

INTERSECTION HOMOLOGY OF LINKAGE SPACES

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ABSTRACT. We consider the moduli spaces $\mathcal{M}_d(\ell)$ of a closed linkage with n links and prescribed lengths $\ell \in \mathbb{R}^n$ in d -dimensional Euclidean space. For $d > 3$ these spaces are no longer manifolds generically, but they have the structure of a pseudomanifold.

We use intersection homology to assign a ring to these spaces that can be used to distinguish the homeomorphism types of $\mathcal{M}_d(\ell)$ for a large class of length vectors in the case of d even. This result is a high-dimensional analogue of the Walker conjecture which was proven by Farber, Hausmann and the author.

1. INTRODUCTION

Configuration spaces of closed linkages in Euclidean space modulo isometry group have occurred in many contexts in recent years. Planar linkages can easily be visualised and the topology of the resulting moduli spaces are now well understood, culminating in the proof of the Walker conjecture by Farber, Hausmann and the author in [3, 20]. Roughly this conjecture states that the cohomology of the linkage space detects the length vector of the linkage. By a *length vector* we simply mean an element $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{R}^n$ such that all entries are positive. The i -th entry ℓ_i describes the length of the i -th link.

For linkages in 3-dimensional Euclidean space, the resulting moduli spaces occur naturally in algebraic geometry and symplectic geometry, see e.g. [17, 14], and the cohomology rings have been calculated in Hausmann and Knutson [11]. Using this description of cohomology, the analogue of the Walker conjecture was proven in [3], with the single exception that for $n = 4$ there exist two different length vectors whose moduli spaces are both the 2-sphere.

Much less is known for linkages in higher-dimensional Euclidean spaces. For linkages in 5-dimensional space, Foth and Lozano obtained analogous results of Kapovich and Millson [14] in a quaternion setting rather than a complex one. Kamiyama [13] obtained an Euler characteristic formula for equilateral linkages in 4-dimensional space, and more recently homology calculations were obtained in [21].

The moduli space we are interested in is

$$\mathcal{M}_d(\ell) = \left\{ (x_1, \dots, x_n) \in (S^{d-1})^n \mid \sum_{i=1}^n \ell_i x_i = 0 \right\} / \mathrm{SO}(d)$$

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where $\mathrm{SO}(d)$ acts diagonally on the product of spheres. In particular we want to know how the topology of $\mathcal{M}_d(\ell)$ depends on the length vector ℓ . Permuting the coordinates of ℓ does not change the topology as we can simply permute the coordinates of $\mathcal{M}_d(\ell)$. It also turns out that small changes of ℓ do not change the topology, provided that ℓ does not admit a *collinear* configuration, that is, if $\mathcal{M}_1(\ell) = \emptyset$. If this is the case, we call the length vector *generic*.

Indeed, the non-generic length vectors are the boundaries of so-called chambers, connected open subsets of \mathbb{R}^n such that any two length vectors in the same chamber admit homeomorphic moduli spaces.

In general, if two length vectors ℓ, ℓ' are in different chambers, even after permuting coordinates, it does not necessarily follow that $\mathcal{M}_d(\ell)$ and $\mathcal{M}_d(\ell')$ are not homeomorphic. In fact, Schoenberg [19] showed that for $d = n$ the moduli space $\mathcal{M}_d(\ell)$ is either a topological disc or empty, from which it can be seen that for $d = n - 1$ the moduli spaces are empty or topologically a sphere. The case where the moduli space is empty is represented by the chamber where one coordinate ℓ_j is bigger than the sum of all other ℓ_i , so in all other cases we always get the same moduli space. Notice that the case $d > n$ reduces to the case $d = n$, as the extra dimensions in \mathbb{R}^d cannot be taken advantage of by linear dependence of the x_1, \dots, x_n .

In the case $d < n - 1$ the topology of the moduli space does depend on the chamber, as can be seen from the homology calculations in [21]. The main result of this paper shows that for a large class of length vectors the topology of the moduli space does recover the chamber of the length vector.

Theorem 1.1. *Let $d \geq 4$ be even, $\ell, \ell' \in \mathbb{R}^n$ be generic, d -normal length vectors. If $\mathcal{M}_d(\ell)$ and $\mathcal{M}_d(\ell')$ are homeomorphic, then ℓ and ℓ' are in the same chamber up to a permutation.*

The notion of d -normal is defined in Section 2, in view of Schoenberg's result it should be pointed out that for $n = d + 1$ there are exactly two chambers up to permutation which contain d -normal length vectors, one with empty moduli space and one where the moduli space is a sphere. If n is large compared to d , d -normality is more common, and we would expect the ratio of all d -normal length vectors in \mathbb{R}^n by all length vectors in \mathbb{R}^n to converge to 1.

The statement of the theorem is known to be true for $d = 2$, as it follows from the proof of the Walker conjecture in [3, 20], and for $d = 3$, as was shown in [3]. Every generic length vector is 2-normal, and there is only one chamber up to permutation so that its length vectors are not 3-normal.

Homology calculations are not enough to obtain Theorem 1.1, and in fact the cases $d = 2, 3$ were obtained using cohomology. If we do not form the quotient by $\mathrm{SO}(d)$ and look instead at a configuration space $E_d(\ell)$ (so that $\mathcal{M}_d(\ell) = E_d(\ell)/\mathrm{SO}(d)$), cohomology is again enough to detect the chamber of ℓ , see Farber and Fromm [2]. A similar result for chain spaces (see Section 2) was obtained in [4].

It is clear from the calculations in [21] that ordinary cohomology is not enough for $d \geq 4$. Instead we use intersection homology in this paper. By letting the perversity vary with the degree of the intersection homology group, we can use the intersection pairing to assign a ring to each moduli space which behaves very similar to the cohomology ring in the case $d = 2$. For even $d \geq 4$ we can explicitly describe this ring and use it to prove Theorem 1.1.

The idea of the ring based on intersection homology is roughly the following. Given $J \subset \{1, \dots, n\}$ with $n \in J$ we can form a new length vector $\ell_J \in \mathbb{R}^{n-|J|+1}$ by merging the links corresponding to the elements of J into one link. This leads to an inclusion $\mathcal{M}_d(\ell_J) \subset \mathcal{M}_d(\ell)$. Furthermore, if $K \cap J = \{n\}$, the intersection $\mathcal{M}_d(\ell_J) \cap \mathcal{M}_d(\ell_K) = \mathcal{M}_d(\ell_{J \cup K})$ is transversal. Each J has a perversity $\mathbf{p}_{|J|}$ such that we have an element $[\mathcal{M}_d(\ell_J)] \in I^{\mathbf{p}_{|J|}} H_*(\mathcal{M}_d(\ell))$ so that these elements behave well with the intersection pairing of Goresky-MacPherson [8, 9]. For even d , these elements turn out to span an exterior algebra which is invariant under homeomorphism. The condition of d -normality ensures that $\mathcal{M}_d(\ell_J)$ is not a disc in which case it would be invisible for homology.

In a forthcoming paper we prove Theorem 1.1 for odd $d \geq 5$.

2. LINKAGE SPACES AND INTERSECTION HOMOLOGY

In order to study $\mathcal{M}_d(\ell)$ it is useful to consider the *chain space*

$$\mathcal{C}_d(\ell) = \left\{ (x_1, \dots, x_{n-1}) \in (S^{d-1})^{n-1} \mid \sum_{i=1}^{n-1} \ell_i x_i = -\ell_n e_1 \right\}$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$ is the usual first coordinate vector. If we let $\mathrm{SO}(d-1)$ act on S^{d-1} by fixing the first coordinate, we see that $\mathrm{SO}(d-1)$ acts diagonally on $\mathcal{C}_d(\ell)$ and

$$\mathcal{M}_d(\ell) \cong \mathcal{C}_d(\ell) / \mathrm{SO}(d-1).$$

We also define

$$\mathcal{N}_d(\ell) = \left\{ (x_1, \dots, x_n) \in (S^{d-1})^n \mid \sum_{i=1}^n \ell_i x_i = 0 \right\} / \mathrm{O}(d)$$

so that $\mathcal{N}_d(\ell) \cong \mathcal{M}_d(\ell) / (\mathbb{Z}/2)$.

Definition 2.1. Let $\ell \in \mathbb{R}^n$ be a length vector. A subset $J \subset \{1, \dots, n\}$ is called *ℓ -short*, if

$$\sum_{j \in J} \ell_j < \sum_{i \notin J} \ell_i.$$

It is called *ℓ -long*, if the complement is ℓ -short, and *ℓ -median*, if it is neither ℓ -short nor ℓ -long. The length vector is called *generic*, if there are no ℓ -median subsets.

For $m \in \{1, \dots, n\}$ the length vector is called *m -dominated*, if $\ell_m \geq \ell_i$ for all $i = 1, \dots, n$.

After permuting the coordinates we can always assume that ℓ is n -dominated. In fact, we can also assume that ℓ is *ordered*, meaning that $\ell_1 \leq \ell_2 \leq \dots \leq \ell_n$. For generic ℓ we can also assume $\ell_n > \ell_{n-1}$, as we can slightly increase the last entry without leaving the chamber.

If ℓ is m -dominated and $k \leq n-3$, we write

$$\mathcal{S}_k(\ell) = \{J \subset \{1, \dots, n\} \mid m \in J, |J| = k+1, J \text{ is } \ell\text{-short}\}.$$

Note that a length vector ℓ can be m -dominated by more than one $m \in \{1, \dots, n\}$. In this case we will form $\mathcal{S}_k(\ell)$ using the maximal m which dominates ℓ . Again,

for generic ℓ we can assume that m is unique. The cardinalities of these sets are denoted by

$$a_k(\ell) = |\mathcal{S}_k(\ell)|.$$

If $J \subset \{1, \dots, n\}$, we define the hyperplane

$$H_J = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{j \in J} x_j = \sum_{j \notin J} x_j \right\}$$

and let

$$\mathcal{H} = \mathbb{R}_{>0}^n - \bigcup_{J \subset \{1, \dots, n\}} H_J,$$

where $\mathbb{R}_{>0}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i > 0\}$. Then \mathcal{H} has finitely many components, which we call *chambers*. It is clear that a length vector ℓ is generic if and only if $\ell \in \mathcal{H}$.

It is shown in [10] that if ℓ and ℓ' are in the same chamber, then $\mathcal{C}_d(\ell)$ and $\mathcal{C}_d(\ell')$ are $O(d-1)$ -equivariantly diffeomorphic. In particular, $\mathcal{M}_d(\ell)$ and $\mathcal{M}_d(\ell')$ are homeomorphic.

It is easy to see that two m -dominated generic length vectors ℓ, ℓ' are in the same chamber if and only if $\mathcal{S}_k(\ell) = \mathcal{S}_k(\ell')$ for all $k = 0, \dots, n-3$.

Definition 2.2. Let $\ell \in \mathbb{R}^n$ be a length vector and $d \geq 2$. Then ℓ is called *d-normal*, if

$$\bigcap_{J \in \mathcal{L}^d(\ell)} J \neq \emptyset$$

where $\mathcal{L}^d(\ell)$ are the subsets $J \subset \{1, \dots, n\}$ with $d-1$ elements that are ℓ -long. If $\mathcal{L}^d(\ell) = \emptyset$, we let the intersection above be $\{1, \dots, n\}$.

So for a length vector to be *not d-normal*, we need a long subset $J \subset \{1, \dots, n\}$ with $d-1$ elements such that J does not contain an element m with ℓ_m maximal. Because then we can replace any element of J with m to get another ℓ -long subset with $d-1$ elements, and the intersection of these sets will be empty.

If ℓ is ordered, then ℓ is *d-normal* if and only if $\{n-d+1, n-d+2, \dots, n-1\}$ is not ℓ -long. For a generic length vector this is equivalent to $\mathcal{S}_{n-d}(\ell) = \emptyset$.

It follows from the definition that every length vector with $n \geq 2$ is 2-normal. Furthermore, there is only one chamber up to permutation which contains non-3-normal length vectors, namely the one containing¹ $\ell = (0, \dots, 0, 1, 1, 1)$. In [3], 4-normal was called normal.

In the case $d = n-1$, there are only two chambers up to permutation which contain *d-normal* length vectors, namely the ones containing $\ell = (1, \dots, 1, n-2)$ and $\ell' = (0, \dots, 0, 1)$. If n is large compared to d , *d-normality* gets more common, and we would expect the ratio of all *d-normal* length vectors in \mathbb{R}^n by all length vectors in \mathbb{R}^n to converge to 1.

¹Technically, this ℓ is not a length vector because of 0-entries. We interpret a 0-entry in a length vector as $\varepsilon > 0$ so small that decreasing it does not change the chamber.

For $d = 2, 3$ and ℓ generic, the spaces $\mathcal{M}_d(\ell)$ are closed manifolds, but for $d \geq 4$ this is no longer the case. But as we will see in Section 3, these spaces are pseudomanifolds for $n > d$. For the precise definition of a pseudomanifold, we refer the reader to [9]. Since we need intersection homology below, we do recall some of the definitions in order to agree on notation. Given an n -dimensional pseudomanifold X^n , there is a stratification

$$\emptyset \subset X^0 \subset X^1 \subset \dots \subset X^{n-2} \subset X^n.$$

A (Goresky-MacPherson) *perversity* is a function $\mathbf{p}: \{2, \dots, n\} \rightarrow \{0, 1, \dots\}$ such that $\mathbf{p}(2) = 0$ and $\mathbf{p}(i) \leq \mathbf{p}(i+1) \leq \mathbf{p}(i) + 1$ for all $i = 2, \dots, n-1$.

Simple examples are the zero-perversity $\mathbf{0}$ and the top perversity \mathbf{t} with $\mathbf{t}(i) = i-2$.

For a perversity \mathbf{p} the intersection homology $I^{\mathbf{p}}H_*(X)$ is the homology of a subcomplex $I^{\mathbf{p}}C_*(X)$ of the ordinary chains $C_*(X)$. If X admits a PL-structure, a PL-chain $\xi \in C_r(X)$ is called \mathbf{p} -allowable, if its support $|\xi| \subset X$ satisfies

$$\dim(|\xi| \cap X^{n-k}) \leq r - k + \mathbf{p}(k)$$

for all $k = 2, \dots, n$. The subcomplex $I^{\mathbf{p}}C_*(X)$ then consists of those ξ for which ξ and $\partial\xi$ are \mathbf{p} -allowable. For more details see [8, 9] and [16].

For normal pseudomanifolds, that is, those for which the link of each stratum is connected, there are canonical isomorphisms

$$I^{\mathbf{t}}H_*(X) \cong H_*(X) \quad \text{and} \quad I^{\mathbf{0}}H_*(X) \cong H^{n-*}(X),$$

see [16, §4.5]. It follows from Lemma 3.4 below that $\mathcal{M}_d(\ell)$ is a normal pseudomanifold whenever it is a pseudomanifold.

