Probabilistic stable rules and Nash equilibrium in two-sided matching problems*

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Abstract

We study *many-to-many* matching with *substitutable* and *cardinally monotonic* preferences. We analyze *stochastic dominance (sd) Nash equilibria* of the game induced by any *probabilistic stable* matching rule. We show that a *unique* match is obtained as the outcome of each sd-Nash equilibrium. Furthermore, *individual-rationality* with respect to the *true* preferences is a necessary and sufficient condition for an equilibrium outcome. In the many-to-one framework, the outcome of each equilibrium in which firms behave *truthfully* is *stable* for the *true* preferences. In the many-to-many framework, we identify an equilibrium in which firms behave truthfully and yet the equilibrium outcome is not stable for the true preferences. However, each *stable* match for the *true* preferences can be achieved as the outcome of such equilibrium.

KEYWORDS: Probabilistic rules, stability, Nash equilibrium, substitutability, cardinal monotonicity.
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1 Introduction

Centralized job matching procedures have received much attention in two-sided matching literature since they were introduced to address market failures (such as uncontrolled

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unraveling of appointment dates, and chaotic recontracting). In centralized matching, each agent submits to the clearinghouse a preference order over agents on the other side, and the clearinghouse then uses an algorithm to produce a match. These procedures are typically deterministic: matches are produced in a context where chance plays no role. Therefore the results do not usually reflect the case in real life situations such as labor markets where lotteries often determine outcomes. *Randomization* is the most common device to ensure *procedural fairness* in an environment where agents have conflicting interests. Hence, it is reasonable to allow for randomization in studies of centralized matching for purposes of achieving equity. Another motivation for studying randomization is that lotteries may be considered to represent the frictions of a decentralized matching process. Decentralized decision making often leads to uncertain outcomes in complex environments. The speed of the mail, the telephone network or the internal structure of firms determine how agents communicate. Therefore, the final match depends on the realization of random events.

We study *probabilistic matching rules* which may be used in centralized matching to achieve procedural fairness. When such a rule is used, agents are faced with a game in which they report a preference order to a clearinghouse which then produces a match. These rules may also appear in decentralized decision making. The game that agents face is as follows: starting from an arbitrary match, at each moment in time a pair of agents from the two sides meets at random. They are matched if this is consistent with their strategies. It may not be in an agent's best interest to behave truthfully. This implies that agents may not report their true preferences in centralized markets and may not act according to their true preferences in decentralized ones. Indeed, no stable matching rule makes it a dominant strategy for all agents to state their true preferences (Roth, 1985).

Another way of stating this result is that no stable matching rule is strategy-proof. This simply says that there is room for agents to benefit from misrepresenting their true preferences when confronted with a game induced by a stable matching rule. Indeed, if all agents but one behaves truthfully, the last agent may gain from this manipulation. What we are really concerned about is not the fact that agents benefit from individually circumventing a stable rule, but rather that stable matches that the rule recommends for the true preferences may not be achieved. Therefore, it is crucial to study equilibria of the game induced by a stable matching rule. Also, the study of incentives proved to be useful for understanding behavior in matching markets with deterministic rules. Therefore, the study of incentives facing agents is a good starting point to understand behavior in matching markets with probabilistic rules. Empirical evidence shows that a stable rule may produce a stable outcome for the true preferences (Roth 1984b, 1990, 1991). Under what conditions do Nash equilibria of the preference revelation game induced by any probabilistic stable rule produce a stable outcome for the true preferences? An answer to the above question will provide a theoretical support for the success of stable matching rules.

In a related paper, Roth (1984c) studies the preference revelation game induced by the man-optimal stable rule in one-to-one matching. He shows that the outcome of each Nash equilibrium in (weakly) undominated strategies is stable for the true preferences. Roth and Vande Vate (1990) prove that starting from an arbitrary match in the marriage problem, the process of allowing randomly selected blocking pairs to create a new match leads to a stable match with probability one. It is argued that since many two-sided matching markets are not centralized and yet are not determined to encounter failures, it is reasonable to think that these markets reach stable outcomes through decentralized decision making. It can also be argued that the process of allowing randomly selected blocking pairs is a good approximation to dynamics in decentralized processes. Roth and Vande Vate (1991) study a one-period game defined by the mentioned process and show that all stable matches can be supported as equilibria in a class of undominated strategies namely *truncations*. A truncation strategy for an agent is a preference ordering that has the same order as in her/its true preference ordering but may have fewer acceptable elements. However, they show that some unstable matches can arise as equilibrium outcomes in this game. They then introduce a multi-period extension of this game and show that all subgame perfect equilibrium outcomes are stable.

We study many-to-many matching problems where each agent can form multiple partnerships. A match is 'stable' if no agent prefers to be matched to a proper subset of its current partners, and no group of firms and workers prefers to deviate by establishing new partnerships only among themselves and possibly dissolving some existing partnerships.[1] This definition is stronger than pairwise-stability that only eliminates blocking by firm-worker pairs. Stability proved to be a crucial property in many entry-level labor markets where workers are matched to firms through a clearinghouse. It has been observed that clearinghouses that adopt stable rules often perform better than those that adopt rules that do not necessarily produce stable matches. Indeed, the weaker stability concept, pairwise-stability, is still of primary interest for many-to-many markets as well (Roth, 1991).

Without any restriction on preferences there are many-to-many problems for which no stable match exists (Roth and Sotomayor, 1990, Example 2.7). We assume that each agent's preferences satisfy *substitutability*. An agent's preferences over groups of partners are substitutable if, once a partner is chosen from a given group of partners, she/it is also chosen from any subset of the given group of partners.[2] We will also refer to *responsiveness*. An agent's preferences over groups of partners are responsive (to her/its preferences over individual partners) if for any two groups that differ in only one partner, the agent prefers the one that contains the preferred partner. Substitutability guarantees the existence of a pairwise-stable match.[3] Hatfield and Kominers (2012) showed that for substitutable preferences, stability and pairwise-stability are equivalent.[4] Thus, when preferences are substitutable, the set of stable matches is non-empty and coincides with the set of pairwise-stable matches. We also impose *cardinal mono-*

[1]This is an adaptation of the stability definition in Hatfield and Kominers (2012).

[2]Substitutability is an adaptation of the gross substitutability property (Kelso and Crawford, 1982) by Roth (1984a) and Roth and Sotomayor (1990) to matching problems without monetary transfers.

[3]The existence of a pairwise-stable match is shown via an algorithm for strict preferences (Roth, 1984a) and via a non-constructive proof for preferences that are not necessarily strict (Sotomayor, 1999).

[4]See Echenique and Oviedo (2006) and Sotomayor (1999) for different formulations of stability in many-to-many matching problems.

tonicity on each agent's preferences. An agent's preferences over groups of partners are cardinally monotonic if whenever the group of partners available to the agent expands, she/it will not choose to have fewer partners.[5] Responsiveness implies substitutability and cardinal monotonicity.

Our model involves the US and UK medical labor markets as special cases. In the US, each student seeks one position and has preferences over positions. In contrast, in the UK each student must find one medical and one surgical position to register as a doctor therefore, she has preferences over pairs of positions. In both the US and UK, each hospital has preferences over groups of students.[6] The top three choices of a hospital with two positions may consist only of females in its rank-order list of individual students but it may indeed wish to employ no more than one female student.[7] Then it may well prefer to employ its first and fourth choices to its first and second choices. Similarly, each student in the UK prefers a pair of one medical and one surgical position to any other pairs of positions and yet the top two choices in her rank-order list of positions may be surgical. While responsiveness is a natural way of extending a rank-order list of individual partners to that of groups of partners, it does not permit the above complex but meaningful relations between the two lists. On the other hand, these relations are admissible by cardinal monotonicity.

We analyze equilibria of the game induced by any probabilistic stable matching rule. The equilibrium concept we study relies on *first-order stochastic dominance*. All rules used in centralized matching markets are ordinal; they elicit only agents' ordinal preferences over their potential partners. In such an environment, an ordinal strategy profile is a *stochastic dominance (sd) Nash equilibrium* if each agent plays her/its best response to the others' strategies for each utility function that is compatible with the true ordinal preferences. The study of ordinal strategies can be justified by the limited information that agents may have about their own utility functions. We show that a *unique* match is achieved as the outcome of an sd-Nash equilibrium of the game induced by any probabilistic stable rule (Proposition 1). Furthermore, a match can be obtained as an equilibrium outcome if and only if it is individually-rational for the true preferences (Propositions 2 and 3). We show that the outcome of each sd-Nash equilibrium in which firms behave truthfully is stable for the true preferences in many-to-one matching (Proposition 4). An implication of this result is that truth-telling by hospitals in equilibrium is a sufficient condition for stability of equilibrium outcomes in the US medical labor market. Nevertheless, the result does not extend to the many-to-many framework. We establish that there are equilibrium misrepresentations that generate a match that is not stable for the true preferences (Example 1). The converse statement holds in many-to-many matching: each stable match is supported as the outcome of an sd-Nash equilibrium in which firms behave truthfully (Proposition 5).

[5] Cardinal monotonicity is called size monotonicity and law of aggregate demand in Alkan and Gale (2003) and Hatfield and Milgrom (2005), respectively.

[6] In the formal description of US hospital-intern market hospitals have preferences over individual students. This suffices to define stability without reference to preferences over groups of students as long as they are assumed to be responsive to preferences over individual students.

[7] Roth (1991) observes that prior to the adoption of a centralized matching procedure the traditional practice among surgeons in Edinburgh was to employ no more than one female student.

