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The Anatomy of a Mathematical Proof: Implications for Analyses with Toulmin's Scheme

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Abstract A model solution to a proof question on an examination is explored and subjected to a detailed analysis in terms of Toulmin's scheme of argumentation. In doing so, the ways in which the scheme has been variously used in the mathematics education and philosophical literature are contrasted. The analysis raises a number of issues concerning the scheme as described by Toulmin and as modified by other authors and suggests ways in which one aspect of the scheme might be reinterpreted. The final analysis of the proof provides insight into what examiners may be expecting from their students in terms of the level of explicit argument they consider essential and the importance played by the focus of the subject matter.

1 The Toulmin scheme and its use in mathematics education

The Toulmin scheme has gained considerable currency in mathematics education research as a way of analysing arguments. It was developed in the 1950s by Stephen Toulmin (Toulmin, 1958), but appears to have come to prominence in mathematics education in the 1990s (Krummheuer, 1995). It has since been used to analyse arguments in primary schools (Evens & Houssart, 2004), secondary schools (Arzarello & Sabena, 2011), undergraduate degrees (Stephan & Rasmussen, 2002) and even arguments from postgraduate mathematicians (Inglis, Mejia-Ramos, & Simpson, 2007).

The scheme was designed to analyse arguments across a range of fields. Toulmin suggests these include arguments delivered in "law courts, professional scientific meetings . . . university seminars . . . engineering design conferences" (Toulmin, Rieke, & Janik, 1984, p. 16) and consists of six components: data (D, the evidence on which the claim is based), qualifier (Q, the degree of confidence in the claim), conclusions (C, the claim which the arguer is putting forward), rebuttal (R, the circumstances in which the claim might not hold), warrant (W, the justification for drawing the conclusion on the basis of the data) and backing (B, "other assurances, without which the warrants themselves would posses[*sic*] neither authority nor currency" (Toulmin, 2003, p. 96).). Toulmin places these items into a scheme (which he calls an "argument pattern") as shown in figure 1.

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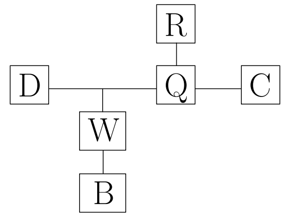

Fig. 1: Toulmin Scheme

While some have criticised the scheme as poorly defined and, in particular, the difficulty in precisely defining some of the these components (Weinstein, 1990), the extent to which it has been used in analysing arguments suggests that it has become a useful tool for researchers.

In undertaking their analyses, many authors adapt Toulmin's scheme to fit their intentions and their data. Inglis et al. (2007) note a particular tendency in mathematics education research using the scheme to omit the qualifier and rebuttal. They suggest that if one has developed what one intends as a finished formal mathematical argument, these may be redundant for the purposes of analysis. However, when one is arguing informally then one would expect to make tentative claims and accept the possibility of the claim being refuted and, therefore, "it is unclear how this omission can be justified in a conceptual framework aimed at the reconstruction of argumentation which may lack logically necessary conclusions" (p. 5)

Other authors make different modifications to the scheme: for example, Langsdorf (2011) includes the possibility of adding a form of backing to the data instead of just to the warrant (seeming to undermine that the argument structure "Given D then C" makes no claim about whether D is the case or whether C would hold in the counterfactual situation in which D was not the case). Prusak, Hershkowitz, and Schwarz (2012) modify the scheme to combine warrant and backing into the single notion of "reason" contending that the nature of their data, gathered from recording peer interactions, makes it difficult to distinguish the two.

The problem of distinguishing aspects of Toumin's argument pattern arises for many authors as they analyse students' discussion. However, Toulmin's scheme was designed to explore the structure of completed *arguments* not the process of *arguing*. He contrasts "trains of reasoning lifted out of their original human contexts and considered apart from them" with "human interactions through which such trains of reasoning are formulated, debated, and/or thrashed out" (Toulmin et al., 1984, pp. 14–15). His scheme was intended only to apply to the former.

Despite this, most literature uses the scheme to analyse the utterances of students in the act of forming arguments (e.g Krummheuer, 1995; Stephan & Rasmussen, 2002; Inglis et al., 2007; Moore-Russo, Conner, & Rugg, 2011). Some even go to considerable lengths to extend and adapt the scheme to fit arguments being formed in the complex situation of a multi-way conversation (Steele, 2005). Some papers do work with completed arguments (normally those which the researchers have developed with some intentional flaw in them) though they still use the scheme for analysing students' statements in the act of arguing rather than for the analysis of the completed argument (e.g. Alcock & Weber, 2005). A small number do analyse decontextualised and completed mathematical arguments, but these appear to be exclusively analyses of well known, formal proofs (e.g. Aberdein, 2005; Pease, Smaill, Colton, & Lee, 2009) and where the analysis is conducted to provide insight into the philosophical underpinnings of mathematical logic and argumentation.

Of course, one could contend that the act of arguing involves the expression of putative arguments, during which much can be understood about students' thinking as their claims, warrants, qualifiers etc. change in response to thought and debate.

The aims of this paper are to undertake an analysis of a completed mathematical argument, explore some aspects of the Toulmin's scheme as described by him and interpreted by other authors, highlight ways in which the scheme supports the analysis of such arguments and ways in which it might be modified to better support such analyses. Finally, the findings of the analysis are briefly outlined in the case of the particular mathematical argument given here, which comes from a model solution to a university mathematics examination question.

2 A problem and a solution

A typical finished (and supposed correct) argument in mathematics is provided by model solutions to proof questions in examinations. While other methods (such as oral assessments) play a significant role in many countries, it is common to evaluate students' understanding of proof through the closed book, written examination, (Iannone & Simpson, 2011). In all these forms of assessment, the production (or reproduction) of a formal proof is an entirely standard feature.

Model solutions to those examination questions can be viewed as communications between examiners and colleagues. In the case we examine, the solution was also available as an exemplar for future students, so might also be seen as a communication between the examiner and future examination candidates about expectations of the nature of a completed argument. Model solutions are meant to form a clear indication of a minimal response which guarantees maximum marks, in the sense that a student handing in an answer which fits the model solution directly *must* be awarded full marks and any omission in an answer (which followed the model solution's form of argument) *could* result in the loss of marks. Of course, the marker might give full marks for alternative arguments, but given the semi-public nature of the model solution, the marker would be obliged to consider a students' response which followed the model solution accurately to be worthy of full credit.

The examination question chosen for analysis is given in figure 2a and is typical of the genre: a definition from the course, a couple of familiar results to prove and a less familiar proof.

