ B_{np}(s,\mathcal{F}_r(\bar{U}),\bar{U}).$$

Thus, for the first term in (4.70) we have

$$|A_{np}(s,\mathcal{F}_r(\bar{U}))|_{\mathbf{V}'} \leq c|\mathcal{F}_k(\bar{U})|_{H^1}. \tag{4.71}$$

For the second term in (4.70) we apply an argument similar to the one used in (4.66)

$$T_2 = |B_{np}(s,\bar{U},\mathcal{F}_r(\bar{U}))|_{\mathbf{V}'} \quad \leq c|B_{np}(s,\bar{U},\mathcal{F}_r(\bar{U}))|_{L^2}$$
$$\leq \frac{C(N)}{\mu}|\tilde{f}\tilde{g}|_{L^2} \qquad \leq \frac{C(N)}{\mu}|\tilde{f}|_{H^1}|\tilde{g}|_{H^1}, \tag{4.72}$$

where $\tilde{f} = \sum_{k\in\mathbb{Z}_\mathcal{M}}|\bar{U}_k|\,|k|^8 e^{ik\cdot x}$ and $\tilde{g} = \sum_{k\in\mathbb{Z}_\mathcal{M}}|\mathcal{F}_{r,k}(\bar{U})|\,|k|^8 e^{ik\cdot x}$.

Relation (4.72) leads us to

$$T_2 \leq \frac{C(N)}{\mu}|\bar{U}|_{H^9}|\mathcal{F}_r(\bar{U})|_{H^9}. \tag{4.73}$$

We only need to find bounds for $|\mathcal{F}_r(\bar{U})|_{H^m}$, $\forall m \geq 1$, in order to conclude. Since

$$\mathcal{F}_r(\bar{U}) = A_r(\bar{U}) + B_r(\bar{U}) + S_r,$$

we find

$$|A_r(\bar{U})|_{H^m} \leq |\bar{U}|_{H^{m+2}},$$
$$|S_r|_{H^m} \leq |S|_{H^m}, \tag{4.74}$$

while for the nonlinear operator B_r we find, using similar arguments as in (4.51)

$$|B_r(\bar{U})|_{H^m} \leq c\left[\sum_{k\in\mathbb{Z}_\mathcal{M}}\left(\sum_{j+l=k}(|j|+|l|)|\bar{U}_j||\bar{U}_l|\right)^2|k|^{2m}\right]^{1/2} \tag{4.75}$$
$$\leq c|h_1 h_2|_{H^m},$$

with $h_1 = \sum_k |k||\bar{U}_k|e^{ik\cdot x}$ and $h_2 = \sum_k |\bar{U}_k|e^{ik\cdot x}$.

For $m = 0, 1$, we have

$$|B_r(\bar{U})|_{H^m} \leq c|h_1|_{H^{m+1}}|h_2|_{H^{m+1}} \tag{4.76}$$
$$\leq c|\bar{U}|_{H^{m+1}}|\bar{U}|_{H^{m+2}},$$

while for $m \geq 2$ we find

$$|B_r(\bar{U})|_{H^m} \leq c|\bar{U}|_{H^m}|\bar{U}|_{H^{m+1}}. \tag{4.77}$$

Returning to T_2, we find

$$T_2 \leq c(N,\mu,|\bar{U}|_{10},|S|_9). \tag{4.78}$$

Similar estimates can be deduced for the term $B_{np}(s,\mathcal{F}_r(\bar{U}),\bar{U})$. Returning to (4.64) and recalling the fact that the renormalized group system is globally well-posed in all Sobolev spaces, provided the initial data is regular enough, we find that we can bound R_ε as

$$|R_\varepsilon|_{\mathbf{V}'} \leq c(N,\mu,|S|_9,|U_0|_{10}). \tag{4.79}$$

Thus, (4.61) can be written as

$$\frac{d}{ds}|W|_{L^2}^2 \leq f_1(s)|W|_{L^2}^2 + g_1(s) \tag{4.80}$$

where $f_1(s) = \varepsilon c\|\tilde{U}^1\|^2\|\tilde{U}^1\|_{H^2}^2$ and $g_1(s) = \varepsilon^2 c|R_\varepsilon|_{\mathbf{V}'}^2$. From (4.79) we know that g_1 is an $L^\infty(\mathbb{R})$-function. Recalling formula (4.53) for $\tilde{U}^1$ as well as relation (4.69), we also find that f_1 is an $L^\infty(\mathbb{R})$-function. Applying Gronwall lemma to (4.80), we conclude with the following result:

