Abstract We consider a two-dimensional model of double-diffusive convection and its time discretisation using a second-order scheme (based on backward differentiation formula for the time derivative) which treats the nonlinear term explicitly. Uniform bounds on the solutions of both the continuous and discrete models are derived (under a timestep restriction for the discrete model), proving the existence of attractors and invariant measures supported on them. As a consequence, the convergence of the attractors and long time statistical properties of the discrete model to those of the continuous one in the limit of vanishing timestep can be obtained following established methods.

Keywords multistep scheme · double-diffusive convection · long-time stability

Mathematics Subject Classification (2000) 65M12 · 35B35 · 35K45

Numerische Mathematik manuscript No.
(will be inserted by the editor)

Long-time dynamics of 2d double-diffusive convection: analysis and/of numerics

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Received: date / Accepted: date

1 Introduction

s:intro

The phenomenon of double-diffusive convection, in which two properties of a fluid are transported by the same velocity field but diffused at different rates, often occurs in nature [13]. Perhaps the best known example is the transport throughout the world's oceans of heat and salinity, which has been recognised as an essential part of climate dynamics [18, 24]. In contrast to simple convections (cf. [3]), double-diffusive convections support a richer set of physical regimes, e.g., a stably stratified initial state rendered unstable by diffusive effects. Although in this paper we shall be referring to the oceanographic case, the mathematical theory is essentially identical for astrophysical [16, 19] and industrial [4] applications.

In this paper, we consider a two-dimensional double-diffusive convection model, which by now-standard techniques [21] can be proved to have a global attractor and invariant measures supported on it, and its temporal discretisation. We use a backward differentiation formula for the time derivative and a fully explicit method for the nonlinearities, resulting in an accurate and efficient numerical scheme. Of central interest, here and in many practical applications, is the ability of the discretised model to capture long-time behaviours of the underlying PDE. This motivates the main aim of this article: to obtain bounds necessary for the convergence of the attractor and associ-

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ated invariant measures of the discretised system to those of the continuous system. We do this using the framework laid down in [22, 23], with necessary modifications for our more complex model.

For motivational concreteness, one could think of our system as a model for the zonally-averaged thermohaline circulation in the world's oceans. Here the physical axes correspond to latitude and altitude, and the fluid is sea water whose internal motion is largely driven by density differentials generated by the temperature T and salinity S, as well as by direct wind forcing on the surface. Both T and S are also driven from the boundary—by precipitation/evaporation and ice melting/formation for the salinity, and by the associated latent heat release and direct heating/cooling for the temperature. Physically, one expects the boundary forcing for T, S and the momentum to have zonal (latitude-dependent) structure, so we include these in our model. Furthermore, one may also wish to impose a quasi-periodic time dependence on the forcing; although this is eminently possible, we do not do so in this paper to avoid technicalities arising from time-dependent attractors.

Taking as our domain $\mathcal{D}_* = [0, L_*] \times [0, H_*]$ which is periodic in the horizontal direction, we consider a temperature field T_* and a salinity field S_*, both transported by a velocity field $\boldsymbol{v}_* = (u_*, w_*)$ which is incompressible, $\nabla_* \cdot \boldsymbol{v}_* = 0$, and diffused at rates κ_T and κ_S, respectively,

$$\begin{aligned} \partial T_*/\partial t_* + \boldsymbol{v}_* \cdot \nabla_* T_* &= \kappa_T \Delta_* T_* \\ \partial S_*/\partial t_* + \boldsymbol{v}_* \cdot \nabla_* S_* &= \kappa_S \Delta_* S_*. \end{aligned} \tag{1.1}$$

Here the star$_*$ denotes dimensional variables. Taking the Boussinesq approximation and assuming that the density is a linear function of T_* and S_*, which is a good approximation for sea water (although not for fresh water near its freezing point), the velocity field evolves according to

$$\partial \boldsymbol{v}_*/\partial t_* + \boldsymbol{v}_* \cdot \nabla_* \boldsymbol{v}_* + \nabla_* p_* = \kappa_v \Delta_* \boldsymbol{v}_* + (\alpha_T T_* - \alpha_S S_*)\boldsymbol{e}_z \tag{1.2}$$

for some positive constants α_T and α_S.

Our system is driven from the boundary by the heat and salinity fluxes (which could be seen to arise from direct contact with air and latent heat release in the case of heat, and from precipitation, evaporation and ice formation/melt in the case of salinity),

$$\partial T_*/\partial n_* = Q_{T*} \quad \text{and} \quad \partial S_*/\partial n_* = Q_{S*}. \tag{1.3}$$

Here n_* denotes the outward normal, $n_* = z_*$ at the top boundary and $n_* = -z_*$ at the bottom boundary. We also prescribe a wind-stress forcing,

$$\partial u_*/\partial n_* = Q_{u*} \tag{1.4}$$

along with the usual no-flux condition $w_* = 0$ on $z_* = 0$ and $z_* = H_*$.

Largely following standard practice, we cast our system in non-dimensional form as follows. Using the scales $\tilde{t}$, $\tilde{l}$, $\tilde{T}$ and $\tilde{S}$, we define the non-dimensional variables $t = t_*/\tilde{t}$, $\boldsymbol{x} = \boldsymbol{x}_*/\tilde{l}$, $\boldsymbol{v} = \boldsymbol{v}_*\tilde{t}/\tilde{l}$, $T = T_*/\tilde{T}$ and $S = S_*/\tilde{S}$, in terms of which our system reads

$$\begin{aligned} \mathfrak{p}^{-1}\big(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\big) &= -\nabla p + \Delta\boldsymbol{v} + (T-S)\boldsymbol{e}_z \\ \partial_t T + \boldsymbol{v}\cdot\nabla T &= \Delta T \\ \partial_t S + \boldsymbol{v}\cdot\nabla S &= \beta\Delta S. \end{aligned} \tag{1.5}$$

q:dUdt

To arrive at this, we have put $\tilde{l} = H_*$ and taken the thermal diffusive timescale for

$$\tilde{t} = \tilde{l}^2/\kappa_T, \tag{1.6}$$

as well as scaled the dependent variables as

$$\tilde{T} = \mathfrak{p}\tilde{l}/(\alpha_T\tilde{t}^2) \quad \text{and} \qquad \tilde{S} = \mathfrak{p}\tilde{l}/(\alpha_S\tilde{t}^2), \tag{1.7}$$

where the non-dimensional *Prandtl number* and *diffusivity ratio* (also known as the *Lewis number* in the engineering literature) are

$$\mathfrak{p} = \kappa_v/\kappa_T \quad \text{and} \quad \beta = \kappa_T/\kappa_S. \tag{1.8}$$

Another non-dimensional quantity is the domain aspect ratio $\xi = L_*/\tilde{l}$. The surface fluxes are non-dimensionalised in the natural way: $Q_T = \mathfrak{p}Q_{T*}/(\alpha_T\tilde{t}^2)$, $Q_S = \mathfrak{p}Q_{S*}/(\alpha_S\tilde{t}^2)$ and $Q_u = Q_{u*}\tilde{t}$.

For clarity and convenience, keeping in mind the oceanographic application, we assume that the fluxes vanish on the bottom boundary $z = 0$,

$$Q_u(x,0) = Q_T(x,0) = Q_S(x,0) = 0. \tag{1.9}$$

q:BC0

For boundedness of the solution in time, the net fluxes must vanish, so (1.9) then implies that the net fluxes vanish on the top boundary $z = 1$,

$$\int_0^\xi Q_u(x,1)\,\mathrm{d}x = \int_0^\xi Q_T(x,1)\,\mathrm{d}x = \int_0^\xi Q_S(x,1)\,\mathrm{d}x = 0. \tag{1.10}$$

q:noflux

These boundary conditions can be seen to imply that the horizontal velocity flux is constant in time, which we take to be zero, viz.,

$$\int_0^1 u(x,z,t)\,\mathrm{d}z = \int_0^1 u(x,z,0)\,\mathrm{d}z \equiv 0 \quad \text{for all } x \in [0,\xi]. \tag{1.11}$$

q:uflux

For some applications (e.g., the classical Rayleigh–Bénard problem), the fluxes on the bottom boundary may not vanish, which must then be balanced by the fluxes on the top boundary,

$$\int_0^\xi [Q_T(x,1) - Q_T(x,0)]\,\mathrm{d}x = 0 \tag{1.12}$$

and similarly for Q_u and Q_S. With some modifications (by subtracting background profiles from u, T and S), the analysis of this paper also apply to this more general case. This involves minimal conceptual difficulty but adds to the clutter, so we do not treat this explicitly here.

Defining the vorticity $\omega := \partial_x w - \partial_z u$, the streamfunction ψ by $\Delta\psi = \omega$ with $\psi = 0$ on $\partial\mathfrak{D}$ (this is consistent with (1.11)), and the Jacobian determinant $\partial(f,g) := \partial_x f \partial_z g - \partial_x g \partial_z f = -\partial(g,f)$, our system reads

$$\begin{aligned} &\mathfrak{p}^{-1}\big\{\partial_t\omega + \partial(\psi,\omega)\big\} = \Delta\omega + \partial_x T - \partial_x S \\ &\partial_t T + \partial(\psi, T) = \Delta T \\ &\partial_t S + \partial(\psi, S) = \beta\Delta S. \end{aligned} \tag{1.13}$$

q:dudt

The boundary conditions are,

$$\partial_z T = Q_T, \quad \partial_z S = Q_S, \quad \omega = Q_u \quad \text{and} \quad \psi = 0 \quad \text{on } \partial\mathfrak{D}. \tag{1.14}$$

q:BC

We note that for the solution to be smooth at $t = 0$, the initial data and the boundary conditions must satisfy a compatibility condition; cf. e.g., [20, Thm. 6.1] in the case of Navier–Stokes equations. In the rest of this paper, we will be working with (1.13)–(1.14) and its discretisation. We assume that ω, T and S all have zero integral over $\mathfrak{D}$ at $t = 0$. Thanks to the no-net-flux condition (1.10), this persists for all $t \geq 0$.

Another dimensionless parameter often considered in studies of (single-species) convection is the *Rayleigh number* Ra. When the top and bottom temperatures are held at fixed values T_1 and T_0, Ra is proportional to $T_0 - T_1$. The relevant parameters in our problem would be $\mathrm{Ra}_T \propto |Q_T|_{L^2(\partial\mathfrak{D})}$ and $\mathrm{Ra}_S \propto |Q_S|_{L^2(\partial\mathfrak{D})}$, but we will not consider them explicitly here; see, e.g., (2.11) in [2]. For notational conciseness, we denote the variables $U := (\omega, T, S)$, the boundary forcing $Q := (Q_u, Q_T, Q_S)$ and the parameters $\pi := (\mathfrak{p}, \beta, \xi)$.

We do not provide details on the convergence of the global attractors and long time statistical properties. Such kind of convergence can be obtained following established methods once we have the uniform estimates derived here. See [10] for the convergence of the global attractors and [23] for the convergence of long time statistical properties.

The rest of this paper is structured as follows. In section 2 we review briefly the properties of the continuous system, setting up the scene and the notation for its discretisation. Next, we describe the time discrete system and derive uniform bounds for the solution. In the appendix, we present an alternate derivation of the boundedness results in [23], without using Wente-type estimates but requiring slightly more regular initial data.

2 Properties of the continuous system

s:cts

In this section, we obtain uniform bounds on the solution of our system and use them to prove the existence of a global attractor $\mathcal{A}$. For the single diffusion case (of T only, without S), this problem has been treated in [6] which we follow in spirit, though not in detail in order to be closer to our treatment of the discrete case.

We start by noting that the zero-integral conditions on ω, T and S imply the Poincaré inequalities

$$|\omega|^2_{L^2(\mathcal{D})} \le c_0\,|\nabla\omega|^2_{L^2(\mathcal{D})}, \tag{2.1}$$

q:cpoi

as well as the equivalence of the norms

$$|\omega|_{H^1(\mathcal{D})} \le c\,|\nabla\omega|_{L^2(\mathcal{D})}, \tag{2.2}$$

q:normeq

with analogous inequalities for T and S. The boundary condition $\psi = 0$ implies that (2.1)–(2.2) also hold for ψ, while an elliptic regularity estimate [7, Cor. 8.7] implies that

$$|\nabla\psi|^2_{L^2(\mathcal{D})} \le c_0\,|\omega|^2_{L^2(\mathcal{D})}. \tag{2.3}$$

q:wpoi

Following the argument in [8], this also holds for functions, such as our T and S, with zero integrals in $\mathcal{D}$.