Let S_n denote the symmetric group on n letters

(i) Define the *signature function* of a permutation. If $\mathrm{sgn} : S_n \to \{\pm 1\}$ is the signature function, prove that sgn is a group homomorphism.
(ii) Prove the if τ is a transposition, then $\mathrm{sgn}(\tau) = -1$.
(iii) Prove that if $g \in S_n$ has order 15 then $\mathrm{sgn}(g) = -1$. Find permutations a and b of order 14 in some sufficiently large permutation group with $\mathrm{sgn}(a) = +1$ and $\mathrm{sgn}(b) = -1$.
(iv) How many elements of S_5 have signature equal to -1?

(a) The question

The *signature function* on S_n is defined thus: take n independent variables $x_1, x_2, \ldots, x_n$ and set $\Delta = \prod_{i<j}(x_i - x_j)$. Now for $\sigma \in S_n$, we let σ act on these variables by $\sigma(x_i) = x_{\sigma(i)}$. Then $\sigma(\Delta) = \pm\Delta$ and we define $\mathrm{sgn}(\sigma)$ to be the sign.

One checks that for σ and τ permutations, $\sigma\tau(\Delta) = \mathrm{sgn}(\sigma\tau)\Delta$ by definition, and $\sigma\tau(\Delta)$ is also equal to
$\sigma(\tau\Delta) = \sigma(\mathrm{sgn}(\tau)\Delta) = \mathrm{sgn}(\tau)\sigma(\Delta) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta$, and hence $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma\tau)$, meaning that sgn is a group homomorphism.

(b) Model solution to part (i)

Fig. 2: Signature function question and partial solution

In this paper, the focus is on part (i) of the question, and particularly the proof provided in the model solution for the second item in this part. The question and model solution as presented here are intended only as examples typical of the genre, not as ideals and there are certainly simpler ways of presenting the key concepts than that given in the model solution. However, these *are* the questions asked and model solutions provided in a real examination setting.

The model solution for part (i) is given in figure 2b and is reproduced verbatim, including one item which may be regarded as a small error or typographical slip: the examiner uses parentheses to indicate the action of the permutations on the polynomial Δ, but at various points neglects this (writing, for example $\sigma(\tau\Delta)$ when $\sigma(\tau(\Delta))$ would match the usage elsewhere). The layout is also as given by the examiner: a rather compressed form which is not unexpected given that it is intended for colleagues to check and for future students to see a minimal, fully accurate solution.

3 A naïve analysis

It is important to note in analysing proofs contained in model solutions that these will normally be put forward by the examiner as a formal mathematical argument. They therefore fit the criteria Inglis et al. (2007) have for omitting the qualifier and rebuttal from the Toulmin scheme since, at most, the qualifier would be ‘necessarily’ and the examiner is arguing on the basis that there could not be a rebuttal. This paper omits these aspects in the subsequent discussion except when needed.

Consider the part of the examination question addressed here: “If sgn : $S_n \to \{\pm 1\}$ is the signature function, prove that is a group homomorphism.”. It would appear that the data from which students are asked to argue is “sgn : $S_n \to \{\pm 1\}$ is the signature function” and the claim sought is “sgn is a group homomorphism”. A warrant for the claim is surely then the proof contained in the model solution and, if Aberdein’s (2005) analysis is followed, the backing is “classical logic”. The Toulmin scheme for this would appear as in figure 3 (with the proof rewritten in a more readable layout and the parenthesis problem corrected).

There are a number of problems with this naive analysis. Most obviously, the same Toulmin diagram would be used to analyse every proof: the statement of the theorem splitting across the data and conclusions, the proof being the warrant and “classical logic” as backing. This would make the scheme of no practical use in analysing the arguments at the level intended here. However, one could contend that the scheme in figure 3 provides an analysis of the *theorem*, not an analysis of the structure of the argument within the proof, as intended.

4 About Warrants - an alternative analysis

An alternative analysis comes from a suggestion of Weber and Alcock (2005). They argue that one might conduct a line-by-line study of a proof, seeking where warrants are required to draw any conclusions stated in the line and identifying what those might be (since, in many cases including the proof analysed here, warrants are often implicit).

Figure 4 highlights each conclusion drawn which appears to need a warrant and indicates a suggested warrant for each of these conclusions.

This analysis exposes some interesting items. One is that there is one unstated conclusion drawn. C7 states $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma\tau)$, but the warrant it draws on involves the polynomial Δ. To obtain a full set of conclusions and warrants C7/W7 needs to be replaced with something like:

C7a: $\mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta = \mathrm{sgn}(\sigma\tau)\Delta$.

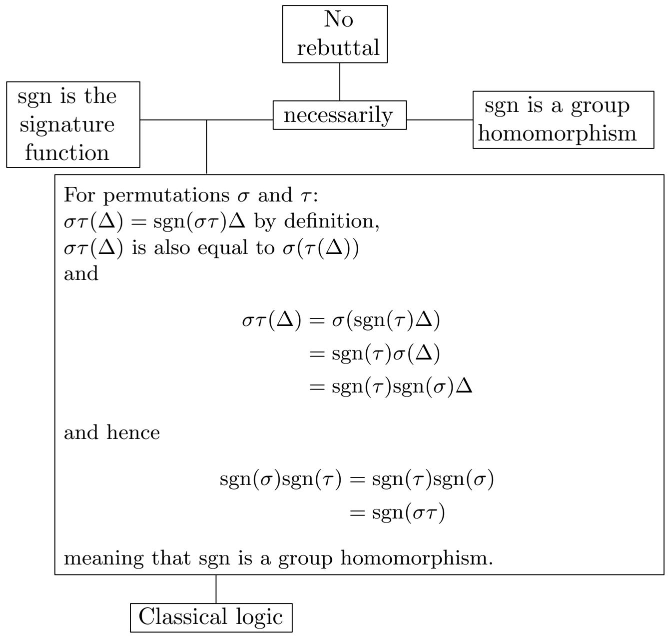

Fig. 3: A naïve analysis

For permutations σ and τ:

C1 $\boxed{\sigma\tau(\Delta) = \mathrm{sgn}(\sigma\tau)\Delta}$ by definition,

C2 $\boxed{\sigma\tau(\Delta) \text{ is also equal to } \sigma(\tau(\Delta))}$

and

$$\begin{aligned}{}^{\text{C3}}\boxed{\sigma\tau(\Delta) = \sigma(\mathrm{sgn}(\tau)\Delta)}&\\ = \boxed{\mathrm{sgn}(\tau)\sigma(\Delta)}&^{\text{C4}}\\ = \boxed{\mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta}&^{\text{C5}}\end{aligned}$$

and hence

$$\begin{aligned}\boxed{\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)}&^{\text{C6}}\\ = \boxed{\mathrm{sgn}(\sigma\tau)}&^{\text{C7}}\end{aligned}$$

meaning that $\boxed{\text{sgn is a group homomorphism}}$.C8

W1: The definition of sgn.
W2: σ and τ permute variables in Δ and so for each $i \in \{1, \ldots, n\}$, $(\sigma\tau)(X_i) = X_{\sigma\tau(i)} = X_{\sigma(\tau(i))} = \sigma(\tau(X_i))$ from the rules about composition of functions.