One of the features of intersection homology is that it satisfies Poincaré duality when using field coefficients. We will also need a Lefschetz duality version for which we require pseudomanifolds with boundary. Basically, a pseudomanifold with boundary X is such that $X - \partial X$ is an n -dimensional pseudomanifold, and ∂X is an $(n-1)$ -dimensional pseudomanifold which has a neighborhood in X stratified homeomorphic to $\partial X \times [0, 1)$, see [7, §4]. For a compact, orientable n -dimensional pseudomanifold with boundary we then get isomorphisms

$$I^{\mathbf{p}}H_i(X; \mathbf{F}) \cong I^{\mathbf{t}-\mathbf{p}}H_{n-i}(X, \partial X; \mathbf{F})$$

for all $i = 0, \dots, n$, assuming that \mathbf{F} is a field, see [7, §4].

3. LINKAGE SPACES AS PSEUDOMANIFOLDS

We want to describe the stratification of $\mathcal{M}_d(\ell)$. This is basically given by $\mathcal{M}_k(\ell)$ where $k < d$. However, the natural map $\mathcal{M}_k(\ell) \rightarrow \mathcal{M}_d(\ell)$ is not injective. But it induces an injection $\mathcal{N}_k(\ell) \rightarrow \mathcal{M}_d(\ell)$.

Definition 3.1. Let $x = [x_1, \dots, x_n] \in \mathcal{M}_d(\ell)$. Then the *rank* of x is defined as the dimension of the vector space spanned by x_1, \dots, x_n .

Clearly, $\text{rank}(x) \leq d$, and since the x_i are linearly dependent, we have $\text{rank}(x) \leq n-1$. We will however be more interested in the case $n > d$, as for $n \leq d$ we get that $\mathcal{M}_d(\ell)$ is contractible or empty.

Furthermore, if $k = \text{rank}(x)$, then x is in the image of the natural map $\mathcal{M}_k(\ell) \rightarrow \mathcal{M}_d(\ell)$.

Lemma 3.2. *Let $\ell \in \mathbb{R}^n$ be a length vector, and $k < d$. Then there is an inclusion $\mathcal{N}_k(\ell) \subset \mathcal{M}_d(\ell)$, and $\mathcal{N}_k(\ell)$ consists of all elements $x \in \mathcal{M}_d(\ell)$ with $\text{rank}(x) \leq k$.*

Proof. First note that if $x \in \mathcal{M}_d(\ell)$ has rank k , we can find an element $A \in \text{SO}(d)$ such that Ax_1, \dots, Ax_n span $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^d$. Hence x is in the image of the natural map $\mathcal{M}_k(\ell) \rightarrow \mathcal{M}_d(\ell)$. Now assume that $x, y \in \mathcal{M}_k(\ell)$ have the same image in $\mathcal{M}_d(\ell)$. Then $\text{rank}(x) = \text{rank}(y) = m \leq k$ for some m .

We can then assume that x_1, \dots, x_n span $\mathbb{R}^m \times \{0\}$ and also y_1, \dots, y_n span $\mathbb{R}^m \times \{0\}$. Now if there is an $A \in \text{SO}(d)$ with $Ax_i = y_i$ for all $i = 1, \dots, n$, we get that A keeps \mathbb{R}^m invariant, and therefore $A|_{\mathbb{R}^m} \in \text{O}(m)$. But this means that x and y represent the same element in $\mathcal{N}_k(\ell)$. \square

In order to understand the local structure of $\mathcal{M}_d(\ell)$, choose $x \in \mathcal{M}_d(\ell)$ with $\text{rank}(x) = k \leq d \leq n$, and represent this point by (x_1, \dots, x_n) . Note that if $d = n$, we have $k < n$, as the coordinates of x are linearly dependent.

Now rotate x_1 into position $e_1 \in \mathbb{R}^d$. Let us assume that $k \geq 2$, which is always the case if ℓ is generic. Then there is another element not contained in $\mathbb{R} \times \{0\}$, and after reordering, we may assume it is x_2 , now using a rotation from $\text{SO}(d-1)$ (fixing the first coordinate), we can assume that $x_2 \in S^1 - S^0$. Repeating this, we can represent x by an element (x_1, \dots, x_n) with

$$\begin{aligned} x_1 &= e_1 \\ x_2 &\in S^1 - S^0 \\ &\vdots \\ x_k &\in S^{k-1} - S^{k-2}. \end{aligned}$$

Since $\text{rank}(x) = k$, we get that $x_{k+1}, \dots, x_n \in S^{k-1} \subset S^{d-1}$. If $k < n-1$, we can assume that the x_{k+1}, \dots, x_n are not collinear: If they are, they cannot be multiples of x_k , since $\sum \ell_i x_i = 0$ and $x_1, \dots, x_{k-1} \in \mathbb{R}^{k-1}$. In that case we can just replace x_k with x_{k+1} .

Also, if $k = n-1$, it follows that x_k and x_{k+1} are not collinear by the same argument.

We can therefore assume that after a permutation of coordinates we have $x_1 = e_1$, $x_i \in S^{i-1} - S^{i-2}$ for $i = 2, \dots, k$, $x_{k+1}, \dots, x_n \in S^{k-1}$ and x_{n-1}, x_n are not collinear. Furthermore, the group $\text{SO}(d-k)$ fixes all x_i .

Let us define

$$(1) \quad \mathbf{d}_d^n = (n-3)(d-1) - \frac{(d-2)(d-3)}{2},$$

which is the dimension of $\mathcal{M}_d(\ell)$ for $n > d$ by Lemma 3.3 below. It then follows easily that the codimension of $\mathcal{N}_{d-k}(\ell)$ in $\mathcal{M}_d(\ell)$ is

$$(2) \quad \mathbf{c}_{d,k}^n = k(n-d) + \frac{k(k-1)}{2}$$

for $k = 2, \dots, d-2$.

Lemma 3.3. *Let $\ell \in \mathbb{R}^n$ be a length vector with $n > d \geq 3$. Let $x \in \mathcal{M}_d(\ell)$ satisfy $\text{rank}(x) \geq d-1$. Then x has a neighborhood homeomorphic to $\mathbb{R}^{\mathbf{d}_d^n}$.*

Proof. We can use the description for x given above the lemma. That is, we can represent x by an element (x_1, \dots, x_n) such that

$$\begin{aligned} x_1 &= e_1 \\ x_2 &\in S^1 - S^0 \\ &\vdots \\ x_{d-1} &\in S^{d-2} - S^{d-3}, \end{aligned}$$

and $x_d, \dots, x_n \in S^{d-1}$ with x_{n-1} and x_n not collinear. If $\text{rank}(x) = d - 1$, we actually have all $x_d, \dots, x_n \in S^{d-2}$, otherwise we can assume that $x_d \in S^{d-1} - S^{d-2}$.

In order to describe points near x , we have to consider points near (x_1, \dots, x_n) , so we can let them vary in small discs D^{d-1} . But notice that for nearby points y we always get $\text{rank}(y) \geq \text{rank}(x)$. After a rotation, we therefore get

$$\begin{aligned} y_1 &= e_1 \\ y_2 &\in S^1 - S^0 \\ &\vdots \\ y_{d-1} &\in S^{d-2} - S^{d-3}. \end{aligned}$$

Furthermore, $y_d, \dots, y_n \in S^{d-1}$, and there are no further rotations possible. The point y_2 can therefore freely vary in a small disc D^1 , y_3 in a small disc D^2 , etc. The points y_d, \dots, y_n can vary in S^{d-1} , but only up to y_{n-2} we can vary them freely. The last two y_{n-1}, y_n have to connect the endpoint of the linkage given by the first $n - 2$ elements to the origin. Since we can assume y_{n-1} and y_n to be not collinear, this is possible near x , and there is a $(d - 2)$ -dimensional sphere of possibilities. The dimension of the neighborhood is therefore

$$1 + 2 + \dots + (d - 2) + (d - 1)(n - 1 - d) + (d - 2)$$

which is easily seen to be $(n - 3)(d - 1) - \frac{(d-2)(d-3)}{2} = \mathbf{d}_d^n$. \square

The following lemma was proven in [5] in the case $d = 5$.

Lemma 3.4. *Let $\ell \in \mathbb{R}^n$ be a length vector, and $2 \leq k < d - 1 \leq n - 1$. If $\text{rank}(x) = k$, then x has a neighborhood homeomorphic to*

$$\mathbb{R}^{m(n,k)} \times ((\mathbb{R}^{d-k})^{n-1-k})/\text{SO}(d-k),$$

where $\text{SO}(d-k)$ acts diagonally on $(\mathbb{R}^{d-k})^{n-1-k}$ in the standard way, and

$$m(n,k) = \frac{(k-1)k}{2} + (n-k-1)(k-1) - 1.$$

Furthermore, if $k < m \leq d - 2$, the points in this neighborhood corresponding to points in $\mathcal{N}_m(\ell)$ are in

$$\mathbb{R}^{m(n,k)} \times ((\mathbb{R}^{m-k})^{n-1-k})/\text{O}(m-k).$$

Proof. The proof is similar to the proof of Lemma 3.3. Represent x by an element $\bar{x} = (x_1, \dots, x_{n-1}) \in \mathcal{C}_d(\ell)$ with $x_i \in S^i - S^{i-1}$ for $i = 1, \dots, k - 1$ and $x_i \in S^{k-1}$ for $i = k, \dots, n - 1$. By the Slice Theorem for compact Lie groups acting on smooth manifolds there is a neighborhood of this point equivariantly diffeomorphic to

$$\text{SO}(d-1) \times_{\text{SO}(d-k)} T_{\bar{x}}\mathcal{C}_d(\ell)/T_{\bar{x}}(\text{SO}(d-1) \cdot \bar{x}).$$

For $T_{\bar{x}}\mathcal{C}_d(\ell)$ note we can vary each of x_i for $i = 1, \dots, n-3$ within S^{d-1} , and the points x_{n-2} and x_{n-1} , which we can assume to be non-collinear, connect up the points $\ell_n e_1 \in \mathbb{R}^d$ and $y = \sum_{i=1}^{n-3} \ell_i x_i$. This gives rise to a sphere of dimension $d-2$. Quotienting the tangent space by the tangent space of the orbit means that there is no variation for x_1, x_2 only varies one-dimensionally in the first two coordinates, up to x_{k-1} which can vary $(k-1)$ -dimensionally in the first k variables. We can therefore write

$$T_{\bar{x}}\mathcal{C}_d(\ell)/T_{\bar{x}}(\text{SO}(d-1) \cdot \bar{x}) = \mathbb{R} \times \mathbb{R}^2 \times \dots \times \mathbb{R}^{k-1} \times (\mathbb{R}^{d-1})^{n-k-2} \times \mathbb{R}^{d-2},$$

with $\text{SO}(d-k)$ acting diagonally in the usual way on the last $(d-k)$ coordinates of $(\mathbb{R}^{d-1})^{n-k-2} \times \mathbb{R}^{d-2}$.

We can therefore write this as

$$T_{\bar{x}}\mathcal{C}_d(\ell)/T_{\bar{x}}(\text{SO}(d-1) \cdot \bar{x}) = \mathbb{R}^{m(n,k)} \times (\mathbb{R}^{d-k})^{n-k-1}$$

with $\text{SO}(d-k)$ acting diagonally on the second factor only. Passing to the $\text{SO}(d-1)$ -quotient of the slice neighborhood gives the result for points of rank k .

To get points in $\mathcal{N}_m(\ell)$ we simply have to make sure the last $d-m$ coordinates stay 0. We have to pass to $\text{O}(m-k)$, since $((\mathbb{R}^{m-k})^{n-1-k})/\text{SO}(m-k) \rightarrow ((\mathbb{R}^{d-k})^{n-1-k})/\text{SO}(d-k)$ is not injective. \square

If we write

$$L(m, n) = \{(x_1, \dots, x_n) \in (\mathbb{R}^m)^n \mid |x_1|^2 + \dots + |x_n|^2 = 1\}/\text{SO}(m)$$

with $\text{SO}(m)$ acting diagonally on $(\mathbb{R}^m)^n$, we see that a neighborhood of a point of rank k is given by $\mathbb{R}^{m(n,k)}$ times the open cone $C(L(d-k, n-k-1))$, with higher rank elements fitting nicely into the stratification of $L(d-k, n-k-1)$ coming from the rank of the x_i in \mathbb{R}^{d-k} .

This implies that for $d \leq n-1$ the space $\mathcal{M}_d(\ell)$ carries the structure of a normal pseudo-manifold with stratification given by

$$\emptyset \subset \mathcal{N}_2(\ell) \subset \mathcal{N}_3(\ell) \subset \dots \subset \mathcal{N}_{d-2}(\ell) \subset \mathcal{M}_d(\ell).$$

We would like to give this pseudo-manifold a piecewise-linear structure. To see that this is possible, note that $\mathcal{C}_d(\ell)$ is a real-analytic manifold with $\text{SO}(d-1)$ acting real-analytically. The submanifolds $\mathcal{C}_k(\ell)$ for $k < d$ are not $\text{SO}(d-1)$ -invariant, but the $\text{SO}(d-1)$ -orbits of these sets are easily seen to be subanalytic $\text{SO}(d-1)$ -invariant closed subsets of $\mathcal{C}_d(\ell)$. By [12, Thm.B] $\mathcal{C}_d(\ell)$ can be given a $\text{SO}(d-1)$ -equivariant triangulation which gives $\mathcal{M}_d(\ell)$ a triangulation such that each $\mathcal{N}_k(\ell)$ is a subcomplex.

4. THE INTERSECTION RING OF A PSEUDOMANIFOLD

Let X be a compact, oriented n -dimensional pseudomanifold. In [8, 9] Goresky-MacPherson define the intersection pairing

$$\cap: I^{\mathbf{p}}H_i(X) \times I^{\mathbf{q}}H_j(X) \rightarrow I^{\mathbf{r}}H_{i+j-n}(X)$$

where \mathbf{p}, \mathbf{q} and \mathbf{r} are perversities such that $\mathbf{p} + \mathbf{q} \leq \mathbf{r}$, and show that it does not depend on the stratification of X . Furthermore, $I^{\mathbf{0}}H_n(X)$ contains a fundamental class $[X]$ which serves as a unit.