Let $\varOmega$ be an H^2 extension of Q_u to $\bar{\mathcal{D}}$ (further requirements will be imposed below) and let $\hat\omega := \omega - \varOmega$; we also define $\varDelta\hat\psi := \hat\omega$ and $\varDelta\varPsi := \varOmega$ with homogeneous boundary conditions. Now $\hat\omega$ satisfies the homogeneous boundary conditions $\hat\omega = 0$ on $\partial\mathcal{D}$, and thus the Poincaré inequality (2.1)–(2.2). Furthermore, let $T_Q \in \dot H^2(\mathcal{D})$ be such that $\partial_z T_Q = Q_T$ on $\partial\mathcal{D}$ (with other constraints to be imposed below) and let $\hat T := T - T_Q$; analogously for S_Q and $\hat S := S - S_Q$. We note that since both $\hat T$ and $\hat S$ have zero integrals over $\mathcal{D}$, they satisfy the Poincaré inequality (2.1)–(2.2).

We start with weak solutions of (1.13). For conciseness, unadorned norms and inner products are understood to be $L^2(\mathcal{D})$, $|\cdot| := |\cdot|_{L^2(\mathcal{D})}$ and $(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\mathcal{D})}$. With $\hat\omega$, $\hat T$ and $\hat S$ as defined above, we have

$$\begin{aligned}
&\partial_t\hat\omega + \partial(\varPsi+\hat\psi, \varOmega+\hat\omega) = \mathfrak{p}\big\{\varDelta\hat\omega + \varDelta\varOmega + \partial_x T_Q + \partial_x\hat T - \partial_x S_Q - \partial_x\hat S\big\}\\
&\partial_t\hat T + \partial(\varPsi+\hat\psi, T_Q+\hat T) = \varDelta T_Q + \varDelta\hat T\\
&\partial_t\hat S + \partial(\varPsi+\hat\psi, S_Q+\hat S) = \beta\,(\varDelta S_Q + \varDelta\hat S).
\end{aligned} \tag{2.4}$$

q:duhdt

On a fixed time interval $[0,T_*)$, a weak solution of (2.4) are

$$\begin{aligned}
&\hat\omega \in C^0(0,T_*;L^2(\mathcal{D})) \cap L^2(0,T_*;H^1_0(\mathcal{D}))\\
&\hat T \in C^0(0,T_*;L^2(\mathcal{D})) \cap L^2(0,T_*;H^1(\mathcal{D}))\\
&\hat S \in C^0(0,T_*;L^2(\mathcal{D})) \cap L^2(0,T_*;H^1(\mathcal{D}))
\end{aligned} \tag{2.5}$$

such that, for all $\tilde\omega \in H^1_0(\mathcal{D})$, $\tilde T$, $\tilde S \in H^1(\mathcal{D})$, the following holds in the distributional sense,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\hat\omega,\tilde\omega) + (\partial(\varPsi+\hat\psi,\varOmega+\hat\omega),\tilde\omega)\\
&\qquad + \mathfrak{p}\,\big\{(\nabla\varOmega + \nabla\hat\omega, \nabla\tilde\omega) - (\partial_x T_Q + \partial_x\hat T, \tilde\omega) + (\partial_x S_Q + \partial_x\hat S, \tilde\omega)\big\} = 0\\
&\frac{\mathrm{d}}{\mathrm{d}t}(\hat T,\tilde T) + (\partial(\varPsi+\hat\psi, T_Q+\hat T),\tilde T) + (\nabla\hat T,\nabla\tilde T) - (\varDelta T_Q,\tilde T) = 0\\
&\frac{\mathrm{d}}{\mathrm{d}t}(\hat S,\tilde S) + (\partial(\varPsi+\hat\psi, S_Q+\hat S),\tilde S) + \beta\,(\nabla\hat S,\nabla\tilde S) - \beta\,(\varDelta S_Q,\tilde S) = 0.
\end{aligned} \tag{2.6}$$

The existence of such solutions can be obtained by Galerkin approximation together with Aubin-Lions compactness argument [20, §3.3], which we do not carry out explicitly here.

Next, we derive L^2 inequalities for T, S and ω. Multiplying (2.4a) by $\hat\omega$ in $L^2(\mathcal{D})$ and noting that $(\partial(\psi,\hat\omega),\hat\omega)=0$, we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\hat\omega|^2 + \mathfrak{p}\,|\nabla\hat\omega|^2 &= -(\partial(\varPsi,\varOmega),\hat\omega) - (\partial(\hat\psi,\varOmega),\hat\omega)\\ &\qquad + \mathfrak{p}\big\{(\varDelta\varOmega,\hat\omega) + (\partial_x T,\hat\omega) - (\partial_x S,\hat\omega)\big\}.\end{aligned} \tag{2.7}$$

We bound the rhs as

$$\begin{aligned}&\big|(\varDelta\varOmega,\hat\omega)\big| = |\nabla\varOmega|\,|\nabla\hat\omega| \le \tfrac{1}{8}\,|\nabla\hat\omega|^2 + 2\,|\nabla\varOmega|^2\\ &\big|(\partial_x T,\hat\omega)\big| = |\partial_x\hat\omega|\,|T| \le \tfrac{1}{8}\,|\nabla\hat\omega|^2 + 2\,|T|^2 \le \tfrac{1}{8}\,|\nabla\hat\omega|^2 + 4c_0|\nabla\hat T|^2 + 4\,|T_Q|^2\\ &\big|(\partial_x S,\hat\omega)\big| = |\partial_x\hat\omega|\,|S| \le \tfrac{1}{8}\,|\nabla\hat\omega|^2 + 2\,|S|^2 \le \tfrac{1}{8}\,|\nabla\hat\omega|^2 + 4c_0|\nabla\hat S|^2 + 4\,|S_Q|^2,\end{aligned}$$

and the “nonlinear” terms as (this defines c_1)

$$\begin{aligned}&\big|(\partial(\hat\psi,\hat\omega),\varOmega)\big| \le c\,|\nabla\hat\psi|_{L^\infty}|\nabla\hat\omega|_{L^2}|\varOmega|_{L^2} \le \frac{c_1}{2}|\varOmega|_{L^2}|\nabla\hat\omega|^2\\ &\big|(\partial(\varPsi,\hat\omega),\varOmega)\big| \le c\,|\nabla\varPsi|_{L^\infty}|\nabla\hat\omega|_{L^2}|\varOmega|_{L^2} \le \frac{\mathfrak{p}}{8}\,|\nabla\hat\omega|^2 + \frac{c}{\mathfrak{p}}\,|\nabla\varPsi|^2_{L^\infty}|\varOmega|^2.\end{aligned} \tag{2.8}$$

This brings us to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\hat\omega|^2 + (\mathfrak{p} - c_1|\varOmega|)|\nabla\hat\omega|^2 &\le 4\mathfrak{p}c_0(|\nabla\hat T|^2 + |\nabla\hat S|^2)\\ &\quad + \frac{c}{\mathfrak{p}}\,|\nabla\varPsi|^2_{L^\infty}|\varOmega|^2 + 4\mathfrak{p}\,(|\nabla\varOmega|^2 + |T_Q|^2 + |S_Q|^2).\end{aligned} \tag{2.9}$$

q:icl2w

As usual, in the above and henceforth, c denotes generic constants which may take different values each time it appears. Numbered constants such as c_0 have fixed values; they are independent of the parameters $\mathfrak{p}$ and β unless noted explicitly.

Now for $\hat S$, we multiply (1.13c), or equivalently,

$$\partial_t\hat S + \partial(\psi,\hat S + S_Q) = \beta\,(\varDelta\hat S + \varDelta S_Q), \tag{2.10}$$

q:dshdt

by $\hat S$ in $L^2(\mathcal{D})$ and use $(\partial(\psi,\hat S),\hat S)=0$ to find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\hat S|^2 + \beta\,|\nabla\hat S|^2 = -(\partial(\varPsi,S_Q),\hat S) - (\partial(\hat\psi,S_Q),\hat S) + \beta\,(\varDelta S_Q,\hat S). \tag{2.11}$$

The last term on the rhs requires some care,

$$\begin{aligned}\big|(\varDelta S_Q,\hat S)\big| &= \big|(Q_S,\hat S)_{L^2(\partial\mathcal{D})} - (\nabla S_Q,\nabla\hat S)\big|\\ &\le c\,|Q_S|_{H^{-1/2}(\partial\mathcal{D})}|\hat S|_{H^{1/2}(\partial\mathcal{D})} + |\nabla Q_S|\,|\nabla\hat S|\\ &\le \tfrac{1}{8}\,|\nabla\hat S|^2 + c\,(\|Q_S\|^2 + |\nabla S_Q|^2)\end{aligned} \tag{2.12}$$

q:BCst

where we have used the trace theorem [1, Thm. 4.12] for the last inequality and denoted $\|Q_S\| := |Q_S|_{H^{-1/2}(\partial\mathcal{D})}$. We note that $|\nabla S_Q|_{L^2(\mathcal{D})}$ ultimately depends on $|Q_S|_{H^{-1/2}(\partial\mathcal{D})}$ plus the constraint (2.16) below. Bounding the "nonlinear" terms as

$$\begin{aligned}\big|(\partial(\Psi,\hat S),S_Q)\big| &\le c\,|\nabla\Psi|_{L^\infty}|\nabla\hat S|_{L^2}|S_Q|_{L^2} \le \frac{\beta}{8}\,|\nabla\hat S|^2 + \frac{c}{\beta}\,|\nabla\Psi|_{L^\infty}^2|S_Q|^2\\ \big|(\partial(\hat\psi,\hat S),S_Q)\big| &\le c\,|\nabla\hat\psi|_{L^\infty}|\nabla\hat S|_{L^2}|S_Q|_{L^2} \le \frac{\beta}{8}\,|\nabla\hat S|^2 + \frac{c}{\beta}\,|\nabla\hat\omega|^2|S_Q|^2,\end{aligned}$$

we arrive at (this defines c_2)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\hat S|^2 + \beta\,|\nabla\hat S|^2 \le{}& \frac{c_2}{8c_0\beta}\,|\nabla\hat\omega|^2|S_Q|^2\\ &+ \frac{c}{\beta}\,|\nabla\Psi|_{L^\infty}^2|S_Q|^2 + c\beta\,(|\nabla S_Q|^2 + \|Q_S\|^2).\end{aligned} \tag{2.13}$$

q:icl2s

Analogously, we have for $\hat T$ (with $\|Q_T\| := |Q_T|_{H^{-1/2}(\partial\mathcal{D})}$),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\hat T|^2 + |\nabla\hat T|^2 \le{}& \frac{c_2}{8c_0}\,|\nabla\hat\omega|^2|T_Q|^2\\ &+ c\,|\nabla\Psi|_{L^\infty}^2|T_Q|^2 + c\,(|\nabla T_Q|^2 + \|Q_T\|^2).\end{aligned} \tag{2.14}$$

q:icl2t

Adding $8\mathfrak{p}c_0$ times (2.14) and $8\mathfrak{p}c_0/\beta$ times (2.13) to (2.9), we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(|\hat\omega|^2 + 8\mathfrak{p}c_0|\hat T|^2 + \frac{8\mathfrak{p}c_0}{\beta}|\hat S|^2\Big) &+ 4\mathfrak{p}c_0\,(|\nabla\hat T|^2 + |\nabla\hat S|^2)\\ &+ \Big(\mathfrak{p} - c_1|\Omega| - c_2\mathfrak{p}|T_Q|^2 - \frac{c_2\mathfrak{p}}{\beta^2}|S_Q|^2\Big)|\nabla\hat\omega|^2\\ \le{}& c\mathfrak{p}\,|\nabla\Psi|_{L^\infty}^2\big(|\Omega|^2/\mathfrak{p}^2 + |T_Q|^2 + |S_Q|^2/\beta^2\big)\\ &+ c\mathfrak{p}\,(|\nabla\Omega|^2 + |\nabla T_Q|^2 + |\nabla S_Q|^2 + \|Q_T\|^2 + \|Q_S\|^2).\end{aligned} \tag{2.15}$$