W3: The definition of sgn.
W4: σ permutes variables in the polynomial Δ, which has no effect on the multiplier $\mathrm{sgn}(\tau)$
W5: The definition of sgn.

W6: Commutativity of multiplication
W7: $\sigma\tau(\Delta) = \mathrm{sgn}(\sigma\tau)\Delta$ and $\sigma\tau(\Delta) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta$ and $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)$

W8: $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma\tau)$.

Fig. 4: The conclusions (C) in the proof with associated suggested warrants (W)

W7a: $\sigma\tau(\Delta) = \text{sgn}(\sigma\tau)\Delta$ and $\sigma\tau(\Delta) = \text{sgn}(\tau)\text{sgn}(\sigma)\Delta$

C7b: $\text{sgn}(\tau)\text{sgn}(\sigma) = \text{sgn}(\sigma\tau)$.
W7b: Division by the (non-zero) polynomial Δ and $\text{sgn}(\sigma)\text{sgn}(\tau) = \text{sgn}(\tau)\text{sgn}(\sigma)$

This shows that not only may many warrants be implicit in the articulation of a proof, but some conclusions may be as well (an issue discussed later).

The analysis also shows, however, that the exclusive focus on warrants suggested by Weber and Alcock is inadequate for our purposes. Doing only this can make pieces of the argument appear to be independent from each other. Being able to explain why each conclusion in a proof has a warrant is a necessary, but not sufficient condition for understanding the proof. One needs also to explain how those lines combine and interdepend. In some cases, this is implicit in the pairs of claims and warrants above. For example, to conclude that for permutations σ and τ and polynomial Δ, (C7a) $\text{sgn}(\tau)\text{sgn}(\sigma)\Delta = \text{sgn}(\sigma\tau)\Delta$ one relies on the warrants established in C1 and C5.

Aberdein (2005) suggests that the structure of an argument might be revealed more clearly by chaining together data-conclusion pairs in a proof. For example, the scheme in figure 5 (redrawn from Aberdein, 2005 p. 293) shows two steps in the proof that an irrational raised to an irrational power can be rational. In this, the *conclusion* of one step becomes the *data* of the next. Such a proposal is also implicit in the analysis by Hoyles and Küchemann (2002) who use chains of Toulmin schemes to distinguish types of arguments pupils gave to a task about the equivalency of statements and Krummheuer (1995) also gives examples of chained and combined arguments.

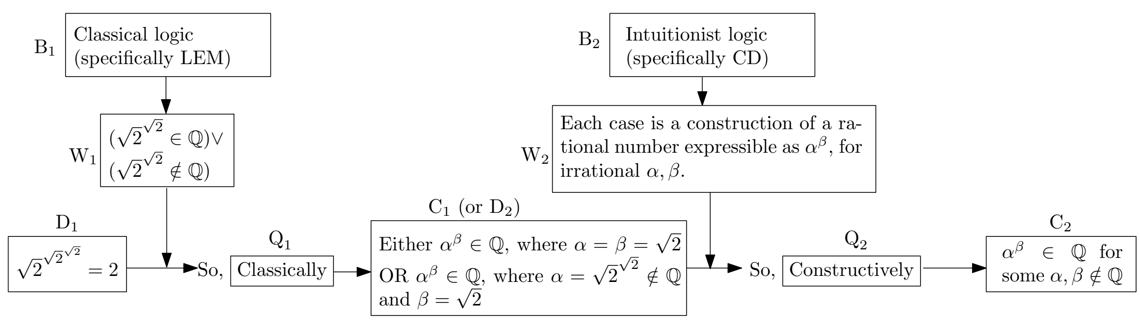

Fig. 5: A chain of sub-arguments

Chaining alone, however, will not suffice for the analysis of the proof here. In the example above, (C7a) $\text{sgn}(\tau)\text{sgn}(\sigma)\Delta = \text{sgn}(\sigma\tau)\Delta$ follows from two conclusions, not just one. In this case (C7a) requires both (C1) $\sigma\tau(\Delta) = \text{sgn}(\sigma\tau)\Delta$ and (C5) $\sigma\tau(\Delta) = \text{sgn}(\tau)\text{sgn}(\sigma)\Delta$. This suggests that this section of the argument should be structured as in figure 6.

Comparing the approach taken by Aberdein and that taken by Weber and Alcock even for this small part of the argument highlights one further issue. In the former, (C1) $\sigma\tau(\Delta) = \text{sgn}(\sigma\tau)\Delta$ is both a conclusion and *data* for a later conclusion while in the latter it is both a conclusion and a *warrant* for a conclusion. This suggests there are, at least, two possible analyses of the argument in terms of the Toulmin scheme one as in figure 6 which accords with the approach taken by Aberdein and another in figure 7 which is suggested by trying to add structure to the warrants found using the approach of Weber and Alcock.

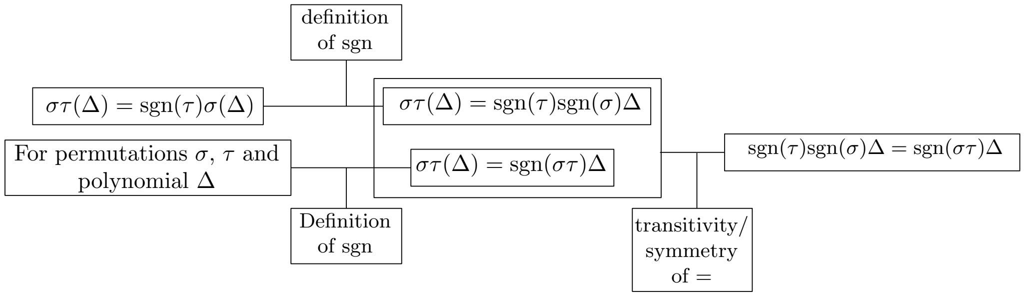

Fig. 6: Combining sub-arguments as in Aberdein

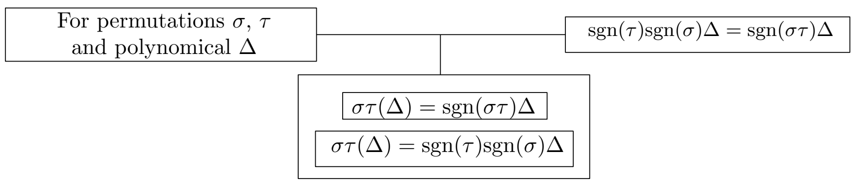

Fig. 7: Combining sub-arguments as in Weber and Alcock

Both of these seem appropriate ways of structuring the argument. One could say "I know that $\sigma\tau(\Delta) = \mathrm{sgn}(\sigma\tau)\Delta$ and $\sigma\tau(\Delta) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta$, so I can conclude $\mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta = \mathrm{sgn}(\sigma\tau)\Delta$ because of the properties of equality" (as the right hand portion of figure 6 might be interpreted) or one could say "Given permutations σ, τ and polynomial Δ, I can conclude $\mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta = \mathrm{sgn}(\sigma\tau)\Delta$ because $\sigma\tau(\Delta) = \mathrm{sgn}(\sigma\tau)\Delta$ and $\sigma\tau(\Delta) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\Delta$" (as figure 7 might be interpreted).