Now let $k, m > 0$ and assume that $\mathbf{p}_0, \dots, \mathbf{p}_k$ is a sequence of perversities such that for all $i, j \geq 0$ with $i + j \leq k$ we have $\mathbf{p}_i + \mathbf{p}_j \leq \mathbf{p}_{i+j}$. For $0 \leq r \leq k$ define

$$I_{\mathbf{p}}.H^{rm}(X) = I^{\mathbf{p}_r}H_{n-rm}(X).$$

The notation \mathbf{p} is used as short-hand for the sequence of perversities. For $r + s \leq k$ the intersection pairing induces a multiplication

$$\cdot : I_{\mathbf{p}}.H^{rm}(X) \times I_{\mathbf{p}}.H^{sm}(X) \rightarrow I_{\mathbf{p}}.H^{(r+s)m}(X)$$

which turns

$$I_{\mathbf{p}}.H^{*m}(X) = \bigoplus_{r=0}^k I_{\mathbf{p}}.H^{rm}(X)$$

into a graded ring with unit, where we treat products with gradings $r + s > k$ as zero. We call this ring the *intersection ring of X with respect to \mathbf{p} and m* . If \mathbf{p}_r is the $\mathbf{0}$ -perversity for all $r \geq 0$, we may choose $k = \infty$.

The subring generated by the elements of $I_{\mathbf{p}}.H^m(X)$ is also a graded ring with unit, and we call it the *reduced intersection ring of X with respect to \mathbf{p} and m* . We denote it by

$$I_{\mathbf{p}}.\tilde{H}^{*m}(X).$$

If a pseudomanifold admits a stratification whose strata have only certain codimensions, a perversity only has to consider these codimensions. The relevant perversities for intersection homology of $\mathcal{M}_d(\ell)$ are therefore given by non-decreasing sequences of integers $(0, p_2, p_3, \dots, p_{d-2})$ for which we have

$$\begin{aligned} p_2 &\leq 2(n-d) - 1 \\ p_{i+1} - p_i &\leq n - d + i \end{aligned}$$

for all $i = 2, \dots, d-3$. The top perversity is thus given by

$$\mathbf{t}_n = (0, \mathbf{c}_{d,2}^n - 2, \dots, \mathbf{c}_{d,d-2}^n - 2).$$

We know from [21, Prop.7.1] that $H_{\mathbf{d}_d^n}(\mathcal{M}_d(\ell)) \cong \mathbb{Z}$, so there is a fundamental class that we can write as $[\mathcal{M}_d(\ell)]$.

Also, if $J \subset \{1, \dots, n-1\}$ we can define a new length vector $\ell_J \in \mathbb{R}^{n-|J|}$ given by

$$\ell_J = (\ell_{i_1}, \dots, \ell_{i_{n-1-|J|}}, \ell_n + \ell_{j_1} + \dots + \ell_{j_{|J|}}),$$

where $J = \{j_1, \dots, j_{|J|}\}$ and $\{i_1, \dots, i_{n-1-|J|}\}$ denotes the complement of J in $\{1, \dots, n-1\}$.

We then get a natural inclusion $\mathcal{M}_d(\ell_J) \hookrightarrow \mathcal{M}_d(\ell)$, and provided that $|J| \leq n-d-1$ we have a fundamental class $[\mathcal{M}_d(\ell_J)] \in H_{\mathbf{d}_d^{n-1-|J|}}(\mathcal{M}_d(\ell_J))$.

Note that if $J \cup \{n\}$ is long, we get $\mathcal{M}_d(\ell_J) = \emptyset$ and the fundamental class is just 0.

Definition 4.1. Let $0 \leq k \leq n-d-1$. The perversity \mathbf{p}_k is defined as

$$\mathbf{p}_k = (0, 2k, 3k, \dots, (d-2)k).$$

Lemma 4.2. Let $\ell \in \mathbb{R}^n$ be a generic length vector and $J \subset \{1, \dots, n-1\}$ satisfy $|J| \leq n-d-1$. Then $[\mathcal{M}_d(\ell_J)]$ represents a well-defined homology class

$$[\mathcal{M}_d(\ell_J)] \in I^{\mathbf{p}_k} H_{\mathbf{d}_d^{n-1-|J|}}(\mathcal{M}_d(\ell)).$$

Proof. We need to show that

$$\dim(\mathcal{M}_d(\ell_J) \cap \mathcal{N}_{d-k}(\ell)) \leq \dim \mathcal{M}_d(\ell_J) - \operatorname{codim} \mathcal{N}_{d-k}(\ell) + k|J|$$

for all $k = 2, \dots, d-2$. Since $\mathcal{M}_d(\ell_J) \cap \mathcal{N}_{d-k}(\ell) = \mathcal{N}_{d-k}(\ell_J)$, this is equivalent to showing

$$(n - |J| - 3)(d - k - 1) - \frac{(d - k - 2)(d - k - 3)}{2} \leq (n - |J| - 3)(d - 1) - \frac{(d - 2)(d - 3)}{2} - k(n - d) - \frac{k(k - 1)}{2} + k|J|.$$

A straightforward calculation shows that this is indeed an equality. \square

The relevant intersection ring for $\mathcal{M}_d(\ell)$ is obtained using $m = d - 1$ and the perversities $\mathbf{p}_0, \dots, \mathbf{p}_{n-d-1}$. To simplify notation and since we are mainly interested in the reduced intersection ring we will write

$$IH^{*(d-1)}(\mathcal{M}_d(\ell))$$

for the reduced intersection ring.

The relevant intersection homology groups are given by $IP^k H_{\mathbf{d}_d^{n-k}}(\mathcal{M}_d(\ell))$, and we have

$$\mathbf{d}_d^{n-k} + \mathbf{d}_d^{n-j} - \mathbf{d}_d^n = \mathbf{d}_d^{n-(k+j)}$$

and the perversities satisfy

$$\mathbf{p}_j + \mathbf{p}_k = \mathbf{p}_{j+k}$$

for $k, j = 0, \dots, n - d - 1$. Recall that products with $j + k \geq n - d$ are considered as 0.

The fundamental class $[\mathcal{M}_d(\ell)] \in IH^0(\mathcal{M}_d(\ell))$ is the unit of both the intersection ring and the reduced intersection ring, which follows immediately from [8, Thm.1].

Remark 4.3. For $d = 3$ we can use the $\mathbf{0}$ -perversity and the intersection ring consists of the cohomology ring $H^*(\mathcal{M}_3(\ell))$ which has been calculated by Hausmann and Knutson [11].

We want to describe the reduced intersection ring of $\mathcal{M}_d(\ell)$ for even $d \geq 4$ and d -normal ℓ . To do this we need a combinatorial description of the chamber.

Definition 4.4. Let Δ be a finite abstract simplicial complex, that is, a collection of subsets of a set $\{x_1, \dots, x_k\}$ which is closed under subsets. The *exterior face ring* $\Lambda_{\mathbf{k}}[\Delta]$ over the commutative ring \mathbf{k} is the quotient of the exterior algebra $\Lambda_{\mathbf{k}}[X_1, \dots, X_k]$ by the ideal generated by elements $X_{i_1} \cdots X_{i_m}$ where $\{x_{i_1}, \dots, x_{i_m}\} \notin \Delta$.

Note that for a length vector $\ell \in \mathbb{R}^n$ the collection

$$\mathcal{S}(\ell) = \bigcup_{k=1}^{n-3} \mathcal{S}_k(\ell)$$

is an abstract simplicial complex with vertex set $\mathcal{S}_1(\ell)$.

The main theorem about the intersection ring of $\mathcal{M}_d(\ell)$ is then given by the following.

Theorem 4.5. *Let $d \geq 4$ be even, $\ell \in \mathbb{R}^n$ a generic, d -normal length vector. Then the reduced intersection ring of $\mathcal{M}_d(\ell)$ with rational coefficients is isomorphic to the exterior face ring*

$$IH^{(d-1)*}(\mathcal{M}_d(\ell); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[\mathcal{S}(\ell)].$$

We need to show that the generators of the exterior face ring correspond exactly to $[\mathcal{M}_d(\ell_{\{j\}})]$, where $\{j\} \in \mathcal{S}_1(\ell)$, and that there are no other relations than the ones described by $\mathcal{S}(\ell)$. Assuming this, we can now prove Theorem 1.1.

Proof of Theorem 1.1. We can assume that both ℓ and ℓ' are ordered. The reduced intersection rings are homeomorphism invariants by [9, §5], so $\Lambda_{\mathbb{Q}}[\mathcal{S}(\ell)] \cong \Lambda_{\mathbb{Q}}[\mathcal{S}(\ell')]$. By [1, Exercise 5.12] there is an isomorphism of simplicial complexes $\mathcal{S}(\ell) \cong \mathcal{S}(\ell')$. But as ℓ and ℓ' are ordered, it follows that $\mathcal{S}(\ell) = \mathcal{S}(\ell')$, compare [3, Lemma 3]. Therefore ℓ and ℓ' are in the same chamber. \square

5. A MORSE FUNCTION FOR LINKAGE SPACES

In [21, §3] the function $F: \mathcal{C}_d(\ell) \rightarrow \mathbb{R}$ given by

$$F(z_1, \dots, z_{n-1}) = p_1(z_{n-1}),$$

where $p_1: \mathbb{R}^d \rightarrow \mathbb{R}$ is projection to the first coordinate, was shown to be a $\mathrm{SO}(d-1)$ -invariant Morse-Bott function whose critical manifolds consist of

- $\mathcal{C}_d(\ell^-)$ as the absolute minimum, $\ell^- = (\ell_1, \dots, \ell_{n-2}, \ell_n - \ell_{n-1})$.
- $\mathcal{C}_d(\ell^+)$ as the absolute maximum, $\ell^+ = (\ell_1, \dots, \ell_{n-2}, \ell_n + \ell_{n-1})$.
- S_J^{d-2} , a $(d-2)$ -dimensional sphere for every subset $J \subset \{1, \dots, n-2\}$ for which $J \cup \{n\}$ is ℓ -short and $J \cup \{n-1, n\}$ is ℓ -long, whose index is $(n-3-|J|)(d-1)$.

The critical spheres S_J can be visualized as in Figure 1 with the first $n-2$ links collinear, and those corresponding to elements of J pointing in the same direction. It was assumed in [21, §3] that ℓ was n -dominated, but this only ensures that

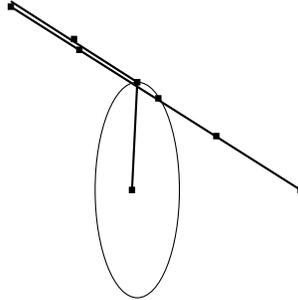


FIGURE 1. The critical manifold S_J obtained by rotating this configuration via $\mathrm{SO}(d-1)$.

(generically) $\ell_n > \ell_{n-1}$ so that ℓ^- is a length vector with all entries positive. However, the techniques of [21, §3] also work if we allow negative entries in a length vector.

We want to replace F by an equivariant Morse-Bott function \bar{F} , all of whose critical manifolds are $(d-2)$ -spheres. If $n = 3$, then $\mathcal{C}_d(\ell)$ is a $(d-2)$ -sphere or empty, so we can use $n = 3$ or 4 as an induction start.

Now consider the Morse-Bott function $F: \mathcal{C}_d(\ell) \rightarrow \mathbb{R}$ for $n > 4$. By the equivariant Morse Lemma, see [22, §4], there is an equivariant normal bundle around the absolute maximum and the absolute minimum so that F is just the sum of quadratics in the normal direction of this bundle. Furthermore, the normal bundle is (non-equivariantly) trivial by [21, Lm.3.3]. As the absolute maximum and minimum are just lower dimensional chain spaces, there exist equivariant Morse-Bott functions $F_1: \mathcal{C}_d(\ell^-) \rightarrow \mathbb{R}$ and $F_2: \mathcal{C}_d(\ell^+) \rightarrow \mathbb{R}$ with only $(d-2)$ -spheres as critical manifolds on the maximum and minimum.

Let $\chi_1, \chi_2: \mathcal{C}_d(\ell) \rightarrow [0, 1]$ be smooth functions so that χ_1 is 1 near $\mathcal{C}_d(\ell^-)$ and 0 outside the Morse-neighborhood of $\mathcal{C}_d(\ell^-)$, and χ_2 is 1 near $\mathcal{C}_d(\ell^+)$ and 0 outside the Morse-neighborhood of $\mathcal{C}_d(\ell^+)$. As we only have to define them on the normal bundle, we can assume them to be $\mathrm{SO}(d-1)$ invariant. We can extend F_1 and F_2 to the normal bundle neighborhood by ignoring the normal direction. Note that then $F + F_i$ turns into a Morse-Bott function on the normal bundle neighborhood, with critical manifolds that of F_i ($i = 1, 2$). Then define $\bar{F}: \mathcal{C}_d(\ell) \rightarrow \mathbb{R}$ by

$$\bar{F}(x) = F(x) + \varepsilon(\chi_1(x)F_1(x) + \chi_2(x)F_2(x)).$$

By choosing $\varepsilon > 0$ small enough we get that the critical points of \bar{F} are exactly the critical manifolds of F_1 , F_2 and the critical manifolds S_J from F . Furthermore, all indices remain the same except for those of F_2 , where the index is increased by $d-1$ coming from the normal direction.

To get the number of critical manifolds of a given index, observe that the number of critical manifolds does not depend on $d \geq 3$, and in the case $d = 3$ we get a perfect Morse function on $\mathcal{M}_3(\ell)$. For $d > 3$ we get that the Morse-Bott function is perfect in the sense that the number of critical manifolds of index $k(d-1)$ agrees with the $(2k)$ -th Betti number of the moduli space $\mathcal{M}_3(\ell)$, and there are no critical manifolds of other indices. The Betti numbers of $\mathcal{M}_3(\ell)$ were determined in [11] and [17], in particular they can be expressed in the $a_k(\ell)$.

We can furthermore assume that different critical manifolds have different values under \bar{F} . Choosing a sequence of regular values x_i such that the interval (x_{i-1}, x_i) contains exactly one critical value, we get a filtration

$$\emptyset \subset \mathcal{M}^0 \subset \mathcal{M}^1 \subset \dots \subset \mathcal{M}^r = \mathcal{M}_d(\ell)$$

by $\mathcal{M}^i = f^{-1}((-\infty, x_i])$, where $f: \mathcal{M}_d(\ell) \rightarrow \mathbb{R}$ is induced by \bar{F} .

Note that \mathcal{M}^i is a pseudomanifold with boundary, and to understand the intersection homology of the pair $(\mathcal{M}^i, \mathcal{M}^{i-1})$, we need to understand the intersection homology of the normal bundle of the critical manifold S^{d-2} relative to its boundary.

Let $N(S^{d-2})$ denote the normal bundle of the critical manifold $S^{d-2} \subset \mathcal{C}_d(\ell)$ and let $p \in S^{d-2}$. The normal space at p is denoted by $N_p(S^{d-2})$.

