Bounding Clique-width via Perfect Graphs *

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Abstract. We continue the study into the clique-width of graph classes defined by two forbidden induced graphs. We present three new classes of bounded clique-width and one of unbounded clique-width. The four new graph classes have in common that one of their two forbidden induced subgraphs is the diamond. To prove boundedness of clique-width for the first three cases we develop a technique based on bounding clique covering number in combination with reduction to subclasses of perfect graphs. We extend our proof of unboundedness for the fourth case to show that Graph Isomorphism is Graph Isomorphism-complete on the same graph class.

Keywords: clique-width, forbidden induced subgraphs, graph class

1 Introduction

Clique-width is a well-known graph parameter and its properties are well studied; see for example the surveys of Gurski [27] and Kamiński, Lozin and Milanič [30]. Computing the clique-width of a given graph is NP-hard, as shown by Fellows, Rosamond, Rotics and Szeider [24]. Nevertheless, many NP-complete graph problems are solvable in polynomial time on graph classes of *bounded* clique-width, that is, classes in which the clique-width of each of its graphs is at most c for some constant c. This follows by combining the fact that if a graph G has clique-width at most c then a so-called $(8^c - 1)$-expression for G can be found in polynomial time [38] together with a number of results [17,31,40], which show that if a q-expression is provided for some fixed q then certain classes of problems can be solved in polynomial time. A well-known example of such a problem is the Colouring problem, which is that of testing whether the vertices of a graph can be coloured with at most k colours such that no two adjacent vertices are

* The research in this paper was supported by EPSRC (EP/K025090/1). The second author is grateful for the generous support of the Graduate (International) Research Travel Award from Simon Fraser University and Dr. Pavol Hell's NSERC Discovery Grant. An extended abstract of this paper appeared in the proceedings of LATA 2015 [19]

coloured alike. Due to these algorithmic implications, it is natural to research whether the clique-width of a given graph class is bounded.

It should be noted that having bounded clique-width is a more general property than having bounded treewidth, that is, every graph class of bounded treewidth has bounded clique-width but the reverse is not true [15]. Clique-width is also closely related to other graph width parameters, e.g. for any class, having bounded clique-width is equivalent to having bounded rank-width [39] and also equivalent to having bounded NLC-width [29]. Moreover, clique-width has been studied in relation to graph operations, such as edge or vertex deletions, edge subdivisions and edge contractions. For instance, Courcelle [16] recently proved that if $\mathcal{G}$ is the class of graphs of clique-width 3 and $\mathcal{G}'$ is the class of graphs obtained from graphs in $\mathcal{G}$ by applying one or more edge contraction operations then $\mathcal{G}'$ has unbounded clique-width.

The classes that we consider in this paper consist of graphs that can be characterized by a family $\{H_1, \ldots, H_p\}$ of forbidden induced subgraphs (such graphs are said to be $(H_1, \ldots, H_p)$-free). The clique-width of such graph classes has been extensively studied in the literature (e.g. [2,3,6,7,8,9,10,11,12,18,21,25,34,35,36,37]). It is straightforward to verify that the class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of the 4-vertex path P_4 (see also [22]). Hence, Dabrowski and Paulusma [22] investigated for which pairs (H_1, H_2) the class of (H_1, H_2)-free graphs has bounded clique-width. In this paper we solve a number of the open cases. The underlying research question is:

What kinds of properties of a graph class ensure that its clique-width is bounded?

As such, our paper is to be interpreted as a further step towards this direction. In particular, we believe there is a clear motivation for our type of research, in which new graph classes of bounded clique-width are identified, because it may lead to a better understanding of the notion of clique-width. It should be noted that clique-width is one of the most difficult graph parameters to deal with. To illustrate this, no polynomial-time algorithms are known for computing the clique-width of very restricted graph classes, such as unit interval graphs, or for deciding whether a graph has clique-width at most c for any fixed $c \geq 4$ (as an aside, such an algorithm does exist for $c = 3$ [14]). Heule and Szeider recently designed a practical computational method (based on a SAT encoding) for determining the clique-width of small graphs and were able to use it to calculate the clique-width of all graphs on at most ten vertices [28].

Rather than coming up with ad hoc techniques for solving specific cases, we aim to develop more general techniques for attacking a number of the open cases simultaneously. Our technique in this paper is obtained by generalizing an approach followed in the literature. In order to illustrate this approach with some examples, we first need to introduce some notation (see Section 2 for all other terminology).

Notation. The disjoint union $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG. The complement of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G. The graphs C_r, K_r and P_r denote the cycle, complete graph and path on r vertices, respectively. The graph $\overline{2P_1 + P_2}$ is called the *diamond*. The graph $K_{1,3}$ is the 4-vertex star, also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{h,i,j}$ be the *subdivided claw* whose three edges are subdivided $h-1$, $i-1$ and $j-1$ times, respectively; note that $S_{1,1,1} = K_{1,3}$. The *clique covering number* $\overline{\chi}(G)$ of a graph G is the smallest number of (mutually vertex-disjoint) cliques such that every vertex of G belongs to exactly one clique.

Our technique. Dabrowski and Paulusma [21] determined all graphs H for which the class of H-free bipartite graphs has bounded clique-width. Such a classification turns out to also be useful for proving boundedness of the clique-width for other graph classes. For instance, in order to prove that $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graphs have bounded clique-width, the given graphs were first reduced to $(P_1 + S_{1,1,2})$-free bipartite graphs [22]. In a similar way, Dabrowski, Lozin, Raman and Ries [20] proved that $(K_3, K_{1,3} + K_2)$-free graphs and $(K_3, S_{1,1,3})$-free have bounded clique-width by reducing to a subclass of bipartite graphs. Note that bipartite graphs are perfect graphs. This motivated us to develop a technique based on perfect graphs that are not necessarily bipartite. In order to do so, we need to combine this approach with an additional tool. This tool is based on the following observation. If the vertex set of a graph can be partitioned into a small number of cliques and the edges between them are sufficiently sparse, then the clique-width is bounded (see also Lemma 8). Our technique can be summarized as follows:

1. Reduce the input graph to a graph that is in some subclass of perfect graphs;
2. While doing so, bound the clique covering number of the input graph.

Another well-known subclass of perfect graphs is the class of chordal graphs. We show that besides the class of bipartite graphs, the class of chordal graphs and the class of perfect graphs itself may be used for Step 1.[1] We explain Steps 1-2 of our technique in detail in Section 3.

Our results. In this paper, we investigate whether our technique can be used to find new pairs (H_1, H_2) for which the clique-width of (H_1, H_2)-free graphs is bounded. We show that this is indeed the case. By applying our technique, we are able to present three new classes of (H_1, H_2)-free graphs of bounded clique-width.[2] By modifying walls via graph operations that preserve unboundedness of clique-width, we are also able to present a new class of (H_1, H_2)-free graphs of unbounded clique-width. Combining our results leads to the following theorem (see also Fig. 1).

[1] To exploit this further, we recently worked on characterizations of the boundedness of clique-width for classes of H-free split graphs [5], H-free chordal graphs [4] and H-free weakly chordal graphs [4]. For this paper, however, we rely only on existing results from the literature.

[2] We do not specify our upper bounds as this would complicate our proofs for negligible gain. This is because in our proofs we apply graph operations that exponentially increase the upper bound of the clique-width, which means that the bounds that could be obtained from our proofs would be very large and far from being tight.

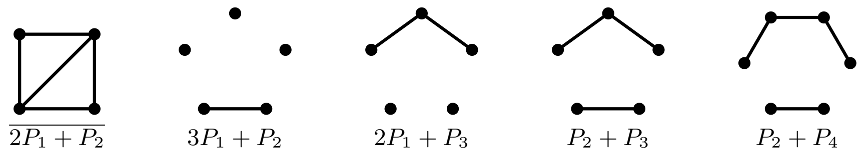

Fig. 1: The graphs in Theorem 1.