This seems to suggest, first, that an analysis of an argument using Toulmin's scheme does not result in a unique structure. That is, a single written proof (as here) might be interpreted in such a way as to produce quite different Toulmin diagrams. Second, it suggests that a statement's status within an argument as data, conclusion, warrant etc. is not always well defined, supporting the contention of Weinstein (1990). Of course, this latter point was clear from both Aberdein (2005) and Hoyles and Küchemann (2002) who allow the same statement to be both data and conclusion; but in this case, the analysis suggests it might also be a warrant. Given that both approaches seem valid, the remaining analysis here generally chains data and conclusion rather than nesting warrants.

There is a further problem with structuring the argument in the proof using the method suggested by Aberdein (2005). The analysis in figure 3 indicates the chain starting with the data "sgn is the signature function", run through a chain of conclusions (which are also data for the following sub-argument) each with a warrant until the final conclusion "sgn is a group homomorphism".

Figure 8 shows the full argument in chain form (which has been reshaped purely to allow it to fit the page). The dotted region in the top left highlights the first part of the second major sub-argument.

It was suggested earlier that the individual arguments identified by following Weber and Alcock (2005) needed to be joined if the structure of the proof is to be revealed. Conversely, the structure produced from such an analysis needs to be able to be split into independent arguments. That is, each data/warrant/conclusion link in the chain must read as potentially an independent argument. In many cases with figure 8, this is the case: for example, the second part of the second major

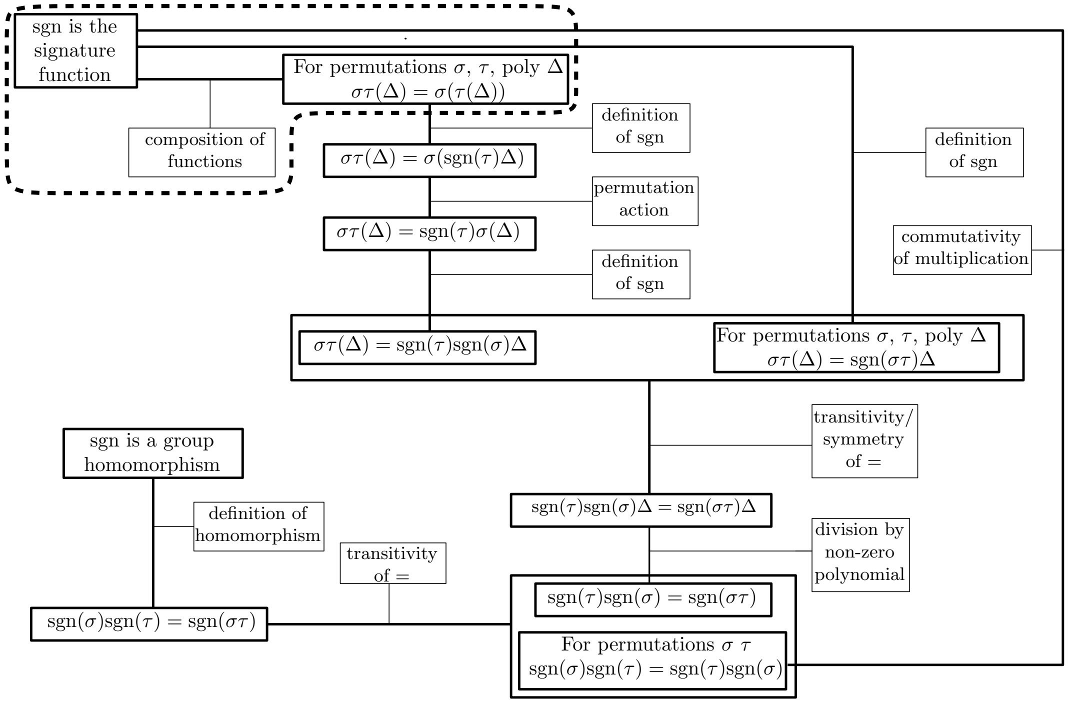

Fig. 8: A proposed chained analysis

sub-argument can be read as "for permutations σ,τ and polynomial Δ, if $\sigma\tau(\Delta) = \sigma(\tau(\Delta))$ we can conclude $\sigma\tau(\Delta)) = \sigma(\mathrm{sgn}(\tau)\Delta)$ from the definition of sgn". This works as an independent argument. The same is true for the first sub-argument: "If sgn is the signature function we can conclude that for permutations σ,τ and polynomial Δ, $\sigma\tau(\Delta) = \mathrm{sgn}(\sigma\tau)\Delta$ from the definition of sgn".

However, the highlighted portion which is the first part of the second sub-argument would read "If sgn is the signature function, we can conclude that for permutations σ,τ and polynomial Δ, $\sigma\tau(\Delta) = \sigma(\tau(\Delta))$ from the rule about composition of function [in W2 above]". This does not act as an independent argument as the conclusions do not depend upon the data.

So, though it is noted later that chaining arguments together is indeed crucial in the analysis of the proof, it is not sufficient.

Returning to the proof, one can see that it has a very simple proof framework (in the sense of Selden & Selden, 1995): the majority of the proof is intend to establish W8: $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma\tau)$ which shows that sgn is a group homomorphism. This would suggest that the outermost part of the analysis of the proof should be structured as in figure 9.

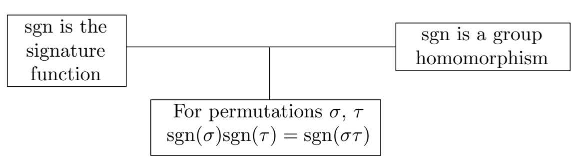

Fig. 9: The outer structure of the proof

However, this leaves unanswered where the remaining argument should go. This brings us to focus on a part of the Toulmin scheme which has been downplayed thus far: the backing.