q:aux00

If we now choose Ω, T_Q and S_Q such that

$$|\Omega|_{L^2} \le \mathfrak{p}/(8c_1), \quad |T_Q|_{L^2}^2 \le 1/(8c_2) \quad \text{and} \quad |S_Q|_{L^2}^2 \le \beta^2/(8c_2), \tag{2.16}$$

q:qc

(given the BC (1.14), this can always be done at the price of making $\nabla\Omega$, ∇T_Q and ∇S_Q large) we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(|\hat\omega|^2 + 8\mathfrak{p}c_0|\hat T|^2 + \frac{8\mathfrak{p}c_0}{\beta}|\hat S|^2\Big) + \frac{\mathfrak{p}}{2}|\nabla\hat\omega|^2 + 4\mathfrak{p}c_0\,(|\nabla\hat T|^2 + |\nabla\hat S|^2) \le \|F\|^2, \tag{2.17}$$

q:icl2

with $\|F\|^2$ denoting the purely "forcing" terms on the rhs of (2.15). Integrating this using the Gronwall lemma, we obtain the uniform bounds, with $|\hat U|^2 = |\hat\omega|^2 + 8\mathfrak{p}c_0|\hat T|^2 + 8\mathfrak{p}c_0|\hat S|^2/\beta$,

$$|\hat U(t)|^2 \le \mathrm{e}^{-\lambda t}|\hat U(0)|^2 + \|F\|^2/\lambda \tag{2.18}$$

q:bdcl2

$$c_3\mathfrak{p}\int_t^{t+1}\big\{|\nabla\hat\omega|^2 + |\nabla\hat T|^2 + |\nabla\hat S|^2\big\}(t')\,\mathrm{d}t' \le \mathrm{e}^{-\lambda t}|U(0)|^2 + (1+1/\lambda)\,\|F\|^2$$

valid for all $t \ge 0$, for some $\lambda(\pi) > 0$. It is clear from (2.18a) that we have an absorbing ball, i.e. $|U(t)|^2 \le M_0(Q;\pi)$ for all $t \ge t_0(|U(0)|;\pi)$.

On to H^1, we multiply (2.4a) by $-\Delta\hat\omega$ in L^2 to find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\hat\omega|^2 + \mathfrak{p}\,|\Delta\hat\omega|^2 &= -(\partial(\nabla\psi,\hat\omega),\nabla\hat\omega) + (\partial(\psi,\Omega),\Delta\hat\omega)\\ &\quad - \mathfrak{p}\,(\Delta\Omega,\Delta\hat\omega) - \mathfrak{p}\,(\partial_x T,\Delta\hat\omega) + \mathfrak{p}\,(\partial_x S,\Delta\hat\omega).\end{aligned} \tag{2.19}$$

Bounding the linear terms in the obvious way, and the nonlinear terms as

$$\begin{aligned}\big|(\partial(\nabla\psi,\hat\omega),\nabla\hat\omega)\big| &\le c\,|\nabla\hat\omega|_{L^4}^2|\nabla^2\psi|_{L^2} \le c\,|\nabla\hat\omega|\,|\Delta\hat\omega|\,|\Delta\psi|\\ &\le \frac{\mathfrak{p}}{8}\,|\Delta\hat\omega|^2 + \frac{c}{\mathfrak{p}}\,|\nabla\hat\omega|^2(|\hat\omega|^2+|\Omega|^2)\\ \big|(\partial(\psi,\Omega),\Delta\hat\omega)\big| &\le \frac{\mathfrak{p}}{8}\,|\Delta\hat\omega|^2 + \frac{c}{\mathfrak{p}}\,|\nabla\Omega|^2\big(|\nabla\hat\omega|^2+|\nabla\Psi|_{L^\infty}^2\big),\end{aligned}$$

we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\hat\omega|^2 + \mathfrak{p}\,|\Delta\hat\omega|^2 &\le \frac{c}{\mathfrak{p}}\,|\nabla\hat\omega|^2(|\hat\omega|^2+|\Omega|^2+|\nabla\Omega|^2) + \frac{c}{\mathfrak{p}}\,|\nabla\Psi|_{L^\infty}^2|\nabla\Omega|^2\\ &\quad + 8\mathfrak{p}\,\big(|\nabla\hat T|^2+|\nabla\hat S|^2+|\nabla T_Q|^2+|\nabla S_Q|^2+|\Delta\Omega|^2\big).\end{aligned} \tag{2.20}$$

q:ich1w

Since $\hat\omega$, $\hat T$ and $\hat S$ have been bounded uniformly in $L^2_{t,1}H^1_x$ in (2.18b), we can integrate (2.20) using the uniform Gronwall lemma to obtain a uniform bound for $|\nabla\hat\omega|^2$,

$$|\nabla\hat\omega(t)|^2 \le M_1(\cdots) \qquad\text{and}\qquad \int_t^{t+1}|\Delta\hat\omega(t')|^2\,\mathrm{d}t' \le \tilde M_1(\cdots). \tag{2.21}$$

q:bdch1w

Similarly, multiplying (2.10) by $-\Delta\hat S$ in L^2, we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\hat S|^2 + \beta\,|\Delta\hat S|^2 &= -\,\beta\,(\Delta S_Q,\Delta\hat S)\\ &\quad - (\partial(\nabla\psi,\hat S),\nabla\hat S) + (\partial(\psi,S_Q),\Delta\hat S).\end{aligned} \tag{2.22}$$

Bounding as we did for $\hat\omega$, we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\hat S|^2 + \beta\,|\Delta\hat S|^2 &\le 8\beta\,|\Delta S_Q|^2\\ &\quad + \frac{c}{\beta}\,|\nabla\hat S|^2(|\hat\omega|^2+|\Omega|^2) + \frac{c}{\beta}\,|\nabla S_Q|^2\big(|\nabla\hat\omega|^2+|\nabla\Psi|_{L^\infty}^2\big),\end{aligned} \tag{2.23}$$

which can be integrated using the uniform Gronwall lemma to obtain

$$|\nabla\hat S(t)|^2 \le M_1(\cdots) \qquad\text{and}\qquad \int_t^{t+1}|\Delta\hat S(t')|^2\,\mathrm{d}t' \le \tilde M_1(\cdots). \tag{2.24}$$

q:bdch1s

Obviously one has the analogous bound for $\hat T$,

$$|\nabla\hat T(t)|^2 \le M_1(\cdots) \qquad\text{and}\qquad \int_t^{t+1}|\Delta\hat T(t')|^2\,\mathrm{d}t' \le \tilde M_1(\cdots). \tag{2.25}$$

q:bdch1t

These bounds allow us to conclude [21] the existence of a global attractor $\mathcal{A}$ and of an invariant measure μ supported on $\mathcal{A}$. The convergence of the global attractors can be deduced following an argument similar to that in [11], while the convergence of the the invariant measures can be inferred from an argument similar to that in [23]. In particular, any generalised long-time average generates an invariant measure in the sense that for any given bounded continuous functional Φ (whose domain is the phase space H and range $\mathbb{R}$), we have

$$\underset{t\to\infty}{\mathrm{LIM}}\, \frac{1}{t}\int_0^t \Phi(\mathbb{S}(t')U_0)\,\mathrm{d}t' = \int_H \Phi(U)\,\mathrm{d}\mu(U) \tag{2.26}$$

where $U(t) = \mathbb{S}(t)U_0$ is the solution of (1.13) with initial data U_0. It is known that $\mathcal{A}$ is unique while μ may depend on the initial data U_0 and the definition of the generalised limit LIM.

Due to the boundary conditions, one cannot simply multiply by $\Delta^2\hat{\omega}$, etc., to obtain a bound in H^2, but following [20, §6.2], one takes time derivative of (1.13a) and uses the resulting bound on $|\partial_t\omega|$ to bound $|\Delta\omega|$, etc. We shall not do this explicitly here, although similar ideas are used for the discrete case below (proof of Theorem 2).

3 Numerical scheme: boundedness

s:disc

Fixing a timestep $k > 0$, we discretise the system (1.13) in time by the following two-step explicit–implicit scheme,

$$\begin{aligned}
&\frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}) \\
&\qquad\qquad = \mathfrak{p}\big\{\Delta\omega^{n+1} + \partial_x T^{n+1} - \partial_x S^{n+1}\big\} \\
&\frac{3T^{n+1} - 4T^n + T^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2T^n - T^{n-1}) = \Delta T^{n+1} \\
&\frac{3S^{n+1} - 4S^n + S^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2S^n - S^{n-1}) = \beta\Delta S^{n+1},
\end{aligned} \tag{3.1}$$

q:ei

plus the boundary conditions (1.14). Writing $U^n = (\omega^n, T^n, S^n)$, we assume that the second initial data U^1 has been obtained from U^0 using some reasonable one-step method, but all we shall need for what follows is that $U^1 \in H^1(\mathcal{D})$. The time derivative term is that of the backward differentiation formula (BDF2) and the explicit form of the nonlinear term is chosen to preserve the order of the scheme. This results in a method that is essentially explicit yet second order in time, and as we shall see below, preserves the important invariants of the continuous system.

Subject to some restrictions on the timestep k, we can obtain uniform bounds and absorbing balls for the solution of the discrete system analogous to those of the continuous system. Our first result is the following:

t:h1 **Theorem 1** *With $Q \in H^{3/2}(\partial\mathcal{D})$, the scheme* (3.1) *defines a discrete dynamical system in $H^1(\mathcal{D}) \times H^1(\mathcal{D})$. Assuming U^0, $U^1 \in H^1(\mathcal{D})$ and the timestep restriction given in* (3.20) *below,*

$$k \le k_1(|U^0|_{H^1}, |U^1|_{H^1}; |Q|_{H^{1/2}(\partial\mathcal{D})}, \pi), \tag{3.2}$$ q:dt

the following bounds hold

$$\begin{aligned} |U^n|_{L^2}^2 &\le 40\,\mathrm{e}^{-\nu nk/4}\big(|U^0|_{L^2}^2 + |U^1|_{L^2}^2\big) + M_0(|Q|_{H^{1/2}(\partial\mathcal{D})};\pi) \\ &\qquad + c(|Q|_{H^{-1/2}(\partial\mathcal{D})};\pi)k\,\mathrm{e}^{-\nu nk/4}\big(|U^0|_{H^1}^2 + |U^1|_{H^1}^2\big), \end{aligned} \tag{3.3}$$ q:l2

$$|U^n|_{H^1}^2 \le N_1(nk; |U^0|_{H^1}, |U^1|_{H^1}, |Q|_{H^{1/2}(\partial\mathcal{D})}, \pi) + M_1(|Q|_{H^{3/2}(\partial\mathcal{D})};\pi), \tag{3.4}$$ q:h1

where $\nu(\pi) > 0$ and $N_1(t;\cdots) = 0$ for $t \ge t_1(|U^0|_{H^1}, |U^1|_{H^1}; Q, \pi)$.

We note that the last term in (3.3) has no analogue in the continuous case; we believe this is an artefact of our proof, but have not been able to circumvent it. Here one can choose the bounds M_0 and M_1 to hold for both the continuous and discrete cases, although the optimal bounds (likely very laborious to compute) may be different.

Unlike in [23], H^2 bounds do not follow as readily due to the boundary conditions, so we proceed by first deriving bounds for $|U^{n+1} - U^n|$, using an approach inspired by [20, §6.2]. We state our result without the transient terms:

t:h2 **Theorem 2** *Assume the hypotheses of Theorem 1. Then for sufficiently large time, $nk \ge t_2(U^0, U^1; Q, \pi)$, one has*

$$|\omega^{n+1} - \omega^n|^2 + |T^{n+1} - T^n|^2 + |S^{n+1} - S^n|^2 \le k^2 M_\delta(|Q|_{H^{3/2}(\partial\mathcal{D})};\pi). \tag{3.5}$$ q:bddU

Furthermore, for large time $nk \ge t_2$ the solution is bounded in H^2 as

$$|\varDelta\omega^n|^2 + |\varDelta T^n|^2 + |\varDelta S^n|^2 \le M_2(|Q|_{H^{3/2}(\partial\mathcal{D})};\pi). \tag{3.6}$$ q:h2

We remark that these difference and H^2 bounds require no additional hypotheses on Q, suggesting that Theorem 1 may be sub-optimal. We also note that using the same method (and one more derivative on Q) one could bound $|U^{n+1} - U^n|_{H^1}$ and $|U^n|_{H^3}$, although we will not need these results here.