We finally argue that Propositions 2 and 4 do not hold without the assumption of cardinal monotonicity. In particular, we identify a many-to-one matching problem with a profile of substitutable preferences that violate cardinal monotonicity and a profile of equilibrium strategies such that the equilibrium outcome is not individually-rational for the true preferences (Example 2). We then identify another many-to-one problem with the same property as above and a profile of equilibrium strategies in which firms act truthfully and yet the equilibrium outcome is not stable for the true preferences (Example 3).

In a related paper, Pais (2008) proves aforementioned equilibrium results in many-to-one matching when each firm's preferences satisfy responsiveness. She also studies sd-Nash equilibria of the sequential game of the sort studied by Roth and Vande Vate (1990). She allows for a broader set of strategies that need not be consistent with a unique preference ordering. However, when concerned with strategies that are compatible with a distinct preference ordering for each play of the game (that corresponds to a sequence of randomly selected pairs of agents), given a profile of stated preferences for agents other than an arbitrary agent v, agent v has a best response that is compatible with a preference ordering. This result extends to many-to-many matching when agents have substitutable and cardinally monotonic preferences. Strategic issues have been the subject of papers that focus on deterministic matching rules in many-to-one matching. Ma (2002) assumes that firms have responsive preferences and studies a refinement of Nash equilibrium based on a class of strategies called *truncations at the match point*. A strategy in truncations at the match point for an agent is a preference ordering that is consistent with her/its true preferences up to her/its current match and that rank as unacceptable all the agents that are less preferred than her/its current match. He shows that a match can arise as the outcome of a strong Nash equilibrium in truncations at the match point if and only if it is stable for the true preferences. In many-to-many matching, Kojima and Ünver (2008) analyze a random procedure of the sort studied by Roth and Vande Vate (1990) in one-to-one matching. When agents on one side have substitutable preferences and those on the other have responsive preferences, they prove that the decentralized process of satisfying randomly chosen blocking pairs converges to a pairwise-stable match. Thus, the decentralized interpretation of our model remains valid in this setup.

The paper is organized as follows. We present the model in Section 2. We impose preference restrictions in Section 3. We introduce the class of probabilistic stable matching rules and the equilibrium concept in Section 4. We present equilibrium results in Section 5. We discuss validity of our results without the assumption of cardinal monotonicity in Section 6. We conclude in Section 7.

2 The Model

Let $F = \{f_1, ..., f_n\}$ and $W = \{w_1, ..., w_m\}$ denote finite sets of **firms** and **workers** respectively. Generic elements of F and W are denoted by f and w respectively while a generic element of $F \cup W$ is denoted by v. "It" refers to a firm and "she" refers to a worker. The set of (possible) partners of agent v is $S_v \equiv W$ if $v \in F$, and

$S_v \equiv F$ if $v \in W$. Each agent v has at most c_v positions to fill. Let $c \equiv (c_v)_{v \in F \cup W}$ denote the list of capacities. Each agent v has a linear preference ordering P_v over 2^{S_v}.[8] We write $P_f : \{w_1, w_2\}, w_2, \emptyset, \{w_2, w_3\}, ..., w_3$, for example, to indicate that f's first choice is being matched to w_1 and w_2, its second choice is being matched to w_2 only, and it prefers remaining unmatched to being matched to any other subset of workers.[9] **Preference profiles** are $(n+m)$ tuples of preference relations denoted by $P \equiv (P_{f_1}, ..., P_{f_n}, P_{w_1}, ..., P_{w_m})$. A **many-to-many matching problem** is a pair (P, c). Let P_{-v} denote the profile $P_{(F \cup W) \setminus \{v\}}$. We sometimes write the profile of preferences P as (P_v, P_{-v}). Let $\mathcal{P}_v$ denote the set of all possible preference relations for agent v and $\mathcal{P} \equiv \prod_{v \in (F \cup W)} \mathcal{P}_v$ be the set of all possible preference profiles. Let R_v denote the **at least as desirable as** relation associated with P_v. For each $v \in F \cup W$ and each $S, S' \subseteq S_v$, $S \ R_v \ S'$ means either $S = S'$ or $S \ P_v \ S'$. Let $v \in F \cup W$ and $S \subseteq S_v$ be given. Let $U_S(P_v) \equiv \{S' \subseteq S_v : S' \ R_v \ S\}$ denote the set of all subsets of S_v that v finds at least as desirable as S.

A **match** is a mapping μ from the set $F \cup W$ to the set of all subsets of $F \cup W$ satisfying the following conditions:

(m1). For each $v \in F \cup W$, $\mu(v) \in 2^{S_v}$ and $|\mu(v)| \leq c_v$;
(m2). For each $(f, w) \in F \times W$, $f \in \mu(w)$ if and only if $w \in \mu(f)$.

Let $\mathcal{M}$ denote the set of all matches. An agent v is **unmatched** at μ if $\mu(v) = \emptyset$ and **matched** otherwise. A matching problem (P, c) is **one-to-one** if for each agent v, $c_v = 1$. A matching problem (P, c) is **many-to-one** if for each worker w, $c_w = 1$. Preferences over partners are extended to preferences over matches in the conventional way: an agent's preferences over matches parallel to her/its preferences over her\its own assignments at the matches. For example, agent v prefers μ to μ' if and only if $\mu(v) \ P_v \ \mu'(v)$.

Let $v \in F \cup W$, $v' \in S_v$ and $P_v \in \mathcal{P}_v$ be given. Agent v' is **acceptable to v** if she/it prefers to be matched to v' rather than remaining unmatched. The set of acceptable partners for v is given by $A(P_v) = \{v' \in S_v : v' \ P_v \ \emptyset\}$.

Let $S \subseteq S_v$ be given. Let $\mathsf{Ch}(S, P_v)$ denote agent v's chosen set in S; her/its most preferred subset of S according to its preference relation P_v. Since preferences are strict, $\mathsf{Ch}(S, P_v)$ is the unique subset S' of S that satisfies the following: for each $S'' \subseteq S, S'' \neq S', S' \ P_v \ S''$. We adapt the definition of stability due to Hatfield and Kominers (2012, Section 2.2) to our model. Match μ is **blocked by an agent** $v \in F \cup W$ at P if $\mathsf{Ch}(\mu(v), P_v) \neq \mu(v)$. Match μ is **blocked by a set of firms and workers** $F' \cup W'$ at P, where $\emptyset \neq F' \subseteq F$ and $\emptyset \neq W' \subseteq W$, if there is a match μ' such that for each $v \in F' \cup W'$,

(b1). $\emptyset \neq (\mu'(v) \setminus \mu(v)) \subseteq F' \cup W'$;
(b2). $\mu'(v) \subseteq \mathsf{Ch}(\mu'(v) \cup \mu(v), P_v)$.

[8] In other words, P_v is transitive, antisymmetric (strict) and total.

[9] With a slight abuse of notation we sometimes write x for a singleton $\{x\}$.

Loosely speaking, the agents in $F' \cup W'$ are strictly better off by establishing new partnerships only among themselves and possibly breaking up some existing partnerships. A match is **individually-rational for $\boldsymbol{P}$** if it is not blocked by any agent. Let $\boldsymbol{IR(P)}$ denote the set of individually-rational matches for P. A match is **stable for $\boldsymbol{P}$** if it is individually-rational for P and is not blocked by any set of firms and workers at P.

Remark 1. In one-to-one matching, the definition of stability eliminates blocking by agents or by sets of firms and workers $F' \cup W'$ with $|F'| = 1$ and $|W'| = 1$. This is referred as pairwise-stability. In many-to-one matching, the definition of stability eliminates blocking by agents and by sets of firms and workers $F' \cup W'$ with $|F'| = 1$ and $|W'| \geq 1$. This is referred as many-to-one stability. Stability implies many-to-one stability and many-to-one stability implies pairwise-stability. In many-to-one matching blocking by a pair (f, w) is equivalent to the following condition: $w \notin \mu(f)$, $f \; P_w \; \mu(w)$ and $w \in \mathsf{Ch}(\mu(f) \cup \{w\}, P_f)$. In many-to-many matching blocking by a pair (f, w) is equivalent to the following condition: $w \notin \mu(f)$, $f \in \mathsf{Ch}(\mu(w) \cup \{f\}, P_w)$ and
$w \in \mathsf{Ch}(\mu(f) \cup \{w\}, P_f)$. ◇

Remark 2. Since (sets of) partners that are less desirable than being unmatched cannot be part of an individually-rational match, it is sufficient to describe each agent's ranking of (sets of) partners that are preferred to being unmatched. For instance,

$$P_f : w_1w_2, w_1, w_3, w_2,$$

indicates that $w_1w_2 \, P_f \, w_1 \, P_f \, w_3 \, P_f \, w_2 \, P_f \, \emptyset$ and that emptyset is preferred to all other sets of partners by f.[10] ◇

3 Preference Restrictions

We now define several restrictions on firms' preferences. Let $v \in F \cup W$.

Substitutability:
For each pair $S, S' \subseteq S_v$ with $S' \subseteq S$ and each $v' \in S'$; if $v' \in \mathsf{Ch}(S, P_v)$ then $v' \in \mathsf{Ch}(S', P_v)$.

Substitutability requires that an agent is willing to continue to be matched to a partner even if some other partners become unavailable.

Cardinal monotonicity:[11]
For each pair $S, S' \subseteq S_v$, with $S' \subseteq S$, $|\mathsf{Ch}(S', P_v)| \leq |\mathsf{Ch}(S, P_v)|$.

The following ordering over 2^W, where $W = \{w_1, w_2, w_3\}$ and $c_f = 3$

$$P_f \quad : \quad w_1w_2w_3, w_1, w_1w_2,$$

[10] With a slight abuse of notation we sometimes write xy for a set $\{x, y\}$.
[11] Cardinal monotonicity was introduced by Alkan (2002).

illustrates that cardinal monotonicity does not imply substitutability. Notice that P_f is cardinally monotonic but not substitutable: $w_2 \in \mathsf{Ch}(W, P_f)$, but $w_2 \notin \mathsf{Ch}(W \backslash \{w_1\}, P_f)$. We now define a well studied preference restriction in many-to-one matching.