Theorem 4.3. *Let μ, L_1, L_2, L_3 be arbitrary fixed positive constants and $U_0 \in (\dot{H}^{10}_{\mathrm{per}}(\mathcal{M}))^3$ $\cap \mathbf{V}$, $S \in (\dot{H}^9_{\mathrm{per}}(\mathcal{M}))^3 \cap \mathbf{V}$. Then there exists a set $\Theta^\mu_3(L_1, L_2, L_3)$ having the Lebesque measure $\mathrm{mes}\, \Theta^\mu_3(L_1, L_2, L_3) \leq \mu$ such that for all Burgers numbers $N \notin \Theta^\mu_3(L_1, L_2, L_3)$, the difference between the exact solution U of (2.6) and the approximate solution $\tilde{U}^1$ in (4.53) satisfies*

$$|\tilde{U}^1(t) - U(t)|^2_{L^2} \leq \varepsilon^2 k' e^{k''t} \qquad \forall t \geq 0,$$

where k' and k'' are constants depending on N, L_1, L_2, L_3, μ, $|U_0|_{H^{10}}$ and $|S|_{H^3}$.

4.2.2. H^m-*error estimates, for $m \geq 1$.* In order to estimate the H^m-norm of the error, we take the L^2-scalar product of equation (4.56) by $(-\Delta)^m W$. We obtain

$$\frac{1}{2}\frac{d}{ds}|W|^2_m + \varepsilon c_1 |W|^2_{m+1} = \varepsilon^2 (R_\varepsilon, (-\Delta)^m W)_{L^2} - \varepsilon b(W, W, (-\Delta)^m W) \\ - \varepsilon b(\tilde{U}^1, W, (-\Delta)^m W) - \varepsilon b(W, \tilde{U}^1, (-\Delta)^m W). \tag{4.81}$$

The first term from the right hand side of (4.81) is estimated as

$$\varepsilon^2 |(R_\varepsilon, (-\Delta)^m W)_{L^2}| \leq \varepsilon^2 |R_\varepsilon|_{m-1} |W|_{m+1} \leq \varepsilon \frac{c_1}{4} |W|^2_{m+1} + c\varepsilon^3 |R_\varepsilon|^2_{m-1}. \tag{4.82}$$

We need to bound R_ε in the H^{m-1}-norm:

$$|R_\varepsilon|_{m-1} \leq |\mathcal{F}_{np}(s, \bar{U})|_{m+1} + |B(e^{-sL}\bar{U}, e^{-sL}\mathcal{F}_{np}(s, \bar{U}))|_{m-1} \tag{4.83}$$
$$+ |B(e^{-sL}\mathcal{F}_{np}(s, \bar{U}), e^{-sL}\bar{U})|_{m-1} + \varepsilon |B(e^{-sL}\mathcal{F}_{np}(s, \bar{U}), e^{-sL}\mathcal{F}_{np}(s, \bar{U}))|_{m-1}$$
$$+ |A_{np}(s, \mathcal{F}_r(\bar{U}))|_{m-1} + |B_{np}(s, \bar{U}, \mathcal{F}_r(\bar{U}))|_{m-1} + |B_{np}(s, \mathcal{F}_r(\bar{U}), \bar{U})|_{m-1}.$$

Using (4.69), we find

$$|\mathcal{F}_{np}|_{m+1} \leq c(N, \mu, |S|_{m+1}, |\bar{U}|_{9+m}) \leq c(N, \mu, |S|_{m+1}, |U_0|_{9+m}), \forall m \geq 1, \tag{4.84}$$

where $c(N, \mu, |S|_{m+1}, |U_0|_{9+m})$ is a constant depending on h, μ, $|S|_{m+1}$ and $|U_0|_{9+m}$. Using (4.65) we know

$$|A_{np}(s, \mathcal{F}_r(\bar{U}))|_{m-1} \leq |\mathcal{F}_r(\bar{U})|_{m+1},$$

and using (4.74)-(4.77), we find

$$|\mathcal{F}_r(\bar{U})|_{m+1} \leq |\bar{U}|_{H^{m+2}} + c|\bar{U}|_{H^m}|\bar{U}|_{H^{m+1}} + |S|_{H^m}$$
$$\leq c(|U_0|_{m+2}, |S|_m).$$