5 About Backing

Toulmin descibes the backing of a warrant as "other assurances, without which the warrants themselves would posses[*sic*] neither authority nor currency" (Toulmin, 2003, p. 96) and they are variously described in the literature as "categorical statements of fact supporting warrants"(Evens & Houssart, 2004, p. 270) or statements which "explain why the warrant has authority" (Stephan & Rasmussen, 2002, p. 462) and provide "further evidence" (Inglis et al., 2007, p. 4). However, it appears that backing is used in different ways by these different authors and, indeed, by Toulmin.

Consider figure 10 which contain two arguments from Toulmin's work (the first redrawn verbatim from (Toulmin et al., 1984, p. 126), the second from (Toulmin, 2003, p. 97) in which, note, Toulmin included no rebuttal). The relationship between "Given the axioms, postulates and definitions of Euclidean geometry" and "Any regular convex solid has equilateral plane figures as its faces, and the angles at any vertex will add up to less than 360°" appears at face value to be very different from the relationship between"the proportion of Roman Catholic Swedes is less than 2%" and "A Swede can be taken to be almost certainly not a Roman Catholic".

Leaving aside the fact that Toulmin omits a condition in the warrant of the first argument (that the figures must be congruent), the backing appears to determine the field in which the argument takes place, it does not tell on the correctness of the warrant. On the other hand, the second warrant does explain why one would be correct to take a Swede to be almost certainly not a Roman Catholic. It is no wonder that Castaneda (1960) notes the lack of clarity about backing: suggesting Toumin "leaves the relationship between [backing] and [warrant] in the dark" (p. 284).

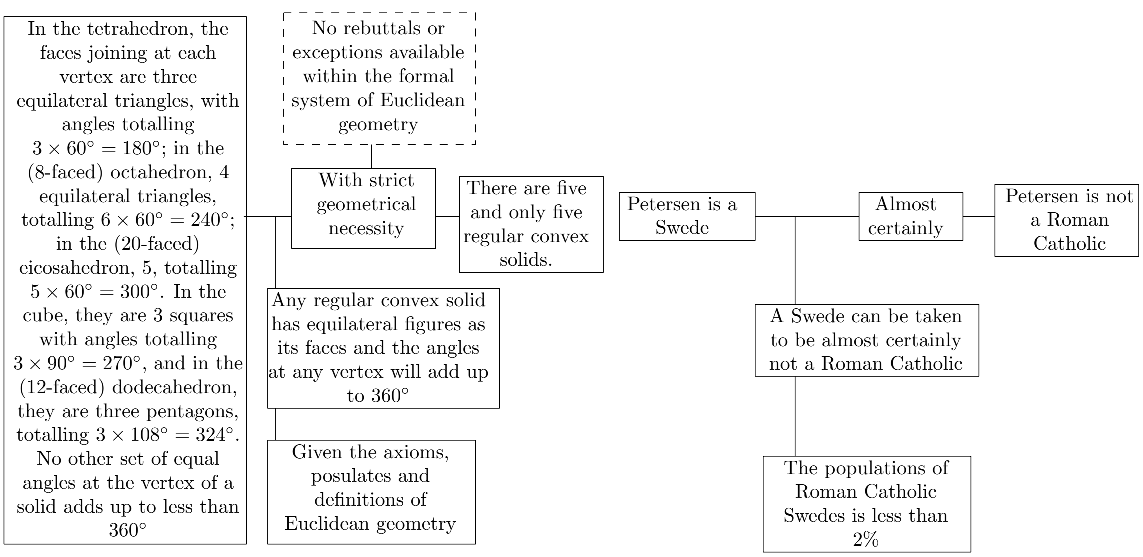

Fig. 10: Toulmin's use of backings

Consider a further three arguments in figure 11 (adapted and redrawn from information in Evens and Houssart (2004, p. 276), Stephan and Rasmussen (2002, p. 465) and Inglis et al. (2007, p. 15) and showing only the quartet data/conclusion/warrant/backing).

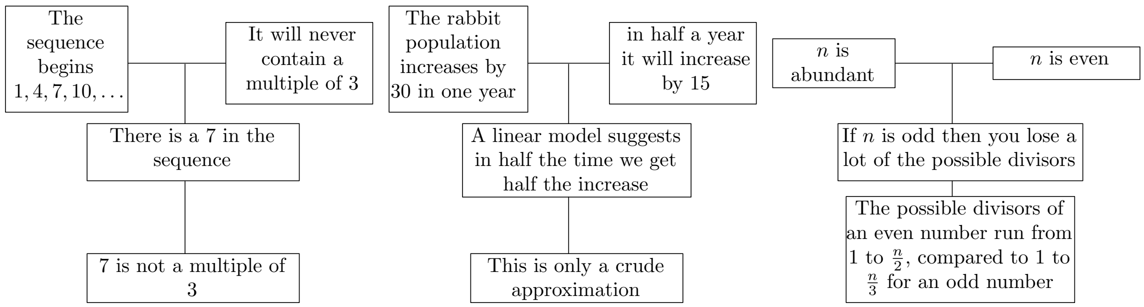

Fig. 11: Three uses of backing

For the first, Evens and Houssart (2004) give an example of pupils arguing that the sequence $(1, 4, 7, 10, \ldots)$ never contains a multiple of 3 with the warrant that it contains 7 and the backing that 7 is not a multiple of 3. While the argument is not one mathematicians would accept, it appears as if the intention is to explain why the warrant applies to an argument about multiples of 3.

The second comes from a paper in which Stephan and Rasmussen (2002) explore arguments constructed in a course on differential equations. In most cases they note that no backings are explicitly identifiable in their data and so do not include a detailed discussion of backings in their analysis. However, they do give this interesting example of a student and instructor constructing an argument about rate of change. Their warrant involves the use of a linear equation (that they knew was a poor model) and is accompanied by statements such as “this is a crude approximation”. One could argue that this statement is backing in support of their warrant: they may be arguing for the acceptance (temporarily) of the warrant (which would be unacceptable in most cases) as their purpose at that stage was to draw a conclusion which is a “crude approximation”.

In the third case, Inglis et al. (2007), investigating postgraduate mathematicians arguing about perfect, abundant and deficient numbers, describe a participant arguing that if n is abundant then n is even, using the warrant that an odd n will have too few divisors and the backing being that the range from which divisors can come for odd numbers is smaller than that for even numbers: the backing appears to play the role of explaining why the warrant might be considered correct.

These three examples suggest three different roles for backing in an argument; or, alternatively, three different types of backing.