Responsiveness:
For each $S \subseteq S_v$ with $|S| < c_v$, and each $v', v'' \in (S_v \backslash S) \cup \{\emptyset\}$; $(S \cup \{v'\})\ P_v\ (S \cup \{v''\})$ if and only if $v'\ P_v\ v''$ and for each S with $|S| > c_v$, $\emptyset\ P_v\ S$.

It is easy to verify that responsiveness implies substitutability and cardinal monotonicity. The following ordering over 2^W, where $W = \{w_1, w_2, w_3\}$ and $c_f = 2$

$$P_f : w_1w_3, w_2w_3, w_1, w_2, w_3,$$

illustrates that the set of responsive preferences is a proper subset of the set of substitutable and cardinally monotonic preferences. A **many-to-one** matching problem with responsive preferences is known as a **college admissions** problem.[12]

Unless otherwise stated, we assume that each agent v's preferences P_v are substitutable and cardinally monotonic.

Remark 3. For each profile of substitutable preferences P, $S(P)$ is non-empty (Theorem 1, Roth, 1984a).[13] Moreover, stability is equivalent to many-to-one stability and pairwise-stability (Hatfield and Kominers, 2012, Proposition 2).[14] $\diamond$

The proof of the existence of a stable match relies on the firm proposing DA algorithm. We will refer to the description of the algorithm in the proof of Proposition 2 below, therefore we now present how it functions in many-to-many matching.

Firm proposing DA algorithm: Let P be a preference profile.
Step 1:
(a) Each firm f proposes to its most preferred set $x_f(1)$ in 2^W.
(b) Each worker w accepts her chosen set in the set $x_w(1) \equiv \{f \in F : w \in x_f(1)\}$ of firms that proposed to her, and rejects the rest.

$\vdots$

Step k:
(a) Each firm f proposes to its most preferred set $x_f(k)$ in 2^W such that f has not been rejected by any $w \in x_f(k)$ in an earlier step.
(b) Each worker w accepts her chosen set in the set consisting of firms that proposed to her in step k and firms that were not rejected in step $k-1$, and rejects the rest.

[12] College admissions was first studied by Gale and Shapley (1962).

[13] Martínez et al. (2004) introduced an algorithm to calculate all stable matches when preferences are substitutable.

[14] Unlike in many-to-one matching with substitutable preferences, pairwise-stability is not equivalent to core-stability in many-to-many matching. Indeed, no logical relation exists between the two concepts (Blair, 1988).

$\vdots$

The algorithm terminates in any step, say t, in which no firm is rejected. It produces the outcome μ_F in which each firm f is assigned its final proposal $\mu_F(f) = x_f(t)$, and each worker w is assigned her chosen set $\mu_F(w) = \mathsf{Ch}(x_w(t), P_w)$ in step t. The worker proposing DA algorithm is symmetrically defined.

Remark 4. Offers remain open: for each firm f, if worker w is contained in $x_f(k-1)$ and is not rejected by w in step $k-1$, then w is contained in $x_f(k)$ (Roth, 1984a, Proposition 2). $\diamond$

In words, Remark 4 says that firms that have not been rejected by w in step $k-1$ propose to w in step k. Thus, the set of firms that propose to w in step k, $x_w(k)$, consists of firms that have proposed in step $k-1$ and have not been rejected by w and those that have proposed in step k but not in $k-1$. Stability of the match produced by the DA algorithm lies in the observation that as firms' preferences are substitutable, each firm repeats its proposal in a subsequent step of the algorithm to a worker who has not rejected its earlier proposal and that as workers' preferences are substitutable, no worker ever regrets having rejected a proposal in an earlier step of the algorithm. For each profile P of substitutable preferences the algorithm produces a stable match that is optimal for the firms in the sense that all firms find it at least as desirable as any other stable match. The worker proposing DA algorithm produces a stable match that is optimal for the workers in the corresponding sense. The optimal stable match for firms is the worst stable match for workers and vice versa (Roth, 1984a). We use the following property of stable matches.

Weak Rural Hospital Theorem [Alkan 2002].[15]

For each profile P of substitutable and cardinally monotonic preferences,

R1. Each firm has the same number of positions filled across stable matches for P.

Remark 3 implies that we need only consider blocking by individual agents and firm-worker pairs. In particular, the terms 'stability' and 'pairwise-stability' can be used interchangeably.

4 Probabilistic matching and equilibrium notions

In practice, matching is often not centralized. Instead, matches are reached through decentralized procedures. Such procedures introduce randomness into what matches

[15]The theorem is first proved for the class of responsive preferences and later for the strictly larger class of substitutable and separable preferences in many-to-one matching problems (Gale and Sotomayor 1985, Roth 1984b, Martínez et al. 2000). Separability: for each $S \subseteq S_v$ with $|S| < c_v$ and each $v' \notin S$; $S \cup \{v'\}$ P_v S if and only if v' P_v $\emptyset$ and for each S with $|S| > c_v$, $\emptyset$ P_v S.

are achieved (the order according to which agents communicate depends on the speed of the mail, the internal structure of firms and the telephone network). One way to model decentralized decision making is to consider a random process that develops a sequence of matches such that each match in the sequence is derived from the previous one by satisfying a randomly selected blocking agent or a blocking pair (See Roth and Vande Vate (1990) for one-to-one matching and Kojima and Ünver (2008) for many-to-many matching). As deterministic rules inherently favor some agents over others, randomness can be introduced in centralized matching to achieve procedural fairness. This is felt most strongly in two-sided matching where the polarization of interests of agents on different sides is reflected in the structure of the set of stable matches. School choice, public housing and on campus housing in American universities are examples of allocation problems that have adopted probabilistic procedures.

We reproduce the definitions and notation in Pais (2008). A **probabilistic** (matching) **rule** $\widetilde{\varphi}$ maps preference profiles to lotteries over the set of matches: $\mathcal{P} \xrightarrow{\widetilde{\varphi}} \Delta\mathcal{M}$. A probabilistic match $\widetilde{\varphi}[P]$ is the image of a preference profile P under a rule. We consider only **probabilistic stable rules** that yield a lottery whose support (abbreviated as supp) is a subset of the set of stable matches for each preference profile P. Formally, for each $P \in \mathcal{P}$, $\text{supp}\widetilde{\varphi}[P] \subseteq S(P)$. Let $\widetilde{\varphi}_v[P]$ denote the probability distribution induced by $\widetilde{\varphi}[P]$ over agent v's achievable partners. Whenever the distribution $\widetilde{\varphi}[P]$ is degenerate, we abuse notation and denote by $\widetilde{\varphi}[P]$ the unique outcome. Similarly, whenever for some agent v, the distribution $\widetilde{\varphi}_v[P]$ is degenerate, we denote by $\widetilde{\varphi}_v[P]$, v's unique partner. A **deterministic rule** φ maps preference profiles to the set of matches: $\mathcal{P} \xrightarrow{\varphi} \mathcal{M}$. We consider only deterministic stable matching rules that yield a unique stable match for each preference profile P. We let $\varphi^F (\varphi^W)$ denote the deterministic stable rule that recommends the firm (worker) optimal stable match for each preference profile. We denote v's assignment at the match $\varphi[P]$ by $\varphi_v[P]$.

We study the game induced by any probabilistic stable rule $\widetilde{\varphi}$ in which agents are called upon to state their preferences. No stable rule makes it a dominant strategy for all workers and firms to state their true preferences (Roth, 1985). This implies that an agent may reveal a different order than her/its true preferences. To understand what outcomes will result when all agents behave in this way, we need to study the manipulation "game" associated with the rule. Consider the following game in which the strategy space for each agent v is the set of all preferences $\mathcal{P}_v$. Each agent announces a preference list $Q_v \in \mathcal{P}_v$ over subsets of agents on the other side and then a match is randomly selected among all matches that are stable for the stated preferences Q. Formally, the set of strategy profiles $\mathcal{P}$ and a probabilistic stable matching rule $\widetilde{\varphi}$ define a mechanism $(\mathcal{P}, \widetilde{\varphi})$. The mechanism together with the true preference profile defines the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Similarly, $(\mathcal{P}, \varphi)$ is a deterministic stable mechanism that induces the game $(\mathcal{P}, \varphi, P)$.

We introduce more notation to define our equilibrium concept. Let $v \in F \cup W$, $S \subseteq S_v$, $P_v \in \mathcal{P}_v$ and $\pi \in \Delta\mathcal{M}$. Let π_v denote the lottery induced by π over v's set of assignments, i.e, over 2^{S_v}. Let $\pi_v(U_S(P_v))$ denote the probability that v obtains a set of partners that is at least as desirable as S according to P_v at π. For each pair $\pi, \pi' \in \Delta\mathcal{M}$ and each $v \in F \cup W$, $\boldsymbol{\pi}$ **stochastically** $\boldsymbol{P_v}$**-dominates** $\boldsymbol{\pi'}$, denoted as $\boldsymbol{\pi \; P_v^{sd} \; \pi'}$, if for each $S \subseteq S_v$, $\pi_v(U_S(P_v)) \geq \pi'_v(U_S(P_v))$.

Next, we define what constitutes a better strategy for an agent. Let $Q \in \mathcal{P}$ and $v \in F \cup W$. Given Q_{-v}, we say that strategy Q_v **stochastically$\boldsymbol{P_v}$-dominates** an alternative strategy Q'_v, if $\widetilde{\varphi}_v[Q_v, Q_{-v}]\ P_v^{sd}\ \widetilde{\varphi}_v[Q'_v, Q_{-v}]$. This means that no agent v can increase the probability of obtaining a set of partners S or a higher ranked set of partners in its list P_v by using Q'_v rather than using Q_v. The following equilibrium notion relies on the criterion of stochastic dominance.