Theorem 1. *The class of* (H_1, H_2)*-free graphs has bounded clique-width if*

(i) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 = 3P_1 + P_2$*;*
(ii) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 = 2P_1 + P_3$*;*
(iii) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 = P_2 + P_3$*.*

The class of (H_1, H_2)*-free graphs has unbounded clique-width if*

(iv) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 = P_2 + P_4$*.*

We prove statements (i)–(iv) of Theorem 1 in Sections 4–7, respectively. In Section 7 we also prove that the Graph Isomorphism problem is Graph Isomorphism-complete for the class of $(\overline{2P_1 + P_2}, P_2 + P_4)$-free graphs[3]. This result was one of the remaining open cases in a line of research initiated by Kratsch and Schweitzer [32], who tried to classify the complexity of the Graph Isomorphism problem in graph classes defined by two forbidden induced subgraphs. The exact number of open cases is not known, but Schweitzer [42] very recently proved that this number is finite. There is a strong connection between classifying the boundedness of clique-width and the tractability of the Graph Isomorphism problem. Indeed, Grohe and Schweitzer [26] recently proved that Graph Isomorphism is polynomial-time solvable on graphs of bounded clique-width. If we assume that Graph Isomorphism cannot be solved in polynomial time (a long-standing open problem), this would mean that for every class of graphs $\mathcal{G}$ on which the Graph Isomorphism problem is Graph Isomorphism-complete, the clique-width of graphs in $\mathcal{G}$ must be unbounded.

Structural consequences. Theorem 1 reduces the number of open cases in the classification of the boundedness of the clique-width for (H_1, H_2)-free graphs to 13 open cases, up to an equivalence relation.[4] Note that the graph H_1 is the diamond in each of the four results in Theorem 1. Out of the 13 remaining

[3] After making our paper available on arXiv, it came to our attention [41] that Kratsch and Schweitzer independently proved this result, together with Theorem 1 (iv). However, they have not yet made these proofs and results publicly available.

[4] Let $H_1, \ldots, H_4$ be four graphs. Then the classes of (H_1, H_2)-free graphs and (H_3, H_4)-free graphs are *equivalent* if the unordered pair $\{H_3, H_4\}$ can be obtained from the unordered pair $\{H_1, H_2\}$ by some combination of the following two operations: complementing both graphs in the pair; or if one of the graphs in the pair is K_3, replacing it with $\overline{P_1 + P_3}$ or vice versa. If two classes are equivalent then one has bounded clique-width if and only if the other one does (see e.g. [22]).

cases, there are still three cases in which H_1 is the diamond, namely when $H_2 \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$. However, for each of these graphs H_2, it is not even known whether the clique-width of the corresponding smaller subclasses of (K_3, H_2)-free graphs is bounded. Of particular note is the class of $(K_3, P_1 + 2P_2)$-free graphs, which is contained in all of the above open cases and for which the boundedness of clique-width is unknown. Settling this case is a natural next step in completing the classification. Note that for K_3-free graphs the clique covering number is proportional to the size of the graph. Another natural research direction is to determine whether the clique-width of $(\overline{P_1 + P_4}, H_2)$-free graphs is bounded for $H_2 = P_2 + P_3$ (the clique-width is known to be unbounded for $H_2 \in \{3P_1 + P_2, 2P_1 + P_3\}$).

Dabrowski, Golovach and Paulusma [18] showed that Colouring restricted to $(sP_1 + P_2, \overline{tP_1 + P_2})$-free graphs is polynomial-time solvable for all pairs of integers s, t. They justified their algorithm by proving that the clique-width of the class of $(sP_1, \overline{tP_1 + P_2})$-free graphs is bounded only for small values of s and t, namely only for $s \leq 2$ or $t \leq 1$ or $s + t \leq 6$. In the light of these two results it is natural to try to classify the clique-width of the class of $(sP_1 + P_2, \overline{tP_1 + P_2})$-free graphs for all pairs (s, t). Theorem 1, combined with the aforementioned classification of the clique-width of $(sP_1, \overline{tP_1 + P_2})$-free graphs and the fact that any class of (H_1, H_2)-free graphs has bounded clique-width if and only if the class of $(\overline{H_1}, \overline{H_2})$-free graphs has bounded clique-width, immediately enables us to do this.

Corollary 1. *The class of $(sP_1 + P_2, \overline{tP_1 + P_2})$-free graphs has bounded clique-width if and only if $s \leq 1$ or $t \leq 1$ or $s + t \leq 5$.*

Algorithmic consequences. Our research was (partially) motivated by a study into the computational complexity of the Colouring problem for (H_1, H_2)-free graphs. As mentioned, Colouring is polynomial-time solvable on any graph class of bounded clique-width. Of the three classes for which we prove boundedness of clique-width in this paper, only the case of $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free (and equivalently[4] $(2P_1 + P_2, \overline{3P_1 + P_2})$-free) graphs was previously known to be polynomial-time solvable [18]. Hence, Theorem 1 gives us four new pairs (H_1, H_2) with the property that Colouring is polynomial-time solvable when restricted to (H_1, H_2)-free graphs, namely if

- $H_1 = 2P_1 + P_2$ and $H_2 = \overline{2P_1 + P_3}$;
- $H_1 = 2P_1 + P_2$ and $H_2 = \overline{P_2 + P_3}$;
- $H_1 = \overline{2P_1 + P_2}$ and $H_2 = 2P_1 + P_3$;
- $H_1 = \overline{2P_1 + P_2}$ and $H_2 = P_2 + P_3$.

There are still 15 potential classes of (H_1, H_2)-free graphs left for which both the complexity of Colouring and the boundedness of their clique-width is unknown [22].

2 Preliminaries

Below we define some graph terminology used throughout our paper. For any undefined terminology we refer to Diestel [23]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote an edge between vertices u and v by uv or vu; all edges in this paper are undirected. For $u \in V(G)$, the set $N(u) = \{v \in V(G) \mid uv \in E(G)\}$ is the *neighbourhood* of u in G. The *degree* of a vertex in G is the size of its neighbourhood. The *maximum degree* of G is the maximum vertex degree. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the *induced* subgraph of G, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. Let H be another graph. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. Let $X \subseteq V(G)$. We write $G \setminus X$ for the graph obtained from G after removing X. A set $M \subseteq E(G)$ is a *matching* if no two edges in M share an end-vertex. We say that two disjoint sets $S \subseteq V(G)$ and $T \subseteq V(G)$ are *complete* to each other if every vertex of S is adjacent to every vertex of T. If no vertex of S is joined to a vertex of T by an edge, then S and T are *anti-complete* to each other. Similarly, we say that a vertex u and a set S not containing u may be complete or anti-complete to each other. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. Recall that G is *$(H_1, \ldots, H_p)$-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write H_1-free instead of (H_1)-free.

The *clique-width* of a graph G, denoted by $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i;
(ii) taking the disjoint union of two labelled graphs G_1 and G_2;
(iii) joining each vertex with label i to each vertex with label j $(i \neq j)$;
(iv) renaming label i to j.

An algebraic term that represents such a construction of G and uses at most k labels is said to be a *k-expression* of G (i.e. the clique-width of G is the minimum k for which G has a k-expression). A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. We say that G is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets B and W. We say that (B, W) is a *bipartition* of G.

Let G be a graph. We define the following two operations. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets X and Y in G, the *bipartite complementation* operation with respect to X and Y acts on G by replacing every edge with one end-vertex in X and the other one in Y by a non-edge and vice versa.

We now state some useful facts for dealing with clique-width. We will use these facts throughout the paper. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k, γ)-*obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [34].

Fact 2. Subgraph complementation preserves boundedness of clique-width [30].

Fact 3. Bipartite complementation preserves boundedness of clique-width [30].

Fact 4. If $\mathcal{G}$ is a class of graphs and $\mathcal{G}'$ is the class of graphs obtained from graphs in $\mathcal{G}$ by recursively deleting all vertices of degree 1, then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}'$ has bounded clique-width [2,34].

The following lemmas are well-known and straightforward to check.

Lemma 1. *The clique-width of a forest is at most* 3*.*

Lemma 2. *The clique-width of a graph of maximum degree at most* 2 *is at most* 4*.*

Let G be a graph. The size of a largest independent set and a largest clique in G are denoted by $\alpha(G)$ and $\omega(G)$, respectively. The chromatic number of G is denoted by $\chi(G)$. We say that G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

We need the following well-known result, due to Chudnovsky, Robertson, Seymour and Thomas.

Theorem 2 (The Strong Perfect Graph Theorem [13]). *A graph is perfect if and only if it is C_r-free and $\overline{C_r}$-free for every odd $r \geq 5$.*

Recall that the clique covering number $\overline{\chi}(G)$ of a graph G is the smallest number of (mutually vertex-disjoint) cliques such that every vertex of G belongs to exactly one clique. If G is perfect, then $\overline{G}$ is also perfect. This follows directly from Theorem 2, but was already known earlier [33]. By definition, $\overline{G}$ can be partitioned into $\omega(\overline{G}) = \alpha(G)$ independent sets. This leads to the following well-known lemma.