The last terms in (4.83) is bounded using (4.67) and (4.68). It only remains to bound $|B(U, \tilde{U})|_{H^m}$ and we do this similarly to (4.66). The most difficult terms in $B(U, \tilde{U})$ are the terms containing $\delta^i_j l_3$ and they are bounded as follows:

$$\left(\sum_{k \in \mathbb{Z}_\mathcal{M}}\left(\sum_{j+l=k}|\delta^i_j l_3||U_j||\tilde{U}_l|\right)^2 |k|^{2m}\right)^{1/2} \leq \left(\sum_{k \in \mathbb{Z}_\mathcal{M}}\left(\sum_{j+l=k}|j||l||U_j||\tilde{U}_l|\right)^2 |k|^{2m}\right)^{1/2} \tag{4.85}$$
$$\leq c|fg|_{H^m} = T_3,$$

with $f = \sum_{k \in \mathbb{Z}_\mathcal{M}}|k||U_k|e^{ik\cdot x}$ and $g = \sum_{k \in \mathbb{Z}_\mathcal{M}}|k||\tilde{U}_k|e^{ik\cdot x}$.

For $m \geq 2$, H^m is a multiplicative algebra and we continue estimating T_3 as

$$T_3 \leq c|f|_m|g|_m \leq |U|_{m+1}|\tilde{U}|_{m+1}.$$

For $m = 0$ we find

$$T_3 \leq c|f|_1|g|_1 \leq c|U|_2|\tilde{U}|_2,$$

and for $m = 1$ we have

$$T_3 \leq c|f|_2|g|_2 \leq c|U|_3|\tilde{U}|_3.$$

We can thus conclude with the following estimate:

$$|B(U, \tilde{U})|_m \leq c|U|_{m+1}|\tilde{U}|_{m+1} \quad \text{for } m \geq 2, \tag{4.86}$$
$$\leq c|U|_{m+2}|\tilde{U}|_{m+2} \quad \text{for } m = 0, 1.$$

Using (4.86) and (4.69), we find

$$|B(e^{-sL}\mathcal{F}_{np}(s, \bar{U}), e^{-sL}\bar{U})|_{m-1} \leq c|e^{-sL}\mathcal{F}_{np}(s, \bar{U})|_m|e^{-sL}\bar{U}|_m$$
$$\leq c|\mathcal{F}_{np}(s, \bar{U})|_m|\bar{U}|_m$$
$$\leq c(N, \mu, |S|_m, |U_0|_{m+8}), \text{ for } m \geq 3,$$

and

$$|B(e^{-sL}\mathcal{F}_{np}(s,\bar{U}), e^{-sL}\bar{U})|_{m-1} \leq c|e^{-sL}\mathcal{F}_{np}(s,\bar{U})|_{m+1}|e^{-sL}\bar{U}|_{m+1}$$
$$\leq c|\mathcal{F}_{np}(s,\bar{U})|_{m+1}|\bar{U}|_{m+1}$$
$$\leq c(N,\mu,|S|_{m+1},|U_0|_{9+m}), \text{ for } m = 1,2.$$

The same arguments are obtained for $|B(e^{-sL}\bar{U}, e^{-sL}\mathcal{F}_{np}(s,\bar{U}))|_{m-1}$. We can thus conclude that

$$\begin{aligned} |R_\varepsilon|_{m-1} &\leq c(N,\mu,|S|_m,|U_0|_{m+8}) && \text{for } m \geq 3, \\ &\leq c(N,\mu,|S|_{m+1},|U_0|_{m+9}) && \text{for } m = 1,2. \end{aligned} \tag{4.87}$$

In order to be able to estimate the last three terms in (4.81), we need to be able to bound terms of the form $b(U,\tilde{U},(-\Delta)^m U^\#)$. We use the following lemma:

Lemma 4.2. *Let $U,\tilde{U} \in \mathbf{V} \cap (\dot{H}_{per}^{m+1}(\mathcal{M}))^3$. Then the following inequality holds:*

$$\begin{aligned} |b(U,\tilde{U},(-\Delta)^m\tilde{U})| &\leq c|U|_m^{1/2}|U|_{m+1}^{1/2}|\tilde{U}|_2|\tilde{U}|_m + c|U|_1^{1/2}|U|_2^{1/2}|\tilde{U}|_m|\tilde{U}|_{m+1} \\ &\quad + c|U|_{m+1}|\tilde{U}_m|\tilde{U}|_{m+1} + c|U|_{m+1}|\tilde{U}|_1^{1/2}|\tilde{U}|_2^{1/2}|\tilde{U}|_m^{1/2}|\tilde{U}|_{m+1}^{1/2}. \end{aligned} \tag{4.88}$$