The profile of strategies Q is a **stochastic-dominance (sd) Nash equilibrium** of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ if for each $v \in F \cup W$, Q_v stochastically P_v-dominates each alternative strategy, given Q_{-v}.[16]

The profile of strategies Q is an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ if, once adopted by the agents, no agent finds any unilateral deviation profitable for each utility function compatible with the true ordinal preferences.

5 Equilibrium Analysis

We now analyze sd-Nash equilibria of the game induced by any probabilistic stable rule. The following lemmas will be used in the proofs of the results. Lemma 1 below shows that each partner of an agent at a stable match is acceptable to the agent.

Lemma 1. *Let $Q \in \mathcal{P}$ and $\mu \in S(Q)$. Then for each $v \in F \cup W$ $\mu(v) \subseteq A(Q_v)$.*

Proof. Let $v \in F \cup W$ and $v' \in \mu(v)$. By $\mu \in S(Q)$, $\mathsf{Ch}(\mu(v), Q_v) = \mu(v)$. Thus, $v' \in \mathsf{Ch}(\mu(v), Q_v)$. By substitutability, $v' \in \mathsf{Ch}(\{v'\}, Q_v)$. Hence, $v'\ Q_v\ \emptyset$ and $v' \in A(Q_v)$. Since v' is arbitrary, $\mu(v) \subseteq A(Q_v)$. □

Let $Q \in \mathcal{P}$ and $\mu \in S(Q)$. Let $Q' \in \mathcal{P}$ differ from Q in that an agent announces her/its partners at μ as her/its most preferred set of partners. Lemma 2 says that $\mu \in S(Q')$.

Lemma 2. *Let $Q \in \mathcal{P}$, $\mu \in S(Q)$ and $v \in F \cup W$. Let $Q' \in \mathcal{P}$ be such that $\mathsf{Ch}(S_v, Q'_v) = \mu(v)$ and for each $v' \in F \cup W, v' \neq v$, $Q'_{v'} = Q_{v'}$. Then, $\mu \in S(Q')$.*

Proof. Let $Q \in \mathcal{P}, \mu \in S(Q)$ and $v \in F \cup W$. Let $Q' \in \mathcal{P}$ be such that $\mathsf{Ch}(S_v, Q'_v) = \mu(v)$ and for each $v' \in F \cup W, v' \neq v$, $Q'_{v'} = Q_{v'}$. We show that $\mu \in S(Q')$. By $\mu \in S(Q)$ and the definition of Q', for each $v' \in F \cup W, v' \neq v$, $\mathsf{Ch}(\mu(v'), Q'_{v'}) = \mu(v')$. By $\mathsf{Ch}(S_v, Q'_v) = \mu(v)$ and the definition of Ch, $\mathsf{Ch}(\mu(v), Q'_v) = \mu(v)$. Hence, $\mu \in IR(Q')$. Suppose that a firm-worker pair (f, w) blocks μ at Q';

$$w \notin \mu(f), f \in \mathsf{Ch}(\mu(w) \cup \{f\}, Q'_w) \text{ and } w \in \mathsf{Ch}(\mu(f) \cup \{w\}, Q'_f). \tag{1}$$

Since $\mu(v)$ is v's most preferred set of partners, $v \notin \{f, w\}$. Then, (1) becomes $w \notin \mu(f)$, $f \in \mathsf{Ch}(\mu(w) \cup \{f\}, Q_w)$ and $w \in \mathsf{Ch}(\mu(f) \cup \{w\}, Q_f)$, contradicting $\mu \in S(Q)$. □

[16] This name is taken from Thomson (2011). The concept was introduced by d'Aspremont and Peleg (1988). It is referred as ordinal Nash equilibrium in the literature.

Since stability implies individual-rationality, Lemma 2 holds for the set of individually-rational matches.

Remark 5. *Let $Q \in \mathcal{P}$, $\mu \in IR(Q)$ and $v \in F \cup W$. Let $Q' \in \mathcal{P}$ be such that $\mathsf{Ch}(S_v, Q'_v) = \mu(v)$ and for each $v' \in F \cup W, v' \neq v$, $Q'_{v'} = Q_{v'}$. Then, $\mu \in IR(Q')$.*

Proposition 1 states that one and only one stable match is achieved as the outcome of each sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$.

Proposition 1. *Let Q be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, a single match is obtained with probability one.*

Proof. Let Q be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Assume, by contradiction that $|\text{supp } \widetilde{\varphi}[Q]| \geq 2$. Then there are $w \in W$ and $\mu, \widehat{\mu} \in \text{supp } \widetilde{\varphi}[Q]$ such that $\mu(w) \neq \widehat{\mu}(w)$. Let $\mu' \in \text{supp } \widetilde{\varphi}[Q]$ be such that for each $\mu \in \text{supp } \widetilde{\varphi}[Q]$, $\mu'(w) \; R_w \; \mu(w)$. Let $Q'_w \in \mathcal{P}$ be such that $A(Q'_w) = \mu'(w)$ and $\mathsf{Ch}(F, Q'_w) = \mu'(w)$. Let $Q' \equiv (Q'_w, Q_{-w})$. By Lemma 2, $\mu' \in S(Q')$. By $A(Q'_w) = \mu'(w)$, Lemma 1 and R1, w is matched to $\mu'(w)$ across stable matches for Q'. Hence, Q_w does not stochastically P_w-dominate Q'_w. □

Propositions 2 and 3 establish that a match can be supported as an equilibrium outcome if and only if it is individually-rational for the true preferences. We provide a complete characterization of sd-Nash equilibria of the game induced by any probabilistic stable rule.

Proposition 2. *Let Q be an sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, $\widetilde{\varphi}[Q] \in \; IR(P)$.*

Proof. By Proposition 1, a unique match is obtained from each sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Let $\mu \equiv \widetilde{\varphi}[Q]$. We prove that $\mu \in IR(P)$. Assume, without loss of generality, that there is $f \in F$ such that $\mathsf{Ch}(\mu(f), P_f) \neq \mu(f)$. Then there is a set of workers $S^G \subsetneq \mu(f)$ such that $S^G \; P_f \; \mu(f)$. Let Q'_f be an alternative strategy for f such that $A(Q'_f) = S^G$ and $\mathsf{Ch}(W, Q'_f) = S^G$. Let $Q' \equiv (Q'_f, Q_{-f})$. Let μ'_F denote the firm-optimal stable match for Q'. It is the match produced by applying to Q' the firm-proposing DA algorithm. We claim that $\mu'_F(f) = S^G$. To prove the claim, it is sufficient to prove that no $\hat{f} \in F$ is ever rejected by any $\hat{w} \in \mu(\hat{f})$ in the course of the algorithm. This, together with $\mathsf{Ch}(W, Q'_f) = S^G \subsetneq \mu(f)$ establishes that $\mu'_F(f) = S^G$. By $A(Q'_f) = S^G$ and R1, f is matched to S^G across stable matches for Q'. Thus, Q_f does not stochastically-P_f dominate Q'_f.

We now argue that no firm is rejected by any of its partners at μ in the course of the algorithm applied to Q'. First note that since $\mu \in S(Q)$ and for each $\hat{w} \in W, Q'_{\hat{w}} = Q_{\hat{w}}$, we have

$$\text{for each } \hat{w} \in W, \; \mathsf{Ch}(\mu(\hat{w}), Q'_{\hat{w}}) = \mu(\hat{w}). \tag{2}$$

Suppose, by the induction hypothesis that no $\hat{f} \in F$ is rejected by any $\hat{w} \in \mu(\hat{f})$ up to step $r-1$ of the algorithm. We show that no $\hat{f} \in F$ is rejected by any $\hat{w} \in \mu(\hat{f})$ in step r.

Assume, by contradiction that there is $\overline{f} \in F$ and $\overline{w} \in \mu(\overline{f})$ such that $\overline{f}$ is rejected by $\overline{w}$ in step r. Then, $\overline{f} \in x_{\overline{w}}(r)$ but $\overline{f} \notin \mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}})$. Each of the following paragraphs begins with a statement and follows with its proof.

(p1). $\mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}}) \neq \emptyset$. Otherwise, by $\overline{f} \in x_{\overline{w}}(r)$ and the definition of Ch, $\overline{f} \notin \mathsf{Ch}(\{\overline{f}\}, Q'_{\overline{w}})$. By $\overline{w} \in \mu(\overline{f})$ and substitutability, $\overline{f} \notin \mathsf{Ch}(\mu(\overline{w}), Q'_{\overline{w}})$, contradicting (2).

(p2). $\mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}})\backslash\mu(\overline{w}) \neq \emptyset$. Otherwise, $\mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}}) \subseteq \mu(\overline{w})$. Let $K \equiv \mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}})$. Notice that $\overline{f} \notin K$. By $K \cup \{\overline{f}\} \subseteq x_{\overline{w}}(r)$ and the definition of Ch, $\mathsf{Ch}(K \cup \{\overline{f}\}, Q'_{\overline{w}}) = K$. Since $K \cup \{\overline{f}\} \subseteq \mu(\overline{w})$ and $\overline{f} \notin K = \mathsf{Ch}(K \cup \{\overline{f}\}, Q'_{\overline{w}})$, by substitutability, $\overline{f} \notin \mathsf{Ch}(\mu(\overline{w}), Q'_{\overline{w}})$. Then $\mathsf{Ch}(\mu(\overline{w}), Q'_{\overline{w}}) \neq \mu(\overline{w})$, contradicting (2). Let $F' \equiv \{f' \in \mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}}) \text{ and } f' \notin \mu(\overline{w})\}$.