Lemma 3 ([33]). *Let G be a perfect graph. Then $\overline{\chi}(G) = \alpha(G)$.*

We say that a graph G is *chordal* if it contains no induced cycle on four or more vertices. By Theorem 2, bipartite graphs and chordal graphs are perfect (for chordal graphs this also follows from a result of Berge [1]).

The following three lemmas give us a number of subclasses of perfect graphs with bounded clique-width. We will make use of these lemmas later on in the proofs as part of our technique.

Lemma 4 ([21]). *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H = sP_1$ *for some* $s \geq 1$
- $H \subseteq_i K_{1,3} + 3P_1$
- $H \subseteq_i K_{1,3} + P_2$
- $H \subseteq_i P_1 + S_{1,1,3}$
- $H \subseteq_i S_{1,2,3}$.

Lemma 5 ([25]). *The class of chordal $(\overline{2P_1 + P_2})$-free graphs has clique-width at most* 3.

Lemma 6 ([20]). *The class of $(K_3, K_{1,3} + P_2)$-free graphs has bounded clique-width.*

Finally, we also need the following lemma, which corresponds to the first lemma of [18] by complementing the graphs under consideration.

Lemma 7 ([18]). *Let $s \geq 0$ and $t \geq 0$. Then every $(\overline{sP_1 + P_2}, tP_1 + P_2)$-free graph is $(K_{s+1}, tP_1 + P_2)$-free or $(\overline{sP_1 + P_2}, (s^2(t-1)+2)P_1)$-free.*

3 The Clique Covering Lemma

In Section 2 we stated several lemmas that can be used to bound the clique-width if we manage to reduce the graphs in the class under consideration to some specific graph class. As we shall see, such a reduction is not always sufficient and the following lemma forms a crucial part of our technique (we use it in the proofs of each of our three main boundedness results).

Lemma 8. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if G is a $(\overline{2P_1 + P_2}, 2P_2 + P_4)$-free graph with $\overline{\chi}(G) \leq k$ then $\mathrm{cw}(G) \leq f(k)$.*

Proof. Let $k \geq 1$. Suppose G is a $(\overline{2P_1 + P_2}, 2P_2 + P_4)$-free graph with $\overline{\chi}(G) \leq k$, that is, $V(G)$ can be partitioned into k cliques $X_1, \ldots, X_k$. By Fact 1, if any of these cliques has less than $k+7$ vertices, we may remove it. If two cliques X_i, X_j are complete to each other then they can be replaced by the single clique $X_i \cup X_j$. After doing this exhaustively, we end up with $k' \leq k$ cliques $Y_1, \ldots, Y_{k'}$, each of which is of size at least $k'+7$ and no two of which are complete to each other.

Suppose a vertex $x \in Y_i$ has two neighbours y_1, y_2 in a different clique Y_j. If x is non-adjacent to some vertex $y_3 \in Y_j$ then $G[y_1, y_2, y_3, x]$ is a $\overline{2P_1 + P_2}$. Thus x must be complete to Y_j. If there is another vertex $x' \in Y_i$ which is complete to Y_j, then every vertex in Y_j has at least two neighbours in Y_i, so Y_i and Y_j must be complete to each other, which we assumed was not the case. Therefore, for any ordered pair (Y_i, Y_j) every vertex of Y_i, except possibly one vertex x, has at most one neighbour in Y_j. By Fact 1, if such vertices x exist, we may delete them, since there are at most $k'(k'-1)$ of them. We obtain a set of cliques $Z_1, \ldots, Z_{k'}$, all of which have size at least $(k'+7) - (k'-1) = 8$. Let $G_Z = G[Z_1 \cup \cdots \cup Z_{k'}]$. We have shown that G has bounded clique-width if and only if G_Z does.

First suppose that $k' \leq 3$. Let G'_Z be the graph obtained from G_Z by complementing the edges in each set Z_i. As G'_Z has maximum degree at most 2, it has clique-width at most 4 by Lemma 2. By Fact 2, G_Z has bounded clique-width if and only if G'_Z does. Hence, G_Z, and thus G, has bounded clique-width.

Now suppose that $k' \geq 4$. If G_Z is a union of disjoint cliques then its clique-width is at most 2. Otherwise, there must be two vertices in different cliques Z_i that are adjacent. Without loss of generality, assume $x_6 \in Z_1$ and $x_7 \in Z_2$ are adjacent. We will show that G_Z (and therefore G) contains an induced $2P_2 + P_4$, two vertices of which are x_6 and x_7. Indeed, since $|Z_1| \geq 8$, there must be a vertex $x_5 \in Z_1$ that is non-adjacent to x_7. Similarly, since $|Z_1| \geq 8$ there must be a vertex $x_8 \in Z_2$ that is non-adjacent to x_5 and x_6. Now $G[x_5, x_6, x_7, x_8]$ is a P_4. Since $|Z_3| \geq 8$, there must be two vertices $x_3, x_4 \in Z_3$ that are non-adjacent to $x_5, \ldots, x_8$. Since $|Z_4| \geq 8$, there must be two vertices $x_1, x_2 \in Z_4$ that are non-adjacent to $x_3, \ldots, x_8$. Now $G[x_1, \ldots, x_8]$ is a $2P_2 + P_4$. This contradiction completes the proof. □

It is easy to see that we can generalize Lemma 8 to be valid for other classes of graphs, for example for any constant $s \geq 2$, the lemma holds for $(\overline{2P_1 + P_2}, 2K_s + P_4)$-free graphs: we repeat all arguments of the proof of Lemma 8 except that instead of constructing cliques $Y_1, \ldots, Y_{k'}$ of size at least $k' + 7$ we construct cliques $Y_1, \ldots, Y_{k'}$ of size at least $k' + 2s + 3$, so the cliques Z_i will each contain at least $2s + 4$ vertices. However, such generalizations are not necessary for the main results of this paper.

4 The Proof of Theorem 1 (i)

Here is the proof of our first main result.

Theorem 1 (i). *The class of $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free graphs has bounded clique-width.*

Proof. Let G be a $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free graph. Applying Lemma 7 we find that G is $(K_3, 3P_1 + P_2)$-free or $(\overline{2P_1 + P_2}, 10P_1)$-free. If G is $(K_3, 3P_1 + P_2)$-free then it has bounded clique-width by Lemma 6, so we may assume it is $(\overline{2P_1 + P_2}, 10P_1, 3P_1 + P_2)$-free.

Suppose G contains a C_5 (respectively C_7) on vertices $v_1, \ldots, v_5$ (respectively $v_1, \ldots, v_7$) in that order. Let S_i be the set of vertices that have i neighbours on the cycle, but are not on the cycle itself. Let v_i and v_j be non-consecutive vertices of the cycle. The set X of vertices adjacent to both v_i and v_j must be independent, otherwise v_i, v_j and two adjacent vertices from X would induce a $\overline{2P_1 + P_2}$. Since G is $10P_1$-free, $|X| \leq 9$. Therefore, by Fact 1, we may delete all such vertices, of which there are at most $9 \times 5 \times 2 \div 2 = 45$ (respectively $9 \times 7 \times 4 \div 2 = 126$). All remaining vertices must be adjacent to at most two vertices of the cycle (so S_i is empty for $i \geq 3$), and if a vertex is adjacent to two vertices of the cycle, these two vertices must be consecutive vertices of the cycle.

Suppose x_1, x_2 are adjacent to two consecutive vertices of the cycle, v_i and v_j, say. Then x_1, x_2 must be adjacent, otherwise $G[v_i, v_j, x_1, x_2]$ would be a $\overline{2P_1 + P_2}$.

Therefore S_2 can be partitioned into at most five (respectively seven) cliques. Let Y be the set of vertices, adjacent to v_1 and none of the other vertices on the cycle. If $x_1, x_2 \in Y$ are non-adjacent then $G[x_1, x_2, v_2, v_4, v_5]$ would be a $3P_1 + P_2$, so Y must be a clique. Therefore S_1 can be partitioned into at most five (respectively seven) cliques. Finally, note that if $x_1, x_2 \in S_0$ are non-adjacent then $G[x_1, x_2, v_1, v_3, v_4]$ is a $3P_1 + P_2$, so S_0 must be a clique. By Fact 1, we may delete the vertices $v_1, \ldots, v_5$ (respectively $v_1, \ldots, v_7$). This leaves a graph whose vertex set can be decomposed into $5 + 5 + 1 = 11$ (respectively $7 + 7 + 1 = 15$) cliques, in which case we are done by Lemma 8.