Lemma 5.1. *$\mathrm{SO}(d-1)$ acts transitively on S^{d-2} , furthermore, for $p \in S^{d-2}$ the stabilizer subgroup is isomorphic to $\mathrm{SO}(d-2)$, and we have $N_p(S^{d-2})$ is $\mathrm{SO}(d-2)$ -equivariantly homeomorphic to $(\mathbb{R}^{d-1})^{n-3}$, where $\mathrm{SO}(d-2)$ acts diagonally on the*

$n-3$ copies of \mathbb{R}^{d-1} , and the action of $\mathrm{SO}(d-2)$ on \mathbb{R}^{d-1} fixes the first coordinate, and is the standard action on the remaining $d-2$ coordinates.

Proof. We start with the critical manifold S_J from F . Elements in S_J have rank 2, and can be visualized as in Figure 1, the transitivity and stabilizer subgroup statements are therefore clear. The normal directions are obtained by varying the first $n-3$ links in S^{d-1} , keeping the $(n-2)$ -th link in S^1 (assuming all coordinates of $p \in S_J$ are in S^1), with $\mathrm{SO}(d-2)$ acting diagonally by fixing the first coordinate. We can therefore assume that the statement holds for the Morse-Bott functions F_1 and F_2 used in the definition of \bar{F} . The critical submanifolds $\mathcal{C}_d(\ell^\pm)$ of F have trivial normal bundle [21, Lemma 3.3], which gives an extra copy of \mathbb{R}^{d-1} on which $\mathrm{SO}(d-2)$ acts in the described way. \square

Let $k_i(d-1)$ be the index of the critical manifold S^{d-2} of \bar{F} whose critical value is in (x_{i-1}, x_i) . Then denote

$$\mathcal{N}_i = (D^{d-1})^{k_i} \times (D^{d-1})^{n-3-k_i} / \mathrm{SO}(d-2),$$

where $D^{d-1} \subset \mathbb{R}^{d-1}$ is the usual closed unit ball. It follows that \mathcal{M}_i is homeomorphic to $\mathcal{M}_{i-1} \cup \mathcal{N}_i$ and with the excision properties for pseudomanifolds with boundary we get

$$I^{\mathbf{p}}H_*(\mathcal{M}_i, \mathcal{M}_{i-1}) \cong I^{\mathbf{p}}H_*(\mathcal{N}_i, \partial_- \mathcal{N}_i),$$

where $\partial_- \mathcal{N}_i = \partial((D^{d-1})^{k_i} \times (D^{d-1})^{n-3-k_i} / \mathrm{SO}(d-2))$ for any perversity.

Let us denote $\mathcal{N}_i^- = (D^{d-1})^{k_i} / \mathrm{SO}(d-2)$ and $\partial \mathcal{N}_i^- = \partial((D^{d-1})^{k_i} / \mathrm{SO}(d-2))$. The obvious inclusion $\mathcal{N}_i^- \subset \mathcal{N}_i$ induces a homotopy equivalence of pairs $(\mathcal{N}_i^-, \partial \mathcal{N}_i^-) \simeq (\mathcal{N}_i, \partial_- \mathcal{N}_i)$, but this does not induce an isomorphism on intersection homology in general. In fact, \mathcal{N}_i^- is not a pseudomanifold for $k_i \leq d-3$.

However, if \mathcal{N}_i^- is a pseudomanifold (with boundary), its dimension is $\mathbf{d}_d^{k_i+3}$, and

$$I^{\mathbf{0}}H_*(\mathcal{N}_i^-, \partial \mathcal{N}_i^-) \cong I^{\mathbf{t}}H_{\mathbf{d}_d^{k_i+3}-*}(\mathcal{N}_i^-)$$

by Lefschetz duality, and since the latter is just ordinary homology of a contractible space, we have

$$I^{\mathbf{0}}H_{\mathbf{d}_d^{k_i}}(\mathcal{N}_i^-, \partial \mathcal{N}_i^-) \cong \mathbb{Z}$$

and all other groups are trivial. Since we are interested in \mathbb{Z} coefficients, we have to be slightly careful with Lefschetz duality and torsion. To see that no torsion occurs, we use [7, Cor.4.4.3], note that since we use the top perversity, the condition of being locally (\mathbf{p}, \mathbb{Z}) -torsion free is trivial. Also, [7, Cor.4.4.3] is stated for Poincaré duality, but because of the way Lefschetz duality is derived from Poincaré duality in [7], the result also holds for Lefschetz duality.

The inclusion $\mathcal{N}_i^- \subset \mathcal{N}_i$ is stratum preserving, but the codimensions of the strata are different. In particular, the inclusion is not placid in the sense of [16, §4.8]. In order to get a homomorphism between intersection homology groups of $(\mathcal{N}_i^-, \partial \mathcal{N}_i^-)$ and $(\mathcal{N}_i, \partial_- \mathcal{N}_i)$ we have to vary the perversities for them. The following lemma gives a criterion for obtaining such a homomorphism; its proof is analogous to the proof of [16, Ex.4.8.2].

Lemma 5.2. *Let X, Y be pseudomanifolds, \mathbf{p}, \mathbf{q} perversities, and $f : X \rightarrow Y$ a stratum preserving map for some stratifications of X and Y . Then f induces a map on homology*

$$f_* : I^{\mathbf{p}}H_*(X) \rightarrow I^{\mathbf{q}}H_*(Y)$$

provided that

$$\mathbf{q}(k) - k \geq \mathbf{p}(m_k) - m_k$$

where m_k is the minimal codimension of a stratum S with $f(S)$ in a stratum of codimension k .

Therefore inclusion induces a map on intersection homology by increasing the perversity for $(\mathcal{N}_i, \partial_- \mathcal{N}_i)$, that is, we have a well defined homomorphism

$$i_* : I^{\mathbf{0}}H_*(\mathcal{N}_i^-, \partial_- \mathcal{N}_i^-) \rightarrow I^{\mathbf{p}_{n-3-k_i}}H_*(\mathcal{N}_i, \partial_- \mathcal{N}_i).$$

In the next section we show that this is indeed an isomorphism.

6. INTERSECTION HOMOLOGY OF MORSE DATA

We modify our notation, for non-negative integers m, k with $m \geq k$ let

$$\begin{aligned} \mathcal{N}^{m,k} &= ((D^{d-1})^k \times (D^{d-1})^{m-k})/\mathrm{SO}(d-2), \\ \partial_- \mathcal{N}^{m,k} &= (\partial((D^{d-1})^k \times (D^{d-1})^{m-k})/\mathrm{SO}(d-2)). \end{aligned}$$

The stratification is given by

$$\emptyset \subset \mathcal{N}_2^{m,k} \subset \dots \subset \mathcal{N}_{d-2}^{m,k} \subset \mathcal{N}^{m,k}$$

where $\mathcal{N}_c^{m,k}$ is the image of $((D^{c-1})^k \times (D^{c-1})^{m-k})/\mathrm{SO}(c-2)$ in $\mathcal{N}^{m,k}$ for all $c = 2, \dots, d-2$.

If l is another non-negative integer with $l \leq m-k$ and $m-l \geq d-2$ we have an inclusion of pairs of pseudomanifolds with boundary

$$(\mathcal{N}^{m-l,k}, \partial_- \mathcal{N}^{m-l,k}) \subset (\mathcal{N}^{m,k}, \partial_- \mathcal{N}^{m,k})$$

which adds zeros into the extra coordinates. Let $s \geq 0$ with $s+l \leq m-d+2$. The inclusion then induces a map on intersection homology

$$i_* : I^{\mathbf{p}_s}H_*(\mathcal{N}^{m-l,k}, \partial_- \mathcal{N}^{m-l,k}) \rightarrow I^{\mathbf{p}_{s+l}}H_*(\mathcal{N}^{m,k}, \partial_- \mathcal{N}^{m,k})$$

by Lemma 5.2. This is an isomorphism.

Lemma 6.1. *Let d, m, l, k, s be non-negative integers with $l \leq m-k$, $m-l \geq d-2$ and $s \leq m-l-d+2$. Then*

$$I^{\mathbf{p}_s}H_*(\mathcal{N}^{m-l,k}, \partial_- \mathcal{N}^{m-l,k}) \cong I^{\mathbf{p}_{s+l}}H_*(\mathcal{N}^{m,k}, \partial_- \mathcal{N}^{m,k})$$

and the isomorphism is induced by inclusion.

The conditions on the integers are to ensure that we do get pseudomanifolds and perversities in the sense of [8].

Proof. There is an obvious retraction $r : (\mathcal{N}^{m,k}, \partial_- \mathcal{N}^{m,k}) \rightarrow (\mathcal{N}^{m-l,k}, \partial_- \mathcal{N}^{m-l,k})$ induced by projection, but this retraction does not preserve the stratification.

Therefore let us assume that $l = 1$. In that case $r^{-1}(\mathcal{N}_c^{m-1,k}) \subset \mathcal{N}_{c+1}^{m,k}$, and we define

$$\mathcal{N}_{c,c+1}^{m,k} = r^{-1}(\mathcal{N}_c^{m-1,k})$$

for all $c = 2, \dots, d-2$. We then have the stratification

$$\emptyset \subset \mathcal{N}_2^{m,k} \subset \mathcal{N}_{2,3}^{m,k} \subset \mathcal{N}_3^{m,k} \subset \dots \subset \mathcal{N}_{d-2}^{m,k} \subset \mathcal{N}_{d-2,d-1}^{m,k} \subset \mathcal{N}^{m,k}$$

and it is easy to check that the retraction is stratum preserving using this stratification. Notice that

$$\dim \mathcal{N}_{c,c+1}^{m,k} = \dim \mathcal{N}_c^{m,k} + 1.$$

To get an appropriate perversity \mathbf{p}'_{s+1} , we need entries for each $\mathcal{N}_{c,c+1}^{m,k}$ which can be at most one less than the entry for $\mathcal{N}_c^{m,k}$, for all $c = 2, \dots, d-2$. We denote these entries by $\mathbf{p}'_{s+1}(c, c+1)$ and set them to be

$$\mathbf{p}'_{s+1}(c, c+1) = \mathbf{p}_{s+1}(c) - 1 = c(s+1) - 1.$$

The other entries are the same:

$$\mathbf{p}'_{s+1}(c) = \mathbf{p}_{s+1}(c)$$

Since intersection homology does not depend on the stratification by [9], we get

$$I^{\mathbf{p}'_{s+1}} H_*(\mathcal{N}^{m,k}, \partial_- \mathcal{N}^{m,k}) \cong I^{\mathbf{p}'_{s+1}} H_*(\mathcal{N}^{m,k}, \partial_- \mathcal{N}^{m,k}).$$

We need to check the conditions of Lemma 5.2 to get an induced map

$$r_* : I^{\mathbf{p}'_{s+1}} H_*(\mathcal{N}^{m,k}, \partial_- \mathcal{N}^{m,k}) \rightarrow I^{\mathbf{p}'_s} H_*(\mathcal{N}^{m-1,k}, \partial_- \mathcal{N}^{m-1,k}).$$

Note that the strata of minimal codimension mapping into $\mathcal{N}_c^{m-1,k} - \mathcal{N}_{c-1}^{m-1,k}$ are $\mathcal{N}_{c,c+1}^{m,k} - \mathcal{N}_c^{m,k}$. The codimension of $\mathcal{N}_c^{m-1,k} - \mathcal{N}_{c-1}^{m-1,k}$ is given by

$$\mathbf{c}_{d,d-c}^{m+2} = (d-c)(m+2-d) + \frac{(d-c)(d-c-1)}{2}$$

using (2). Therefore, the codimension of $\mathcal{N}_{c,c+1}^{m,k} - \mathcal{N}_c^{m,k}$ is $\mathbf{c}_{d,d-c}^{m+3} - 1 = \mathbf{c}_{d,d-c}^{m+2} + c - 1$.

It follows that

$$\mathbf{p}_s(c) - \mathbf{c}_{d,d-c}^{m+2} = \mathbf{p}'_{s+1}(c, c+1) - (\mathbf{c}_{d,d-c}^{m+2} + c - 1)$$

which fits exactly the condition needed for Lemma 5.2 to apply. To see that r_* is the inverse isomorphism for i_* , notice that the obvious strong deformation retraction between the identity and $i \circ r$ is also stratum preserving, and can therefore be used to construct the isomorphism, compare [6, Prop.2.1].

This finishes the case $l = 1$. For $l > 1$ simply iterate this argument l times. \square

We saw in the previous section how to calculate $I^0 H_*(\mathcal{N}^{k,k}, \partial_- \mathcal{N}^{k,k})$ using ordinary homology and cohomology. We will also need calculations for other perversities.

To simplify notation, write $\mathcal{N}^k = \mathcal{N}^{k,k}$ and $\partial \mathcal{N}^k = \partial_- \mathcal{N}^{k,k}$. In [21, §6] a relative CW-complex structure for $(\mathcal{N}^k, \partial \mathcal{N}^k)$ was given and used to calculate rational homology groups. We want to use this structure to also do intersection homology calculations. However, one has to be careful with using arbitrary CW-decompositions for intersection homology calculations, compare [18, Appendix]. We therefore repeat the construction of the CW-structure and justify its use for intersection homology calculations.

Recall that $\mathcal{N}^k = (D^{d-1})^k / \text{SO}(d-2)$ where $\text{SO}(d-2)$ acts on D^{d-1} as the subgroup of $\text{SO}(d-1)$ which fixes the first coordinate. A typical element of \mathcal{N}^k is represented by (x_1, \dots, x_k) with each $x_i \in D^{d-1}$. Using an element of $\text{SO}(d-2)$ we can assume

that $x_1 \in D^2 \times \{0\} \subset D^{d-1}$ with the second coordinate non-negative. Let us introduce the notation

$$D_+^c = \{(y_1, \dots, y_c, 0, \dots, 0) \in D^{d-1} \mid y_c \geq 0\},$$

for $c = 2, \dots, d-2$, so that $x_1 \in D_+^2$. We will also think of $D^c \subset D^{d-1}$ for all $c = 1, \dots, d-2$, occupying the first c coordinates.

Using an element of $\text{SO}(d-3)$, which is understood to fix the first two coordinates of D^{d-1} , we can assume that $x_2 \in D_+^3$. Continuing this way we can represent the element of \mathcal{N}^k by

$$(x_1, \dots, x_k) \in D_+^2 \times D_+^3 \times \dots \times D_+^{d-2} \times (D^{d-1})^{k-(d-3)}.$$

To get a relative CW-structure for $(\mathcal{N}^k, \partial\mathcal{N}^k)$ we start with a k -cell $(D^1)^k$, which gives the cell structure for \mathcal{N}_2^k . To cover the elements of \mathcal{N}_3^k , we need cells of the form $(D^1)^l \times D_+^2 \times (D^2)^{k-1-l}$ for $l = 0, \dots, k-1$.