(p3). $f \notin F'$. Otherwise, $f \notin \mu(\overline{w})$ implies that f does not state $\overline{w}$ acceptable at Q'_f. In any step k of the algorithm $\overline{w}$ can not be part of f's chosen set in the set of workers who have not rejected f prior to step k. This contradicts $f \in \mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}})$.

(p4). For each $f' \in F', \overline{w} \in \mathsf{Ch}(\mu(f') \cup \{\overline{w}\}, Q'_{f'})$. First, $\mathsf{Ch}(x_{f'}(r) \cup \mu(f'), Q'_{f'}) = x_{f'}(r)$. This is because $x_{f'}(r)$ is f''s chosen set in the set of workers who have not rejected f' prior to step r and because by the induction hypothesis, no worker in $\mu(f')$ rejects f' prior to step r. Notice that $f' \in x_{\overline{w}}(r)$. Therefore, $\overline{w} \in x_{f'}(r)$. By substitutability, $\overline{w} \in \mathsf{Ch}(\mu(f') \cup \{\overline{w}\}, Q'_{f'})$.

(p5). $\mathsf{Ch}(\mu(\overline{w}) \cup x_{\overline{w}}(r), Q'_{\overline{w}})\backslash\mu(\overline{w}) \neq \emptyset$. Otherwise, $\mathsf{Ch}(\mu(\overline{w}) \cup x_{\overline{w}}(r), Q'_{\overline{w}}) \subseteq \mu(\overline{w})$. By the definition of Ch, $\mathsf{Ch}(\mu(\overline{w}), Q'_{\overline{w}}) = \mathsf{Ch}(\mu(\overline{w}) \cup x_{\overline{w}}(r), Q'_{\overline{w}}) \subseteq \mu(\overline{w})$. This, together with (2) implies that $\mathsf{Ch}(\mu(\overline{w}) \cup x_{\overline{w}}(r), Q'_{\overline{w}}) = \mu(\overline{w})$. By $\overline{f} \in \mu(\overline{w}) \cap x_{\overline{w}}(r)$ and substitutability, $\overline{f} \in \mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}})$, contradicting our assumption.

(p6) There is a firm $f' \in F'$ such that $f' \in \mathsf{Ch}(\mu(\overline{w}) \cup \{f'\}, Q'_{\overline{w}})$. By (p5), there is $f' \in F$ such that $f' \in \mathsf{Ch}(\mu(\overline{w}) \cup x_{\overline{w}}(r), Q'_{\overline{w}})$ and $f' \in x_{\overline{w}}(r)\backslash\mu(\overline{w})$. By substitutability, $f' \in \mathsf{Ch}(x_{\overline{w}}(r), Q'_{\overline{w}})$ (hence $f' \in F'$), and $f' \in \mathsf{Ch}(\mu(\overline{w}) \cup \{f'\}, Q'_{\overline{w}})$, as desired.

By (p3), $f' \neq f$. Since $Q'_{f'} = Q_{f'}$ and $Q'_{\overline{w}} = Q_{\overline{w}}$, (p4) and (p6) establish that $f' \in \mathsf{Ch}(\mu(\overline{w}) \cup \{f'\}, Q_{\overline{w}})$ and $\overline{w} \in \mathsf{Ch}(\mu(f') \cup \{\overline{w}\}, Q_{f'})$, contradicting $\mu \in S(Q)$.

$\square$

Proposition 3. *Let $\mu \in IR(P)$ and let $\widetilde{\varphi}$ be a probabilistic stable rule. Then, there is an sd-Nash equilibrium Q of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ that supports μ.*

Proof. Let $Q \in \mathcal{P}$ be such that for each $v \in F \cup W$, $\mathsf{Ch}(S_v, Q_v) = \mu(v)$ and $A(Q_v) = \mu(v)$. By repeatedly applying Remark 5 to preference profile P, we obtain $\mu \in IR(Q)$. We next show that there is no blocking pair for μ at Q. Assume, by contradiction that there is a blocking pair (f, w) for μ at Q; $w \notin \mu(f)$, $f \in \mathsf{Ch}(\mu(w) \cup \{f\}, Q_w)$ and $w \in \mathsf{Ch}(\mu(f) \cup \{w\}, Q_f)$. By substitutability, $w \in \mathsf{Ch}(\{w\}, Q_f)$. Thus, $w \; Q_f \; \emptyset$ and $w \in A(Q_f)$, contradicting $w \notin \mu(f) = A(Q_f)$. Hence, $\mu \in S(Q)$. Since for each $v \in F \cup W$, $A(Q_v) = \mu(v)$, by R1, each agent v is matched to $\mu(v)$ across stable matches for Q. Thus, $S(Q) = \{\mu\}$ and μ is reached with probability one. We next show that no agent can profitably deviate. Let $v \in F \cup W$ and Q'_v be an alternative strategy. Let $Q' \equiv (Q'_v, Q_{-v})$. Let $\mu' \in S(Q')$. Since only agents in $\mu(v)$ find v acceptable at Q', by Lemma 1, $\mu'(v) \subseteq \mu(v)$. By individual-rationality of μ for P, $\mu(v) \; R_v \; \mu'(v)$. Since μ' is arbitrary, agent v cannot benefit from deviating. Hence, Q is an sd-Nash equilibrium. □

In many-to-one matching, we provide a sufficient condition for stability of the outcome of each sd-Nash equilibrium in the game induced by any probabilistic stable rule. To this end, we turn our attention to sd-Nash equilibria where firms behave truthfully. Each sd-Nash equilibrium where firms behave truthfully generates a stable match for the true preferences. However, Example 1 shows that truthtelling by firms in equilibrium is not sufficient for stability of the equilibrium outcome in many-to-many matching.

Proposition 4. *Let (P, c) be a many-to-one matching problem, i.e, for each $w \in W, c_w = 1$. Let $Q \equiv (P_F, Q_W)$ be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, $\widetilde{\varphi}[Q] \in S(P)$.*

Proof. By Proposition 1, a unique match is obtained from each sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Let $\mu \equiv \widetilde{\varphi}[Q]$. By Proposition 2, $\mu \in IR(P)$. We prove that $\mu \in S(P)$. Assume, by contradiction that $\mu \notin S(P)$. Suppose (f, w) blocks μ at P;

$$w \notin \mu(f), \; f \; P_w \; \mu(w) \text{ and } w \in \mathsf{Ch}(\mu(f) \cup \{w\}, P_f). \tag{3}$$

Let Q'_w be an alternative strategy for w such that for each $v, v' \in (F \backslash \{f\}) \cup \{\emptyset\}$, $f \; Q'_w \; v'$ and $v \; Q'_w \; v'$ if and only if $v \; Q_w \; v'$. Let $Q' \equiv (Q'_w, Q_{-w})$. If w is matched to f with positive probability at $\widetilde{\varphi}[Q']$, then Q_w does not stochastically P_w-dominate Q'_w and hence, Q is not an sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Suppose w is not matched to f with positive probability at $\widetilde{\varphi}[Q']$.

Let $\mu' \in \text{supp}\widetilde{\varphi}[Q']$. We show that $\mu' \in S(Q)$. Assume, by contradiction that $\mu' \notin S(Q)$. Since the definitions of Q', Q and of $\widetilde{\varphi}$ and $\mu'(w) \neq f$ ensure $\mu' \in IR(Q)$, then there is a blocking pair (f', w') for μ' at Q; $w' \notin \mu'(f')$, $f' \; Q_{w'} \; \mu'(w')$ and $w' \in \mathsf{Ch}(\mu'(f') \cup \{w'\}, Q_{f'})$. This implies that (f', w') blocks μ' at Q' unless $w' = w$ and $\mu'(w') = f$, contradicting the assumption that w is not matched to f with positive probability at $\widetilde{\varphi}[Q']$. Hence, $\mu' \in S(Q)$.

We next show that μ is the firm-optimal stable match for Q. Suppose not. Then, there are $\overline{\mu} \in S(Q)$ and $\overline{f} \in F$ such that $\overline{\mu}(\overline{f})\ Q_{\overline{f}}\ \mu(\overline{f})$. Since $P_{\overline{f}} = Q_{\overline{f}}$, then $\overline{\mu}(\overline{f})\ P_{\overline{f}}\ \mu(\overline{f})$. Let $\overline{Q}_{\overline{f}}$ be an alternative strategy for $\overline{f}$ such that $A(\overline{Q}_{\overline{f}}) = \overline{\mu}(\overline{f})$ and $\mathsf{Ch}(W, \overline{Q}_{\overline{f}}) = \overline{\mu}(\overline{f})$. Let $\overline{Q} \equiv (\overline{Q}_{\overline{f}}, Q_{-\overline{f}})$. By Lemma 2, $\overline{\mu} \in S(\overline{Q})$. By $A(\overline{Q}_{\overline{f}}) = \overline{\mu}(\overline{f})$, Lemma 1 and R1, $\overline{f}$ is matched to $\overline{\mu}(\overline{f})$ across stable matches for $\overline{Q}$ and in particular across matches in supp$\widetilde{\varphi}[\overline{Q}]$. Since $\overline{\mu}(\overline{f})\ P_{\overline{f}}\ \mu(\overline{f})$, then $\overline{f}$ has a profitable deviation when the other agents act according to $Q_{-\overline{f}}$. Hence, μ is the firm-optimal stable match for Q.

We next show that $w \notin \mathsf{Ch}(\mu'(f) \cup \{w\}, P_f)$. Assume, by contradiction that $w \in \mathsf{Ch}(\mu'(f) \cup \{w\}, P_f)$. Since $Q'_f = Q_f = P_f$, then $w \in \mathsf{Ch}(\mu'(f) \cup \{w\}, Q'_f)$. Since w is not matched to f with positive probability at $\widetilde{\varphi}[Q']$, then by the definition of Q'_w, $f\ \ Q'_w\ \ \mu'(w)$, contradicting $\mu' \in S(Q')$. Hence, $w \notin \mathsf{Ch}(\mu'(f) \cup \{w\}, P_f)$.