We may therefore assume that G contains no induced C_5 or C_7. Since G is $(3P_1 + P_2)$-free it contains no odd cycle on nine or more vertices. Since it is $\overline{C_5}$-free (because $\overline{C_5} = C_5$), and $\overline{2P_1 + P_2}$-free, it contains no induced complements of odd cycles of length 5 or more. By Theorem 2 we find that G must be perfect. Then G has clique covering number at most $\alpha(G)$ by Lemma 3. Since G is $10P_1$-free, $\alpha(G) \leq 9$. Applying Lemma 8 completes the proof. $\square$

5 The Proof of Theorem 1 (ii)

In this section we prove the second of our four main results.

Theorem 1 (ii). *The class of $(\overline{2P_1 + P_2}, 2P_1 + P_3)$-free graphs has bounded clique-width.*

Proof. Let G be a $(\overline{2P_1 + P_2}, 2P_1 + P_3)$-free graph. We need the following claim.

Claim 1. Let C and I be a clique and independent set of G, respectively, with $C \cap I = \emptyset$. Then there is a set $S \subseteq C \cup I$ containing at most four vertices, such that every edge with one end-vertex in C and the other one in I is incident to at least one vertex of S (see also Fig. 2).

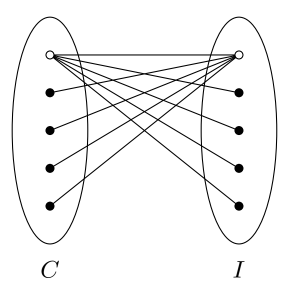

Fig. 2: An example of a graph satisfying Claim 1. In this example S consists of the two white vertices. The edges in the clique C are not shown.

We prove Claim 1 as follows. Assume $|I|, |C| \geq 5$, as otherwise we can simply set S to equal either I or C respectively. Since G is $\overline{2P_1 + P_2}$-free, every vertex in I must be adjacent to zero, one or all vertices of C. Since G is $\overline{2P_1+P_2}$-free, at most one vertex z of I can be complete to C. If such a vertex z exists, let $I' = I \setminus \{z\}$, and add z to S, otherwise let $I' = I$ and leave S empty. Now $|I'| \geq 4$ and every vertex of I' has at most one neighbour in C. It remains to show that it is possible to disconnect I' and C by deleting at most three vertices (which we add to S). If a vertex x in C has two neighbours and two non-neighbours in I', then these four vertices, together with x would induce a $2P_1+P_3$ in G. If some vertex of C is adjacent to all but at most one vertex of I', then since each vertex of I' has at most one neighbour in C, deleting at most two vertices in C will disconnect I' and C. We may therefore assume that each vertex in C has at most one neighbour in I'. Therefore the edges between I' and C form a matching. If there are no edges between C and I' then we are done. Suppose $x \in I'$ is adjacent to $y \in C$. Since $|C| \geq 5$, we can choose $y' \in C$ which is not adjacent to x. Since $|I'| \geq 4$, we can choose $x', x'' \in I'$ which are non-adjacent to y and y'. However, then $G[x', x'', x, y, y']$ is a $2P_1+P_3$, which is a contradiction. This completes the proof of Claim 1.

Now suppose G contains a C_4, say on vertices v_1, v_2, v_3, v_4 in order. Let X be the set of vertices non-adjacent to v_1, v_2, v_3 and v_4. For $i \in \{1,2,3,4\}$ let W_i be the set of vertices adjacent to v_i, but non-adjacent to all other vertices of the cycle. For $i \in \{1,2\}$ let V_i be the set of vertices not on the cycle that are adjacent to precisely v_{i-1} and v_{i+1} on the cycle (throughout this part of the proof we interpret subscripts modulo 4). For $i \in \{1,2,3,4\}$, let Y_i be the set of vertices adjacent to precisely v_i and v_{i+1} on the cycle. No vertex can be adjacent to three or more vertices of the cycle, otherwise this vertex together with three of its neighbours on the cycle would induce a $\overline{2P_1+P_2}$ in G.

If $x, y \in W_i \cup X$ are non-adjacent then $G[x, y, v_{i+1}, v_{i+2}, v_{i+3}]$ is a $2P_1+P_3$. Therefore $W_i \cup X$ is a clique. If $x, y \in Y_i$ are non-adjacent then $G[v_i, v_{i+1}, x, y]$ is a $\overline{2P_1+P_2}$. Therefore Y_i is a clique. If $x, y \in V_i$ are adjacent then $G[x, y, v_{i-1}, v_{i+1}]$ is a $\overline{2P_1+P_2}$, so V_i is an independent set. This means that the vertex set of G can be partitioned into a cycle on four vertices, eight cliques and two independent sets. By Claim 1, after deleting the original cycle (four vertices) and at most $4 \times 2 \times 8 = 48$ vertices (which we may do by Fact 1), we obtain a graph whose vertex set is partitioned into eight cliques and two independent sets such that the two independent sets are not in the same components as the cliques. The components containing the cliques have bounded clique-width by Lemma 8. The two independent sets form a bipartite $(2P_1+P_3)$-free graph, which has bounded clique-width by Lemma 4. This completes the proof for the case where G contains a C_4.

We may now assume that G is $(C_4, \overline{2P_1+P_2}, 2P_1+P_3)$-free. Because G is $(2P_1+P_3)$-free, it cannot contain a cycle on eight or more vertices. Suppose it contains a cycle on vertices $v_1, \ldots, v_k$ in order, where $k \in \{5,6,7\}$. Let X be the set of vertices with no neighbours on the cycle, W_i be the set of vertices adjacent to v_i, but no other vertices on the cycle, V_i be the set of vertices adjacent to v_i and v_{i+1}, but no other vertices of the cycle and if v_i and v_j are not consecutive vertices of the cycle, let $V_{i,j}$ be the set of vertices adjacent to both v_i and v_j.

(Throughout this part of the proof we interpret subscripts modulo k. Note that a vertex may be in more than one set $V_{i,j}$.)

The set $X \cup W_i$ must be a clique, otherwise two non-adjacent vertices in $X \cup W_i$ together with $v_{i+1}, v_{i+2}, v_{i+3}$ would form a $2P_1 + P_3$. The set V_i must be a clique, as otherwise two non-adjacent vertices in V_i, together with v_i and v_{i+1} would from a $\overline{2P_1 + P_2}$. The set $V_{i,j}$ cannot contain two vertices, otherwise these two vertices, together with v_i and v_j, would form a C_4 or a $\overline{2P_1 + P_2}$, depending on whether the two vertices were non-adjacent or adjacent, respectively. We delete all vertices from all the $V_{i,j}$ sets; we may do so by Fact 1 as there are at most $\frac{1}{2}k(k-3)$ of such vertices. In this way we obtain a graph that can be partitioned into at most $2k$ cliques. Therefore G has bounded clique-width by Lemma 8.

Finally, we may assume that G contains no induced cycle on four or more vertices. In other words, we may assume that G is chordal. It remains to recall that $(\overline{2P_1 + P_2})$-free chordal graphs have bounded clique-width by Lemma 5. This completes the proof. $\square$

6 The Proof of Theorem 1 (iii)

In this section we prove the third of our four main results, namely that the class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs has bounded clique-width. We first establish, via a series of lemmas, that we may restrict ourselves to graphs in this class that are also (C_4, C_5, C_6, K_5)-free.

Lemma 9. *The class of those $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs that contain a K_5 has bounded clique-width.*

Proof. Let G be a $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graph. Let X be a maximal (by set inclusion) clique in G containing at least five vertices. Since X is maximal and $(\overline{2P_1 + P_2})$-free, every vertex not in X has at most one neighbour in X. By Fact 4 we may therefore assume that every component of $G \setminus X$ contains at least two vertices.