Following the approach of [23], these uniform bounds (along with the uniform convergence results that follow from them) then give us the convergence of long-time statistical properties of the discrete dynamical system (3.1) to those of the continuous system (1.13).

Proof (Proof of Theorem 1) Central to our approach is the idea of G-stability for multistep methods [9, §V.6]. First, for f, $g \in L^2(\mathcal{D})$ and $\nu k \in [0,1]$, we define the norm

$$[\![f,g]\!]_{\nu k}^2 = \frac{|f|_{L^2}^2}{2} + \frac{5+\nu k}{2}\,|g|_{L^2}^2 - 2(f,g)_{L^2}. \tag{3.7}$$ q:gndef

Note that our notation is slightly different from that in [11, 23]. Since both eigenvalues of the quadratic form are finite and positive for all $\nu k \in [0,1]$, this norm is equivalent to the L^2 norm, i.e. there exist positive constants c_+ and c_-, independent of $\nu k \in [0,1]$, such that

$$c_-(|f|_{L^2}^2 + |g|_{L^2}^2) \le [\![f,g]\!]_{\nu k}^2 \le c_+(|f|_{L^2}^2 + |g|_{L^2}^2) \tag{3.8}$$

q:equivn

for all $f,\, g \in L^2(\mathcal{D})$; computing explicitly, we find

$$c_- = \frac{6-\sqrt{32}}{4} \qquad \text{and} \qquad c_+ = \frac{7+\sqrt{41}}{4}. \tag{3.9}$$

q:c+-

As in [23], an important tool for our estimates is an identity first introduced in [9] for $\nu k = 0$; the following form can be found in [11, proof of Lemma 6.1]: for $f,\, g,\, h \in L^2(\mathcal{D})$ and $\nu k \in [0,1]$,

$$\begin{aligned}(3h - 4g + f, h)_{L^2} &+ \nu k\, |h|_{L^2}^2 \\ &= [\![g,h]\!]_{\nu k}^2 - \frac{1}{1+\nu k}[\![f,g]\!]_{\nu k}^2 + \frac{|f-2g+(1+\nu k)h|_{L^2}^2}{2(1+\nu k)}.\end{aligned} \tag{3.10}$$

q:hs00

The fact that (3.1) forms a discrete dynamical system in $H^1 \times H^1$ can be seen by writing

$$(3 - 2k\Delta)T^{n+1} = 4T^n - T^{n-1} - 2k\,\partial(2\psi^n - \psi^{n-1}, 2T^n - T^{n-1}) \tag{3.11}$$

and inverting: given U^{n-1} and $U^n \in H^1(\mathcal{D})$, the Jacobian is in H^{-1}, which, with the Neumann BC $\partial_z T^{n+1} = Q_T \in H^{1/2}(\partial\mathcal{D})$, gives $T^{n+1} \in H^1$. Similarly for S^{n+1} and, since now $T^{n+1},\, S^{n+1} \in H^1$ and $\omega^{n+1} = Q_u \in H^{1/2}(\partial\mathcal{D})$, for ω^{n+1}. Therefore $(U^{n-1}, U^n) \in H^1 \times H^1$ maps to $(U^n, U^{n+1}) \in H^1 \times H^1$.

Let $\hat\omega^n := \omega^n - \Omega$, $\hat T^n := T^n - T_Q$ and $\hat S^n := S^n - S_Q$ be defined as in the continuous case, i.e. $\Omega,\, T_Q,\, S_Q \in H^2(\mathcal{D})$ satisfying the boundary conditions $\Omega = Q_u$, $\partial_z T_Q = Q_T$ and $\partial_z S_Q = Q_S$, and the constraint (3.29), which is essentially (2.16). The scheme (3.1) then implies

$$\begin{aligned}
&\frac{3\hat\omega^{n+1} - 4\hat\omega^n + \hat\omega^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\hat\omega^n - \hat\omega^{n-1} + \Omega) \\
&\qquad\qquad = \mathfrak{p}\big\{\Delta\hat\omega^{n+1} + \Delta\Omega + \partial_x T^{n+1} - \partial_x S^{n+1}\big\} \\
&\frac{3\hat T^{n+1} - 4\hat T^n + \hat T^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\hat T^n - \hat T^{n-1} + T_Q) = \Delta\hat T^{n+1} + \Delta T_Q \\
&\frac{3\hat S^{n+1} - 4\hat S^n + \hat S^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\hat S^n - \hat S^{n-1} + S_Q) = \beta(\Delta\hat S^{n+1} + \Delta S_Q)
\end{aligned} \tag{3.12}$$

q:eih

where we have kept some ψ^n, T^n and S^n for now. We start by deriving difference inequalities for $\hat\omega^n$, $\hat T^n$ and $\hat S^n$. In order to bound terms of the form $|\nabla\hat\psi^n|_{L^\infty}^2 \le c\,|\hat\omega^n|_{H^{1/2}}^2$, we assume for now the uniform bound

$$|\hat\omega^n|_{H^{1/2}}^2 \le k^{-1/2} M_\omega(\cdots) \qquad \text{for all } n = 0, 1, 2, \cdots \tag{3.13}$$

q:whalf

where M_ω will be fixed in (3.31) below. We also assume for clarity that $k \le 1$.

Multiplying (3.12a) by $2k\hat\omega^{n+1}$ in $L^2(\mathcal{D})$ and using (3.10), we find

$$\begin{aligned}
&[\![\hat\omega^n,\hat\omega^{n+1}]\!]_{\nu k}^2 - \nu k\,|\hat\omega^{n+1}|^2 + 2\mathfrak{p}k\,|\nabla\hat\omega^{n+1}|^2 + \frac{|(1+\nu k)\hat\omega^{n+1} - 2\hat\omega^n + \hat\omega^{n-1}|^2}{2\,(1+\nu k)}\\
&= \frac{[\![\hat\omega^{n-1},\hat\omega^n]\!]_{\nu k}^2}{1+\nu k} - 2k\,(\partial(2\psi^n - \psi^{n-1},\hat\omega^{n+1}),(1+\nu k)\hat\omega^{n+1} - 2\hat\omega^n + \hat\omega^{n-1})\\
&\qquad + 2k\,(\partial(2\hat\psi^n - \hat\psi^{n-1},\hat\omega^{n+1}),\mathit{\Omega}) + 2k\,(\partial(\mathit{\Psi},\hat\omega^{n+1}),\mathit{\Omega})\\
&\qquad + 2\mathfrak{p}k\,\big\{(\mathit{\Delta\Omega},\hat\omega^{n+1}) + (\hat\omega^{n+1},\partial_x T^{n+1}) - (\hat\omega^{n+1},\partial_x S^{n+1})\big\}.
\end{aligned}\tag{3.14}$$

q:c00

where $\nu > 0$ will be set below. We bound the last terms as in the continuous case,

$$\begin{aligned}
2\,\big|(\mathit{\Delta\Omega},\hat\omega^{n+1})\big| &\le \tfrac18\,|\nabla\hat\omega^{n+1}|^2 + 8\,|\nabla\mathit{\Omega}|^2\\
2\,|(\partial_x T^{n+1},\hat\omega^{n+1})| &\le \tfrac18\,|\nabla\hat\omega^{n+1}|^2 + 16c_0\,|\nabla\hat T^{n+1}|^2 + 16\,|T_Q|^2\\
2\,|(\partial_x S^{n+1},\hat\omega^{n+1})| &\le \tfrac18\,|\nabla\hat\omega^{n+1}|^2 + 16c_0\,|\nabla\hat S^{n+1}|^2 + 16\,|S_Q|^2\\
2\,\big|(\partial(\mathit{\Psi},\hat\omega^{n+1}),\mathit{\Omega})| &\le \frac{\mathfrak{p}}{8}\,|\nabla\hat\omega^{n+1}|^2 + \frac{c}{\mathfrak{p}}\,|\nabla\mathit{\Psi}|_{L^\infty}^2\,|\mathit{\Omega}|^2,
\end{aligned}$$

and the previous one as

$$\begin{aligned}
2\,|(\partial(2\hat\psi^n - \hat\psi^{n-1},\hat\omega^{n+1}),\mathit{\Omega})| &\le c\,|2\nabla\hat\psi^n - \nabla\hat\psi^{n-1}|_{L^\infty}|\nabla\hat\omega^{n+1}|_{L^2}|\mathit{\Omega}|_{L^2}\\
&\le \frac{\mathfrak{p}}{8}\,|\nabla\hat\omega^{n+1}|^2 + \frac{c}{\mathfrak{p}}\,(|\nabla\hat\omega^{n-1}|^2 + |\nabla\hat\omega^n|^2)|\mathit{\Omega}|^2.
\end{aligned}\tag{3.15}$$

Taking $\nu = \mathfrak{p}/(8c_0)$ for now, we can bound the second term in (3.14) using the third. Using (3.13), we then bound the first nonlinear term as

$$\begin{aligned}
&2\,|(\partial(2\psi^n - \psi^{n-1},\hat\omega^{n+1}),(1+\nu k)\hat\omega^{n+1} - 2\hat\omega^n + \hat\omega^{n-1})|\\
&\quad\le \frac{\mathfrak{p}}{8}\,|\nabla\hat\omega^{n+1}|^2 + \frac{c}{\mathfrak{p}}\,|2\nabla\psi^n - \nabla\psi^{n-1}|_{L^\infty}^2\,|(1+\nu k)\hat\omega^{n+1} - 2\hat\omega^n + \hat\omega^{n-1}|^2\\
&\quad\le \frac{\mathfrak{p}}{8}\,|\nabla\hat\omega^{n+1}|^2 + c_3\,(k^{-1/2}M_\omega + |\nabla\mathit{\Psi}|_{L^\infty}^2)\,\frac{|(1+\nu k)\hat\omega^{n+1} - 2\hat\omega^n + \hat\omega^{n-1}|^2}{4\mathfrak{p}}.
\end{aligned}\tag{3.16}$$

q:c01

Recalling that the validity of (3.8) and (3.9) demands $k \le 1/\nu$, which we henceforth assume, we have $2(1+\nu k) \le 4$. This then implies that k times the last term in (3.16) can be majorised by the fourth term in (3.14) if k is small enough that

$$c_3 k^{1/2} M_\omega \le \mathfrak{p}/2 \quad\text{and}\quad c_3 k\,|\nabla\mathit{\Psi}|_{L^\infty}^2 \le \mathfrak{p}/2.\tag{3.17}$$

All this brings us to [cf. (2.9)]

$$\begin{aligned}
[\![\hat\omega^n,\hat\omega^{n+1}]\!]_{\nu k}^2 + \mathfrak{p}k\,|\nabla\hat\omega^{n+1}|^2 &\le \frac{[\![\hat\omega^{n-1},\hat\omega^n]\!]_{\nu k}^2}{1+\nu k}\\
&\quad + \frac{ck}{\mathfrak{p}}\,(|\nabla\hat\omega^{n-1}|^2 + |\nabla\hat\omega^n|^2)|\mathit{\Omega}|^2 + 16c_0\mathfrak{p}k\,(|\nabla\hat T^{n+1}|^2 + |\nabla\hat S^{n+1}|^2)\\
&\quad + ck\,(|\nabla\mathit{\Psi}|_{L^\infty}^2\,|\mathit{\Omega}|^2/\mathfrak{p} + \mathfrak{p}\,|T_Q|^2 + \mathfrak{p}\,|S_Q|^2 + \mathfrak{p}\,|\nabla\mathit{\Omega}|^2).
\end{aligned}\tag{3.18}$$

q:idl2w

For $\hat{S}^n$, we multiply (3.12c) by $2k\hat{S}^{n+1}$ in $L^2(\mathcal{D})$ and use (3.10) to find

$$\begin{aligned}
&[\![\hat{S}^n,\hat{S}^{n+1}]\!]_{\nu k}^2 - \nu k\,|\hat{S}^{n+1}|^2 + 2\beta k\,|\nabla\hat{S}^{n+1}|^2 + \frac{|(1+\nu k)\hat{S}^{n+1} - 2\hat{S}^n + \hat{S}^{n-1}|^2}{2\,(1+\nu k)}\\
&\quad = \frac{[\![\hat{S}^{n-1},\hat{S}^n]\!]_{\nu k}^2}{1+\nu k} - 2k\,(\partial(2\psi^n - \psi^{n-1},\hat{S}^{n+1}),(1+\nu k)\,\hat{S}^{n+1} - 2\hat{S}^n + \hat{S}^{n-1})\\
&\quad + 2k\,(\partial(2\hat{\psi}^n - \hat{\psi}^{n-1},\hat{S}^{n+1}),S_Q) + 2k\,(\partial(\Psi,\hat{S}^{n+1}),S_Q) + 2\beta k\,(\Delta S_Q,\hat{S}^{n+1}).
\end{aligned}$$