Continuing, the cells needed for \mathcal{N}_c^k , $c = 2, \dots, d-2$, are of the form

$$(D^1)^{l_1} \times D_+^2 \times (D^2)^{l_2} \times \dots \times D_+^{c-1} \times (D^{c-1})^{k-(c-2)-l_1-\dots-l_{c-2}},$$

and to cover $\mathcal{N}^k - \mathcal{N}_{d-2}^k$ we require the cells

$$(D^1)^{l_1} \times D_+^2 \times (D^2)^{l_2} \times \dots \times D_+^{d-2} \times (D^{d-1})^{k-(d-3)-l_1-\dots-l_{d-3}}.$$

Writing D^l as a $(d-2)$ -dimensional column vector $\begin{pmatrix} * \\ 0 \end{pmatrix}$ with the entry $*$ used $(l-1)$

times, and D_+^l as $\begin{pmatrix} * \\ + \\ 0 \end{pmatrix}$ with the entry $*$ used $(l-2)$ times, we can write each cell

as a $(d-2) \times k$ matrix with entries $0, *$ and $+$, compare [21, §5]. Note that the k -dimensional cell $(D^1)^k$ is represented by the zero matrix.

The boundary operator of the corresponding chain complex $C_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ has been described explicitly in [21, §6]. Roughly, on the level of matrices it is obtained by summing over all matrices obtained by replacing $+$ with 0 , and coefficients either 0 or ± 2 , where $\begin{matrix} + & * \\ 0 & * \end{matrix}$ always turns into 0 , and $\begin{matrix} + & * \\ 0 & 0 \end{matrix}$ turns into $\varepsilon \cdot \begin{matrix} 0 & + \\ 0 & 0 \end{matrix}$ with $\varepsilon \in \{0, \pm 2\}$.

This is justified as $D_+^l \times D^l$ has as part of its boundary $D^{l-1} \times D^l$ which can be thought of as $D^{l-1} \times D_+^l \cup D^{l-1} \times D_-^l$. Using an element of $\text{SO}(d-2)$ we can map $D^{l-1} \times D_-^l$ to $D^{l-1} \times D_+^l$, but this map affects the orientations of other discs, so that on the level of the chain complex the boundary contribution of the cell $D^{l-1} \times D_+^l$ may be 0 or ± 2 .

Note that a cell $(D^1)^{l_1} \times D_+^2 \times \dots \times (D^{d-1})^{k-L}$ intersected with \mathcal{N}_c^k is another cell $(D^1)^{l_1} \times D_+^2 \times \dots \times (D^{c-1})^{k-L'}$, so the CW-structure could be considered ‘flag-like’. We use this to subdivide the cell structure to a flag-like triangulation without changing the chain homotopy type of the corresponding intersection homology complex.

We begin by subdividing D^{d-1} . For $i = 1, \dots, d-1$ let $\varepsilon_i \in \{-, 0, +\}$ and

$$D_{\varepsilon_1 \dots \varepsilon_{d-1}}^{d-1} = \{(y_1, \dots, y_{d-1}) \in D^{d-1} \mid y_i \sim \varepsilon_i\},$$

where $y_i \sim 0$ means $y_i = 0$, $y_i \sim +$ means $y_i \geq 0$ and $y_i \sim -$ means $y_i \leq 0$. If $2 \leq c \leq d-2$, we can also write $D_{\varepsilon_1 \dots \varepsilon_{c-1}}^{c-1} = D_{\varepsilon_1 \dots \varepsilon_{c-1} 0 \dots 0}^{d-1}$.

We then get a subdivision of the previous cell structure using cells of the form

$$D_{\varepsilon_1 1 \dots \varepsilon_1 c_1 - 1}^{c_1 - 1} \times \dots \times D_{\varepsilon_k 1 \dots \varepsilon_k c_k - 1}^{c_k - 1}$$

subject to the condition that $c_1 \geq 2$, with $\varepsilon_1 c_1 - 1 = +$ if $c_1 > 2$, $c_i - c_{i-1} \in \{0, 1\}$ and if $c_i > c_{i-1}$, then $\varepsilon_i c_i - 1 = +$.

Let us denote by $C'_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ the cellular chain complex of this subdivision. We can form the subcomplexes $IPC_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ and $IPC'_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ using the standard definition of \mathbf{p} -allowable chains. Then subdivision induces a chain map

$$i: IPC_*(\mathcal{N}^k, \partial\mathcal{N}^k) \rightarrow IPC'_*(\mathcal{N}^k, \partial\mathcal{N}^k).$$

To obtain a chain map $p: IPC'_*(\mathcal{N}^k, \partial\mathcal{N}^k) \rightarrow IPC_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ note that each cell in the subdivision complex satisfies $D_{\varepsilon_1 \dots \varepsilon_{c-1}}^{c-1} \subset D^{c-1}$, and if $\varepsilon_{c-1} = +$, $D_{\varepsilon_1 \dots \varepsilon_{c-1}}^{c-1} \subset D_+^{c-1}$.

We can find a cellular homotopy $H: D^{c-1} \times I \rightarrow D^{c-1}$ between the identity and a map H_1 which is cellular when viewed as a map from the subdivision of D^{c-1} to the original cell. Furthermore, we obtain another homotopy $H_+: D_+^{c-1} \times I \rightarrow D_+^{c-1}$ which does the same for D_+^{c-1} .

This can be done using induction on c , basically by sliding the cell $D_{+ \dots +}^{c-1}$ over D^{c-1} (or D_+^{c-1}) while all other $c-1$ -dimensional cells map into lower skeleta. These homotopies are stratum preserving, so can be used to define a chain map $p: IPC'_*(\mathcal{N}^k, \partial\mathcal{N}^k) \rightarrow IPC_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ with $p \circ i = \text{id}_{IPC_*}$ and $i \circ p$ chain homotopic to $\text{id}_{IPC'_*}$.

Note that the construction of the homotopies to give the chain homotopy equivalences is similar to the proof in [18, Appendix].

The subdivision cell complex is regular in the sense that the attaching maps are homeomorphisms onto their image, and it has the same flag-like property as the original cell complex. If we subdivide this subdivision further by a flag-like triangulation, we can use the flag-like property to see that $IPC'_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ has the correct chain homotopy type for intersection homology by a similar argument as above, compare also [18, Appendix].

It remains to calculate the homology of $IPC_*(\mathcal{N}^k, \partial\mathcal{N}^k)$.

Each cell σ is represented by a symbolic $(d-2) \times k$ matrix, whose non-zero rows (except the last one) are of the form $(0 \dots 0 + * \dots *)$, with the last one of the form $(0 \dots 0 * \dots *)$. More precisely, if we denote the number of non-zero entries in the i -th row, $i = 1, \dots, d-2$, by k_i , we have $0 \leq k_1 \leq k$, $0 \leq k_i \leq \max\{0, k_{i-1} - 1\}$ for $i = 2, \dots, d-3$, and $k_{d-2} = \max\{0, k_{d-3} - 1\}$.

The dimension of the cell σ is then given by

$$\dim \sigma = k + k_1 + \dots + k_{d-2}.$$

Also note that σ is a cell in \mathcal{N}_c^k if and only if $k_i = 0$ for $i \geq c-1$, for all $c = 2, \dots, d-2$. Furthermore,

$$\dim(\sigma \cap \mathcal{N}_c^k) = k + k_1 + \dots + k_{c-2}.$$

If \mathbf{p} is a perversity, the condition for the cell σ to be \mathbf{p} -allowable is then simply given by

$$\mathbf{c}_{d,d-c}^{k+3} - \mathbf{p}(d-c) \leq k_{c-1} + \cdots + k_{d-2}.$$

Recall that $\mathbf{c}_{d,d-c}^{k+3}k$ is the codimension of \mathcal{N}_c^k and $\mathbf{p}(d-c)$ is the entry corresponding to this stratum.

If we look at the perversity \mathbf{p}_s with $s \in \{0, \dots, k+2-d\}$, we can get a simpler criterion for allowability.

Lemma 6.2. *Let $s \in \{0, \dots, k+2-d\}$. For the cell σ to be \mathbf{p}_s -allowable, we need*

$$k-i+1-s \leq k_i \leq k-i+1$$

for all $i = 1, \dots, d-3$.

Proof. Note that $k_i \leq k-i+1$ is satisfied anyway. So assume that $k-i+1-s > k_i$ for some $i \in \{1, \dots, d-3\}$. Then

$$k-j+1-s > k_j$$

for all $j \geq i$. Write $c = d-i+1$. Then

$$\begin{aligned} k_{d-c-1} + k_{d-c} + \cdots + k_{d-2} &< ck - (d-c-1) - \cdots - (d-2) + c - cs \\ &= ck - cd + 2 + \cdots + (c-1) + c + (c+1) + c - cs \\ &= c(k+3-d) + 1 + 2 + \cdots + (c-1) - cs \\ &= \mathbf{c}_{d,c}^{k+3} - \mathbf{p}_s(c) \end{aligned}$$

which would contradict \mathbf{p}_s -allowability. The same calculation also shows that the inequality is sufficient for allowability. \square

For $s = 0$ this means that only the top-dimensional cell is allowable. As its boundary is zero, this confirms our previous calculation of $I^0 H_*(\mathcal{N}^k, \partial\mathcal{N}^k)$.

The remaining case of interest for us is when $s = 1$. Recall that we assume $k \geq d-2$, so that \mathcal{N}^k is a pseudomanifold with boundary. In order to get that \mathbf{p}_1 is a perversity, we actually need $k \geq d-1$. The dimension of \mathcal{N}^k is given by \mathbf{d}_d^{k+3} , compare (1).

Proposition 6.3. *Let $k \geq d-1$.*

If d is odd, then

$$I^{\mathbf{p}_1} H_{\mathbf{d}_d^{k+3}-r}(\mathcal{N}^k, \partial\mathcal{N}^k) = \begin{cases} \mathbb{Z} & r = 0 \\ \mathbb{Z}/2 & r = 2l + 1, l = 1, \dots, (d-3)/2 \\ 0 & \text{otherwise} \end{cases}$$

If d is even, then

$$I^{\mathbf{p}_1} H_{\mathbf{d}_d^{k+3}-r}(\mathcal{N}^k, \partial\mathcal{N}^k) = \begin{cases} \mathbb{Z} & r = 0 \\ \mathbb{Z}/2 & r = 2l, l = 1, \dots, (d-4)/2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We claim that $I^{\mathbf{p}_1} C_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ is generated by one cell each in dimensions $\mathbf{d}_d^{k+3} - r$ for $r = 0, 2, 3, \dots, d-2$.

To see this note that for a cell to be \mathbf{p}_1 -allowable, Lemma 6.2 gives the existence of $i_0 \in \{1, \dots, d-3, d-1\}$ such that

$$k_i = \begin{cases} k-i+1 & i < i_0 \\ k-i & i \geq i_0 \end{cases}$$

The case $i_0 = d-2$ is omitted, as $k_{d-2} = k_{d-3} - 1$, and $i_0 = d-1$ corresponds to the top-dimensional cell. The matrices of the non-top-dimensional, \mathbf{p}_1 -allowable cells each have one occurrence of a submatrix $\begin{pmatrix} + & * & * \\ 0 & 0 & + \end{pmatrix}$, while all other occurrences of $+$ are in a submatrix of the form $\begin{pmatrix} + & * \\ 0 & + \end{pmatrix}$. Therefore the boundary of such a cell in $C_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ can involve at most one other cell, which is also \mathbf{p}_1 -allowable. The dimension of such a cell, depending on i_0 , is easily seen to be $\mathbf{d}_d^{k+3} - (d-1-i_0)$ which proves the claim.

Let us denote such a cell by σ_{i_0} with $i_0 \in \{1, \dots, d-3, d-1\}$. Clearly $\partial(\sigma_{d-1}) = 0$ which shows that the top-dimensional homology is \mathbb{Z} . But for $i_0 \leq d-3$, we get $\partial(\sigma_{i_0}) = \varepsilon_{i_0} \sigma_{i_0-1}$ with $\varepsilon_{i_0} \in \{0, \pm 2\}$. This boundary has been analyzed in detail in [21, §8], and it is shown there that ε depends on the difference of the number of non-zero entries in the i_0 -th row with the number of non-zero-entries in the last row. More precisely, if the difference of these numbers, taken from the matrix of σ_{i_0-1} , is l , the coefficient is $1 + (-1)^l$.

In particular, we have $\partial(\sigma_{d-3}) = 2\sigma_{d-4}$, interpreting $\sigma_0 = 0$, in case $d = 4$. As the coefficients are alternating between 0 and 2 from then on, the chain complex $IP^1 C_*(\mathcal{N}^k, \partial\mathcal{N}^k)$ is of the form

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \longrightarrow \dots \xrightarrow{\cdot \varepsilon_1} \mathbb{Z}$$

As $\varepsilon_1 = 2$ for odd d and $\varepsilon_1 = 0$ for even d , the result follows. \square

7. GENERATORS FOR THE REDUCED INTERSECTION RING

The reduced intersection ring $IH^{(d-1)*}(\mathcal{M}_d(\ell))$ is generated by $IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell))$, so we begin by calculating this group.

In Section 5 we obtained a $SO(d-1)$ -invariant Morse-Bott function $\bar{F} : \mathcal{C}_d(\ell) \rightarrow \mathbb{R}$ with all critical manifolds spheres of dimension $d-2$ and of index $k(d-1)$ for some $k \in \{0, \dots, n-3\}$. Furthermore, the number of such spheres with index $k(d-1)$ is equal to the $(2k)$ -th Betti number of $\mathcal{M}_3(\ell)$. The Betti numbers of $\mathcal{M}_3(\ell)$ are well known, see [11, 17]. In particular, for $\mathcal{M}_3(\ell) \neq \emptyset$ we have exactly one absolute maximum, and the number of critical manifolds of index $(n-4)(d-1)$ is equal to $1 + a_1(\ell)$ by [21, Prop.4.4]. Recall the $a_k(\ell)$ are defined in Section 2.

The Morse function \bar{F} induces the filtration

$$\emptyset \subset \mathcal{M}^0 \subset \mathcal{M}^1 \subset \dots \subset \mathcal{M}^m = \mathcal{M}_d(\ell)$$

as in Section 5.

Recall that we need $n \geq d+2$ for \mathbf{p}_1 to be a perversity. If $n = d+1$, $\mathcal{M}_d(\ell)$ is either empty or a sphere of dimension \mathbf{d}_d^{d+1} , so we do not need to consider this case.