Suppose there is a P_3 in $G \setminus X$, say on vertices x_1, x_2, x_3 in that order. Since $|X| \geq 5$, we can find $y_1, y_2 \in X$ none of which are adjacent to any of x_1, x_2, x_3. Then $G[y_1, y_2, x_1, x_2, x_3]$ is a $P_2 + P_3$. Hence $G \setminus X$ is P_3-free and must therefore be a union of disjoint cliques $X_1, \ldots, X_k$. Suppose there is only at most one such clique. Then $\overline{G}$ is a $(2P_1 + P_2)$-free bipartite graph, and so G has bounded clique-width by Fact 2 and Lemma 4. From now on we assume that $k \geq 2$, that is, $G \setminus X$ contains at least two cliques.

Suppose that some vertex $x \in X$ is adjacent to a vertex $y \in X_i$. We claim that x can have at most one non-neighbour in any X_j. First suppose $j \neq i$. For contradiction, assume that x is non-adjacent to $z_1, z_2 \in X_j$, where $j \neq i$. Since $|X| \geq 5$ and each vertex that is not in X has at most one neighbour in X, there must be a vertex $x' \in X$ that is non-adjacent to y, z_1 and z_2. Then $G[z_1, z_2, x', x, y]$ is a $P_2 + P_3$, a contradiction. Now suppose $j = i$. Since $k \geq 2$, there must be another clique X_j with $j \neq i$. Since X_j must contain at least two vertices and x can have at most one non-neighbour in X_j, there must be a neighbour y' of x in X_j. By the same argument as above, x can therefore have at most one non-neighbour in X_i. We conclude that if some vertex x has a neighbour in $\{X_1, \ldots, X_k\}$ then it has at most one non-neighbour in each X_j.

As every vertex in every X_i has at most one neighbour in X, this means that at most two vertices in X have a neighbour in $X_1 \cup \cdots \cup X_k$. Therefore, by deleting at most two vertices of X, we obtain a graph which is a disjoint union of cliques and therefore has clique-width at most 2. Therefore by Fact 1, the clique-width of G is bounded, which completes the proof. □

Lemma 10. *The class of those* $(\overline{2P_1 + P_2}, P_2 + P_3, K_5)$*-free graphs that contain an induced* C_5 *has bounded clique-width.*

Proof. Let G be a $(\overline{2P_1 + P_2}, P_2 + P_3, K_5)$-free graph containing a C_5, say on vertices v_1, v_2, v_3, v_4, v_5 in order. Let Y be the set of vertices adjacent to v_1 and v_2 (and possibly other vertices on the cycle). If $y_1, y_2 \in Y$ are non-adjacent then $G[v_1, v_2, y_1, y_2]$ would be a $\overline{2P_1 + P_2}$. Therefore Y is a clique. Since G is K_5-free and $\{v_1, v_2\}$ is complete to Y, it follows that Y contains at most two vertices. Therefore by Fact 1 we may assume that no vertex in G has two consecutive neighbours on the cycle. This also means that no vertex has three or more neighbours on the cycle. For $i \in \{1, 2, 3, 4, 5\}$, let V_i be the set of vertices not on the cycle that are adjacent to v_{i-1} and v_{i+1}, but non-adjacent to all other vertices of the cycle (subscripts are interpreted modulo 5 throughout this proof). Suppose there are two vertices x, y, both of which are adjacent to the same vertex on the cycle, say v_1, and non-adjacent to all other vertices of the cycle. If x and y are adjacent, then $G[x, y, v_2, v_3, v_4]$ is a $P_2 + P_3$, otherwise $G[v_3, v_4, x, v_1, y]$ is a $P_2 + P_3$. This contradiction means that there is at most one vertex whose only neighbour on the cycle is v_1. By Fact 1, we may therefore assume that there is no vertex with exactly one neighbour on the cycle. Let X be the set of vertices with no neighbours on the cycle. Note that every vertex not on the cycle is either in X or in some set V_i (see also Fig. 3).

Now X must be an independent set, since if two vertices in $x_1, x_2 \in X$ are adjacent, then $G[x_1, x_2, v_1, v_2, v_3]$ would be a $P_2 + P_3$. Also, V_i must be an independent set, since if $x, y \in V_i$ are adjacent then $G[x, y, v_{i-1}, v_{i+1}]$ is a $\overline{2P_1 + P_2}$.

We say that two sets V_i and V_j are *consecutive* (respectively *opposite*) if v_i and v_j are distinct adjacent (respectively non-adjacent) vertices of the cycle. We say that a set X or V_i is *large* if it contains at least three vertices, otherwise it is *small*. We say that a bipartite graph with bipartition classes A and B is a *matching* (*co-matching*) if every vertex in A has at most one neighbour (non-neighbour) in B, and vice versa.

We now prove a series of claims about the edges between these sets.

1. $G[V_i \cup X]$ *is a matching.* Indeed if some vertex x in V_i (respectively X) is adjacent to two vertices y_1, y_2 in X (respectively V_i), then $G[v_{i+2}, v_{i+3}, y_1, x, y_2]$ is a $P_2 + P_3$.

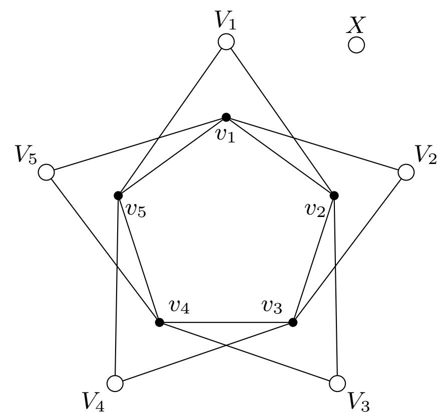

Fig. 3: The graph G. The black points are the vertices of the C_5 and the circles are (possibly empty) sets of vertices, which we will show are independent. Note that G may contain edges between these independent sets that are not represented in this figure.

2. *If V_i and V_j are opposite then $G[V_i \cup V_j]$ is a matching.* Suppose for contradiction that $x \in V_1$ is adjacent to two vertices $y, y' \in V_3$. Then $G[v_2, x, y, y']$ would be a $\overline{2P_1 + P_2}$, a contradiction.
3. *If V_i and V_j are consecutive then $G[V_i \cup V_j]$ is a co-matching.* Suppose for contradiction that $x \in V_1$ is non-adjacent to two vertices $y, y' \in V_2$. Then $G[x, v_5, y, v_3, y']$ is a $P_2 + P_3$, a contradiction.
4. *If V_i is large then X is anti-complete to $V_{i-2} \cup V_{i+2}$.* Suppose for contradiction that V_3 is large and $x \in X$ has a neighbour $y \in V_1$. Then since V_3 is large and both $G[X \cup V_3]$ and $G[V_1 \cup V_3]$ are matchings, there must be a vertex $z \in V_3$ that is non-adjacent to both x and y. Then $G[x, y, v_3, v_4, z]$ is a $P_2 + P_3$, a contradiction.
5. *If V_i is large then V_{i-1} is anti-complete to V_{i+1}.* Suppose for contradiction that V_2 is large and $x \in V_1$ has a neighbour $y \in V_3$. Since V_2 is large and each vertex in $V_1 \cup V_3$ has at most one non-neighbour in V_2, there must be a vertex $z \in V_2$ that is adjacent to both x and y. Now $G[x, y, v_2, z]$ is a $\overline{2P_1 + P_2}$, a contradiction.
6. *If V_{i-1}, V_i, V_{i+1} are large then V_i is complete to $V_{i-1} \cup V_{i+1}$.* Suppose for contradiction that V_1, V_2, V_3 are large and some vertex $x \in V_1$ is non-adjacent to a vertex $y \in V_2$. Since V_3 is large and $G[V_2 \cup V_3]$ is a co-matching, there must be two vertices $z, z' \in V_3$, adjacent to y. By the previous claim, since V_2 is large, z, z' must be non-adjacent to x. Therefore $G[x, v_5, z, y, z']$ is a $P_2 + P_3$, which is a contradiction.

By Fact 1 we may delete the vertices $v_1, \ldots, v_5$ and all vertices in every small set X or V_i. Let G' be the graph obtained from the resulting graph by complementing the edges between any two consecutive V_i, V_j. By Fact 3, G' has bounded clique-width if and only if G does. If at most three of $V_1, \ldots, V_5, X$ are large, then G' has maximum degree at most 2 and we are done by Lemma 2. We may therefore assume that at least four of $V_1, \ldots, V_5, X$ are large, so at least three of $V_1, \ldots, V_5$ are large.