Bounding the last term as in (2.12) and everything else as with $\hat{\omega}^n$, and taking (this also takes care of $\hat{T}^n$ below)

$$\nu = \min\{\mathfrak{p},\beta,1\}/(8c_0) \tag{3.19}$$ q:nu

$$k \le \min\Big\{\frac{\min\{\mathfrak{p}^2,\beta^2,1\}}{(2c_3 M_\omega)^2},\frac{\min\{\mathfrak{p},\beta,1\}}{2c_3|\nabla\Psi|_{L^\infty}^2},\frac{1}{\nu}\Big\}, \tag{3.20}$$ q:k1

we arrive at

$$\begin{aligned}
[\![\hat{S}^n,\hat{S}^{n+1}]\!]_{\nu k}^2 + \beta k\,|\nabla\hat{S}^{n+1}|^2 &\le \frac{[\![\hat{S}^{n-1},\hat{S}^n]\!]_{\nu k}^2}{1+\nu k} + \frac{ck}{\beta}\,(|\nabla\hat{\omega}^{n-1}|^2 + |\nabla\hat{\omega}^n|^2)|S_Q|^2\\
&\quad + \frac{ck}{\beta}\,|\nabla\Psi|_{L^\infty}^2|S_Q|^2 + c\beta k\,(|\nabla S_Q|^2 + \|Q_S\|^2).
\end{aligned} \tag{3.21}$$ q:idl2s

Similarly, for $\hat{T}^n$ we have

$$\begin{aligned}
[\![\hat{T}^n,\hat{T}^{n+1}]\!]_{\nu k}^2 + k\,|\nabla\hat{T}^{n+1}|^2 &\le \frac{[\![\hat{T}^{n-1},\hat{T}^n]\!]_{\nu k}^2}{1+\nu k} + ck\,(|\nabla\hat{\omega}^{n-1}|^2 + |\nabla\hat{\omega}^n|^2)|T_Q|^2\\
&\quad + ck\,|\nabla\Psi|_{L^\infty}^2|T_Q|^2 + ck\,(|\nabla T_Q|^2 + \|Q_T\|^2).
\end{aligned} \tag{3.22}$$ q:idl2t

Adding $16\mathfrak{p}c_0$ times (3.22) and $16\mathfrak{p}c_0/\beta$ times (3.21) to (3.18), and writing

$$\begin{aligned}
&[\![\hat{U}^n,\hat{U}^{n+1}]\!]_{\nu k}^2\\
&\quad := [\![\hat{\omega}^n,\hat{\omega}^{n+1}]\!]_{\nu k}^2 + 16\mathfrak{p}c_0[\![\hat{T}^n,\hat{T}^{n+1}]\!]_{\nu k}^2 + 16\mathfrak{p}c_0[\![\hat{S}^n,\hat{S}^{n+1}]\!]_{\nu k}^2/\beta,
\end{aligned} \tag{3.23}$$

we have

$$\begin{aligned}
&[\![\hat{U}^n,\hat{U}^{n+1}]\!]_{\nu k}^2 + \mathfrak{p}k\,\big(|\nabla\hat{\omega}^{n+1}|^2 + 8c_0|\nabla\hat{T}^{n+1}|^2 + 8c_0|\nabla\hat{S}^{n+1}|^2/\beta\big) \qquad (3.24)\\
&\qquad \le \frac{[\![\hat{U}^{n-1},\hat{U}^n]\!]_{\nu k}^2}{1+\nu k} + k\,\|F_1\|^2(|\nabla\hat{\omega}^{n-1}|^2 + |\nabla\hat{\omega}^n|^2) + k\,\|F_2\|^2
\end{aligned}$$ q:idl2

where

$$\begin{aligned}
\|F_1\|^2 &:= c_4\mathfrak{p}\,\big(|\Omega|^2/\mathfrak{p}^2 + |T_Q|^2 + |S_Q|^2/\beta^2\big) \qquad (3.25)\\
\|F_2\|^2 &:= |\nabla\Psi|_{L^\infty}^2\|F_1\|^2 + c\mathfrak{p}\,\big(|\nabla T_Q|^2 + |\nabla S_Q|^2 + |\nabla\Omega|^2 + |Q_T\|^2 + \|Q_S\|^2\big).
\end{aligned}$$

In order to integrate this difference inequality, we consider a three-term recursion of the form

$$x_{n+1} + \mu y_{n+1} \le (1+\delta)^{-1}x_n + \varepsilon y_n + \varepsilon y_{n-1} + r_n. \tag{3.26}$$ q:3tr

For $\mu > 0$, $\delta \in (0,1]$ and $\varepsilon \in (0, \mu/8]$, we have

$$x_n + \mu y_n \le \frac{x_{n-m} + \mu y_{n-m}}{(1+\delta)^m} + \frac{\varepsilon\, y_{n-m-1}}{(1+\delta)^{m-1}} + \sum\nolimits_{j=1}^{m} \frac{r_{n-j}}{(1+\delta)^{j-1}} \tag{3.27}$$

(which follows readily by induction) and in particular

$$x_{n+1} + \mu y_{n+1} \le \frac{x_1 + \mu y_1}{(1+\delta)^n} + \frac{\varepsilon\, y_0}{(1+\delta)^{n-1}} + \sum\nolimits_{j=1}^{n} \frac{r_j}{(1+\delta)^{n-j}}. \tag{3.28}$$

q:3tb

In order to apply the bound (3.28) of (3.26) to (3.24), we demand that Ω, T_Q and S_Q be small enough that

$$|\Omega|_{L^2}^2 \le \mathfrak{p}^2/(32c_4), \quad |T_Q|_{L^2}^2 \le 1/(32c_4) \quad \text{and} \quad |S_Q|_{L^2}^2 \le \beta^2/(32c_4). \tag{3.29}$$

q:qd

We note that, up to parameter-independent constants, these conditions are identical to those in the continuous case (2.16). Using the fact that $(1+x)^{-1} \le \exp(-x/2)$ for $x \in (0,1]$, we integrate (3.24) to find a bound uniform in nk,

$$\begin{aligned} &[\![\hat U^n, \hat U^{n+1}]\!]_{\nu k}^2 + \mathfrak{p}k\,|\nabla\hat\omega^{n+1}|^2 \\ &\qquad \le \mathrm{e}^{-\nu nk/2}\big\{[\![\hat U^0, \hat U^1]\!]_{\nu k}^2 + \mathfrak{p}k\,(|\nabla\hat\omega^0|^2 + |\nabla\hat\omega^1|^2)\big\} + \frac{2}{\nu}\,\|F_2\|^2. \end{aligned} \tag{3.30}$$

q:bdl2

Using (3.8)–(3.9), (3.3) follows.

The hypothesis (3.13) can now be recovered by interpolation,

$$\begin{aligned} |\hat\omega^n|_{H^{1/2}}^2 &\le c\,|\hat\omega^n|\,|\nabla\hat\omega^n| \le c\,[\![\hat U^{n-1}, \hat U^n]\!]_{\nu k}|\nabla\hat\omega^n| \\ &\le c\,(\mathfrak{p}k)^{-1/2}\big\{[\![\hat U^0, \hat U^1]\!]_{\nu k}^2 + \mathfrak{p}\,(|\nabla\hat\omega^0|^2 + |\nabla\hat\omega^1|^2) + 2\,\|F_2\|^2/\nu\big\} \end{aligned} \tag{3.31}$$

q:whalf1

and replacing $[\![\hat U^0, \hat U^1]\!]_{\nu k}^2$ by its sup over $\nu k \in (0,1]$. Summing (3.24) and using (3.29), we find (discarding terms on the lhs)

$$\begin{aligned} &k\sum\nolimits_{j=n+1}^{n+m}\Big\{\frac{\mathfrak{p}}{2}\,|\nabla\hat\omega^j|^2 + 8c_0\,|\nabla\hat T^j|^2 + \frac{8c_0}{\beta}\,|\nabla\hat S^j|^2\Big\} \\ &\qquad \le [\![\hat U^{n-1}, \hat U^n]\!]_{\nu k}^2 + 2k\,\|F_1\|^2(|\nabla\hat\omega^{n-1}|^2 + |\nabla\hat\omega^n|^2) + mk\,\|F_2\|^2. \end{aligned} \tag{3.32}$$

q:bdl2h1

From (3.30) and (3.32), it is clear that there exists a $t_0(|\nabla U^0|, |\nabla U^1|, Q; \pi)$ such that, whenever $nk \ge t_0$,

$$|\hat U^n|^2 \le M_0(Q;\pi) \quad \text{and} \quad k\sum\nolimits_{j=n}^{n+\lfloor 1/k\rfloor} |\nabla\hat U^j|^2 \le \tilde M_0(Q;\pi). \tag{3.33}$$

q:bdl2u

We redefine M_0 and $\tilde M_0$ to bound $|U^n|^2$ and $\sum_j |\nabla U^j|^2$ as well.

On to H^1, we multiply (3.12a) by $-2k\Delta\hat\omega^{n+1}$ in L^2 to get

$$\begin{aligned}
&[\![\nabla\hat\omega^n,\nabla\hat\omega^{n+1}]\!]_{\nu k}^2 - \nu k\,|\nabla\hat\omega^{n+1}|^2 + \frac{|(1+\nu k)\nabla\hat\omega^{n+1} - 2\nabla\hat\omega^n + \nabla\hat\omega^{n-1}|^2}{2\,(1+\nu k)}\\
&\quad = \frac{[\![\nabla\hat\omega^{n-1},\nabla\hat\omega^n]\!]_{\nu k}^2}{1+\nu k} - 2\mathfrak{p}k\,|\Delta\hat\omega^{n+1}|^2\\
&\quad + 2\mathfrak{p}k\,(\partial_x S^{n+1} - \partial_x T^{n+1} - \Delta\Omega, \Delta\hat\omega^{n+1}) \qquad (3.34)\\
&\quad - 2k\,(\partial(2\psi^n - \psi^{n-1}, \nabla\hat\omega^{n+1}), (1+\nu k)\nabla\hat\omega^{n+1} - 2\nabla\hat\omega^n + \nabla\hat\omega^{n-1})\\
&\quad - 2k\,(\partial(2\nabla\hat\psi^n - \nabla\hat\psi^{n-1}, 2\hat\omega^n - \hat\omega^{n-1}), \nabla\hat\omega^{n+1})\\
&\quad - 2k\,(\partial(\nabla\Psi, 2\hat\omega^n - \hat\omega^{n-1}), \nabla\hat\omega^{n+1}) + 2k\,(\partial(2\psi^n - \psi^{n-1}, \Omega), \Delta\hat\omega^{n+1}).
\end{aligned}$$

Labelling the last four "nonlinear" terms by ①, $\cdots$,④, we bound them as

$$\begin{aligned}
① &\le ck\,|2\nabla\psi^n - \nabla\psi^{n-1}|_{L^\infty}|\nabla^2\hat\omega^{n+1}|_{L^2}|(1+\nu k)\nabla\hat\omega^{n+1} - 2\nabla\hat\omega^n + \nabla\hat\omega^{n-1}|_{L^2}\\
&\le \frac{\mathfrak{p}k}{8}|\Delta\hat\omega^{n+1}|^2 + \frac{c_3 k^{1/2}}{4\mathfrak{p}}\big(M_\omega + |\nabla\Psi|_{L^\infty}^2\big)|\nabla((1+\nu k)\hat\omega^{n+1} - 2\hat\omega^n + \hat\omega^{n-1})|^2\\
② &\le ck\,|2\hat\omega^n - \hat\omega^{n-1}|_{L^4}|\nabla^2\hat\omega^{n+1}|_{L^2}|2\hat\omega^n - \hat\omega^{n-1}|_{L^4}\\
&\le \frac{\mathfrak{p}k}{8}\,|\Delta\hat\omega^{n+1}|^2 + \frac{ck}{\mathfrak{p}}\,|2\hat\omega^n - \hat\omega^{n-1}|^2|2\nabla\hat\omega^n - \nabla\hat\omega^{n-1}|^2\\
③ &\le ck\,|\Omega|_{L^\infty}|\nabla^2\hat\omega^{n+1}|_{L^2}|2\hat\omega^n - \hat\omega^{n-1}|_{L^2}\\
&\le \frac{\mathfrak{p}k}{8}\,|\Delta\hat\omega^{n+1}|^2 + \frac{ck}{\mathfrak{p}}\,|\Omega|_{L^\infty}^2|2\hat\omega^n - \hat\omega^{n-1}|^2\\
④ &\le ck\,|2\nabla\psi^n - \nabla\psi^{n-1}|_{L^\infty}|\nabla\Omega|_{L^2}|\Delta\hat\omega^{n+1}|_{L^2}\\
&\le \frac{\mathfrak{p}k}{8}\,|\Delta\hat\omega^{n+1}|^2 + \frac{ck}{\mathfrak{p}}\,|\nabla\Omega|^2\big(|\nabla\Psi|_{L^\infty}^2 + |2\nabla\hat\omega^n - \nabla\hat\omega^{n-1}|^2\big).
\end{aligned}$$