Furthermore,

$$b(U,(-\Delta)^{m/2}\tilde{U},(-\Delta)^{m/2}\tilde{U}) = 0 \qquad \forall U \in \mathbf{V}, \quad \forall \tilde{U} \in \mathbf{V} \cap \left(\dot{H}_{per}^{m+1}(\mathcal{M})\right)^3. \tag{4.89}$$

Proof. Relation (4.89) is obvious. For (4.88), we estimate as follows:

$$\begin{aligned} \left|b(U,\tilde{U},(-\Delta)^m\tilde{U})\right| &= \left|i\sum_{j+l+k=0}\sum_{n=1}^2(l_n - \frac{j_n}{j_3}l_3)U_{n,j}\tilde{U}_l(-\Delta)^m\tilde{U}_k\right| \\ &\leq \sum_{j+l+k=0}\sum_{n=1}^2(|l_n| + \frac{|j_n|}{|j_3|}|l_3|)|U_j||\tilde{U}_l||k|^{2m}|\tilde{U}_k| \\ &\leq \text{ (using (4.89) in order to subtract } |l|^m) \\ &\leq c\sum_{j+l+k=0}(|\tilde{l}| + \frac{|\tilde{j}||l_3|}{|j_3|})|U_j||\tilde{U}_l||k|^m(|k|^m - |l|^m)|\tilde{U}_k| \\ &\leq c\sum_{j+l+k=0}(|\tilde{l}| + \frac{|\tilde{j}||l_3|}{|j_3|})|U_j||\tilde{U}_l||k|^m(|j+l|^m - |l|^m)|\tilde{U}_k| \\ &\leq c\sum_{j+l+k=0}(|\tilde{l}| + \frac{|\tilde{j}||l_3|}{|j_3|})|U_j||\tilde{U}_l|(|j|^m + |l|^{m-1}|j|)|k|^m|\tilde{U}_k| \\ &= \sum\nolimits_1 + \sum\nolimits_2. \end{aligned} \tag{4.90}$$

For the first sum, we find

$$\begin{aligned} \sum\nolimits_1 &= c\sum_{j+l+k=0}|\tilde{l}|(|j|^m + |l|^{m-1}|j|)|k|^m|U_j||\tilde{U}_l||\tilde{U}_k| \\ &\leq c\sum_{j+l+k=0}|j|^m|l||k|^m|U_j||\tilde{U}_l||\tilde{U}_k| + c\sum_{j+l+k=0}|j||l|^m|k|^m|U_j||\tilde{U}_l||\tilde{U}_k| \\ &\leq c|U|_{m+\frac{1}{2}}|\tilde{U}|_2|\tilde{U}|_m + c|U|_{\frac{3}{2}}|\tilde{U}|_{m+1}|\tilde{U}|_m \\ &\leq c|U|_m^{1/2}|U|_{m+1}^{1/2}|\tilde{U}|_2|\tilde{U}|_m + c|U|_1^{1/2}|U|_2^{1/2}|\tilde{U}|_m|\tilde{U}|_{m+1}. \end{aligned} \tag{4.91}$$

The second sum is bounded as

$$\begin{aligned} \sum\nolimits_2 &= c\sum_{j+l+k=0}\frac{|\tilde{j}||l_3|}{|j_3|}|l|^{m-1}|j||k|^m|U_j||\tilde{U}_l||\tilde{U}_k| \\ &\quad + c\sum_{j+l+k=0}\frac{|\tilde{j}||l_3|}{|j_3|}|j|^m|k|^m|U_j||\tilde{U}_l||\tilde{U}_k| \\ &= \sum\nolimits_2' + \sum\nolimits_2''. \end{aligned}$$

For $\sum_2'$ we find

$$
\begin{aligned}
\sum\nolimits_2' &\le c \sum_{\tilde{j}+\tilde{l}+\tilde{k}=0} \Big(\sum_{j_3\neq 0} \frac{|\tilde{j}||j|}{|j_3|}|U_j|\Big)\Big(\sum_{l_3}|l|^{2m}|\tilde{U}_l|^2\Big)^{1/2}\Big(\sum_{k_3}|k|^{2m}|\tilde{U}_k|^2\Big)^{1/2} \\
&\le c \sum_{\tilde{j}+\tilde{l}+\tilde{k}=0} \Big(\sum_{j_3\neq 0} \frac{1}{|j_3|^2}\Big)^{1/2}\Big(\sum_{j_3\neq 0}|j|^4|U_j|^2\Big)^{1/2}\Big(\sum_{k_3}|k|^{2m}|\tilde{U}_k|^2\Big) \\
&= c \int_{\mathcal{M}'} f g^2 \, d\mathcal{M}',
\end{aligned}
\tag{4.92}
$$