5.1 Type 1: Backing for the warrant’s validity

Another way of thinking about backings is as ways to “provide support for warrants …[which] …become relevant when a warrant is challenged” (Verheij, 2005, p. 358). There appear to be three ways in which one may challenge a warrant. The first is that it does not apply: it is not relevant or is invalid. For example, suppose someone argued according to the scheme in figure 12. The statement in the warrant here is true and the form of the argument that is being used is appropriate for mathematics, but it does not allow one to validly conclude that permutations commute under sgn, since the warrant does not involve commutativity of composition of permutations.

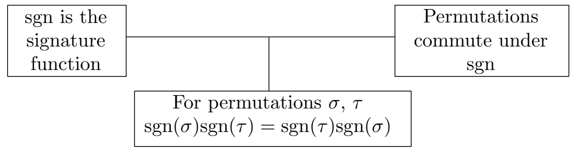

Fig. 12: An invalid warrant

In the first argument outlined in figure 11, from Evens and Houssart (2004), one can see that the pupils' backing is an attempt to argue for the relevance of their warrant. They are not trying to explain why it is correct to state that 7 is in the sequence, nor explain that the form of argument they are using is appropriate to a mathematics classroom, they are trying to show why it is relevant to the question of divisibility by 3.

An example of a similar form of backing is given by Arzarello and Sabena (2011) where a pupil makes a claim to classify three graphs as a function, its derivative and its antiderivatives. She supports her claim with various observations (such as "where [the] green [graph] has a stationary point, [the] blue [graph] has zeroes" p. 201) and backs this with a table of calculus theorems which pertain (such as the derivative at a stationary point is zero). She is not justifying the form of her argumentation nor explaining the reason her statement is correct. Instead, the theorems support the warrant by explaining why the link between stationary points in one graph and zeroes in another is relevant to the relationship between functions and their derivatives.

In the outer structure of our proof analysis (figure 9), the reason why the warrant that for all permutations σ,τ, $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma\tau)$ shows that sgn is a group homomorphism is that it shows sgn fits the definition of a group homomorphism. This form of backing will be called *backing for the warrant's validity*.

5.2 Type 2: Backing for the warrant's field

A second way of challenging a warrant is to contend that the form of argumentation is not acceptable in the field. For example, suppose someone argued as in figure 13. The warrant here is true and is relevant (at least in the sense that it is an example of permutations which commute under sgn) but it is not acceptable in mathematics to argue that a single example demonstrates a general rule.

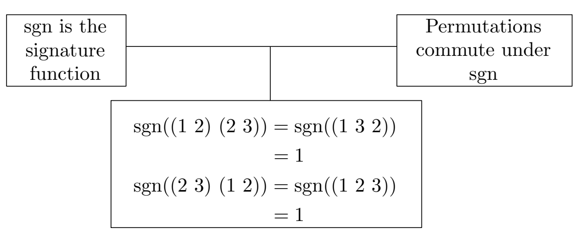

Fig. 13: An warrant from the wrong argumentation field

Equally, one could challenge the warrant in first argument outlined in figure 11, from Evens and Houssart (2004), this way: a single example of an element in the sequence not being divisible by 3 is not an appropriate form of arguing in mathematics that all elements have this property. One might instead imagine a pupil who gave a warrant like "$7 = 2 \times 3 + 1$" and backed it with a statement like "they're all like this"; this backing, one might contend, indicates a generic argument which may be acceptable in that classroom (though may not be acceptable in others).

In the second argument in figure 11, from Stephan and Rasmussen (2002), the student and instructor may be warding off a challenge to their warrant by suggesting that the kind of argument they are producing is a "crude approximation" which should be accepted in these circumstances. That is, the argument is one which, temporarily at least, is from an appropriate field.

Similarly, in the argument put forward by Aberdein in figure 5, since his intention is to highlight the forms of argument which are acceptable or not within intuitionistic logic (or, as he puts it, "make the guilty steps explicit", p. 292), his backings highlight the logical field in which the warrants are acceptable.

In the analysis of our proof, the conclusion follows from the data because of the warrant as the argument is a deduction. This form of backing will be called *backing for the warrant's field*.

5.3 Type 3: Backing for the warrant's correctness

The third way in which a warrant may be challenged would be to contend that it is incorrect, so backing might provide support for a warrant in showing that the warrant itself is correct. For example, suppose someone argued according to the scheme in figure 14. The warrant here is relevant (in the sense that, if it was the case, it would justify drawing the conclusion) and the form of the argument attempted is appropriate for mathematics (again, being a deduction). However, the statement in the warrant is not true.

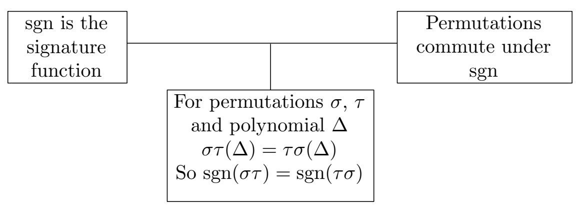

Fig. 14: An incorrect warrant

In the third argument outlined in figure 11, from Inglis et al. (2007), the mathematician is not explaining the form of argumentation they are making, nor are they explaining why it is relevant to drawing the conclusion. What they appear to be doing is explaining why it is that odd numbers "lose a lot of divisors". That is, why the warrant may be the case.

Similarly in Toulmin's second argument in figure 10 the statement "the proportion of Roman Catholic Swedes is less than 2%" explains why it is the case that "a Swede can be taken to be almost certainly not a Roman Catholic".

Forman, Larreamendy-Joerns, Stein, and Brown (1998) describes pupils counting the number of 1cm^2 grid squares covering an irregular plane figure, who were asked to give a result in square millimetres. They interpret one pupil's answer in terms of Toulmin's categories with the data as "17 grid squares", the conclusion as "170mm", the warrant as "conversion: 17×10" and backing that "for every one centimetre, there is 10 millimetres" (p. 537). One could argue that the backing is an attempt to explain why the conversion factor should be multiplication by 10. That is the backing is an attempt to show that the warrant is correct.

In our analysis of the proof, the reason that it is the case that for all permuations σ,τ, $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma\tau)$ is the remainder of the proof. This form of backing will be called *backing* *for the warrant's correctness.*

These three forms of backing support warrants in three different ways: the first ensures that it applies to this situation, the second that it is an acceptable form of argument in the situation and the third that it is the case.

Weber and Alcock (2005) claim that "If the warrant of an argumentation is plausible, but not socially agreed upon by the mathematical community, backing for this warrant is required and the proof is said to have a 'gap' in it." (p. 37). This seems to conflate two forms of backing: gaps in proofs should be filled by showing that the warrant used to argue across the gap is correct while the reliance on a warrant that is not socially agreed on by mathematicians requires a backing about the form or field of argument used (just as the student and instructor in Stephan and Rasmussen (2002) backed an argument as a "crude approximation").