First suppose there is an edge in G between a vertex in X and a vertex in V_i for some i. Then V_{i-2}, V_{i+2} must be small (and as such we already removed them). Consequently, V_{i-1}, V_i, V_{i+1} must be large. However, in this case, every large V_j is either complete or anti-complete to every other large $V_{j'}$ in G and X is anti-complete to $V_{i-1} \cup V_{i+1}$ in G. Therefore G' has maximum degree at most 1 implying that G', and thus G, has bounded clique-width by Lemma 2.

Now suppose that there are no edges in G between any vertex in X and any vertex in V_i for all i. Since X is an independent set, every vertex in X forms a component in G of size 1. We can therefore delete every vertex in X without affecting the clique-width of G. That is, in this case we may assume that X is not large. In this case, as stated above, we may assume that at least four of $V_1, \ldots, V_5$ are large. We may without loss of generality assume that these sets are $V_1, \ldots, V_4$, whereas V_5 may or may not be large. If V_5 is large, then every large V_i is either complete or anti-complete to every other large V_j in G. If V_5 is small (and as such not in G') then the same holds with the possible exception of V_1 and V_4. Hence G' has maximum degree at most 1 implying that G', and thus G, has bounded clique-width by Lemma 2. This completes the proof. $\square$

Lemma 11. *The class of those $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5)$-free graphs that contain an induced C_4 has bounded clique-width.*

Proof. Suppose that G is a $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5)$-free graph containing a C_4, say on vertices v_1, v_2, v_3, v_4 in order. Let Y be the set of vertices adjacent to v_1 and v_2 (and possibly other vertices on the cycle). If $y_1, y_2 \in Y$ are non-adjacent then $G[v_1, v_2, y_1, y_2]$ would be a $\overline{2P_1 + P_2}$. Therefore Y is a clique. Since G is K_5-free, there are at most four such vertices. Therefore by Fact 1 we may assume that no vertex in G has two consecutive neighbours on the cycle. For $i \in \{1, 2\}$ let V_i be the set of vertices outside the cycle adjacent to v_{i+1} and v_{i+3} (where $v_5 = v_1$). For $i \in \{1, 2, 3, 4\}$ let W_i be the set of vertices whose unique neighbour on the cycle is v_i. Let X be the set of vertices with no neighbours on the cycle (see also Fig. 4).

We first prove the following properties:

(i) V_i are independent sets for $i = 1, 2$.
(ii) W_i are independent sets for $i = 1, 2, 3, 4$.
(iii) X is an independent set.
(iv) X is anti-complete to W_i for $i = 1, 2, 3, 4$.
(v) Without loss of generality $W_3 = \emptyset$ and $W_4 = \emptyset$.
(vi) Without loss of generality W_1 is anti-complete to W_2.

To prove Property (i), if $x, y \in V_i$ are adjacent then $G[x, y, v_{i+1}, v_{i+3}]$ is a $\overline{2P_1 + P_2}$. For $i = 1, \ldots, 4$, the set $W_i \cup X$ must also be independent, since if

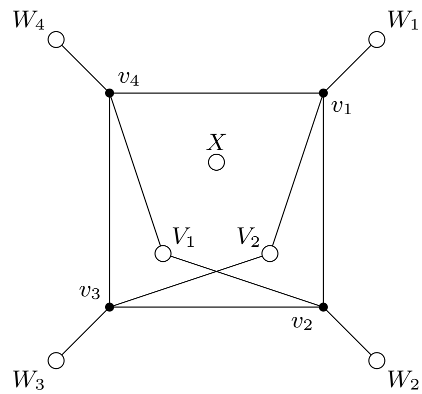

Fig. 4: The graph G. The black points are the vertices of the C_4 and the circles are (possibly empty) sets of vertices, which we will show are independent. Note that G may contain edges between these independent sets that are not represented in this figure.

$x, y \in W_1 \cup X$ were adjacent then $G[x, y, v_2, v_3, v_4]$ would be a $P_2 + P_3$. This proves Properties (ii)–(iv).

To prove Property (v), suppose that $x \in W_1$ and $y \in W_3$ are adjacent. In that case $G[v_1, v_2, v_3, y, x]$ would be a C_5. This contradiction means that no vertex of W_1 is adjacent to a vertex of W_3. Now suppose that $x, x' \in W_1$ and $y \in W_3$. Then $G[y, v_3, x, v_1, x']$ would be a $P_2 + P_3$ by Property (ii). Therefore, if both W_1 and W_3 are non-empty, then they each contain at most one vertex and we can delete these vertices by Fact 1. Without loss of generality we may therefore assume that W_3 is empty. Similarly, we may assume W_4 is empty. Hence we have shown Property (v).

We are left to prove Property (vi). Suppose that $x \in W_1$ is adjacent to $y \in W_2$. Then x cannot have a neighbour in V_2. Indeed, suppose for contradiction that x has a neighbour $z \in V_2$. Then $G[x, z, y, v_1]$ is a $\overline{2P_1 + P_2}$ if y and z are adjacent, and $G[x, y, v_2, v_3, z]$ is a C_5 if y and z are not adjacent. By symmetry, y cannot have a neighbour in V_1. Now y must be complete to V_2. Indeed, if y has a non-neighbour $z \in V_2$ then $G[x, y, z, v_3, v_4]$ is a $P_2 + P_3$. By symmetry, x is complete to V_1. Recall that $W_1 \cup X$ is an independent set by Properties (ii)–(iv). We conclude that any vertex in W_1 with a neighbour in W_2 is complete to V_1 and anti-complete to $V_2 \cup X$. Similarly, any vertex in W_2 with a neighbour in W_1 is complete to V_2 and anti-complete to $V_1 \cup X$.

Let W_1^* (respectively W_2^*) be the set of vertices in W_1 (respectively W_2) that have a neighbour in W_2 (respectively W_1). Then, by Fact 3, we may apply two bipartite complementations, one between W_1^* and $V_1 \cup \{v_1\}$ and the other between W_2^* and $V_2 \cup \{v_2\}$. After these operations, G will be split into two disjoint parts with no edges between them: $G[W_1^* \cup W_2^*]$ and $G \setminus (W_1^* \cup W_2^*)$, both of which are induced subgraphs of G. The first of these is a bipartite $(P_2 + P_3)$-free graph and therefore has bounded clique-width by Lemma 4. We therefore only need to consider the second graph $G \setminus (W_1^* \cup W_2^*)$. In other words, we may assume without loss of generality that W_1 is anti-complete to W_2. This proves Property (vi).

If a vertex in X has no neighbours in $V_1 \cup V_2$ then it is an isolated vertex by Property (iv) and the definition of the set X. In this case we may delete it without affecting the clique-width. Hence, we may assume without loss of generality that every vertex in X has at least one neighbour in $V_1 \cup V_2$. We partition X into three sets X_0, X_1, X_2 as follows. Let X_1 (respectively X_2) denote the set of vertices in X with at least one neighbour in V_1 (respectively V_2), but no neighbours in V_2 (respectively V_1). Let X_0 denote the set of vertices in X adjacent to at least one vertex of V_1 and at least one vertex of V_2.

Let $G^* = G[V_1 \cup V_2 \cup W_1 \cup W_2 \cup X_1 \cup X_2]$. We prove the following additional properties:

(vii) G^* is bipartite.
(viii) Without loss of generality $X_0 \neq \emptyset$.
(ix) Every vertex in V_1 that has a neighbour in X is complete to V_2.
(x) Every vertex in V_2 that has a neighbour in X is complete to V_1.
(xi) Every vertex in X_0 has exactly one neighbour in V_1 and exactly one neighbour in V_2.
(xii) Without loss of generality, every vertex in $V_1 \cup V_2$ has at most one neighbour in X_0.
(xiii) Without loss of generality, V_1 is anti-complete to W_2.
(xiv) Without loss of generality, V_2 is anti-complete to W_1.