with $f = \sum_{\tilde{k}}(\sum_{k_3\neq 0}|\tilde{k}|^4|U_k|^2)^{1/2}e^{i\tilde{k}\cdot\tilde{x}}$, $g = \sum_{\tilde{k}}(\sum_{\tilde{k}_3}|k|^{2m}|\tilde{U}_k|^2)e^{i\tilde{k}\cdot\tilde{x}}$, $\tilde{x}=(x_1,x_2)\in\mathcal{M}'$.
We continue estimating the terms as follows

$$
\begin{aligned}
\sum\nolimits_2' &\le c|f|_{L^2(\mathcal{M}')}|g|^2_{L^4(\mathcal{M}')} \le c|U|_{m+1}|g|_{L^2(\mathcal{M}')}|g|_{H^1(\mathcal{M}')} \\
&\le c|U|_{m+1}|\tilde{U}|_m|\tilde{U}|_{m+1}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\sum\nolimits_2'' &\le c \sum_{\tilde{j}+\tilde{l}+\tilde{k}=0}\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\Big)^{1/2}\Big(\sum_{j_3\neq 0}|j|^{2(m+1)}|U_j|^2\Big)^{1/2}\Big(\sum_{l_3}|l|^2|\tilde{U}_l|^2\Big)^{1/2} \\
&\quad \Big(\sum_{k_3}|k|^{2m}|\tilde{U}_k|^2\Big)^{1/2} \\
&\le c|U|_{m+1}|\tilde{U}|_1^{1/2}|\tilde{U}|_2^{1/2}|\tilde{U}|_m^{1/2}|\tilde{U}|_{m+1}^{1/2}.
\end{aligned}
$$

$\square$

Using (4.88), we find

$$
\begin{aligned}
|b(W,W,(-\Delta)^m W)| &\le c|W|_2|W|_m^{3/2}|W|_{m+1}^{1/2} + c|W|_1^{1/2}|W|_2^{1/2}|W|_m|W|_{m+1} \\
&\quad + c|W|_m|W|_{m+1}^2 + c|W|_1^{1/2}|W|_2^{1/2}|W|_m^{1/2}|W|_{m+1}^{3/2},
\end{aligned}
\tag{4.93}
$$

and

$$
\begin{aligned}
|b(\tilde{U}^1,W,(-\Delta)^m W)| &\le c|\tilde{U}^1|_m^{1/2}|\tilde{U}^1|_{m+1}^{1/2}|W|_2|W|_m + c|\tilde{U}^1|_1^{1/2}|\tilde{U}^1|_2^{1/2}|W|_m|W|_{m+1} \\
&\quad + c|\tilde{U}^1|_{m+1}|W|_m|W|_{m+1} + c|\tilde{U}^1|_{m+1}|W|_1^{1/2}|W|_2^{1/2}|W|_m^{1/2}|W|_{m+1}^{1/2}.
\end{aligned}
\tag{4.94}
$$

For $b(W,\tilde{U}^1,(-\Delta)^m W)$, we proceed similarly,

$$
\begin{aligned}
|b(W,\tilde{U}^1,(-\Delta)^m W)| &\le c \sum_{j+l+k=0}(|\tilde{l}| + \frac{|\tilde{j}||l_3|}{|j_3|})|W_j||\tilde{U}_l^1||k|^m|W_k|(|j|^m + |l|^m) \\
&\le c \sum_{j+l+k=0}|j||l|(|j|^m + |l|^m)|W_j||\tilde{U}_l^1||k|^m|W_k| \\
&\le c|\tilde{U}^1|_2|W|_m^{1/2}|W|_{m+1}^{3/2} + c|\tilde{U}^1|_{m+1}|W|_2|W|_m^{1/2}|W|_{m+1}^{1/2}.
\end{aligned}
\tag{4.95}
$$

We can now return to (4.81). We need to distinguish the cases $m=1, m=2$ from the case $m>2$.
For $m=1$ we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{ds}|W|_1^2 + \frac{3\varepsilon c_1}{4}|W|_2^2 &\le c\varepsilon^3|R_\varepsilon|^2 + c\varepsilon|W|_1^{3/2}|W|_2^{3/2} + c\varepsilon|W|_1|W|_2^2 \\
&\quad + c\varepsilon|\tilde{U}^1|_2|W|_1|W|_2 + c\varepsilon|\tilde{U}^1|_2|W|_1^{1/2}|W|_2^{3/2} \\
&\le \frac{\varepsilon c_1}{4}|W|_2^2 + c\varepsilon^3|R_3|^2 + c\varepsilon|W|_1|W|_2^2 \\
&\quad + c\varepsilon|\tilde{U}^1|_2^4|W|_1^2.
\end{aligned}
\tag{4.96}
$$