Cobb (2002) gives an interesting example which might be interpreted as a confusion between backing types. Pupils were undertaking an analysis of the life of different brands of battery given some data on the time twenty named-brand batteries had lasted. One pupil gave a warrant for her choice which depended on how many of the ten longest lasting batteries were from a given brand. When challenged on the argument, the backing that "half of 20 is 10, so that's how I chose it" (p. 195) as given.Cobb notes "Crucially, this backing did not refer to the issue under investigation, that of deciding which of the two sets of batteries lasted longer" (p. 195). One could suggest that the pupil was backing the correctness of the argument while the challenge was to its validity. Confusion between types of backing may account for many classroom misunderstandings.

However, when one analyses mathematicians' proofs, as here, it is likely that the warrant field is determined: such arguments should be deductive and it is only when one wishes to delve in to the particular form of logic used (as Aberdein, 2005) that the backing for the warrant field would be interesting. However, in analysing the arguments used amongst students and in classrooms, all three types of backing are of interest - the forms of argument permitted in a classroom are clearly a part of the classroom's sociomathematical norms (in the sense of Yackel & Cobb, 1996).

6 Final analysis of the proof

In terms, then, of the Toulmin scheme with the added distinction of these three types of backing, the final analysis of the proof is given in figure 15.

The warrant is valid as it is an instance of the definition of a group homomorphism, it is from the appropriate field in that it is a deduction and it is correct, which follows from the rest of the proof. Of course, in this final part, the remainder of the proof contains steps which need warrants of their own and backings (potentially of all three types) of their own. This is not, however, an infinite regress of backings supporting warrants' correctness. Toulmin notes that, in some cases, backings (interpreted as backings for the correctness of warrants) are axioms.

Even without going down further levels of that (finite) regress, the analysis in figure 15 provides some useful insight. Most noticeable is that only a small number of the warrants are explicit. In all there are ten different places in which the analysis notes warrants. In the proof given by the examiner, only two are made explicit. The warrant for $\sigma\tau(\Delta) = \mathrm{sgn}(\sigma\tau)\Delta$ is directly referred to ("by definition") and, while split by other expressions, the overarching warrant "for permutations σ and τ, ..., $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \ \ldots\ \mathrm{sgn}(\sigma\tau)$" is also clearly expressed. Everywhere else the reader must infer the warrants.

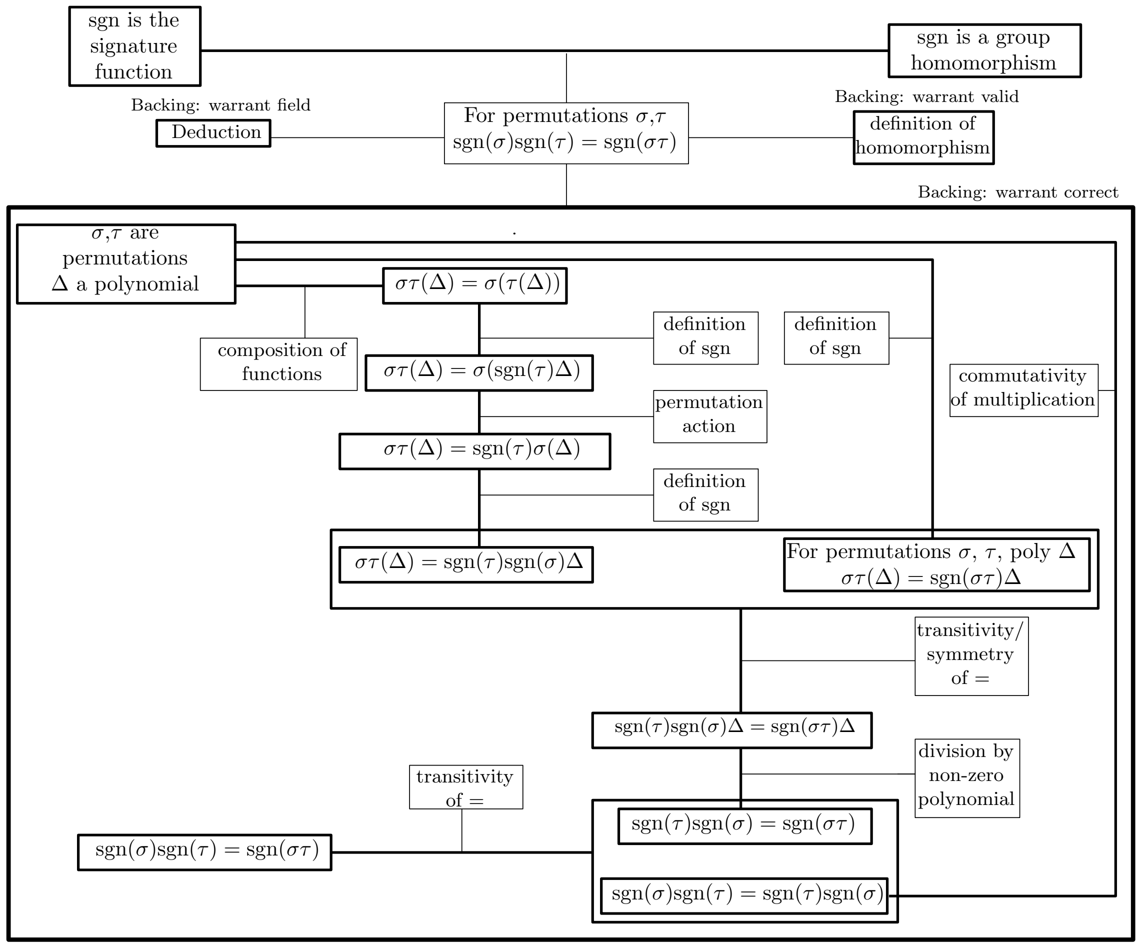

Fig. 15: A final analysis

Given that a model solution is an indication of the minimal response which can obtain maximum marks, it suggests that examiners are not necessarily looking for students to give much justification for their steps below the main warrant required to draw the conclusion.

Moreover, as noted earlier, one piece of the argument — the division by the (non-zero) polynomial Δ — is only implicit in the examiner's model solution and yet other, arguably simpler steps are made explicit. For example, the examiner could have written " and hence $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma\tau)$ ..." in the penultimate line of the proof (as shown in figure 3), leaving the claim based on the commutativity of multiplication implicit and yet the claim $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)$ is made explicit (even if the warrant itself is not). One might ask why it is acceptable to omit the step involving division of a polynomial as implicit but important to have commutativity of multiplication explicit. One interpretation is that, as Hazzan (1999) notes, commutativity is an issue that may only become explicitly a focus of attention when encountering abstract algebra for the first time, so it is important that students can demonstrate that they know when it does and does not apply, while division by a (non-zero) polynomial is more peripheral to the study of group theory.