Bounding the linear term in the obvious fashion and again using (3.19)–(3.20), we arrive at

$$\begin{aligned}
&[\![\nabla\hat\omega^n,\nabla\hat\omega^{n+1}]\!]_{\nu k}^2 + \mathfrak{p}k\,|\Delta\hat\omega^{n+1}|^2 \qquad (3.35)\\
&\quad\le [\![\nabla\hat\omega^{n-1},\nabla\hat\omega^n]\!]_{\nu k}^2\big[1 + c\mathfrak{p}^{-1}k\,(M_0 + |\nabla\Omega|^2)\big] + 8\mathfrak{p}k\,\big(|\nabla\hat T^{n+1}|^2 + |\nabla\hat S^{n+1}|^2\big)\\
&\qquad + c\mathfrak{p}^{-1}k\,\big(M_0|\Omega|_{L^\infty}^2 + |\nabla\Omega|^2|\nabla\Psi|_{L^\infty}^2\big) + 8\mathfrak{p}k\,\big(|\Delta\Omega|^2 + |\nabla T_Q|^2 + |\nabla S_Q|^2\big)
\end{aligned}$$ q:idh1w

valid for large times $nk \ge t_0$.

Noting that, for $x_n \ge 0$, $r_n \ge 0$ and $b > 0$,

$$x_{n+1} \le (1+b)\,x_n + r_n \quad\Rightarrow\quad x_{n+m} \le (1+b)^m\big(x_n + \textstyle\sum_{j=n}^{n+m-1} r_j\big), \qquad (3.36)$$ q:gw1

we can obtain a uniform H^1 bound from (3.33) and (3.35) as follows. Borrowing an argument from [5], we conclude from (3.33) that there exists an $n_* \in \{n + \lfloor 1/k\rfloor, \cdots, n + \lfloor 2/k\rfloor - 1\}$ such that

$$|\nabla\hat\omega^{n_*}|^2 + |\nabla\hat\omega^{n_*+1}|^2 \le \tfrac14\,\tilde M_0(Q;\pi) \;\Rightarrow\; [\![\nabla\hat\omega^{n_*},\nabla\hat\omega^{n_*+1}]\!]_{\nu k}^2 \le c_5\,\tilde M_0. \qquad (3.37)$$ q:ah100

(In other words, in any sequence of non-negative numbers, one can find two consecutive terms whose sum is no greater than four times the average.) Taking $n_* \in \{\lceil t_0/k \rceil, \cdots, \lceil (t_0+1)/k \rceil - 1\}$ and integrating (3.35) using (3.36) with $m = \lfloor 2/k \rfloor$ and (3.33) to bound the $|\nabla \hat{T}^n|^2$ and $|\nabla \hat{S}^n|^2$ on the rhs, we find

$$[\![\nabla\hat{\omega}^n, \nabla\hat{\omega}^{n+1}]\!]_{\nu k}^2 \le M_1(Q;\pi) \tag{3.38}$$ q:bdh1w

for all $n \in \{n_*, \cdots, n_* + \lfloor 2/k \rfloor - 1\}$. We then find a $n_{**} \in \{n_* + \lfloor 1/k \rfloor, \cdots, n_* + \lfloor 2/k \rfloor - 1\}$ that satisfies (3.37) and repeat the argument to find that (3.38) also holds for all $n \in \{n_{**}, \cdots, n_{**} + \lfloor 2/k \rfloor - 1\}$. Since $n_{**} \ge n_* + \lfloor 1/k \rfloor$, with each iteration we increase the time of validity of (3.38) by at least 1 using no further assumptions, implying that (3.38) in fact holds for all $n \ge n_*$, i.e. whenever $nk \ge t_0 + 1$.

Similarly for $\hat{S}^n$, we multiply (3.12c) by $-2k\Delta\hat{S}^{n+1}$ in L^2 to find after a similar computation

$$\begin{aligned}
[\![\nabla\hat{S}^n, \nabla\hat{S}^{n+1}]\!]_{\nu k}^2 + \beta k\,|\Delta\hat{S}^{n+1}|^2 &\le [\![\nabla\hat{S}^{n-1}, \hat{S}^n]\!]_{\nu k}^2 \big(1 + ck\beta^{-1}M_0\big) \\
&\quad + \frac{ck}{\beta}\,(M_0 + |\nabla S_Q|^2)\big(|\nabla\Psi|_{L^\infty}^2 + |\nabla\hat{\omega}^{n-1}|^2 + |\nabla\hat{\omega}^n|^2\big) \\
&\quad + \frac{ck}{\beta}\,M_0|\Omega|_{L^\infty}^2 + 8\beta k\,|\Delta S_Q|^2.
\end{aligned} \tag{3.39}$$ q:idh1s

Arguing as we did with $\hat{\omega}^n$, we conclude that (redefining M_1 as needed) one has

$$[\![\nabla\hat{S}^n, \nabla\hat{S}^{n+1}]\!]_{\nu k}^2 \le M_1(Q;\pi) \qquad \text{whenever } nk \ge t_0 + 1. \tag{3.40}$$

Obviously the same bound applies to $\hat{T}^n$,

$$[\![\nabla\hat{T}^n, \nabla\hat{T}^{n+1}]\!]_{\nu k}^2 \le M_1(Q;\pi) \qquad \text{whenever } nk \ge t_0 + 1. \tag{3.41}$$

As we did with M_0, we redefine M_1 to bound $[\![\nabla\omega^n, \nabla\omega^{n+1}]\!]_{\nu k}^2$, etc., as well as $[\![\nabla\hat{\omega}^n, \nabla\hat{\omega}^{n+1}]\!]_{\nu k}^2$. □

Proof (Proof of Theorem 2) Let $\delta U^n := U^n - U^{n-1} = \hat{U}^n - \hat{U}^{n-1}$. We first prove that $|\delta U^n|^2 \le kM$ for all large n, and then use this result to prove (3.5).

Writing $3\omega^{n+1} - 4\omega^n + \omega^{n-1} = 3\delta\omega^{n+1} - \delta\omega^n$ and using the identity

$$2\,(3\delta\omega^{n+1} - \delta\omega^n, \delta\omega^{n+1}) = 3\,|\delta\omega^{n+1}|^2 - \tfrac{1}{3}\,|\delta\omega^n|^2 + \tfrac{1}{3}\,|3\delta\omega^{n+1} - \delta\omega^n|^2, \tag{3.42}$$ q:idd1

we multiply (3.1a) by $4k\delta\omega^{n+1}$,

$$\begin{aligned}
3\,|\delta\omega^{n+1}|^2 + \tfrac{1}{3}\,|3\delta\omega^{n+1} - \delta\omega^n|^2 &= \tfrac{1}{3}\,|\delta\omega^n|^2 \\
&\quad + 4\mathfrak{p}k\,(\Delta\omega^{n+1}, \delta\omega^{n+1}) + 4\mathfrak{p}k\,(\partial_x T^{n+1} - \partial_x S^{n+1}, \delta\omega^{n+1}) \\
&\quad - 4k\,(\partial(2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}), \delta\omega^{n+1}).
\end{aligned} \tag{3.43}$$

For the dissipative term, we integrate by parts using the fact that $\delta\omega^{n+1} = 0$ on the boundary to write it as

$$-2\,(\Delta\omega^{n+1}, \delta\omega^{n+1}) = |\nabla\omega^{n+1}|^2 - |\nabla\omega^n|^2 + |\nabla\delta\omega^{n+1}|^2. \tag{3.44}$$ q:idd2

We bound the nonlinear term as

$$\begin{aligned}4\,\big|(\partial(2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}), \delta\omega^{n+1})\big| &\\ \le c\,|2\nabla\psi^n - \nabla\psi^{n-1}|_{L^\infty}|2\nabla\omega^n - \nabla\omega^{n-1}|_{L^2}|\delta\omega^{n+1}|_{L^2} &\\ \le \tfrac{1}{8}\,|\delta\omega^{n+1}|^2 + c\,|2\nabla\omega^n - \nabla\omega^{n-1}|^4. & \qquad (3.45)\end{aligned}$$

Bounding the buoyancy terms by Cauchy–Schwarz, we arrive at

$$\begin{aligned}2\,|\delta\omega^{n+1}|^2 + \tfrac{1}{3}\,|3\delta\omega^{n+1} - \delta\omega^n|^2 + 2\mathfrak{p}k\,|\nabla\omega^{n+1}|^2 + 2\mathfrak{p}k\,|\nabla\delta\omega^{n+1}|^2 &\\ \le \tfrac{1}{3}\,|\delta\omega^n|^2 + 2\mathfrak{p}k\,|\nabla\omega^n|^2 + ck^2\,|2\nabla\omega^n - \nabla\omega^{n-1}|^4 &\\ + c\mathfrak{p}^2k^2\,\big(|\partial_xT^{n+1}|^2 + |\partial_xS^{n+1}|^2\big) &\\ \le \tfrac{1}{3}\,|\delta\omega^n|^2 + c(\pi)\big(kM_1 + k^2M_1^2\big). &\end{aligned} \tag{3.46}$$

It is now clear that, since $\delta\omega^1$ is bounded in L^2, we have for large nk

$$|\delta\omega^n|^2 \le k\,c(\pi)(M_1 + kM_1^2). \tag{3.47}$$

Similarly for $\hat{S}^n$, we multiply (3.12c) by $4k\delta\hat{S}^{n+1}$ to find

$$\begin{aligned}3\,|\delta\hat{S}^{n+1}|^2 + \tfrac{1}{3}\,|3\delta\hat{S}^{n+1} - \delta\hat{S}^n|^2 = \tfrac{1}{3}\,|\delta\hat{S}^n|^2 + 4k\beta\,(\Delta\hat{S}^{n+1} + \Delta S_Q, \delta\hat{S}^{n+1}) &\\ - 4k\,(\partial(2\psi^n - \psi^{n-1}, 2\hat{S}^n - \hat{S}^{n-1} + S_Q), \delta\hat{S}^{n+1}). & \qquad (3.48)\end{aligned}$$

Bounding the nonlinear term as we did for ω^n,

$$\begin{aligned}4\,\big|(\partial(2\psi^n - \psi^{n-1}, 2\hat{S}^n - \hat{S}^{n-1} + S_Q), \delta\hat{S}^{n+1})\big| & \qquad (3.49)\\ \le \tfrac{1}{8}\,|\delta\hat{S}^{n+1}|^2 + c\,|2\nabla\omega^n - \nabla\omega^{n-1}|^2(|2\nabla\hat{S}^n - \nabla\hat{S}^{n-1}|^2 + |\nabla S_Q|^2), &\end{aligned}$$

and the linear terms as we did with ω^n, we arrive at

$$\begin{aligned}2\,|\delta\hat{S}^{n+1}|^2 + \tfrac{1}{3}\,|3\delta\hat{S}^{n+1} - \delta\hat{S}^n|^2 + 2\beta k\,|\nabla\hat{S}^{n+1}|^2 + 2\beta k\,|\nabla\delta\hat{S}^{n+1}|^2 &\\ \le \tfrac{1}{3}\,|\delta\hat{S}^n|^2 + 2\beta k\,|\nabla\hat{S}^n|^2 + ck^2|2\nabla U^n - \nabla U^{n-1}|^4 &\\ + c(\beta)k^2(|\nabla S_Q|^4 + |\Delta S_Q|^2), &\end{aligned} \tag{3.50}$$

whence

$$|\delta\hat{S}^n|^2 \le k\,c(\pi)(M_1 + kM_1^2) \qquad \text{for large } nk. \tag{3.51}$$