Thus, we obtain

$$
\begin{aligned}
\frac{d}{ds}|W|_1^2 + \varepsilon(c_1 - c|W|_1)|W|_2^2 &\le \varepsilon^3 c'(N,\mu,|S|_2,|U_0|_{10}) \\
&\quad + \varepsilon c(N,\mu,|S|_2,|U_0|_{10})|W|_1^2,
\end{aligned}
\tag{4.97}
$$

where in (4.97) we used (4.53) and (4.69).

As long as $|W(s)|_1 \leq \dfrac{c_1}{2c}$, applying the Gronwall lemma to (4.97) we find

$$|W(s)|_1^2 \leq \varepsilon^2 c'(N, \mu, |S|_2, |U_0|_{10}) e^{\varepsilon s c(N, \mu, |S|_2, |U_0|_{10})}. \tag{4.98}$$

For every $T > 0$ we can find an $\varepsilon_T > 0$ such that for all $\varepsilon \leq \varepsilon_T$ we have $|W(s)|_1 \leq c_1/2c$, which implies that estimate (4.98) holds globally on the interval $[0, T]$. We proceed similarly for $m = 2$. We can thus conclude with the following theorem on the error estimates in H^1 or H^2.

Theorem 4.4. *Let $\mu > 0$, L_1, L_2, L_3 be positive, fixed constants and let $m \in \{1, 2\}$. If $U_0 \in \left(\dot{H}_{per}^{m+9}(\mathcal{M})\right)^3 \cap \mathbf{V}$ and $S \in \left(\dot{H}_{per}^{m+9}(\mathcal{M})\right)^3 \cap \mathbf{V}$, there exists a set $\Theta_3^\mu(L_1, L_2, L_3)$ having the Lebesque measure $\mathrm{mes}\, \Theta_3^\mu(L_1, L_2, L_3) \leq \mu$ such that, for all Burgers numbers $N \notin \Theta_3^\mu(L_1, L_2, L_3)$, we have*

For all $T > 0$ there exists $\varepsilon_T > 0$ such that for all $\varepsilon \leq \varepsilon_T$, the error between the exact solution U of (2.6) and the approximate solution $\tilde{U}^1$ given by (4.53) satisfies

$$|\tilde{U}^1(t) - U(t)|_m \leq \varepsilon^2 k e^{tk'}, \quad \forall t \in [0, T], \tag{4.99}$$

where k and k' are constants depending on N, μ, $|S|_{m+1}$, $|U_0|_{m+9}$, L_1, L_2 and L_3.

For $m > 2$, we proceed similarly and after using Lemma 4.2 and relations (4.87), (4.95) we can conclude to the following theorem:

Theorem 4.5. *Let $\mu > 0$, L_1, L_2, L_3 be positive, fixed constants and let $m \in \{3, 4, \cdots\}$. If $U_0 \in \left(\left(\dot{H}_{per}^{m+9}(\mathcal{M})\right)^3 \cap \mathbf{V}\right.$ and $S \in \left(\dot{H}_{per}^{m+9}(\mathcal{M})\right)^3 \cap \mathbf{V}$, there exists a set $\Theta_3^\mu(L_1, L_2, L_3)$ having the Lebesque measure $\mathrm{mes}\, \Theta_3^\mu(L_1, L_2, L_3) \leq \mu$ such that, for all Burgers numbers $N \notin \Theta_3^\mu(L_1, L_2, L_3)$, we have:*

For all $T > 0$ there exists $\varepsilon_T > 0$ such that for all $\varepsilon \leq \varepsilon_T$, the error between the exact solution U of (2.6) and the approximate solution $\tilde{U}^1$ given by (4.53) satisfies

$$|\tilde{U}^1(t) - U(t)|_m \leq \varepsilon^2 k e^{tk'}, \qquad \forall t \in [0, T], \tag{4.100}$$

where k and k' are constants depending on N, μ, $|S|_m$, $|U_0|_{m+8}$, L_1, L_2 and L_3.