We now complete the proof. By substitutability, $w \notin \mathsf{Ch}(\mu(f) \cup \mu'(f) \cup \{w\}, P_f)$. Let $K \equiv \mathsf{Ch}(\mu(f) \cup \mu'(f) \cup \{w\}, P_f)$. Thus, $w \notin K$. We show that $K \nsubseteq \mu(f)$. Assume, by contradiction that $K \subseteq \mu(f)$. Then, by the definition of Ch, $\mathsf{Ch}(\mu(f) \cup \{w\}, P_f) = K$. Thus, $w \notin \mathsf{Ch}(\mu(f) \cup \{w\}, P_f)$, contradicting (3). Hence, $K \nsubseteq \mu(f)$. This, together with $w \notin K$ implies that $(K \cap \mu'(f)) \backslash \mu(f) \neq \emptyset$. Let $\overline{w} \in (K \cap \mu'(f)) \backslash \mu(f)$. By substitutability, $\overline{w} \in \mathsf{Ch}(\mu(f) \cup \{\overline{w}\}, P_f)$. Since $Q_f = P_f$, then

$$\overline{w} \in \mathsf{Ch}(\mu(f) \cup \{\overline{w}\}, Q_f). \tag{4}$$

Since μ is the firm-optimal stable match for Q, then workers unanimously find μ the worst among all stable matches for Q. This, together with $\mu' \in S(Q)$ implies that

$$\mu'(\overline{w}) = f\ Q_{\overline{w}}\ \mu(\overline{w}). \tag{5}$$

Statements (4) and (5) imply that $(f, \overline{w})$ blocks μ at Q, contradicting $\mu \in S(Q)$. □

Example 1: Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let for each $v \in F \cup W \backslash \{w_4\}$, $c_v = 2$ and $c_{w_4} = 1$. Let true preference profile $P \in \mathcal{P}$ be as follows.

$P_{f_1}:\ \ w_1w_3, w_1w_2, w_2w_3, w_1, w_3, w_2,$
$P_{f_2}:\ \ w_2w_4, w_3w_4, w_2w_3, w_4,$
$P_{f_3}:\ \ w_2w_3, w_1w_2, w_1w_3, w_2, w_3, w_1,$

$P_{w_1}:\ \ f_1f_3, f_1, f_3,$
$P_{w_2}:\ \ f_1f_3, f_2f_3, f_1f_2, f_3, f_1, f_2,$
$P_{w_3}:\ \ f_1f_2, f_2f_3, f_1f_3, f_2, f_1, f_3,$
$P_{w_4}:\ \ f_2,$

Consider the game $(\mathcal{P}, \varphi^F, P)$ induced by the probabilistic matching rule that assigns probability one to the firm-optimal stable match for each preference profile.

Let Q be a preference profile such that for each $v \in F \cup W \backslash \{w_3\}$ $Q_v = P_v$ and $Q_{w_3}: f_2f_3, f_2, f_3$. The match $\mu \equiv \varphi^F[Q]$ is the firm-optimal stable match for Q shown below.

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2 & w_3w_4 & w_2w_3 \end{pmatrix}.$$

Notice that since (f_1, w_3) is a blocking pair for μ at P, then μ is not stable for P. We now prove that Q is an sd-Nash equilibrium of the game $(\mathcal{P}, \varphi^F, P)$. Since firm f_3 and workers w_2 and w_4 are assigned at μ to their most preferred partners according to their true preferences, we only need to consider deviations for firms f_1 and f_2 and workers w_1 and w_3. Since w_3 finds any subsets of partners that include f_1 unacceptable at Q_{w_3}, firm f_1 cannot deviate to obtain w_1w_3. Thus, firm f_1 cannot improve upon μ by deviating.

We next consider deviations for f_2. Firm f_2 can benefit from deviation only if it obtains w_2w_4. Let Q'_{f_2} be an alternative strategy for f_2. Let $\mu_2 \equiv \varphi^F(Q'_{f_2}, Q_{-f_2})$. We show that firm f_2 can not be assigned w_2w_4 at μ_2. Assume, by contradiction that f_2 is assigned w_2w_4 at μ_2. Then there are three possibilities for w_2's assignment at μ_2. 1) Worker w_2 is assigned f_2f_3 at μ_2. Since w_3 finds any subsets of partners that include f_1 unacceptable at Q_{w_3}, f_1 is either assigned w_1 or unmatched at μ_2. In either case (f_1, w_2) blocks μ_2 at (Q'_{f_2}, Q_{-f_2}), contradicting stability of μ_2. 2) Worker w_2 is assigned f_1f_2 at μ_2. Then f_3 is assigned w_1w_3 or w_1 only or w_3 only at μ_2. In all cases (f_3, w_2) blocks μ_2 at (Q'_{f_2}, Q_{-f_2}), contradicting stability of μ_2. 3) Worker w_2 is assigned f_2 only at μ_2. Then as before f_3 is assigned w_1w_3 or w_1 only or w_3 only at μ_2. In all cases (f_3, w_2) blocks μ_2 at (Q'_{f_2}, Q_{-f_2}), contradicting stability of μ_2. Hence, firm f_2 can not be assigned w_2w_4 at μ_2. Since Q'_{f_2} is arbitrary, then firm f_2 cannot improve upon μ by deviating.

We now consider deviations for w_1. Worker w_1 can benefit from deviation only if she obtains f_1f_3. Let Q'_{w_1} be an alternative strategy for w_1. Let $\mu_1 \equiv \varphi^F(Q'_{w_1}, Q_{-w_1})$. We show that w_1 can not be assigned f_1f_3 at μ_1. Assume, by contradiction that w_1 is assigned f_1f_3 at μ_1. Then there are three possibilities for f_3's assignment at μ_1. 1) Firm f_3 is assigned w_1w_2 at μ_1. Then w_3 is either assigned f_2 or unmatched at μ_1. In either case (f_3, w_3) blocks μ_1 at (Q'_{w_1}, Q_{-w_1}), contradicting stability of μ_1. 2) Firm f_3 is assigned w_1w_3 at μ_1. Then w_2 is assigned f_1f_2 or f_1 only or f_2 only at μ_1. In all cases (f_3, w_2) blocks μ_1 at (Q'_{w_1}, Q_{-w_1}), contradicting stability of μ_1. 3) Firm f_3 is assigned to w_1 only at μ_1. Then w_3 is either assigned f_2 or unmatched at μ_1. In either case (f_3, w_3) blocks μ_1 at (Q'_{w_1}, Q_{-w_1}), contradicting stability of μ_1. Hence, worker w_1 can not be assigned f_1f_3 at μ_1. Since Q'_{w_1} is arbitrary, worker w_1 cannot benefit from deviating at Q.

We finally consider deviations for w_3. Worker w_3 can benefit from deviation only if she obtains f_1f_2. Let Q'_{w_3} be an alternative strategy for w_3. Let $\mu_3 \equiv \varphi^F(Q'_{w_3}, Q_{-w_3})$. We show that w_3 can not be assigned f_1f_2 at μ_3. Assume, by contradiction that w_3 is assigned f_1f_2 at μ_3. Then there are three possibilities for f_2's assignment at μ_3. Firm f_2 is assigned w_3w_4 at μ_3. If w_2 is assigned f_1 or f_3 only, then (f_2, w_2) blocks μ_3 at (Q'_{w_3}, Q_{-w_3}), contradicting stability of μ_3. Thus, w_2 is assigned f_1f_3 at μ_3. This implies that f_1 is assigned w_2w_3 at μ_3. Then, w_1 is either assigned f_3 or unmatched at μ_3. In either case (f_1, w_1) blocks μ_3 at (Q'_{w_3}, Q_{-w_3}), contradicting stability of μ_3. Firm f_2 is assigned w_2w_3 at μ_3. Then, w_4 is unmatched at μ_3. Thus, (f_2, w_4) blocks μ_3 at (Q'_{w_3}, Q_{-w_3}), contradicting stability of μ_3. 3) Firm f_2 is assigned w_3 at μ_3. Again w_4 is unmatched at μ_3. Thus, (f_2, w_4) blocks μ_3 at (Q'_{w_3}, Q_{-w_3}), contradicting stability of

μ_3. $\Diamond$

The example illustrates that even when each agent has responsive preferences, the result does not hold either. We next question the existence of an sd-Nash equilibrium where firms behave truthfully. Proposition 5 states the converse result that each stable match for the true preferences can be achieved as the outcome of an sd-Nash equilibrium in which firms behave truthfully. An immediate implication of the result is that workers can obtain any jointly achievable match as the outcome of the game induced by any probabilistic stable rule.