Lemma 7.1. *Let $d \geq 4$, $n \geq d + 2$ and $\ell \in \mathbb{R}^n$ a generic length vector such that $\mathcal{M}_d(\ell) \neq \emptyset$. Then*

$$\begin{aligned} I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}^{m-1}) &\cong \mathbb{Z}^{1+a_1(\ell)}, \\ I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}-1}(\mathcal{M}^{m-1}) &\cong 0. \end{aligned}$$

Proof. For $l \leq m - 1$ we have the long exact sequence

$$\begin{aligned} \cdots \longrightarrow I^{\mathbb{P}^1} H_{r+1}(\mathcal{M}^l, \mathcal{M}^{l-1}) \longrightarrow I^{\mathbb{P}^1} H_r(\mathcal{M}^{l-1}) \longrightarrow I^{\mathbb{P}^1} H_r(\mathcal{M}^l) \longrightarrow \\ I^{\mathbb{P}^1} H_r(\mathcal{M}^l, \mathcal{M}^{l-1}) \longrightarrow \cdots \end{aligned}$$

and $I^{\mathbb{P}^1} H_{r+1}(\mathcal{M}^l, \mathcal{M}^{l-1}) \cong I^{\mathbb{P}^1} H_r(\mathcal{N}^{n-3, k_l}, \partial_- \mathcal{N}^{n-3, k_l})$ using the notation of Section 6. Here k_l refers to the index of the critical point contained in $\mathcal{M}^l - \mathcal{M}^{l-1}$. Then $k_l \leq n - 4$ as there is only one critical point of index $(n - 3)(d - 1)$, and it is contained in $\mathcal{M}^m - \mathcal{M}^{m-1}$.

We need to show that

$$I^{\mathbb{P}^1} H_r(\mathcal{N}^{n-3, k_l}, \partial_- \mathcal{N}^{n-3, k_l}) = 0$$

for $r = \mathbf{d}_d^{n-1} - 1, \mathbf{d}_d^{n-1}, \mathbf{d}_d^{n-1} + 1$ if $k_l < n - 4$, and

$$I^{\mathbb{P}^1} H_r(\mathcal{N}^{n-3, n-4}, \partial_- \mathcal{N}^{n-3, n-4}) = \begin{cases} \mathbb{Z} & r = \mathbf{d}_d^{n-1} \\ 0 & r = \mathbf{d}_d^{n-1} \pm 1 \end{cases}$$

The latter follows directly from Lemma 6.1. For the former, we use Lemma 6.1 and Lefschetz duality for pseudomanifolds with boundary to get

$$\begin{aligned} I^{\mathbb{P}^1} H_r(\mathcal{N}^{n-3, k_l}, \partial_- \mathcal{N}^{n-3, k_l}) &\cong I^0 H_r(\mathcal{N}^{n-4, k_l}, \partial_- \mathcal{N}^{n-4, k_l}) \\ &\cong I^{\mathbf{t}} H_{\mathbf{d}_d^{n-1}-r}(\mathcal{N}^{n-4, k_l}, \partial_+ \mathcal{N}^{n-4, k_l}) \end{aligned}$$

This is the ordinary homology group $H_{\mathbf{d}_d^{n-1}-r}(\mathcal{N}^{n-4, n-4-k_l}, \partial_- \mathcal{N}^{n-4, n-4-k_l})$. The homology of this pair has been calculated in [21, §7]. In particular, as $k_l < n - 4$, $\partial_- \mathcal{N}^{n-4, n-4-k_l} \neq \emptyset$, so the homology vanishes in degrees $0, \pm 1$. Again we do not get any torsion from Lefschetz duality, as [7, Cor.4.4.3] applies.

Therefore no homology occurs in degree $\mathbf{d}_d^{n-1} - 1$, and the rank of $I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}^l)$ increases by one exactly when there is a critical point of index $(n - 4)(d - 1)$ in $\mathcal{M}^l - \mathcal{M}^{l-1}$. As there are exactly $1 + a_1(\ell)$ such critical points, the result follows. \square

Corollary 7.2. *Let $d \geq 5$ be odd, $n \geq d + 2$ and $\ell \in \mathbb{R}^n$ a generic length vector with $\mathcal{M}_d(\ell) \neq \emptyset$. Then*

$$I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell)) \cong \mathbb{Z}^{1+a_1(\ell)}.$$

Proof. We have the long exact sequence

$$\begin{aligned} \cdots \longrightarrow I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}+1}(\mathcal{M}_d(\ell), \mathcal{M}^{m-1}) \longrightarrow I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}^{m-1}) \longrightarrow \\ I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell)) \longrightarrow I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell), \mathcal{M}^{m-1}) \longrightarrow \cdots \end{aligned}$$

Combining Proposition 6.3 with Lemma 7.1, this sequence reduces to

$$\mathbb{Z}/2 \longrightarrow \mathbb{Z}^{1+a_1(\ell)} \longrightarrow I^{\mathbb{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell)) \longrightarrow 0$$

As $\mathbb{Z}/2$ necessarily has to map to 0, the result follows. \square

It can easily be shown that $IP^1 H_{\mathbf{d}_d^{n-1}+1}(\mathcal{M}^{n-1}) = 0$, so we also get

$$IP^1 H_{\mathbf{d}_d^{n-1}+1}(\mathcal{M}_d(\ell)) \cong \mathbb{Z}/2$$

for odd d . However, we have no real use for this result.

For even d the long exact sequence turns into

$$\mathbb{Z} \longrightarrow \mathbb{Z}^{1+a_1(\ell)} \longrightarrow IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell)) \longrightarrow 0$$

and we will see that the first map is non-trivial. This will actually simplify our calculation of the reduced intersection ring, at least after using rational coefficients. For d odd we can only handle certain special cases. A particular case is given in the following example.

Example 7.3. Let $\ell^n = (1, \dots, 1, n-2) \in \mathbb{R}^n$. This is the unique generic and ordered length vector with $a_1(\ell) = 0$. The space $\mathcal{M}_d(\ell^n)$ is also known as the shape space Σ_{d-1}^{n-1} of [15], see [21, Prop.A.1].

Let d be odd and $n = d + k$ with $k \geq 1$. For $s \leq k - 1$ we claim that

$$IP^s H_{\mathbf{d}_d^{n-s}}(\mathcal{M}_d(\ell^n)) \cong \mathbb{Z}.$$

Furthermore, the reduced intersection ring agrees with the unreduced intersection ring, and we have

$$IH^{(d-1)*}(\mathcal{M}_d(\ell^{d+k})) \cong \mathbb{Z}[X]/X^k$$

with the degree of X equal to $d - 1$.

The proof is by induction on k . We will also show that $IP^s H_{\mathbf{d}_d^{n-s}}(\mathcal{M}_d(\ell^n))$ is generated by the fundamental class

$$[\mathcal{M}_d(\ell^{n-s})] \in IP^s H_{\mathbf{d}_d^{n-s}}(\mathcal{M}_d(\ell^n))$$

and that

$$IP^s H_{\mathbf{d}_d^{n-s-1}}(\mathcal{M}_d(\ell^n)) = 0.$$

For $s = 0$ and all k the fundamental class part is a standard result for pseudomanifolds, and the latter part follows from Section 6.

For $k \geq 2$, note that $(\ell^{d+k})^- = \ell^{d+k-1}$ and $(\ell^{d+k})^+$ satisfies $\mathcal{M}_d((\ell^{d+k})^+) = \emptyset$. The $SO(d-1)$ -equivariant Morse function $F : \mathcal{C}_d(\ell^{d+k}) \rightarrow \mathbb{R}$ from Section 5 has two critical manifolds, the absolute minimum $\mathcal{C}_d(\ell^{d+k-1})$ and one absolute maximum S^{d-2} . We thus get a filtration of $\mathcal{M}_d(\ell^{d+k})$ of the form

$$\emptyset \subset \mathcal{M}^{d+k} \subset \mathcal{M}_d(\ell^{d+k})$$

where \mathcal{M}^{d+k} has the homotopy type of $\mathcal{M}_d(\ell^{d+k-1})$. In fact, using the same technique as in the proof of Lemma 6.1, we see that

$$IP^s H_*(\mathcal{M}_d(\ell^{d+k-1})) \cong IP^{s+1} H_*(\mathcal{M}^{d+k}).$$

In particular,

$$[\mathcal{M}_d(\ell^{d+k-s-1})] \in IP^{s+1} H_{\mathbf{d}_d^{d+k-s-1}}(\mathcal{M}^{d+k})$$

is a generator and

$$IP^{s+1} H_{\mathbf{d}_d^{d+k-s-1}-1}(\mathcal{M}^{d+k}) = 0.$$

We have the long exact sequence

$$\begin{aligned} \cdots \longrightarrow I^{\mathbf{P}^{s+1}} H_{r+1}(\mathcal{M}_d(\ell^{d+k}), \mathcal{M}^{d+k}) &\longrightarrow I^{\mathbf{P}^{s+1}} H_r(\mathcal{M}^{d+k}) \longrightarrow \\ &I^{\mathbf{P}^{s+1}} H_r(\mathcal{M}_d(\ell^{d+k})) \longrightarrow I^{\mathbf{P}^{s+1}} H_r(\mathcal{M}_d(\ell^{d+k}), \mathcal{M}^{d+k}) \end{aligned}$$

Note that

$$I^{\mathbf{P}^{s+1}} H_r(\mathcal{M}_d(\ell^{d+k}), \mathcal{M}^{d+k}) \cong I^{\mathbf{P}^{s+1}} H_r(\mathcal{N}^{d+k-3}, \partial \mathcal{N}^{d+k-3})$$

in the notation of Section 6. By Lemma 6.2, the minimal dimensional cell which is \mathbf{p}_{s+1} -allowable is obtained by choosing each $k_i = d+k-3-i-s$ for $i = 1, \dots, d-2$. The dimension of this cell is

$$\begin{aligned} d+k-3 + (d+k-3-1-s) + \cdots + (d+k-3-(d-2)-s) = \\ (d+k-3-s-1)(d-1) + (s+1) + 1 + \cdots + (d-3) = \\ \mathbf{d}_d^{d+k-s-1} + (s+1) \end{aligned}$$

In particular, for $s > 0$ and $r \leq \mathbf{d}_d^{d+k-s-1}$ inclusion induces an isomorphism

$$I^{\mathbf{P}^{s+1}} H_r(\mathcal{M}^{d+k}) \cong I^{\mathbf{P}^{s+1}} H_r(\mathcal{M}_d(\ell^{d+k}))$$

and this even holds for $s = 0$ as in Corollary 7.2. This finishes our induction step. It remains to calculate the intersection ring. Of course, X corresponds to the fundamental class

$$[\mathcal{M}_d(\ell^{d+k-1})] \in I^{\mathbf{P}^1} H_{\mathbf{d}_d^{d+k-1}}(\mathcal{M}_d(\ell^{d+k})).$$

Note that we think of $\mathcal{M}_d(\ell^{d+k-1}) \subset \mathcal{M}_d(\ell^{d+k})$ as those points $[x_1, \dots, x_n]$ with $x_{n-1} = -x_n$. But we could also fix a different coordinate to point in the opposite direction of the last entry: let

$$\mathcal{M}_d(\ell^{d+k-1})' = \{[x_1, \dots, x_n] \in \mathcal{M}_d(\ell^{d+k}) \mid x_{n-2} = -x_n\}.$$

Clearly this is homeomorphic to $\mathcal{M}_d(\ell^{d+k-1})$ by permuting coordinates. From Lemma 7.4 below it follows that

$$[\mathcal{M}_d(\ell^{d+k-1})] = [\mathcal{M}_d(\ell^{d+k-1})'].$$

Therefore, as $\mathcal{M}_d(\ell^{d+k-1})$ and $\mathcal{M}_d(\ell^{d+k-1})'$ are transverse in the sense of [8],

$$\begin{aligned} X^2 &= [\mathcal{M}_d(\ell^{d+k-1})] \cdot [\mathcal{M}_d(\ell^{d+k-1})] \\ &= [\mathcal{M}_d(\ell^{d+k-1})] \cdot [\mathcal{M}_d(\ell^{d+k-1})'] \\ &= [\mathcal{M}_d(\ell^{d+k-1}) \cap \mathcal{M}_d(\ell^{d+k-1})'] \\ &= [\mathcal{M}_d(\ell^{d+k-2})] \end{aligned}$$

This means that X^2 is a generator of $I^{\mathbf{P}^2} H_{\mathbf{d}_d^{d+k-2}}(\mathcal{M}_d(\ell^{d+k}))$, and we can iterate this argument until $X^k = 0$.

Lemma 7.4. *Let $d \geq 3$, $n \geq d+2$ and $\ell^n = (1, \dots, 1, n-2) \in \mathbb{R}^n$. Denote*

$$\begin{aligned} \mathcal{M} &= \{[x_1, \dots, x_n] \in \mathcal{M}_d(\ell^n) \mid x_{n-1} = -x_n\} \\ \mathcal{M}' &= \{[x_1, \dots, x_n] \in \mathcal{M}_d(\ell^n) \mid x_{n-2} = -x_n\}. \end{aligned}$$

Then $[\mathcal{M}] = [\mathcal{M}'] \in I^{\mathbf{P}^1} H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell^n))$.

Proof. The idea is the following: if $[x_1, \dots, x_n] \in \mathcal{M}$, then x_{n-2} is linearly independent of x_{n-1} , unless $[x_1, \dots, x_n] \in \mathcal{M} \cap \mathcal{M}'$. Note that each x_i for $i < n$ has to be close to $-x_n$ by the particular form of the length vector. We can therefore flip the position of x_{n-2} and x_{n-1} through a 1-dimensional parameter. This will define a homotopy $f : \mathcal{M} \times [0, 1] \rightarrow \mathcal{M}_d(\ell^n)$ relative to $\mathcal{M} \cap \mathcal{M}'$ between the inclusion of \mathcal{M} and the inclusion of \mathcal{M}' .

More precisely, let $x = [x_1, \dots, x_n] \in \mathcal{M}$. Assume that $x_n = e_1$, $x_{n-1} = -e_1$ and $x_{n-2} \in S_-^1 = \{(y_1, y_2, 0, \dots, 0) \in S^{d-1} \mid y_1, y_2 \leq 0\}$. Let $l_x = \|x_1 + \dots + x_{n-3}\| > 0$ and $\ell_x = (l_x, 1, 1, n-2)$. It is easy to see that $\mathcal{C}_2(\ell_x)$ is a point for $x \in \mathcal{M} \cap \mathcal{M}'$ and a circle otherwise.