5. **Appendix.** As announced above, in this section we present an approach (adapting an idea of Babin, Mahalov and Nicolenko [2]) that allows us to avoid the three-waves interactions. We want to see in which conditions the scenario

$$\omega_j^\pm + \omega_l^\pm + \omega_k^\pm = 0, \text{ where } j + l = k,$$

never happens, and to estimate the term

$$I = \frac{e^{is(\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)} - 1}{i(\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)}, \tag{5.1}$$

term that appears in $\mathcal{F}_{np}$ from (3.11).

We start by estimating $(\omega_j^+ + \omega_l^+ + \omega_k^+)^{-1}$, assuming ω_j^+, ω_l^+, $\omega_k^+ > 0$, the other cases being treated similarly. Then

$$\frac{1}{|\omega_j^+ + \omega_l^+ - \omega_k^+|} = \left| \frac{(\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)(-\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)(\omega_j^\pm - \omega_l^\pm + \omega_k^\pm)}{(\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)(\omega_j^\pm + \omega_l^\pm - \omega_k^\pm)(-\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)} \right|$$
$$= \frac{|(\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)(-\omega_j^\pm + \omega_l^\pm + \omega_k^\pm)(\omega_j^\pm - \omega_l^\pm + \omega_k^\pm)|}{|P(\lambda)|} j_3^4 l_3^4 k_3^4, \tag{5.2}$$

where $\lambda = N^2$ and $P(\lambda) = \lambda^2(x_j^2 + x_l^2 + x_k^2 - 2x_j x_l - 2x_l x_k) - 2x j_3^2 k_3^2 k_3^2 (x_j + x_l + x_k) - 3j_3^4 l_3^4 k_3^4$. Here we wrote $x_j = |\tilde{j}|^2 l_3^2 k_3^2$ and similarly for x_l and x_k.

The discriminant of this quadratic polynomial is

$$\Delta = 2j_3^4 l_3^4 k_3^4 \left\{ (x_k - x_l)^2 + (x_k - x_j)^2 + (x_l - x_j)^2 \right\} \geq 0. \tag{5.3}$$

Thus, $P(\lambda) = 0$ has no more than two solutions for each fixed (j, l) and this implies thta the set of Burgers numbers N for which $\omega_j^+ + \omega_l^+ - \omega_k^+ = 0$ is at most countable. In what folows we denote the solutions of $P(\lambda) = 0$ by $\lambda_\pm(j, l)$.

To estimate I, we distinguish two cases:

Case 1:   If $|\omega_l^+ - \omega_k^+| \leq \dfrac{|\omega_j^+|}{2}$, then $\dfrac{1}{|\omega_j^+ + \omega_l^+ - \omega_k^+|} \leq \dfrac{2}{|\omega_j^+|} \leq 2.$

Case 2:   If $|\omega_l^+ - \omega_k^+| \geq \dfrac{|\omega_j^+|}{2}$, the estimate is more delicate. We define the three-wave quasi-resonant set $\Theta_3^\mu(L_1, L_2, L_3)$:

Given $\mu > 0$ and a sequence of positive numbers $\{\xi_{j,l}\}$ with $\sum\limits_{j,l \in \mathbb{Z}_\mathcal{M}} \xi_{j,l} \leq 1$, we define the three-wave quasi-resonant set $\Theta_3^\mu(L_1, L_2, L_3)$ as

$$\Theta_3^\mu(L_1, L_2, L_3) = \bigcup_{j,l \in \mathbb{Z}_\mathcal{M}} \left\{ N : 2|N - N^*(j, l, L_1, L_2, L_3)| \leq \mu \xi_{j,l} \right\}. \tag{5.4}$$

where $N^*(j, l, L_1, L_2, L_3) := \sqrt{\lambda_\pm(j, l)}$. The set $\Theta_3^\mu(L_1, L_2, L_3)$ is of Lebesque measure mes $\Theta_3^\mu(L_1, L_2, L_3) \leq \mu$, for all L_1, L_2, L_3.

A small neighborhood of $\lambda_\pm(j, l)$ is defined by $|P(\lambda)| \leq \delta$, with $\delta > 0$ small.