Proposition 5. *Let $\mu \in S(P)$ and $\widetilde{\varphi}$ be a probabilistic stable matching rule. Then, there is an sd-Nash equilibrium $Q = (P_F, Q_W)$ of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ that supports μ.*

Proof. Let $Q \in \mathcal{P}$ be such that for each $w \in W$, $A(Q_w) = \mu(w)$ and $\mathsf{Ch}(F, Q_w) = \mu(w)$ and for each $f \in F$, $Q_f = P_f$. We show that $\mu \in S(Q)$. Since $\mu \in S(P)$ and $Q_F = P_F$, then for each $f \in F$, $\mathsf{Ch}(\mu(f), Q_f) = \mu(f)$. By $\mathsf{Ch}(F, Q_w) = \mu(w)$ and the definition of Ch, for each $w \in W$, $\mathsf{Ch}(\mu(w), Q_w) = \mu(w)$. Thus, $\mu \in IR(Q)$. Suppose that a firm-worker pair (f', w') blocks μ at Q;

$$w' \notin \mu(f'),\ f' \in \mathsf{Ch}(\mu(w') \cup \{f'\}, Q_{w'}) \text{ and } w' \in \mathsf{Ch}(\mu(f') \cup \{w'\}, Q_{f'}). \tag{6}$$

By substitutability, $f' \in \mathsf{Ch}(\{f'\}, Q_{w'})$. Thus, $f'\ Q_{w'}\ \emptyset$. Then, $f' \in A(Q_{w'})$ which implies that $f' \in \mu(w')$, contradicting (6). Hence, $\mu \in S(Q)$. Since for each $w \in W$, $A(Q_w) = \mu(w)$, by R1, each worker w is matched to $\mu(w)$ across stable matches for Q. Hence, $S(Q) = \{\mu\}$ and μ is reached with probability one.

We now prove that Q is an sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Let $w \in W$ and Q'_w be an alternative strategy for w. Let $S \subseteq F$ be such that $S\ P_w\ \mu(w)$. Let $Q' \equiv (Q'_w, Q_{-w})$. We show that w can not be matched to S at any stable match for Q'. Assume, by contradiction that there is $\mu' \in S(Q')$ such that $\mu'(w) = S$.

We first show that $\mathsf{Ch}(\mu'(w) \cup \mu(w), P_w) \backslash \mu(w) \neq \emptyset$. Suppose not. Then, $\mathsf{Ch}(\mu'(w) \cup \mu(w), P_w) \subseteq \mu(w)$. By the definition of Ch, $\mathsf{Ch}(\mu'(w) \cup \mu(w), P_w) = \mathsf{Ch}(\mu(w), P_w)$. By $\mu \in S(P)$, $\mathsf{Ch}(\mu(w), P_w) = \mu(w)$. Thus, $\mathsf{Ch}(\mu'(w) \cup \mu(w), P_w) = \mu(w)$, contradicting $\mu'(w) = S\ P_w\ \mu(w)$.

Now let $f \in \mathsf{Ch}(\mu'(w) \cup \mu(w), P_w) \backslash \mu(w)$. Then, $f \in \mu'(w) \backslash \mu(w)$. By substitutability, $f \in \mathsf{Ch}(\mu(w) \cup \{f\}, P_w)$. By $\mu \in S(P)$,

$$w \notin \mathsf{Ch}(\mu(f) \cup \{w\}, P_f). \tag{7}$$

By the definition of Ch, $\mathsf{Ch}(\mu(f) \cup \{w\}, P_f) = \mathsf{Ch}(\mu(f), P_f)$. This, together with $\mu \in S(P)$ implies that $\mathsf{Ch}(\mu(f) \cup \{w\}, P_f) = \mu(f)$. By (7) and $Q'_f = Q_f = P_f$,

$$w \notin \mathsf{Ch}(\mu(f) \cup \{w\}, Q'_f) = \mu(f). \tag{8}$$

We next show that $\mu(f) \backslash \mu'(f) = \emptyset$. Suppose not. Then $\mu(f) \subseteq \mu'(f)$. This, together with $f \in \mu'(w) \backslash \mu(w)$ implies that $\mu(f) \cup \{w\} \subseteq \mu'(f)$. By (8) and substitutability, $w \notin \mathsf{Ch}(\mu'(f), Q'_f)$, contradicting $\mu' \in S(Q')$. Thus, $\mu(f) \backslash \mu'(f) \neq \emptyset$.

Let $\overline{w} \in \mu(f) \backslash \mu'(f)$. Notice that $\overline{w} \neq w$ and $\mu'(\overline{w}) \neq \mu(\overline{w})$. Since $A(Q'_{\overline{w}}) = \mu(\overline{w})$, and $\mu' \in S(Q')$, by Lemma 1, $\mu'(\overline{w}) \subseteq A(Q'_{\overline{w}}) = \mu(\overline{w})$. By $\mu \in S(Q)$, $Q'_{\overline{w}} = Q_{\overline{w}}$

and $\overline{w} \in \mu(f)$, we have $f \in \mu(\overline{w}) = \mathsf{Ch}(\mu(\overline{w}), Q'_{\overline{w}})$. Since $\mu'(\overline{w}) \cup \{f\} \subseteq \mu(\overline{w})$, by substitutability,

$$f \in \mathsf{Ch}(\mu'(\overline{w}) \cup \{f\}, Q'_{\overline{w}}). \tag{9}$$

Since only workers in $\mu(f) \cup \{w\}$ find f acceptable at Q', by Lemma 1, $\mu'(f) \subseteq (\mu(f) \cup \{w\})$. Noting that $\overline{w} \in \mu(f)$, we have $(\mu'(f) \cup \{\overline{w}\}) \subseteq (\mu(f) \cup \{w\})$. By (8) and substitutability,

$$\overline{w} \in \mathsf{Ch}(\mu'(f) \cup \{\overline{w}\}, Q'_f). \tag{10}$$

Statements (9) and (10) imply that $(f, \overline{w})$ blocks μ' at Q', contradicting $\mu' \in S(Q')$. Hence, w cannot get matched to f at any stable match for Q'. This implies that w cannot improve upon $\mu(w)$ by deviating.

Now let $f \in F$. The only workers willing to get matched to f are those in $\mu(f)$. Moreover, by individual-rationality of μ for P, for each $S \subseteq \mu(f)$, $\mu(f)\ R_f\ S$. Hence, f cannot improve upon $\mu(f)$ by deviating. □

6 Discussion

We now inquire validity of our results without the assumption of cardinal monotonicity. All our proofs rely on the substitutability assumption and property R1. This property does not hold without further assumptions on preferences. Hatfield and Milgrom (2005) establish cardinal monotonicity as a maximal domain for R1 in many-to-one matching. Klijn and Yazici (2014) complements this result by establishing cardinal monotonicity as a maximal domain for R1 in many-to-many matching. Precisely, if some agent's preferences violate cardinal monotonicity but do not necessarily satisfy substitutability, then there are substitutable and cardinally monotonic preferences for the others such that R1 fails. In Example 2 below we identify a profile of substitutable preferences that violate cardinal monotonicity, a probabilistic stable rule and an equilibrium strategy profile such that the equilibrium outcome is not individually-rational for the true preferences. In our construction each worker has a capacity 1, therefore, the example applies to many-to-one matching as well. In Example 3 below we identify a many-to-one problem with a profile of substitutable preferences that violate cardinal monotonicity, a probabilistic stable rule and an equilibrium strategy profile where firms behave truthfully such that the equilibrium outcome is not stable for the true preferences.

Example 2: Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let for each $v \in F \cup W \backslash \{f_1, f_2\}$, $c_v = 1$ and $c_{f_1} = 2$, $c_{f_2} = 3$. Let $\mathcal{P}$ be the class of substitutable preferences. Let true preference profile $P \in \mathcal{P}$ be as follows.

$P_{f_1}:\quad w_1, w_1w_3,$
$P_{f_2}:\quad w_2, w_1w_3w_4, w_1w_3, w_1w_4, w_3w_4, w_1, w_3, w_4,$
$P_{f_3}:\quad w_4, w_2,$

$P_{w_1}: \quad f_1, f_2$
$P_{w_2}: \quad f_2, f_3$
$P_{w_3}: \quad f_1, f_2$
$P_{w_4}: \quad f_3, f_2$

Notice that firm f_2's preferences violate cardinal monotonicity: $w_1 \; P_{f_2} \; \emptyset$ but $w_2 \; P_{f_2} \; w_1w_2$. Consider the game $(\mathcal{P}, \widetilde{\varphi}, P)$ induced by the probabilistic matching rule $\widetilde{\varphi}$ described below.

$$\text{For each } P \in \mathcal{P},\; \widetilde{\varphi}[P] = \begin{cases} \varphi^F[P] & \text{if } P_{f_1} : w_1w_3, w_1, w_3, \\ \varphi^W[P] & \text{otherwise.} \end{cases}$$

In words, $\widetilde{\varphi}[P]$ assigns probability one to the firm-optimal stable match if f_1 states the particular strategy mentioned above and to the worker-optimal stable match otherwise. Let strategy profile $Q \in \mathcal{P}$ be as follows.

$Q_{f_1}: \quad w_1w_3, w_1, w_3,$
$Q_{f_2}: \quad w_2, w_1w_3w_4, w_1w_3, w_1w_4, w_3w_4, w_1, w_3, w_4,$
$Q_{f_3}: \quad w_4, w_2,$

$Q_{w_1}: \quad f_2, f_1,$
$Q_{w_2}: \quad f_3, f_2,$
$Q_{w_3}: \quad f_2, f_1,$
$Q_{w_4}: \quad f_2, f_3,$

It is easy to verify that

$$\widetilde{\varphi}[Q] = \varphi^F[Q] = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_3 & w_2 & w_4 \end{pmatrix}.$$

We argue that Q is an sd-Nash equilibrium. Notice that each agent but f_1 is assigned her/its most preferred (set of) partners according to P. Thus, each agent but f_1 has no interest in deviating. For each alternative strategy that f_1 states $\widetilde{\varphi}$ recommends the worker-optimal stable match for the corresponding strategy profile. Indeed, for each such profile of strategies $\widetilde{\varphi}$ recommends the following match.

$$\text{For each } Q'_{f_1} \in \mathcal{P}_{f_1},\; Q'_{f_1} \neq Q_{f_1},\; \widetilde{\varphi}[Q'_{f_1}, Q_{-f_1}] = \begin{pmatrix} f_1 & f_2 & f_3 \\ \emptyset & w_1w_3w_4 & w_2 \end{pmatrix}.$$

Firm f_1 prefers w_1w_3 to being unmatched. Thus, it can not improve upon $\widetilde{\varphi}[Q](f_1)$ by deviating. $\Diamond$

Example 3: Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$. Let for each $w \in W$, $c_w = 1$ and each $f \in F$, $c_f = 3$. Let $\mathcal{P}$ be the class of substitutable preferences. Let true preference profile $P \in \mathcal{P}$ be as follows.