Obviously a similar bound holds for $\delta\hat{T}^n$, so we conclude that

$$|\delta U^n|^2 \le k\,c(\pi)(M_1 + kM_1^2) =: k\tilde{M}_\delta \qquad \text{for large } nk. \tag{3.52}$$ q:delU

By taking difference of (3.1a), we find

$$\frac{3\delta\omega^{n+1} - 4\delta\omega^n + \delta\omega^{n-1}}{2k} + \partial(2\psi^{n-1} - \psi^{n-2}, 2\delta\omega^n - \delta\omega^{n-1}) \qquad (3.53)$$
$$+ \partial(2\delta\psi^n - \delta\psi^{n-1}, 2\omega^n - \omega^{n-1}) = \mathfrak{p}\big\{\Delta\delta\omega^{n+1} + \partial_x\delta T^{n+1} - \partial_x\delta S^{n+1}\big\}.$$

q:del0

Multiplying this by $2k\delta\omega^{n+1}$ and using (3.10), we have

$$[\![\delta\omega^n, \delta\omega^{n+1}]\!]_{\nu k}^2 - \nu k\,|\delta\omega^{n+1}|^2 + \frac{|(1+\nu k)\delta\omega^{n+1} - 2\delta\omega^n + \delta\omega^{n-1}|^2}{2(1+\nu k)} + kI$$
$$= \frac{[\![\delta\omega^{n-1}, \delta\omega^n]\!]_{\nu k}^2}{1+\nu k} - 2\mathfrak{p}k\,|\nabla\delta\omega^{n+1}|^2 + 2\mathfrak{p}k\,(\partial_x\delta T^{n+1} - \partial_x\delta S^{n+1}, \delta\omega^{n+1}). \qquad (3.54)$$

Here $I = I_1 + I_2$ denotes the nonlinear terms, which we bound as

$$\begin{aligned} |I_1| &\le c\,|2\nabla\psi^{n-1} - \nabla\psi^{n-2}|_{L^\infty}|\nabla\delta\omega^{n+1}|_{L^2}|2\delta\omega^n - \delta\omega^{n-1}|_{L^2} \\ &\le \frac{\mathfrak{p}}{8}\,|\nabla\delta\omega^{n+1}|^2 + \frac{c}{\mathfrak{p}}\,|2\nabla\omega^{n-1} - \nabla\omega^{n-2}|^2|2\delta\omega^n - \delta\omega^{n-1}|^2 \\ |I_2| &\le c\,|2\nabla\delta\psi^n - \nabla\delta\psi^{n-1}|_{L^4}|2\omega^n - \omega^{n-1}|_{L^4}|\nabla\delta\omega^{n+1}|_{L^2} \\ &\le \frac{\mathfrak{p}}{8}\,|\nabla\delta\omega^{n+1}|^2 + \frac{c}{\mathfrak{p}}\,|2\delta\omega^n - \delta\omega^{n-1}|^2|2\nabla\omega^n - \nabla\omega^{n-1}|^2. \end{aligned} \qquad (3.55)$$

Bounding the linear terms as

$$\big|(\partial_x\delta T^{n+1} - \partial_x\delta S^{n+1}, \delta\omega^{n+1})\big| \le \tfrac{1}{4}\,|\nabla\delta\omega^{n+1}|^2 + 2\,|\delta T^{n+1}|^2 + 2\,|\delta S^{n+1}|^2 \qquad (3.56)$$

and using (3.52), we obtain

$$\begin{aligned} &[\![\delta\omega^n, \delta\omega^{n+1}]\!]_{\nu k}^2 + \mathfrak{p}k\,|\nabla\delta\omega^{n+1}|^2 \\ &\qquad\le \frac{1}{1+\nu k}[\![\delta\omega^{n-1}, \delta\omega^n]\!]_{\nu k}^2 + k^2 c(\pi)\tilde{M}_\delta(1+M_1). \end{aligned} \qquad (3.57)$$

q:del1

Integrating this and the analogous expressions for δT^n and δS^n, we obtain (3.5) for nk large.

To prove (3.6), we note that (3.1b) implies

$$\begin{aligned} |\Delta T^{n+1}| &\le |\partial(2\psi^n - \psi^{n-1}, 2T^n - T^{n-1})| + \frac{|3\delta T^{n+1} - \delta T^n|}{2k} \\ &\le c\,|2\nabla\omega^n - \nabla\omega^{n-1}|\,|2\nabla T^n - \nabla T^{n-1}| + \frac{3|\delta T^{n+1}| + |\delta T^n|}{2k}. \end{aligned} \qquad (3.58)$$

Since the right-hand side has been bounded (independently of k for the first term and by Mk for the second) on the attractor $\mathcal{A}_k$, it follows that $|\Delta T^n|$ is uniformly bounded on $\mathcal{A}_k$ as well. Clearly similar H^2 bounds also hold for S^n and ω^n, proving (3.6) and the Theorem. □

For convenience, we recap our main notations:

c_0	Poincaré constant
$\pi = (\mathfrak{p}, \beta, \xi)$	Prandtl, Froude numbers, aspect ratio
$U = (\omega, T, S)$	non-dimensional variables; see (1.13)
$Q = (Q_u, Q_T, Q_S)$	BC for U in (1.14), with norm
$\\|Q_T\\| = \|Q_T\|_{H^{-1/2}(\partial\mathcal{D})}$	$\\|Q_S\\| = \|Q_S\|_{H^{-1/2}(\partial\mathcal{D})}$
(Ω, T_Q, S_Q)	H^2 extension of Q into $\bar{\mathcal{D}}$: (2.3), (2.16)
$(\hat\omega, \hat T, \hat S) = U - (\Omega, T_Q, S_Q)$	homogeneous variables, cf. (2.4)
$M_0, M_1, \tilde M_0, \tilde M_1, M_\omega$	bounds: (2.18), (2.21)–(2.25), (3.13)
$[\![\cdot,\cdot]\!]_{\nu k}$	G-norm: (3.7)

Also, $\Delta\psi := \omega$, $\Delta\hat\psi := \hat\omega$ and $\Delta\Psi := \Omega$, all with homogeneous BC.

A 2d Navier–Stokes equations

In this appendix we present an alternate derivation of the boundedness results in [23], without using the Wente-type estimate of [15] but requiring slightly more regular initial data. In principle these could be obtained following the proofs of Theorems 1 and 2 above, but the computation is much cleaner in this case (mostly due to the periodic boundary conditions) so we present it separately.

The system is the 2d Navier–Stokes equations

$$\frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}) = \mu\Delta\omega^{n+1} + f^n \tag{A.1}$$

q:adwdt

with periodic boundary conditions. It is clear that ω^n has zero integral over $\mathcal{D}$, and we define ψ^n uniquely by the zero-integral condition. These imply (2.1)–(2.2), which we will use below without further mention. Assuming that the initial data ω^0, $\omega^1 \in H^{1/2}$ (in fact, we only need H^ϵ for any $\epsilon > 0$, but will write $H^{1/2}$ for concreteness), we derive uniform bounds for ω^n in L^2, H^1 and H^2.

Assuming for now the uniform bound

$$|\omega^n|^2_{H^{1/2}} \le k^{-1/2} M_\omega(\cdots) \qquad \text{for } n \in \{2, 3, \cdots\}, \tag{A.2}$$

q:ahalf

we multiply (A.1) by $2k\omega^{n+1}$ in L^2, use (3.10) and estimate as before,

$$\begin{aligned}
[\![\omega^n, \omega^{n+1}]\!]^2_{\nu k} &- \nu k\,|\omega^{n+1}|^2 + 2\mu k\,|\nabla\omega^{n+1}|^2 + \frac{|(1+\nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}|^2}{2(1+\nu k)} \\
&= \frac{[\![\omega^{n-1}, \omega^n]\!]^2_{\nu k}}{1+\nu k} + 2k\,(f^n, \omega^{n+1}) \\
&\quad - 2k\,(\partial(2\psi^n - \psi^{n-1}, \omega^{n+1}), (1+\nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}) \\
&\le \frac{[\![\omega^{n-1}, \omega^n]\!]^2_{\nu k}}{1+\nu k} + \frac{\mu k}{2}\,|\nabla\omega^{n+1}|^2 + \frac{ck}{\mu}\,|f^n|^2_{H^{-1}} \\
&\quad + \frac{ck}{\mu}\,|2\nabla\psi^n - \nabla\psi^{n-1}|^2_{L^\infty}\,|(1+\nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}|^2,
\end{aligned} \tag{A.3}$$

giving (as before, we require $k \le 1/\nu$)

$$\begin{aligned}
[\![\omega^n, \omega^{n+1}]\!]^2_{\nu k} - \nu k\,|\omega^{n+1}|^2 + \frac{3\mu k}{2}\,|\nabla\omega^{n+1}|^2 &\le \frac{[\![\omega^{n-1}, \omega^n]\!]^2_{\nu k}}{1+\nu k} + \frac{ck}{\mu}\,|f^n|^2_{H^{-1}} \\
&\quad + |(1+\nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}|^2 \big(c_3 k^{1/2} M_\omega/\mu - \tfrac14\big).
\end{aligned} \tag{A.4}$$

Setting $\nu = \mu/(2c_0)$ and imposing the timestep restriction

$$k \le k_0 := \min\{\mu^2/(4c_3M_\omega)^2, 1/\nu\}, \tag{A.5}$$ q:adt

this gives

$$[\![\omega^n,\omega^{n+1}]\!]_{\nu k}^2 + \mu k\,|\nabla\omega^{n+1}|^2 \le \frac{[\![\omega^{n-1},\omega^n]\!]_{\nu k}^2}{1+\nu k} + \frac{ck}{\mu}\,|f^n|_{H^{-1}}^2. \tag{A.6}$$ q:aidl2

Integrating using the Gronwall lemma, we arrive at the L^2 bound

$$\begin{aligned}[\![\omega^{n+1},\omega^{n+2}]\!]_{\nu k}^2 + \mu k\,|\nabla\omega^{n+2}|^2 &\le \mathrm{e}^{-\nu nk/2}[\![\omega^0,\omega^1]\!]_{\nu k}^2 + \frac{c}{\mu^2}\sup\nolimits_j|f^j|_{H^{-1}}^2\\ &\le [\![\omega^0,\omega^1]\!]_{\nu k}^2 + \frac{c}{\mu^2}\sup\nolimits_j|f^j|_{H^{-1}}^2 =: M_0.\end{aligned} \tag{A.7}$$ q:abdl2

The hypothesis (A.2) is now recovered by interpolation as before,

$$\begin{aligned}|\omega^n|_{H^{1/2}}^2 &\le c\,|\omega^n|\,|\nabla\omega^n| \le c\,[\![\omega^{n-1},\omega^n]\!]_{\nu k}|\nabla\omega^n|\\ &\le c\,(\mu k)^{-1/2}\big([\![\omega^0,\omega^1]\!]_{\nu k}^2 + (1/\mu+1/\mu^2)\sup\nolimits_j|f^j|_{H^{-1}}^2\big).\end{aligned} \tag{A.8}$$

Summing (A.6), we find

$$\mu k\sum\nolimits_{j=n+1}^{n+\lfloor 1/k\rfloor}|\nabla\omega^j|^2 \le [\![\omega^{n-1},\omega^n]\!]_{\nu k}^2 + c_\mu\sup\nolimits_j|f^j|_{H^{-1}}^2. \tag{A.9}$$ q:abdl2h1

It is clear that both bounds (A.7) and (A.9) can be made independent of the initial data for sufficiently large time, $nk \ge t_0(\omega^0,\omega^1;f,\mu)$.