For δ small, we have

$$\begin{aligned} \delta &\simeq \left| \frac{d\lambda}{d\delta}(0) \right|^{-1} |\lambda(\delta) - \lambda_\pm(j, l)| \\ &\simeq 2N^*(j, l, L_1, L_2, L_3)|N - N^*(j, l, L_1, L_2, L_3)| \left| \frac{d\lambda}{d\delta}(0) \right|^{-1}. \end{aligned} \tag{5.5}$$

Using the quadratic formula, we obtain the derivative at $\delta = 0$

$$\left| \frac{d\lambda}{d\delta} \right| = \frac{1}{\sqrt{\Delta}} = \frac{1}{j_3^2 l_3^2 k_3^2 \sqrt{2\{(x_k - x_l)^2 + (x_k - x_j)^2 + (x_l - x_j)^2\}}}.$$

Since $|(\omega_k^+)^2 - (\omega_l^+)^2|j_3^2 k_3^2 l_3^2 = N^2(x_k - x_l)$, we obtain

$$\left| \frac{d\lambda}{d\delta}(0) \right| = \frac{N^2}{j_3^4 l_3^4 k_3^4 \sqrt{2\{\left((\omega_k^+)^2 - (\omega_l^+)^2\right)^2 + \left((\omega_k^+)^2 - (\omega_j^+)^2\right)^2 + \left((\omega_j^+)^2 - (\omega_l^+)^2\right)^2\}}}.$$

We are in the case $|\omega_l^+ - \omega_k^+| \geq \dfrac{|\omega_j^+|}{2}$, so

$$|(\omega_k^+)^2 - (\omega_l^+)^2| = |\omega_k^+ - \omega_l^+||\omega_k^+ + \omega_l^+| \geq \frac{|\omega_j^+|}{2}|\omega_k^+ + \omega_l^+|.$$

Since $|(\omega_k^+)^2 - (\omega_l^+)^2| \geq |\omega_j^+| \geq 1$, we find

$$\left| \frac{d\lambda}{d\delta}(0) \right| \leq \frac{N^2}{j_3^4 l_3^4 k_3^4 \sqrt{2}}.$$

For $\delta << 1$ we have $P(\lambda) = \delta$ and using (5.5), we find

$$\begin{aligned} \frac{1}{|\omega_j^+ + \omega_l^+ - \omega_k^+|} &= \frac{|(\omega_j^+ + \omega_l^+ + \omega_k^+)(-\omega_j^+ + \omega_l^+ + \omega_k^+)(\omega_j^+ - \omega_l^+ + \omega_k^+)|}{\delta} j_3^4 l_3^4 k_3^4 \\ &\simeq \frac{|(\omega_j^+ + \omega_l^+ + \omega_k^+)(-\omega_j^+ + \omega_l^+ + \omega_k^+)(\omega_j^+ - \omega_l^+ + \omega_k^+)|}{2N^*(j, l, L_1, L_2, L_3)|N - N^*(j, l, L_1, L_2, L_3)|} \left| \frac{d\lambda}{d\delta}(0) \right| j_3^4 l_3^4 k_3^4 \\ &\leq \frac{L_3^3 N^2(|k|_N + |l|_N + |j|_N)^3}{2\sqrt{2} N^*(j, l, L_1, L_2, L_3)|N - N^*(j, l, L_1, L_2, L_3)|}. \end{aligned} \tag{5.6}$$

Since in this paper we are not interested in studying the limit cases $N \to 0$ or $N \to \infty$, we can continue to bound I' as

$$I' \leq \frac{C(N, L_3)(|k| + |l| + |j|)^3}{\mu \xi_{j,l}}. \tag{5.7}$$

We can now choose $\xi_{j,l}$ as follows: for any $\gamma > 0$ fixed, we take

$$\xi_{j,l} = |j|^{-3-\gamma}|l|^{-3-\gamma}\left(\sum_{j,l \in \mathbb{Z}_\mathcal{M}} |j|^{-3-\gamma}|l|^{-3-\gamma} \right)^{-1}. \tag{5.8}$$

Introducing (5.8) in (5.7), we find that

$$I' \leq C(N, L_3, \gamma) \frac{(|k| + |l| + |j|)^3}{\mu} |j|^{3+\gamma}, \forall N \notin \Theta_3^\mu(L_1, L_2, L_3).$$

We can thus conclude with the following result:

Theorem 5.1. *Let $\mu > 0$ and $\gamma > 0$, then for every domain $\mathcal{M}$ and every Burgers N such that $N \notin \Theta_3^\mu(L_1, L_2, L_3)$ we have*

$$\omega_j^\pm + \omega_l^\pm + \omega_k^\pm \neq 0 \,\forall\, j, l, k \in \mathbb{Z}_\mathcal{M} \text{ with } j + l + k = 0,$$

and

$$\frac{1}{|\omega_j^\pm + \omega_l^\pm + \omega_k^\pm|} \leq \max\left(2, C(N, L_3, \gamma)\frac{(|k| + |l| + |j|)^3}{\mu}|j|^{3+\gamma}|l|^{3+\gamma}\right). \tag{5.9}$$

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