$P_{f_1}:\quad w_6w_8, w_2w_3w_8, w_2w_3w_6, w_1w_2w_3, \boldsymbol{w_2w_3w_7}, w_3w_8, w_2w_8, w_1w_8, w_2w_6, w_3w_6, w_1w_6, w_2w_3, w_1w_3, w_1w_2, w_3w_7, w_2w_7, w_1w_7, w_8, w_6, w_2, w_3, w_7, w_1,$

$P_{f_2}:\quad w_3w_4, w_4w_5w_8, w_3w_5w_8, \boldsymbol{w_5w_6w_8}, w_4w_5, w_4w_8, w_3w_5, w_3w_8, w_5w_8, w_5w_6, w_6w_8, w_3, w_4, w_5, w_8, w_6,$

$P_{f_3}:\quad w_2w_5, w_2w_4w_6, w_5w_6, w_4w_5, w_2w_4, w_4w_6, \boldsymbol{w_1w_4}, w_1w_5, w_2w_6, w_1w_2, w_1w_6, w_2, w_5, w_6, w_4, w_1,$

$P_{w_1}:\quad f_1, \boldsymbol{f_3},$
$P_{w_2}:\quad \boldsymbol{f_1}, f_3,$
$P_{w_3}:\quad \boldsymbol{f_1}, f_2,$
$P_{w_4}:\quad \boldsymbol{f_3}, f_2,$
$P_{w_5}:\quad \boldsymbol{f_2}, f_3,$
$P_{w_6}:\quad \boldsymbol{f_2}, f_3, f_1,$
$P_{w_7}:\quad \boldsymbol{f_1},$
$P_{w_8}:\quad \boldsymbol{f_2}, f_1,$

Notice that f_1's preferences violate cardinal monotonicity: $|\mathsf{Ch}(W, P_{f_1})| = |\{w_6, w_8\}| = 2$ and $|\mathsf{Ch}(W\backslash\{w_6\}, P_{f_1})| = |\{w_2, w_3, w_8\}| = 3$. Indeed, each firm's preferences violate cardinal monotonicity. Consider the game $(\mathcal{P}, \widetilde{\varphi}, P)$ induced by the probabilistic matching rule $\widetilde{\varphi}$ described below.

$$\text{For each } P \in \mathcal{P},\ \widetilde{\varphi}[P] = \begin{cases} \varphi^W[P] & \text{if } P_{w_1} : f_3, \\ \varphi^F[P] & \text{otherwise.} \end{cases}$$

In words, $\widetilde{\varphi}[P]$ assigns probability one to the worker-optimal stable match if worker w_1 has the particular preference relation mentioned above and to the firm-optimal stable match otherwise. Let strategy profile $Q \in \mathcal{P}$ be such that for each $v \in F \cup W\backslash\{w_1\}$ $Q_v = P_v$ and worker w_1 has the following strategy.

$Q_{w_1}:\quad f_3,$

It is easy to verify that

$$\mu \equiv \widetilde{\varphi}[Q] = \varphi^W[Q] = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2w_3w_7 & w_5w_6w_8 & w_1w_4 \end{pmatrix}.$$

Notice that (f_1, w_1) blocks μ (shown in bold) at P. We argue that Q is an sd-Nash equilibrium. Notice that each worker but w_1 is assigned her most preferred partner according to P. Thus, only w_1 may have an interest in deviating. For each alternative strategy that w_1 states $\widetilde{\varphi}$ recommends the firm-optimal stable match for the corresponding strategy profile. Indeed, for each such profile of strategies w_1 remains unmatched.

$$\text{For each } Q'_{w_1} \in \mathcal{P}_{w_1},\ Q'_{w_1} \neq Q_{w_1},\ \widetilde{\varphi}[Q'_{w_1}, Q_{-w_1}] = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_6w_8 & w_3w_4 & w_2w_5 \end{pmatrix}.$$

Worker w_1 prefers f_3 to being unmatched. Thus, she can not improve upon $\widetilde{\varphi}[Q](w_1)$ by deviating.

For each possible firm deviation $\widetilde{\varphi}$ recommends the worker-optimal stable match for the corresponding strategy profile. We first consider deviations for f_1. Let Q'_{f_1} be an alternative strategy for f_1. First notice that each firm receives proposals from its partners at μ at the end of the first step of the worker proposing DA algorithm applied to (Q'_{f_1}, Q_{-f_1}). Furthermore, each of f_2 and f_3 holds its partners at μ at the end of the first step. Firm f_1 rejects at least one of its partners at μ in the first step. Otherwise, it holds each of its partners at μ at the end of the first step. Since no worker is rejected, the algorithm stops. Firm f_1 is matched to its partners at μ and cannot benefit from deviating. Firm f_1 can receive a more preferred set of partners than those it is matched at μ according to its true preferences only if it rejects either w_7 only or each of its partners at μ in the first step. 1) Firm f_1 rejects w_7 only in the first step. Worker w_7 does not make any further proposals. The algorithm stops and f_1 is matched to w_2w_3. 2) Firm f_1 rejects each of its partners at μ, w_2, w_3, w_7, in the first step. Workers w_2 and w_3 propose respectively to f_3 and f_2 whereas w_7 makes no proposal in the second step. Firm f_2 holds $w_3w_5w_8$ and rejects w_6. Firm f_3 holds w_2w_4 and rejects w_1. In the next step w_6 proposes to f_3 and w_1 makes no proposal. Firm f_3 holds $w_2w_4w_6$ and issues no rejection. Since no worker is rejected, the algorithm stops. Firm f_1 is left unmatched.

We next consider deviations for f_2. Let Q'_{f_2} be an alternative strategy for f_2. First notice that each firm receives proposals from its partners at μ at the end of the first step of the worker proposing DA algorithm applied to (Q'_{f_1}, Q_{-f_1}). Furthermore, each of f_1 and f_3 holds its partners at μ at the end of the first step. Using the same reasoning above we conclude that f_1 rejects at least one of its partners at μ in the first step. Firm f_2 can receive a more preferred set of partners than those it is matched at μ according to its true preferences only if it rejects either w_6 only or each of its partners at μ in the first step. 1) Firm f_2 rejects w_6 only in the first step. Worker w_6 proposes to f_3 in the second step. Firm f_3 holds w_4w_6 and rejects w_1 who makes no proposal in the next step. The algorithm stops and f_2 is matched to w_5w_8. 2) Firm f_2 rejects each of its partners at μ, w_5, w_6, w_8, in the first step. Workers w_5 and w_6 propose to f_3 and w_8 proposes to f_1 in the second step. Firm f_3 holds w_5w_6 and rejects w_4 and w_1. Firm f_1 holds $w_2w_3w_8$ and rejects w_7. In the next step w_4 proposes to f_2 whereas w_1 and w_7 make no proposals. Firm f_2 holds w_4 and issues no rejection. Since no worker is rejected, the algorithm stops. Firm f_2 is matched to w_4.

We next consider deviations for f_3. Let Q'_{f_3} be an alternative strategy for f_3. First notice that each firm receives proposals from its partners at μ at the end of the first step of the worker proposing DA algorithm applied to (Q'_{f_1}, Q_{-f_1}). Furthermore, each of f_1 and f_2 holds its partners at μ at the end of the first step. As before f_3 rejects at least one of its partners at μ in the first step. Firm f_3 can receive a more preferred set of partners than those it is matched at μ according to its true preferences only if it rejects either w_1 only or each of its partners at μ in the first step. 1) Firm f_3 rejects w_1 only in the first step. Worker w_1 does not make any further proposals. The algorithm stops and f_3 is matched to w_3. 2) Firm f_3 rejects each of its partners at μ in the first step. Worker w_4 proposes to f_2 whereas w_1 makes no proposal in the second step. Firm

f_2 holds $w_4w_5w_8$ and rejects w_6. In the next step w_6 proposes to f_3. Firm f_3 holds w_6 and issues no rejection. Since no worker is rejected, the algorithm stops. Firm f_3 is matched to w_6. $\diamondsuit$

7 Conclusion

We analyze sd-Nash equilibria of the game induced by any probabilistic stable matching rule in many-to-many matching when each agent has substitutable and cardinally monotonic preferences. Our first result is that a unique match is achieved as the outcome of each sd-Nash equilibrium whereas multiple matches may arise with positive probability in the game as the outcome of truthful behavior. The second result establishes individual-rationality with respect to the true preferences as a necessary and sufficient condition for a match to be achieved as the outcome of an sd-Nash equilibrium. Stochastically dominant Nash equilibria where firms behave truthfully always lead to stable matches for the true preferences in many-to-one matching. This result is not carried over to the many-to-many matching framework. Conversely, each stable match for the true preferences is supported as the outcome of an sd-Nash equilibrium where firms behave truthfully.

Pais (2008) examines the connection between equilibria of the game induced by a probabilistic stable rule with those induced by a deterministic stable rule in the college admissions problem. In particular, each sd-Nash equilibrium of the game induced by any probabilistic stable rule is a Nash equilibrium of the game induced by some deterministic stable rule. A partially converse result is the following. Let Q be an sd-Nash equilibrium of the game induced by the firm-optimal stable rule and of the game induced by the worker-optimal stable rule. Then Q is an sd-Nash equilibrium of the game induced by any probabilistic stable rule. With obvious modifications, the proofs of all these results remain valid in many-to-many matching when each agent has substitutable and cardinally monotonic preferences.

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