7 Summary and Conclusions

The analysis of this model solution in terms of the expanded Toulmin scheme does, then, provide some insight into the argumentation which examiners expect of their students. Given the widespread use of Toulmin's scheme in analysing arguments and, predominantly, in analysing the act of arguing in classrooms it is important to note how the scheme's components function in such analyses and how aspects can become conflated.

The Toulmin scheme has been used extensively in the literature, but not always in the ways Toulmin intended. However, mathematics education researchers have tended to find as much value in using it to analyse *arguing* as philosophers have in analysing classical *arguments*. In studying proofs such as model examination solutions, the naïve interpretation of the Toulmin scheme has little value in itself and, while the suggestion of Weber and Alcock (2005) to look at warrants needed at each line of the proof expands considerably on this, it does not focus the analysis of the *structure* of a proof. Combining this with chaining and other combinations of partial arguments adds more, but a fuller analysis is not available until one looks carefully at the notion of backing.

Existing literature has used backing in what one might now contend are three different ways: the first ensures the warrant applies to the situation, the second that it is an acceptable form of argument in the context and the third that it is actually correct. The form of argument for a mathematician writing something they call a proof is likely to be fixed (except for situations in which one may be interested in the precise axiom system in which the argument is positioned). However, in the classroom, the form of argument is an important aspect of the sociomathematical norms, and teachers and pupils may have to cope with shifting norms where some arguments may be allowed to be generic or heuristic but others must be formal. One role of teachers and the wider mathematical educational system may be to clarify the acceptable forms of argument in mathematics, such as justifications for new inferences being based on old ones (Fukawa-Connelly, 2012).

In addition to the difficulty of enculturation into those sociomathematical norms, further confusion can occur in the classroom when there is a lack of clarity about which of the different forms of backing being sought: recall the reinterpretation of the interesting example given by Cobb (2002). A teacher may be asking a pupil to explain why their warrant applies to the situation, but the pupil may defend themselves by giving evidence that their warrant is correct. This need not mean that a student is not capable of giving an appropriate form of backing for the validity of their warrant, just that they took the enquiry to be a challenge to its correctness.

The more detailed analysis permitted by the expanded Toulmin scheme allows us to focus on what examiners value: the highest level of warrant appears to be essential, as does providing at least some warrant for major steps, but other justifications for proof steps are not as highly valued. Calculation steps which are directly in the focus of the module (for example, those resting on commutativity) must be made clear, but those which are more peripheral (for example, those resting on division by a non-zero polynomial) need not. One could easily imagine a different course in which a model solution would not make explicit the commutativity of multiplication, but required that students were entirely explicit about steps involving the division by a polynomial (and were clear that the polynomial was not zero).

This expanded scheme thus helps provide insight into the sub-fields of argumentation, even within the overall field of arguing 'mathematically'.

This leads to one final point about the proof explored in this paper.

At first sight, the complexity of the structure revealed by the full analysis of the argument may make one wonder about the rationale for the relatively small number of marks available for it: the mark scheme indicates that, with the definition, students could attain a maximum of five marks from a total of twenty for the question, which was one of four equally weighted questions which had to be completed in two hours. However, if one looks at the steps, most of them are standard steps which one would expect to be used repeatedly in mathematical arguments: use of definitions, rules of arithmetic etc. The only parts of the analysis which arguably include concepts from the module are the definition of the signature function itself, the definition of homomorphism and an understanding of how a permutation acts on a polynomial.

That said, the analysis also shows the number of different lower level ideas which need to be meshed with these main concepts into an intricate structure to enable the proof to be constructed. For students to be successful on even the very small part of their mathematics assessment presented here, they need not only the concepts of the course, but also a fluent grasp of how to co-ordinate them with multiple basic proof steps.

It is perhaps unsurprising, then, that so many seem to resort to memorisation.

References

Aberdein, A. (2005). The uses of argument in mathematics. *Argumentation*, *19*(3), 287–301.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior*, *24*(2), 125–134.

Arzarello, F., & Sabena, C. (2011). Semiotic and theoretic control in argumentation and proof activities. *Educational Studies in Mathematics*, *77*(2-3), 189–206.

Castaneda, H. N. (1960). On a proposed revolution in logic. *Philosophy of Science*, *27*(3), 279–292.

Cobb, P. (2002). Reasoning with tools and inscriptions. *Journal of the Learning Sciences*, *11*(2-3), 187–215.

Evens, H., & Houssart, J. (2004). Categorizing pupils' written answers to a mathematics test question: 'I know but I can't explain'. *Educational Research*, *46*(3), 269–282.

Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and Instruction*, *8*(6), 527–548.

Fukawa-Connelly, T. (2012). Classroom sociomathematical norms for proof presentation in undergraduate in abstract algebra. *The Journal of Mathematical Behavior*, *31*(3), 401–416.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, *11*(1), 38–42.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics*, *40*(1), 71–90.

Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, *51*(3), 193–223.

Iannone, P., & Simpson, A. (2011). The summative assessment diet: how we assess in mathematics degrees. *Teaching Mathematics and its Applications*, *30*(4), 186–196.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*(1), 3–21.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning, interactions in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.

Langsdorf, L. (2011). Argumentation as contextual logic: an appreciation of backing in Toulmin's model. *Cogency: Journal of Reasoning and Argumentation*, *3*(1), 51–78.

Moore-Russo, D., Conner, A., & Rugg, K. I. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics*, *76*(1), 3–21.

Pease, A., Smaill, A., Colton, S., & Lee, J. (2009). Bridging the gap between argumentation theory and the philosophy of mathematics. *Foundations of Science*, *14*(1-2), 111–135.

Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics*, *79*(1), 19–40.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*(2), 123–151.

Steele, M. D. (2005). Comparing knowledge bases and reasoning structures in discussions of mathematics and pedagogy. *Journal of Mathematics Teacher Education*, *8*(4), 291–328.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*, *21*(4), 459–490.

Toulmin, S. E. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

Toulmin, S. E. (2003). *The uses of argument (second edition)*. Cambridge: Cambridge University Press.

Toulmin, S. E., Rieke, R. D., & Janik, A. (1984). *An introduction to reasoning (second edition)*. New York: Macmillan.

Verheij, B. (2005). Evaluating arguments based on Toulmin's scheme. *Argumentation*, *19*(3), 347–371.

Weber, K., & Alcock, L. (2005). Using warranted implications to understand and validate proofs. *For the Learning of Mathematics*, *25*(1), 34–51.

Weinstein, M. (1990). Towards an account of argumentation in science. *Argumentation*, *4*(3), 269–298.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477.