Property (vii) can be seen has follows. Because G is $(P_2 + P_3, C_5)$-free, G^* has no induced odd cycles of length at least 5. Suppose, for contradiction, that G^* is not bipartite. Then it must contain an induced C_3. Now V_1, V_2, W_1, W_2, X_1 and X_2 are independent sets, so at most one vertex of the C_3 can be in any one of these sets. The set X_1 is anti-complete to V_2, W_1, W_2 and X_2 (by definition of V_2 and Properties (iii) and (iv)). Hence no vertex of the C_3 can be in X_1. Similarly, no vertex of the C_3 be be in X_2. The sets W_1 and W_2 are anti-complete to each other by Property (vi), so the C_3 must therefore consist of one vertex from each of V_1 and V_2, along with one vertex from either W_1 or W_2. However, in this case, these three vertices, along with either v_1 or v_2, respectively would induce a $\overline{2P_1 + P_2}$ in G, which would be a contradiction. Hence we have proven Property (vii).

We now prove Property (viii). Suppose X_0 is empty. Then, since G^* is $(P_2 + P_3)$-free and bipartite (by Property (vii)), it has bounded clique-width by Lemma 4. Hence, G has bounded clique-width by Fact 1, since we may delete v_1, v_2, v_3 and v_4 to obtain G^*. This proves Property (viii).

We now prove Property (ix). Let $y_1 \in V_1$ have a neighbour $x \in X$. Suppose, for contradiction, that y_1 has a non-neighbour $y_2 \in V_2$. Then $G[x, y_2, v_1, v_2, y_1]$ is a C_5 if x is adjacent to y_2 and $G[x, y_1, v_1, y_2, v_3]$ is a $P_2 + P_3$ if x is non-adjacent to y_2, a contradiction. This proves Property (ix). By symmetry, Property (x) holds.

We now prove Property (xi). By definition, every vertex in X_0 has at least one neighbour in V_1 and at least one neighbour in V_2. Suppose, for contradiction, that a vertex $x \in X_0$ has two neighbours $y, y' \in V_1$. By definition, x must also have a neighbour $z \in V_2$. Then z must be adjacent to both y and y' by Property (x). However, then $G[x, z, y, y']$ is a $\overline{2P_1 + P_2}$ by Property (i), a contradiction.This proves Property (xi).

We now prove Property (xii). Suppose a vertex $y \in V_1$ has two neighbours $x, x' \in X_0$. If there is another vertex $z \in X_0$ then z must have a unique neighbour z' in V_1. If z' is a different vertex from y then $G[z, z', x, y, x']$ would be a $P_2 + P_3$ by Properties (i) and (iii). Thus $z' = y$, that is, every vertex in X_0 must be adjacent to y and to no other vertex of V_1. By Fact 1, we may delete y. In the resulting graph no vertex of X would have neighbours in both V_1 and V_2. So X_0 would become empty, in which case we can argue as in the proof of Property (viii). This proves Property (xii).

We now prove Property (xiii). First, for $i \in \{1, 2\}$, suppose that a vertex $y \in V_i$ is adjacent to a vertex $x \in X$. Then y can have at most one non-neighbour in W_i. Indeed, suppose for contradiction that $z, z' \in W_i$ are non-neighbours of y. Then $G[x, y, z, v_i, z']$ is a $P_2 + P_3$ by Properties (ii) and (vi), a contradiction. We claim that at most one vertex of W_2 has a neighbour in V_1. Suppose, for contradiction, that W_2 contains two vertices w and w' adjacent to (not necessarily distinct) vertices z and z' in V_1, respectively. Since $X_0 \neq \emptyset$ by Property (viii), there must be a vertex $y \in V_2$ with a neighbour in X_0. As we just showed that such a vertex y can have at most one non-neighbour in W_2, we may assume without loss of generality that y is adjacent to w. Since y has a neighbour in X, it must also be adjacent to z by Property (x). Now $G[w, z, y, v_2]$ is a $\overline{2P_1 + P_2}$, which is a contradiction. Therefore at most one vertex of W_2 has a neighbour in V_1 and similarly, at most one vertex of W_1 has a neighbour in V_2. By Fact 1, we may delete these vertices if they exist. This proves Properties (xiii) and (xiv).

For $i = 1, 2$ let V_i' be the set of vertices in V_i that have a neighbour in X_0. We show two more properties:

(xv) Every vertex in $W_1 \cup X_1$ is adjacent to either none, precisely one or all vertices of V_1'.

(xvi) Every vertex of $W_2 \cup X_2$ is adjacent to either none, precisely one or all vertices of V_2'.

We prove Property (xv) as follows. Suppose a vertex $x \in X_1 \cup W_1$ has at least two neighbours in $z, z' \in V_1$. We claim that x must be complete to V_1'. Suppose, for contradiction, that x is not adjacent to $y \in V_1'$. By definition, y has a neighbour $y' \in X_0$. Then $G[y, y', z, x, z']$ is a $P_2 + P_3$ by Properties (i), (iii) and (iv), a contradiction. This proves Property (xv). Property (xvi) follows by symmetry.

Let W_i' and X_i' be the sets of vertices in W_i and X_i respectively that are adjacent to precisely one vertex of V_i'. We delete v_1, v_2, v_3 and v_4, which we may do by

Fact 1. We do a bipartite complementation between V_1' and those vertices in $W_1 \cup X_1$ that are complete to V_1'. We also do this between V_2' and those vertices in $W_2 \cup X_2$ that are complete to V_2'. Finally, we perform a bipartite complementation between V_1' and $V_2 \setminus V_2'$ and also between V_2' and $V_1 \setminus V_1'$. We may do all of this by Fact 3. Afterwards, Properties (i)–(vi), (ix), (x), (xiii)–(xvi) and the definitions of V_1', V_2', W_1', W_2', X_1, X_2 imply that there are no edges between the following two vertex-disjoint graphs:

1. $G[W_1' \cup W_2' \cup X_1' \cup X_2' \cup V_1' \cup V_2' \cup X_0]$ and
2. $G \setminus (W_1' \cup W_2' \cup X_1' \cup X_2' \cup V_1' \cup V_2' \cup X_0 \cup \{v_1, v_2, v_3, v_4\})$

Both of these graphs are induced subgraphs of G. The second of these graphs does not contain any vertices of X_0. So it is bipartite by Property (vii) and therefore has bounded clique-width, as argued before (in the proof of Property (viii)).

Now consider the first graph, which is $G[W_1' \cup W_2' \cup X_1' \cup X_2' \cup V_1' \cup V_2' \cup X_0]$. By Fact 3, we may complement the edges between V_1' and V_2'. This yields a new graph G'. By definition of V_1', V_2' and Properties (ix) and (x), we find that V_1' is anti-complete to V_2' in G'. Hence, by definition of V_1', V_2' and Properties (i), (iii), (xi) and (xii), we find that $G'[V_1' \cup V_2' \cup X_0]$ is a disjoint union of P_3's. For $i \in \{1, 2\}$, every vertex in $W_i' \cup X_i'$ is adjacent to precisely one vertex in V_i' by definition. As the last bipartite complementation operation did not affect these sets, this is still the case in G'. By Properties (ii)–(iv) and (vi), we find that $W_1' \cup W_2' \cup X_0 \cup X_1' \cup X_2'$ is an independent set. Then, by also using Properties (xiii) and (xiv) together with the definitions of X_1 and X_2, we find that no vertex in $W_i' \cup X_i'$ has any other neighbour in G' besides its neighbour in V_i'. Therefore G' is a disjoint union of trees and thus has bounded clique-width by Lemma 1. We conclude that G has bounded clique-width. This completes the proof of Lemma 11. □

Lemma 12. *The class of those $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5, C_4)$-free graphs that contain an induced C_6 has bounded clique-width.*

Proof. Let G be a $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5, C_4)$-free graph containing a C_6, say on vertices $v_1, v_2, v_3, v_4, v_5, v_6$ in order. Let Y be the set of vertices adjacent to v_1 and v_2 (and possibly other vertices on the cycle). If $y_1, y_2 \in Y$ are non-adjacent then $G[v_1, v_2, y_1, y_2]$ would be a $\overline{2P_1 + P_2}$. Therefore Y must be a clique. Since G is K_5-free, Y contains at most four vertices. Therefore by Fact 1 we may assume that no vertex in G has two consecutive neighbours on the cycle. Suppose there are two vertices x and x', both of which are adjacent to two non-consecutive vertices of the cycle v_i and v_j. Then if x and x' are adjacent, $G[x, x', v_i, v_j]$ would be a $\overline{2P_1 + P_2}$, otherwise $G[x, v_i, x', v_j]$ would be a C_4, a contradiction. Thus for every two non-adjacent vertices on the cycle, there can be at most one vertex adjacent to both of them. By Fact 1 we may delete all such vertices. We conclude that every other vertex which is not on the cycle can be adjacent to at most one vertex on the cycle. Suppose x is adjacent to v_1, but not v_2, v_3, v_4, v_5, v_6. Then $G[x, v_1, v_3, v_4, v_5]$ would be a $P_2 + P_3$. Therefore no vertex which is not on the cycle can have a neighbour on the cycle. If two vertices x and x' are not adjacent to any vertex of the cycle then they cannot be adjacent, otherwise $G[x, x', v_1, v_2, v_3]$ would be a P_2+P_3. Therefore the remaining graph is composed of a C_6 and zero or more isolated vertices. Hence, G has bounded clique-width. This completes the proof. $\square$