We think of $\mathcal{C}_2(\ell_x)$ as a subspace of $\mathcal{C}_d(\ell^n)$, where the first link corresponds to a rotation of $x_1 + \dots + x_{n-3}$ in the plane, and the second and third link correspond to x_{n-2} and x_{n-1} . Indeed, denote $z_x = (x_1 + \dots + x_{n-3})/l_x \in S^1$. Then define $i_x : \mathcal{C}_2(\ell_x) \rightarrow \mathcal{C}_d(\ell^n)$ by

$$i_x(a, b, c) = ((az_x^{-1})x_1, \dots, (az_x^{-1})x_{n-3}, b, c)$$

where we think of S^1 acting on S^{d-1} by rotation of the first two coordinates, that is, as $\text{SO}(2)$. It is easy to see that this is indeed an inclusion.

There is a unique point $(a, b, c) \in \mathcal{C}_2(\ell_x)$ such that $c = -e_1$, $b \in S_-^1$. Write $b = \exp((\pi + u_x)i)$ with $u_x \geq 0$. As $\mathcal{C}_2(\ell_x)$ is either a circle or a point, there is a unique map $h_x : [0, 1] \rightarrow \mathcal{C}_2(\ell_x)$ with $p_3(h_x(t)) = \exp((\pi + u_x t)i)$, where $p_3 : \mathcal{C}_2(\ell_x) \rightarrow S^1$ is projection to the third coordinate.

It is now straightforward to check that $f : \mathcal{M} \times [0, 1] \rightarrow \mathcal{M}_d(\ell^n)$ given by

$$f(x, t) = [i_x(h_x(t))]$$

is a well defined map which satisfies $f(x, 0)$ inclusion, and $f(x, 1) = q(x)$, where $q : \mathcal{M} \rightarrow \mathcal{M}_d(\ell^n)$ is inclusion followed by flipping the $(n-2)$ -nd and $(n-1)$ -st coordinates. As f is stratum preserving, f_0 and f_1 induce the same map on intersection homology

$$f_{0*} = f_{1*} : I^0 H_*(\mathcal{M}) \rightarrow I^{\mathbb{P}^1} H_*(\mathcal{M}_d(\ell^n))$$

which implies that $[\mathcal{M}] = [\mathcal{M}']$ in the latter group. \square

For even d Example 7.3 is a bit different. This is related to the following lemma.

Lemma 7.5. *Let $d \geq 4$ be even and $\ell \in \mathbb{R}^n$ an ordered, generic length vector with $n \geq d+2$. Then*

$$2[\mathcal{M}_d(\ell^-)] = 2[\mathcal{M}_d(\ell^+)] \in I^{\mathbb{P}^1} H_{d^{n-1}}(\mathcal{M}_d(\ell)).$$

Proof. Let

$$\tilde{\mathcal{C}}_d(\ell) = \{(x_1, \dots, x_{n-1}) \in \mathcal{C}_d(\ell) \mid x_{n-2}, x_{n-1} \in S^1\}$$

where $S^1 \subset S^{d-2}$ is the standard inclusion using the first two coordinates. For generic ℓ this is a submanifold of $\mathcal{C}_d(\ell)$, and the projection $p : \tilde{\mathcal{C}}_d(\ell) \rightarrow S^1$ to the last coordinate has $-1 \in S^1$ as a regular value. In particular, for $\theta \in S^1$ close to -1 , we get that

$$\tilde{\mathcal{C}}_d^\theta(\ell) = p^{-1}(\{\theta\})$$

is diffeomorphic to $p^{-1}(\{-1\})$. Furthermore, this diffeomorphism can be chosen to be $\mathrm{SO}(d-2)$ -equivariant, where $\mathrm{SO}(d-2)$ fixes the first two coordinates.

Fix $\theta_0 \in S^1$ close enough to -1 so that this diffeomorphism exists. This gives rise to a $\mathrm{SO}(d-2)$ -equivariant map

$$\tilde{H}: \tilde{\mathcal{C}}_d^{\theta_0}(\ell) \times [0, 1] \rightarrow \mathcal{C}_d(\ell)$$

which at time t is inclusion to $p^{-1}(\{\theta_t\})$, where $\theta_t \in S^1$ starts at θ_0 and ends at -1 .

Define

$$\mathcal{M}_d^{\theta_0}(\ell) = \tilde{\mathcal{C}}_d^{\theta_0}(\ell)/\mathrm{SO}(d-2),$$

which is easily seen to be a pseudo-manifold of dimension \mathbf{d}_d^{n-1} that represents an element $[\mathcal{M}_d^{\theta_0}(\ell)] \in I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell))$. Furthermore, we get the stratum preserving homotopy

$$H: \mathcal{M}_d^{\theta_0}(\ell) \times [0, 1] \rightarrow \mathcal{M}_d(\ell)$$

starting with the inclusion and ending with a surjection $H_1: \mathcal{M}_d^{\theta_0}(\ell) \rightarrow \mathcal{M}_d(\ell^-)$. For a generic set of points in $\mathcal{M}_d(\ell^-)$ the map H_1 is a 2 : 1-map: if $x_{n-2} \neq \pm x_{n-1}$ for $x = [x_1, \dots, x_{n-1}] \in \mathcal{M}_d(\ell^-)$ there are two points in $\mathcal{M}_d^{\theta_0}(\ell)$ send to x coming from the $\mathbb{Z}/2$ -action on $\mathcal{M}_d^{\theta_0}(\ell)$ which flips the second coordinate.

For even d the $\mathbb{Z}/2$ -action is orientation preserving, as we can rotate the second and the last coordinate by an angle of π , and for a generic point $x \in \mathcal{M}_d^{\theta_0}(\ell)$ there are $d-2$ elements which do not have a non-zero entry in the last coordinate. This means that H_1 is a degree 2 map from $\mathcal{M}_d^{\theta_0}(\ell)$ to $\mathcal{M}_d(\ell^-)$, which implies that

$$[\mathcal{M}_d^{\theta_0}(\ell)] = 2[\mathcal{M}_d(\ell^-)].$$

For η near $+1 \in S^1$ we can do a similar construction, showing that

$$[\mathcal{M}_d^{\eta_0}(\ell)] = 2[\mathcal{M}_d(\ell^+)].$$

Let $J \subset S^1$ be the interval in the upper half plane with endpoints θ_0 and η_0 . Then $p^{-1}(J) \subset \tilde{\mathcal{C}}_d(\ell)$ is a cobordism between $\tilde{\mathcal{C}}_d^{\theta_0}(\ell)$ and $\tilde{\mathcal{C}}_d^{\eta_0}(\ell)$. Passing to the quotient under the $\mathrm{SO}(d-2)$ action shows that $[\mathcal{M}_d^{\theta_0}(\ell)] = [\mathcal{M}_d^{\eta_0}(\ell)]$. \square

Example 7.6. For ℓ^n from Example 7.3 with even d we have $\mathcal{M}_d((\ell^n)^+) = \emptyset$, so over the rationals $I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell^n); \mathbb{Q}) = 0$, and the rational intersection ring is trivial.

For $d = 4$ we can actually show that

$$I\mathbb{P}^1 H_{\mathbf{d}_4^{n-1}}(\mathcal{M}_4(\ell^n)) = \mathbb{Z}/2\mathbb{Z}.$$

In [15, §5.2] it is shown that $H_{\mathbf{d}_4^{n-1}}(\Sigma_3^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\Sigma_3^{n-2} \approx \mathcal{M}_4(\ell^{n-1})$ represents the generator. The natural map

$$I\mathbb{P}^1 H_{\mathbf{d}_4^{n-1}}(\mathcal{M}_4(\ell^n)) \longrightarrow H_{\mathbf{d}_4^{n-1}}(\Sigma_3^{n-1})$$

is therefore surjective, and since the former is generated by $[\mathcal{M}_4(\ell^{n-1})]$ which has order 2, it has to be an isomorphism.

We can now repeat the argument of Example 7.3 to show that

$$I\mathbb{P}^s H_{\mathbf{d}_4^{n-s}}(\mathcal{M}_4(\ell^n)) \cong \mathbb{Z}/2\mathbb{Z}$$

generated by $[\mathcal{M}_4(\ell^{n-s})]$ for $s = 1, \dots, n-5$, and the unreduced intersection ring satisfies

$$IH^{3*}(\mathcal{M}_4(\ell)) \cong \mathbb{Z}[X]/\langle X^{n-4}, 2X \rangle.$$

We expect the analogous statement to hold also for $d \geq 6$ even, but the homology calculations for Σ_{d-1}^{n-1} are somewhat more involved, compare [15, §5].

Recall the length vector ℓ_J for $J \subset \{1, \dots, n-1\}$ defined in Section 4. To simplify notation, we will write

$$Y_i = [\mathcal{M}_d(\ell_{\{i\}})] \in IH^{d-1}(\mathcal{M}_d(\ell))$$

for $i = 1, \dots, n-1$. Note that if $\{i, n\}$ is ℓ -long, then $Y_i = 0$. If $J \cup \{n\}$ is ℓ -short, we also write

$$Y_J = Y_{i_1} \cdots Y_{i_k} \in IH^{(d-1)k}(\mathcal{M}_d(\ell))$$

for $J = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$. By the properties of the intersection product we have $Y_J = \pm[\mathcal{M}_d(\ell_J)]$.

Proposition 7.7. *Let $\ell \in \mathbb{R}^n$ be a generic and ordered length vector with $n \geq d+2$, $d \geq 4$ and $k = a_1(\ell)$. Then Y_1, \dots, Y_k are linearly independent elements of $IH^{d-1}(\mathcal{M}_d(\ell))$.*

Proof. This follows directly from Lemma 8.1 below, the proof of this Lemma does not require any further material from this section. \square

Theorem 7.8. *Let $d \geq 4$ be even, $n \geq d+2$ and $\ell \in \mathbb{R}^n$ a generic, ordered, d -normal length vector. Then $IH^{d-1}(\mathcal{M}_d(\ell); \mathbb{Q})$ is generated by $Y_1, \dots, Y_{a_1(\ell)}$.*

Proof. The proof is by induction on n , and it starts with $n = d+2$. Since ℓ is d -normal, we have $a_2(\ell) = 0$. We distinguish the cases $a_1(\ell) < n-1$ and $a_1(\ell) = n-1$. If $a_1(\ell) < n-1$, then $\mathcal{M}_d(\ell^+) = \emptyset$. By Lemma 7.1, $I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}^{m-1})$ has rank $1 + a_1(\ell)$. Furthermore, $[\mathcal{M}_d(\ell^-)]$ is one of the generators. By Lemma 7.5 it represents 0 in $I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell); \mathbb{Q})$, and as the natural map

$$I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}^{m-1}; \mathbb{Q}) \rightarrow I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell); \mathbb{Q})$$

is surjective, the result follows from Proposition 7.7.

If $a_1(\ell) = n-1$, we use the original Morse-Bott function F , so that we have the minimum given by $\mathcal{M}_d(\ell^-)$ and the maximum given by $\mathcal{M}_d(\ell^+)$. As ℓ^+ is d -normal, it has to be $(1, \dots, 1, d-1) \in \mathbb{R}^{d+1}$. Let $\mathcal{M} \subset \mathcal{M}_d(\ell)$ be the inverse image of a regular value slightly smaller than the maximum, so that \mathcal{M} has the homotopy type of $\mathcal{M}_d(\ell) - \mathcal{M}_d(\ell^+)$. The standard Morse-Bott function on $\mathcal{M}_d(\ell^+)$ has two critical manifolds, $\mathcal{M}_d(\ell^{+-})$ and one absolute maximum $x \in \mathcal{M}_d(\ell^+)$. In $\mathcal{M}_d(\ell) - \mathcal{M}$ the minimum $\mathcal{M}_d(\ell^{+-})$ has index $d-1$, so it represents the generator of $I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell) - \{x\}, \mathcal{M})$. However, with a construction as in the proof of Lemma 7.5 this element vanishes in $I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell), \mathcal{M}; \mathbb{Q})$. It follows that $I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell); \mathbb{Q}) \cong I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}; \mathbb{Q})$.

Recall that the Morse-Bott function F has a critical manifold S_J for every $J \subset \{1, \dots, n-2\}$ such that $J \cup \{n\}$ is short, while $J \cup \{n-1, n\}$ is long. In particular, we get this for $\{i\}$ for all $i = 1, \dots, n-2$. Therefore

$$I\mathbb{P}^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}; \mathbb{Q}) \cong \mathbb{Q}^{n-1}$$

with one generator being $[\mathcal{M}_d(\ell^-)]$ and the others coming from the critical points. As Y_1, \dots, Y_{n-1} are linearly independent by Proposition 7.7, they have to be a basis.

For the induction step, let $n > d + 2$. We again use the Morse-Bott function F and the subspace \mathcal{M} , which is a pseudomanifold with boundary. Let $\mathcal{N} = \mathcal{M}_d(\ell) - \mathcal{M}^\circ$, also a pseudomanifold with boundary $\partial\mathcal{N} = \partial\mathcal{M}$. If $a_1(\ell) < n - 1$, then $\mathcal{N} = \mathcal{N}^{n-3}$. Also, the rank of $IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}; \mathbb{Q})$ is $1 + a_1(\ell)$ by Lemma 7.1, and one of the generators is given by $[\mathcal{M}_d(\ell^-)]$. This generator dies in $IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell); \mathbb{Q})$ by Lemma 7.5. Hence the rank of this group is $a_1(\ell)$, and the result follows by Proposition 7.7.

If $a_1(\ell) = n - 1$, the inclusion $\mathcal{M}_d(\ell^+) \subset \mathcal{N}$ is a homotopy equivalence which induces an isomorphism on intersection homology if we add \mathbf{p}_1 to the perversity.

Recall the top perversity $\mathbf{t}_n = (0, \mathbf{c}_{d,2}^n - 2, \dots, \mathbf{c}_{d,d-2}^n - 2)$. It is easy to check that $\mathbf{t}_n - \mathbf{p}_1 = \mathbf{t}_{n-1}$. Using Lefschetz duality for pseudomanifolds with boundary, we get

$$\begin{aligned} IP^1 H_*(\mathcal{M}_d(\ell), \mathcal{M}; \mathbb{Q}) &\cong IP^1 H_*(\mathcal{N}, \partial\mathcal{N}; \mathbb{Q}) \\ &\cong I^{\mathbf{t}_n - \mathbf{p}_1} H_{\mathbf{d}_d^{n-*}}(\mathcal{N}; \mathbb{Q}) \\ &\cong I^{\mathbf{t}_{n-1}} H_{\mathbf{d}_d^{n-*}}(\mathcal{N}; \mathbb{Q}) \\ &\cong I^{\mathbf{t}_{n-1} - \mathbf{p}_1} H_{\mathbf{d}_d^{n-*}}(\mathcal{M}_d(\ell^+); \mathbb{Q}) \\ &\cong IP^1 H_{*-(d-1)}(\mathcal{M}_d(\ell^+); \mathbb{Q}). \end{aligned}$$

The long exact sequence of $(\mathcal{M}_d(\ell), \mathcal{M})$ thus turns into

$$\begin{aligned} \dots \longrightarrow IP^1 H_{\mathbf{d}_d^{n-2+1}}(\mathcal{M}_d(\ell^+); \mathbb{Q}) \longrightarrow IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}; \mathbb{Q}) \longrightarrow \\ IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell); \mathbb{Q}) \longrightarrow IP^1 H_{\mathbf{d}_d^{n-2}}(\mathcal{M}_d(\ell^+); \mathbb{Q}) \longrightarrow IP^1 H_{\mathbf{d}_d^{n-1-1}}(\mathcal{M}; \mathbb{Q}) \end{aligned}$$

and the last term is 0 by Lemma 7.1.