For the H^1 estimate, we multiply (A.1) by $-2k\Delta\omega^{n+1}$ in L^2 and use (3.10). Writing the nonlinear term as

$$\begin{aligned}N_1 := (\partial(2\psi^n-\psi^{n-1},2\omega^n-\omega^{n-1}),\Delta\omega^{n+1})&\\ = (\partial(2\nabla\psi^n-\nabla\psi^{n-1},\nabla\omega^{n+1}),2\omega^n-\omega^{n-1})&\\ -(\partial(2\psi^n-\psi^{n-1},\nabla\omega^{n+1}),\nabla((1+\nu k)\omega^{n+1}-2\omega^n+\omega^{n-1}))&\end{aligned} \tag{A.10}$$

and bounding the terms as

$$\begin{aligned}|N_1| &\le c\,|2\omega^n-\omega^{n-1}|_{L^4}|\nabla^2\omega^{n+1}|_{L^2}|2\omega^n-\omega^{n-1}|_{L^4}\\ &\quad + c\,|2\nabla\psi^n-\nabla\psi^{n-1}|_{L^\infty}|\nabla^2\omega^{n+1}|_{L^2}|\nabla((1+\nu k)\omega^{n+1}-2\omega^n+\omega^{n-1})|_{L^2}\\ &\le \frac{\mu}{2}\,|\Delta\omega^{n+1}|^2 + \frac{c}{\mu}\,|2\omega^n-\omega^{n-1}|^2|2\nabla\omega^n-\nabla\omega^{n-1}|^2\\ &\quad + \frac{ck^{-1/2}}{\mu}\,M_\omega\,|\nabla((1+\nu k)\omega^{n+1}-2\omega^n+\omega^{n-1})|^2,\end{aligned} \tag{A.11}$$

we find the differential inequality, using the bound (A.7),

$$\begin{aligned}[\![\nabla\omega^n,\nabla\omega^{n+1}]\!]_{\nu k}^2 + \mu k\,|\Delta\omega^{n+1}|^2 &\le [\![\nabla\omega^{n-1},\nabla\omega^n]\!]_{\nu k}^2\big(1+ck\,M_0/\mu\big)\\ &\quad + |\nabla((1+\nu k)\omega^{n+1}-2\omega^n+\omega^{n-1})|^2\big(c_3k^{1/2}M_\omega/\mu-\tfrac14\big) + ck\,|f^n|^2/\mu.\end{aligned} \tag{A.12}$$ q:aidh1

Using the earlier timestep restriction (A.5), we can suppress the second term on the r.h.s. Thanks to (A.9), for any $n \in \{0,1,\cdots\}$ we can find $n_* \in \{n,\cdots,n+\lfloor 1/k\rfloor\}$ such that $[\![\nabla\omega^{n_*},\nabla\omega^{n_*+1}]\!]_{\nu k}^2 \le c(\mu)\,\big([\![\omega^0,\omega^1]\!]_{\nu k}^2 + \sup_j|f^j|_{H^{-1}}^2\big)$. Arguing as before, we can use this to integrate (A.12) to give us a uniform H^1 bound

$$[\![\nabla\omega^n,\nabla\omega^{n+1}]\!]_{\nu k}^2 \le M_1(|\nabla\omega^0|,|\nabla\omega^1|;\mu,\sup\nolimits_j|f^j|) \tag{A.13}$$ q:abdh1

valid for all $n \in \{0,1,\cdots\}$. Moreover, M_1 can be made independent of the initial data $|\nabla\omega^0|$, $|\nabla\omega^1|$ for sufficiently large n; in fact, we do not even need ω^0, $\omega^1 \in H^1$, although we still need them to be in H^ϵ for the timestep restriction (A.5). Summing (A.12) and using (A.13), we find

$$\mu k \textstyle\sum_{j=n+1}^{n+\lfloor 1/k\rfloor} |\Delta\omega^j|^2 \le \tilde{M}_1(\sup_j|f^j|;\mu) \qquad \text{for all } nk \ge t_1(\omega^0,\omega^1,f;\mu). \tag{A.14}$$

Similarly, for the H^2 estimate, we multiply (A.1) by $2k\Delta^2\omega^{n+1}$ in L^2 and write the nonlinear term as

$$\begin{aligned} N_2 &:= (\partial(2\psi^n-\psi^{n-1},2\omega^n-\omega^{n-1}),\Delta^2\omega^{n+1}) \\ &= -(\partial(2\nabla\psi^n-\nabla\psi^{n-1},2\omega^n-\omega^{n-1}),\nabla\Delta\omega^{n+1}) \\ &\quad -(\partial(2\psi^n-\psi^{n-1},2\nabla\omega^n-\nabla\omega^{n-1}),\nabla\Delta\omega^{n+1}). \end{aligned} \tag{A.15}$$

Bounding this as

$$\begin{aligned} |N_2| &\le c\,|2\omega^n-\omega^{n-1}|_{L^\infty}|2\nabla\omega^n-\nabla\omega^{n-1}|_{L^2}|\nabla\Delta\omega^{n+1}|_{L^2} \\ &\quad + c\,|2\nabla\psi^n-\nabla\psi^{n-1}|_{L^\infty}|2\nabla^2\omega^n-\nabla^2\omega^{n-1}|_{L^2}|\nabla\Delta\omega^{n+1}|_{L^2} \\ &\le \frac{\mu}{2}\,|\nabla\Delta\omega^{n+1}|^2 + \frac{c}{\mu}\,|2\nabla\omega^n-\nabla\omega^{n-1}|^2[\![\Delta\omega^{n-1},\Delta\omega^n]\!]_{\nu k}^2, \end{aligned} \tag{A.16}$$

we arrive at the differential inequality

$$\begin{aligned} &[\![\Delta\omega^n,\Delta\omega^{n+1}]\!]_{\nu k}^2 + \mu k\,|\nabla\Delta\omega^{n+1}|^2 \\ &\qquad \le [\![\Delta\omega^{n-1},\Delta\omega^n]\!]_{\nu k}^2\big(1+ckM_1/\mu\big) + ck|\nabla f^n|^2/\mu. \end{aligned} \tag{A.17}$$

As with (A.12), this can be integrated to obtain the uniform bound

$$[\![\Delta\omega^n,\Delta\omega^{n+1}]\!]_{\nu k}^2 \le M_2(\sup_j|\nabla f^j|;\mu) \tag{A.18}$$ q:aidh2

valid whenever $nk \ge t_2(\omega^0,\omega^1,f;\mu)$.

To bound the difference $\delta\omega^n := \omega^n-\omega^{n-1}$, we write (A.1) as

$$\frac{3\delta\omega^{n+1}-\delta\omega^n}{2k} + \partial(2\psi^n-\psi^{n-1},2\omega^n-\omega^{n-1}) = \mu\Delta\omega^{n+1}+f^n. \tag{A.19}$$

Multiplying by $4k\delta\omega^{n+1}$ and using (3.42) and (3.44), we find

$$\begin{aligned} 3|\delta\omega^{n+1}|^2 &+ \tfrac{1}{3}|\delta\omega^{n+1}-\delta\omega^n|^2 = \tfrac{1}{3}|\delta\omega^n|^2 \\ &+ 2\mu k|\nabla\omega^n|^2 - 2\mu k|\nabla\omega^{n+1}|^2 - 2\mu k|\nabla\delta\omega^{n+1}|^2 \\ &- 4k(\partial(2\psi^n-\psi^{n-1},2\omega^n-\omega^{n-1}),\delta\omega^{n+1}) + 4k(f^n,\delta\omega^{n+1}). \end{aligned} \tag{A.20}$$

Bounding the nonlinear term and suppressing harmless terms, we arrive at

$$\begin{aligned} 2|\delta\omega^{n+1}|^2 &\le \tfrac{1}{3}|\delta\omega^n|^2 + 2\mu k|\nabla\omega^n|^2 \\ &\quad + ck^2|2\nabla\psi^n-\nabla\psi^{n-1}|_{L^\infty}^2|2\nabla\omega^n-\nabla\omega^{n-1}|^2 + \frac{ck^2}{\mu}|f^n|_{H^{-1}}^2. \end{aligned} \tag{A.21}$$

Since the r.h.s. has been bounded uniformly for large nk, we conclude that

$$|\delta\omega^n|^2 \le k\hat{M}_0(f,\mu) \tag{A.22}$$

for nk sufficiently large. Arguing as in (3.53)–(3.57), we can improve the bound on $|\delta\omega^n|$ to $\mathsf{O}(k)$.

Acknowledgements Wang's work is supported in part by grants from the National Science Foundation and a planning grant from Florida State University. We thank the referee for a careful reading of the manuscript and for constructive comments.

References

ms-fournier:ss 1. R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Elsevier, 2002.

almforth-al:06 2. N. J. Balmforth, S. A. Ghadge, A. Kettapun, and S. D. Mandre, *Bounds on double-diffusive convection*, J. Fluid Mech., 569 (2006), pp. 29–50.

rasekhar:hdhms 3. S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, Dover, 1961.

hen-johnson:84 4. C. F. Chen and D. H. Johnson, *Double-diffusive convection: a report on an Engineering Foundation Conference*, J. Fluid Mech., 138 (1984), pp. 405–416.

coti-tone:12 5. M. Coti Zelati and F. Tone, *Multivalued attractors and their approximation: applications to the Navier–Stokes equations*, Numer. Math., 122 (2012), p. 421–441. arXiv:1111:4368.

nley-temam:87b 6. C. Foias, O. P. Manley, and R. M. Temam, *Attractors for the Bénard problem: existence and physical bounds on their fractal dimension*, Nonlin. Anal., 11 (1987), pp. 939–967.

trudinger:epde 7. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, 1977.

gtw3:dodu 8. S. Gottlieb, F. Tone, C. Wang, X. Wang, and D. Wirosoetisno, *Long time stability of a classical efficient scheme for two dimensional Navier–Stokes equations*, SIAM J. Numer. Anal., 50 (2012), pp. 126–150. arXiv:1105.4349.

r-wanner:sode2 9. E. Hairer and G. Wanner, *Solving ordinary differential equations II: stiff and differential-algebraic problems*, Springer-Verlag, 2 ed., 1996.

hill-suli:95a 10. A. T. Hill and E. Süli, *Upper semicontinuity of attractors for linear multistep methods approximating sectorial evolution equations*, Math. Comput., 64 (1995), pp. 1097–1122.

hill-suli:00 11. ———, *Approximation of the global attractor for the incompressible Navier–Stokes equations*, IMA J. Numer. Anal., 20 (2000), pp. 633–667.

sia-ma-wang:08 12. C. H. Hsia, T. Ma, and S. Wang, *Bifurcation and stability of two-dimensional double-diffusive convection*, Comm. Pure Appl. Anal., 7 (2008), pp. 23–48.

pert-turner:81 13. H. E. Huppert and J. S. Turner, *Double-diffusive convection*, J. Fluid Mech., 106 (1981), pp. 299–329.

ju:02 14. N. Ju, *On the global stability of a temporal discretization scheme for the Navier–Stokes equations*, IMA J. Numer. Anal., 22 (2002), pp. 577–597.

kim:09 15. N. Kim, *Large friction limit and the inviscid limit of 2d Navier–Stokes equations under Navier friction condition*, SIAM J. Math. Anal., 41 (2009), pp. 1653–1663.

mirouh-al:12 16. G. M. Mirouh, P. Garaud, S. Stellmach, A. L. Traxler, and T. S. Wood, *A new model for mixing by double-diffusive convection (semi-convection) I. The conditions for layer formation*, Astrophys. J., 750 (2012), p. 18 pp.

radko-stern:00 17. T. Radko and M. E. Stern, *Finite-amplitude salt fingers in a vertically bounded layer*, J. Fluid Mech., 425 (2000), pp. 133–160.

schmitt:94 18. R. W. Schmitt, *Double diffusion in oceanography*, Ann. Rev. Fluid Mech., 26 (1994), pp. 255–285.

spiegel:69 19. E. A. Spiegel, *Semiconvection*, Comm. Astroph. Space Phys., 1 (1969), pp. 57–60.

temam:nsenfa 20. R. Temam, *Navier–Stokes equations and nonlinear functional analysis*, SIAM, 2 ed., 1995.

temam:iddsmp 21. ———, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, 2 ed., 1997.

wxm:10 22. X. Wang, *Approximation of stationary statistical properties of dissipative dynamical systems: time discretization*, Math. Comput., 79 (2010), pp. 259–280.

wxm:12 23. ———, *An efficient second order in time scheme for approximating long time statistical properties of the two dimensional Navier–Stokes equations*, Numer. Math., 121 (2012), pp. 753–779. arXiv:1108.5409.

ing-holland:99 24. A. J. Weaver, C. M. Bitz, A. F. Fanning, and M. M. Holland, *Thermohaline circulation: high-latitude phenomena and the difference between the Pacific and Atlantic*, Annu. Rev. Earth Planet. Sci., 27 (1999), pp. 231–285.