We now use Lemmas 9–12 and the fact that $(\overline{2P_1+P_2}, P_2+P_3, C_4, C_5, C_6)$-free graphs are chordal graphs, and so have bounded clique-width by Lemma 5, to obtain:

Theorem 1 (iii). *The class of* $(\overline{2P_1+P_2}, P_2+P_3)$*-free graphs has bounded clique-width.*

Proof. Suppose G is a $(\overline{2P_1+P_2}, P_2+P_3)$-free graph. By Lemmas 9–12, we may assume that G is $(\overline{2P_1+P_2}, P_2+P_3, K_5, C_5, C_4, C_6)$-free. Because G is (P_2+P_3)-free, it contains no induced cycles of length 7 or more. Hence G is chordal, that is, it is a $(\overline{2P_1+P_2})$-free chordal graph, in which case the clique-width of G is bounded by Lemma 5. This completes the proof of the theorem. $\square$

7 The Proof of Theorem 1 (iv)

To prove our fourth main result we need the well-known notion of a *wall*. We do not formally define this notion but instead refer to Fig. 5, in which three examples of walls of different height are depicted.

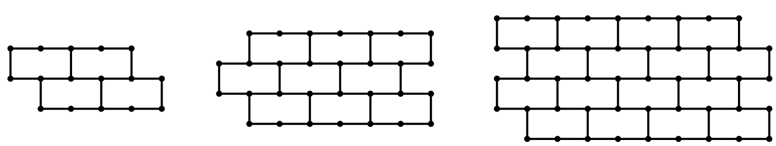

Fig. 5: Walls of height 2, 3, and 4, respectively.

The class of walls is well known to have unbounded clique-width; see for example [30]. We need a more general result. The *subdivision* of an edge uv in a graph replaces uv by a new vertex w with edges uw and vw. A k*-subdivided wall* is a graph obtained from a wall after subdividing each edge exactly k times for some constant $k \geq 0$. The following lemma is well known.

Lemma 13 ([35]). *For any constant* $k \geq 0$*, the class of* k*-subdivided walls has unbounded clique-width.*

Theorem 1 (iv). *The class of* $(\overline{2P_1+P_2}, P_2+P_4)$*-free graphs has unbounded clique-width.*

Proof. Let $n \geq 2$ and let G_n be a wall of height n. Note that G_n is a connected bipartite graph. Let A and C be its two bipartition classes. We subdivide every edge in G_n exactly once to obtain a 1-subdivided wall. Let B be the set of new vertices introduced by this operation. We then apply a bipartite complementation between A and C, which results in a graph G'_n. The set of graphs $\{G'_n\}_{n\geq 2}$ has unbounded clique-width by Lemma 13 and Fact 3. Hence it suffices to prove that G'_n is $(\overline{2P_1+P_2}, P_2+P_4)$-free. We do this using three observations.

(i) A and C are independent sets in G'_n that are complete to each-other, in other words, $G'_n[A \cup C]$ is a complete bipartite graph.
(ii) B is an independent set and every vertex of B has exactly one neighbour in A and exactly one neighbour in C.
(iii) No two vertices of B have the same neighbourhood.

We now prove that G'_n is $\overline{2P_1+P_2}$-free. For contradiction, suppose that G'_n contains an induced subgraph H isomorphic to $\overline{2P_1+P_2}$. Since $G'_n[A \cup C]$ is complete bipartite, any triangle in G'_n must contain a vertex of B. Since the vertices of B have degree 2, this means that the two degree-2 vertices of H must be in B. As $G'_n[A \cup C]$ is complete bipartite, one of the degree-3 vertices of H is in A and the other one is in C. This implies that the two degree-2 vertices in H have the same neighbourhood. Since both of these vertices belong to B, this is a contradiction.

It remains to prove that G'_n is (P_2+P_4)-free. For contradiction, suppose that G'_n contains an induced subgraph H isomorphic to P_2+P_4. Let H_1 and H_2 be the connected components of H isomorphic to P_2 and P_4, respectively. Since $G'_n[A \cup C]$ is complete bipartite, H_2 must contain at least one vertex of B. Since the two neighbours of any vertex of B are adjacent, any vertex of B in H_2 must be an end-vertex of H_2. Then, as A and C are independent sets, H_2 contains a vertex of both A and C. As H_1 can contain at most one vertex of B (because B is an independent set), H_1 contains a vertex $u \in A \cup C$. However, $G'_n[A \cup C]$ is complete bipartite and H_2 contains a vertex of both A and C. Hence, u has a neighbour in H_2, which is not possible. This completes the proof of Theorem 1 (iv). □

We finish this section with one more result. A *dominating* vertex in a graph G is a vertex adjacent to all other vertices of G. We need the following two well-known observations (see e.g. [32]).

Lemma 14. *Let G'_1 and G'_2 be the graphs obtained from two graphs G_1 and G_2, respectively, by adding a dominating vertex. Then G'_1 and G'_2 are isomorphic if and only if G_1 and G_2 are.*

Lemma 15. *Let G'_1 and G'_2 be the graphs obtained from subdividing every edge of two graphs G_1 and G_2, respectively, exactly once. Then G'_1 and G'_2 are isomorphic if and only if G_1 and G_2 are.*

We are now ready to prove our last result. Recall that Graph Isomorphism was recently shown to be polynomial-time solvable on graphs of bounded clique-width [26]. Hence, if Graph Isomorphism is not solvable in polynomial time on general graphs, then combining this result with the following theorem would imply Theorem 1 (iv).

Theorem 3. Graph Isomorphism *is* Graph Isomorphism*-complete for the class of* $(\overline{2P_1 + P_2}, P_2 + P_4)$*-free graphs.*

Proof. Let G_1 and G_2 be arbitrary graphs. For $i = 1, 2$ we modify G_i as follows. First, add four dominating vertices. (Note that these added vertices are pairwise adjacent.) This ensures that the graph has minimum degree at least 3. Let A_i be the set of vertices in the resulting graph. Subdivide every edge once and let C_i be the set of new vertices. Note that this results in a bipartite graph with bipartition classes A_i and C_i. Subdivide each edge in this modified graph and let B_i be the set of new vertices. Call the resulting graph G_i'. Finally, apply a bipartite complementation between A_i and C_i. Let G_i'' be the resulting graph. Now in the graph G_i'', the sets of vertices A_i, B_i and C_i satisfy the three observations from the proof of Theorem 1 (iv) and G_1'' and G_2'' are therefore $(\overline{2P_1 + P_2}, P_2 + P_4)$-free by exactly the same arguments.

We claim that G_1 and G_2 are isomorphic if and only if G_1'' and G_2'' are. In order to see this, we first use Lemmas 14 and 15 to deduce that G_1 and G_2 are isomorphic if and only if G_1' and G_2' are. It remains to show that G_1' and G_2' are isomorphic if and only if G_1'' and G_2'' are. Note that for $i = 1, 2$, every vertex in A_i has degree at least 3 in both G_i' and G_i'', every vertex of B_i has degree exactly 2 in both G_i' and G_i'' and every vertex of C_i has degree exactly 2 in G_i' and degree at least 3 in G_i''. Now a vertex is in B_i if and only if it is adjacent to a vertex of degree at least 3 in G_i' if and only if it is of degree exactly 2 in G_i''. A vertex in G_i' or G_i'' is in A_i if and only if it is adjacent to at least three vertices of degree 2. Hence, every isomorphism from G_1' to G_2' and every isomorphism from G_1'' and G_2'' maps the vertices of A_1, B_1 and C_1 to the vertices of A_2, B_2 and C_2, respectively. The claim follows since for $i = 1, 2$ the graph G_i'' is obtained from G_i' by adding all edges between A_i and C_i. □

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