By induction, the rank of $IP^1 H_{\mathbf{d}_d^{n-2}}(\mathcal{M}_d(\ell^+); \mathbb{Q})$ is $a_1(\ell^+)$. Note that $\{i, n-1\}$ is ℓ^+ -short if and only if $\{i, n-1, n\}$ is ℓ -short. The rank of $IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}; \mathbb{Q})$ is $1 + c_{n-4}$, where c_{n-4} is the number of critical points of index $(n-4)(d-1)$ of F . These critical points correspond to sets $\{i\}$ with $\{i, n\}$ ℓ -short, but $\{i, n-1, n\}$ ℓ -long. Hence $1 + c_{n-4} + a_1(\ell^+) = a_1(\ell) = n - 1$. Therefore the rank of $IP^1 H_{\mathbf{d}_d^{n-1}}(\mathcal{M}_d(\ell))$ can be at most $n - 1$. By Proposition 7.7, it has to be $n - 1$. \square

8. CALCULATION OF THE INTERSECTION RING

Whenever $J \subset \{1, \dots, n-1\}$ satisfies $J \cup \{n\}$ is ℓ -short, we have an element $Y_J \in IH^{|J|(d-1)}(\mathcal{M}_d(\ell))$. We want these to be linearly independent. In order to do this, we construct explicitly the Poincaré dual of $\mathcal{M}_d(\ell_J)$.

Lemma 8.1. *Let $n \geq d + 1 \geq 5$ and $\ell \in \mathbb{R}^n$ be a generic, ordered, d -normal length vector, and $J \subset \{1, \dots, n-1\}$ with $J \cup \{n\}$ being ℓ -short. Then there exists an element $Z_J \in I^0 H_{|J|(d-1)}(\mathcal{M}_d(\ell))$ with*

$$Z_J \cdot Y_K = \begin{cases} 1 & J = K \\ 0 & \text{else} \end{cases}$$

Proof. We want to construct an embedding $(S^{d-1})^{|J|} \rightarrow \mathcal{M}_d(\ell) - \mathcal{N}_{d-2}(\ell)$ which has empty intersection with $\mathcal{M}_d(\ell_K)$ for every $K \subset \{1, \dots, n-1\}$ with $|K| = |J|$, $K \cup \{n\}$ ℓ -short and $K \neq J$, and which intersects $\mathcal{M}_d(\ell_J)$ transversely in exactly one point. By the standard properties of the intersection pairing this will prove the Lemma.

After reordering, we can assume that $J = \{k+1, \dots, n-1\}$. By d -normality, we have $k \geq d \geq 4$. By genericity we can assume that $\ell_1 < \ell_2 < \dots < \ell_k < \ell_n$. We have

$$\ell_1 + \dots + \ell_{k-1} - \ell_k > \ell_{k+1} + \dots + \ell_{n-1} - \ell_n$$

as

$$\ell_1 + \dots + \ell_{k-1} + \ell_n > \ell_{k+1} + \dots + \ell_{n-1} + \ell_k$$

which is true because $J \cup \{n\}$ is ℓ -short and $\ell_n > \ell_k$.

Let $B \subset \mathbb{R}^d$ be the closed ball of radius $\ell_{k+1} + \dots + \ell_{n-1}$ centered at $-\ell_n \cdot e_1$. We want to find a map $f: B \rightarrow (S^{d-1})^k$ with $A \circ f = \text{id}_B$, where $A: (S^{d-1})^k \rightarrow \mathbb{R}^d$ is given by

$$A(x_1, \dots, x_k) = \sum_{i=1}^k \ell_i x_i.$$

The map $F: (S^{d-1})^{|J|} \rightarrow \mathcal{C}_d(\ell)$ given by

$$(3) \quad F(x_{k+1}, \dots, x_{n-1}) = \left(f \left(-\ell_n \cdot e_1 - \sum_{i=k+1}^{n-1} \ell_i x_i \right), x_{k+1}, \dots, x_{n-1} \right)$$

is then nearly the map that we need.

In order to construct f , let us first consider the case $d = 2$. We begin by constructing a ‘Snake charmer’, a map $\gamma: [0, 1] \rightarrow (S^{d-1})^k$ such that

$$(4) \quad A \circ \gamma(t) = (tC_1 + (1-t)C_2) \cdot e_1$$

where

$$-(\ell_1 + \dots + \ell_k) < C_1 < -(\ell_{k+1} + \dots + \ell_n)$$

and

$$\ell_{k+1} + \dots + \ell_{n-1} - \ell_n < C_2 < \ell_1 + \dots + \ell_{k-1} - \ell_k.$$

To do this, we can start in the position $(e_1, \dots, e_1, -e_1) \in (S^{d-1})^k$ and then start to rotate the $(k-1)$ -th coordinate counterclockwise into the upper half plane. At the same time, the k -th coordinate rotates counterclockwise so that the robot arm $\ell_{k-1}x_{k-1} + \ell_k x_k$ remains on the first axis. We continue this until the $(k-1)$ -th coordinate is nearly rotated to $-e_1$. After that we rotate the $(k-2)$ -th coordinate counterclockwise into the upper half plane and rotate the k -th and $(k-1)$ -th coordinate so that the robot arm consisting of the last three coordinates remains on the first axis. Here the k -th and $(k-1)$ -th coordinates are rotated by the same amount, so that these two links remain stiff. When the $(k-2)$ -th coordinate nearly reached the $-e_1$ position, we start to rotate the $(k-3)$ -th coordinate counterclockwise, using the last three coordinates to keep the robot arm on the first axis. We continue this until all coordinates are near $-e_1$. After reparametrisation, we have the desired snake charmer.

We actually do not want any of the links to point to e_1 . So rather than starting with $(e_1, \dots, e_1, -e_1)$ we start at a position $(x^*, \dots, x^*, y^*) \in (S^{d-1})^k$ with x^* in the upper half plane so close to e_1 that the resulting snake charmer still satisfies (4). Figure 2 indicates this map. The outer circle has radius $\ell_1 + \dots + \ell_k$, so represents

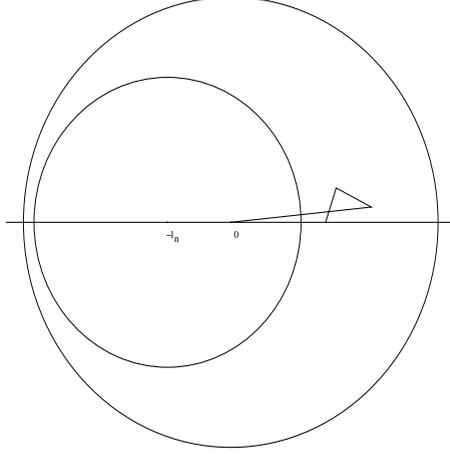


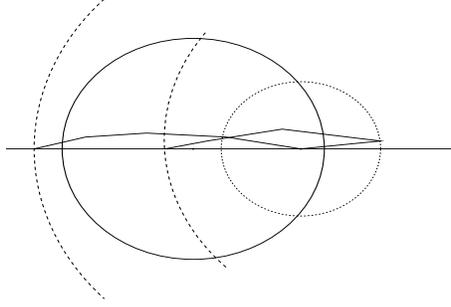
FIGURE 2. Snake charming along the x -axis.

all the points the robot arm in the first k coordinates could reach, and the inner circle bounds B .

This construction also provides numbers $0 = t_0 < t_1 < \dots < t_{k-1} = 1$ such that on the interval $[t_i, t_{i+1}]$ the first $k - (i + 2)$ coordinates are fixed, the $k - (i + 1)$ -th coordinate rotates from a position near e_1 to a position near $-e_1$, and all other coordinates have negative scalar product with e_1 . We refer to $A \circ \gamma$ as the robot arm R .

We now want to extend γ to a map $\Gamma: [0, 1] \times [-1, 1] \rightarrow (S^{d-1})^k$ so that $A \circ \Gamma$ restricts to a homeomorphism of a closed subset $\tilde{B} \subset [0, 1] \times [-1, 1]$ to $B \subset \mathbb{R}^2$. Consider the interval $[t_{k-2}, 1] \subset [0, 1]$. On this interval we want to rotate the robot arm R into the plane. To do this rotation, consider $[-1, 1] \subset S^1$ as those points with first coordinate non-positive (and 0 corresponding to $-e_1 \in S^1$). We need a map $h: [-1, 1] \rightarrow \text{SO}(2)$ with $h(x) \cdot (-e_1) = x$, which is no problem as $\text{SO}(2)$ can be identified with S^1 via the action. We could then define Γ on $[t_{k-2}, 1] \times [-1, 1]$ by $\Gamma(t, x) = h(x) \cdot \gamma(t)$. Doing this will make it difficult to extend Γ to $[0, 1] \times [-1, 1]$, and also the first coordinate can be rotated to e_1 .

Note that there is a unique point $t^* \in (t_{k-2}, 1)$ with the first coordinate of $\gamma(t^*)$ equal to e_2 . From this point on we do not rotate the first coordinate by the same amount as the other coordinates, and by the time $t = t_{k-2}$, only the coordinates $2, \dots, k$ will rotate via h , the first coordinate will be fixed in the position x^* . This way we can assure that the first coordinate is always different from e_1 . The other coordinates are also different from e_1 , as they start with negative scalar product with e_1 and rotate by at most an angle of $\pi/4$. Figure 3 indicates the movement of the robot arm near $t = 1$ and near $t = t_{k-2}$. The solid circle bounds B , the dotted circle indicates the movements of the first coordinate of the robot arm R .

FIGURE 3. Rotating along the origin and along x^* .

By elementary geometry, the map $A \circ \Gamma$ maps a closed subset of $[t_{k-2}, 1] \times [-1, 1]$ homeomorphically onto $B \cap A \circ \Gamma([t_{k-2}, 1] \times [-1, 1])$.

We now extend Γ to $[t_{k-3}, t_{k-2}] \times [-1, 1]$ in basically the same way. We first rotate all of the coordinates $2, \dots, k$, eventually rotating the second coordinate less, until it is no longer rotated at time $t = t_{k-3}$.

After finitely many steps we have a map $\Gamma: [0, 1] \times [-1, 1] \rightarrow (S^{d-1})^k$ and a closed subset $\tilde{B} \subset [0, 1] \times [-1, 1]$ such that $A \circ \Gamma$ maps \tilde{B} homeomorphically onto B .

Note that we were assuming $d = 2$, and we need the analogous result for $d \geq 4$. In order to do this, replace $[-1, 1]$ by $D^{d-1} = \{(x_1, \dots, x_d) \in S^{d-1} \mid x_1 \leq 0\}$. The map $p: \text{SO}(d) \rightarrow S^{d-1}$ given by $p(A) = A \cdot (-e_1)$ is a fiber bundle, so there exists a section $h: D^{d-1} \rightarrow \text{SO}(d)$ which then can be used to define Γ .

The resulting map $F: (S^{d-1})^{|J|} \rightarrow \mathcal{C}_d(\ell)$ given by (3) then intersects $\mathcal{C}_d(\ell_J)$ transversely in exactly one point, and has empty intersection with $\mathcal{C}_d(\ell_K)$ for all $K \subset \{1, \dots, n-1\}$ with $|K| = |J|$ and $K \neq J$. The point of intersection is given by $F(e_1, \dots, e_1)$.

Consider the induced map $\bar{F}: (S^{d-1})^{|J|} \rightarrow \mathcal{M}_d(\ell)$. With the current construction we have that γ has image in $(S^1)^k$. This means that the image of \bar{F} will intersect lower strata of $\mathcal{M}_d(\ell)$. However, if we modify the robot arm γ slightly by using the higher dimensions, the map $A \circ \Gamma$ will remain injective and we can repeat the construction so that the image of \bar{F} is in $\mathcal{M}_d(\ell) - \mathcal{N}_{d-2}(\ell)$. This is using that $k \geq d$. The transverse intersection of F with $\mathcal{C}_d(\ell_J)$ induces a transverse intersection of \bar{F} with $\mathcal{M}_d(\ell_J)$ in exactly one point, so that \bar{F} induces an element $Z_J \in I^0 H_{|J|(d-1)}(\mathcal{M}_d(\ell))$ which has the desired properties. \square

The fact that the Z_J can be defined with $\mathbf{0}$ -perversity means that the Y_J remain linearly independent in ordinary homology. The reduced intersection ring for even d can now be determined.

Proof of Theorem 4.5. We can assume that ℓ is ordered. By Theorem 7.8 the reduced intersection ring is generated by Y_1, \dots, Y_k with $k = a_1(\ell) = |\mathcal{S}_1(\ell)|$. As d is even, we get that

$$Y_i \cdot Y_j = -Y_j \cdot Y_i$$

for all $i, j \leq k$ by [8, §2.4]. Therefore

$$IH^{(d-1)*}(\mathcal{M}_d(\ell); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[Y_1, \dots, Y_k]/I$$

for some ideal I . If $K \subset \{1, \dots, n-1\}$ is such that $K \cup \{n\}$ is ℓ -long, then $\mathcal{M}_d(\ell_K) = \emptyset$, so $Y_K = 0$, which means that $Y_K \in I$. It remains to show that there are no other relations in I . But by Lemma 8.1 all Y_J with $J \in \mathcal{S}(\ell)$ are linearly independent, which means that I is indeed generated by Y_K with $K \cup \{n\}$ long. \square

Remark 8.2. The statement of Theorem 1.1 is true in the case $d = 3$, but the condition of 3-normality can be replaced by $n > 4$. So in fact there are only two chambers which have homeomorphic linkage spaces but are not the same chamber up to permutation. For $d = 4$ one may therefore ask whether 4-normality can be replaced by $n > 5$, or if one can give an example of length vectors from different chambers with $n > 5$ with homeomorphic linkage spaces.

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