

**Unitarity violation at the Wilson-Fisher fixed point in  $4 - \epsilon$  dimensions**Matthijs Hogervorst,<sup>1,2,3</sup> Slava Rychkov,<sup>3,2,4</sup> and Balt C. van Rees<sup>3,5</sup><sup>1</sup>*C.N. Yang Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794-3840, USA*<sup>2</sup>*Laboratoire de Physique Théorique de l'École normale supérieure (LPTENS), 75231 Paris Cedex 05, France*<sup>3</sup>*CERN, Theory Department, CH-1211 Geneva 23, Switzerland*<sup>4</sup>*Sorbonne Universités, UPMC Univ Paris 06, Faculté de Physique, 75005 Paris, France*<sup>5</sup>*Centre for Particle Theory & Department of Mathematical Sciences, Durham University, Durham DH1 3LE, United Kingdom*

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We consider the continuation of free and interacting scalar field theory to noninteger spacetime dimension  $d$ . We find that the correlation functions in these theories are necessarily incompatible with unitarity (or with reflection positivity in Euclidean signature). In particular, the theories contain negative-norm states unless  $d$  is a positive integer. These negative-norm states can be obtained via the operator product expansion from simple positive-norm operators, and are therefore an integral part of the theory. At the Wilson-Fisher fixed point the nonunitarity leads to the existence of complex anomalous dimensions. We demonstrate that they appear already at leading order in the epsilon expansion.

DOI: [10.1103/PhysRevD.93.125025](https://doi.org/10.1103/PhysRevD.93.125025)**I. INTRODUCTION**

Unitarity, or its absence, is a key structural property of any quantum field theory (QFT). The purpose of this paper is to present a novel mechanism by which unitarity can be violated for QFTs defined in *noninteger* spacetime dimensions. This mechanism was first pointed out in our recent work [1], and here it will be explained in detail. The mechanism is general, but here it will be demonstrated using the example of the scalar  $\phi^4$  theory in  $d$  dimensions, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{4!}g\mu^{4-d}\phi^4. \quad (1.1)$$

As is well known, in  $d = 4 - \epsilon$  dimensions, this theory has a conformal IR fixed point, called the Wilson-Fisher (WF) fixed point [2]. The fixed point coupling  $g = g_*$  is given at the lowest order by

$$g_*/(4\pi)^2 = \epsilon/3 + O(\epsilon^2). \quad (1.2)$$

Unitarity is violated, we believe, for all noninteger  $d$  in the usually considered range  $2 \leq d \leq 4$ .<sup>1</sup> However, for most of

<sup>1</sup>Continuation to a larger range  $1 \leq d \leq 4$  is also sometimes considered (see Ref. [3] for a recent discussion). The theories in the range  $1 < d < 2$  are also expected to be nonunitary. For the free scalar theory this is obvious, as the dimension of  $\phi$  is negative and violates the scalar primary unitarity bound  $\max(0, (d-2)/2)$ .

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the paper we will focus on  $d = 4 - \epsilon$ ,  $\epsilon \ll 1$ . On the one hand, this will be enough to illustrate the general mechanism. On the other hand, since the theory in this range is weakly coupled, it will be possible to buttress general arguments by explicit perturbative calculations.

The mechanism responsible for these unitarity violations makes use of operators that are present in the spectrum for all noninteger  $d$ , but that decouple for certain integer  $d$ . Such operators are known as *evanescent operators* in the QFT literature [4]. In standard QFT computations, one is interested in computing observables at integer  $d$ , typically  $d = 4$ . After regulating the theory using dimensional regularization and passing to  $d = 4 - \epsilon$  dimensions, such evanescent operators can appear, as has been shown in theories with fermions [5,6].<sup>2</sup> Although these operators themselves decouple in the limit  $\epsilon \rightarrow 0$ , they leave an imprint on four-dimensional (4D) observables. A well-known example of this phenomenon is furnished by the QCD next-to-leading-order anomalous dimensions of four-Fermi operators, responsible for the hadronic weak decays [8].

In contrast to all the works just cited, here we will not take the limit  $\epsilon \rightarrow 0$ . This is because we are primarily interested in the WF fixed point, which becomes trivial in this limit. Rather we will keep  $\epsilon$  small but finite. Our main point will be then to show that some of the evanescent operators give rise to states that have negative norm, implying that the  $\phi^4$  theory in noninteger dimensions, and *a fortiori* the WF fixed point, is not unitary. In passing, the present work shows that evanescent operators already occur in theories with a single scalar field, whereas

<sup>2</sup>Evanescent operators are also necessary to renormalize theories whose action involves the epsilon tensor, like the WZNW model [7].

previous QFT computations only encountered such operators when nonscalar fields were present.

There is a simple reason that evanescent operators in the WF fixed point have not been noticed up to now. This is because most computations have focused on low-dimension operators such as  $\phi$ ,  $\phi^2$  and  $\phi^4$ , which are obviously present in  $d = 4$ . As we will see, to discover the evanescent operators in the scalar theory and to demonstrate their various properties, one has to go pretty high up in operator dimension.<sup>3</sup> For instance, in this paper we will work with all scalar operators below the cutoff  $\Delta_{\max} = 23$  in operator dimension. The number of operators grows exponentially fast, and as a result there are thousands of operators below this cutoff.

To our knowledge, such a systematic investigation of the Hilbert space and spectrum of WF was taken only once before us, by Kehrein *et al.* [9–11]. These papers used a different methodology that did not take evanescent operators into account. As a result, they did not notice the presence of negative-norm states in  $4 - \epsilon$  dimensions. In other aspects there are some similarities between our works, and a more detailed comparison will be given in Sec. IV C.

The remainder of this paper is organized as follows. Section II discusses evanescent operators in the free scalar theory. We will show that their existence implies that the theory in noninteger dimensions has negative-norm states, violating unitarity. In addition we mention how these operators fit into representations of the conformal algebra, and we also give a systematic counting of all the evanescent operators around  $d = 4$  up to  $\Delta_{\max} = 23$ .

From Sec. III onwards we consider the interacting theory at the WF fixed point in  $4 - \epsilon$  dimensions. We begin with a review of the computation of one-loop anomalous dimensions in Sec. III. In Sec. IV we discuss the anomalous dimensions of the evanescent operators and show that there is no reason for them to be real valued. We substantiate this claim with an explicit computation in Sec. IV B: there exist four scalar operators with  $\Delta = 23$  (in  $d = 4$ ) that get a complex anomalous dimension at one loop. We also briefly explain how the evanescent operators can affect the general computation of higher-loop anomalous dimensions. The implications of our results for the structure of four-point functions are discussed in Appendix B.

Our final Sec. V offers some general comments, including an overview of the expected structure of the WF fixed point for any  $2 \leq d \leq 4$  that stems from our results. In Sec. V C we consider the effect of evanescent operators on numerical studies of the crossing symmetry equations.

<sup>3</sup>This is different from the fermionic case where already at the four-Fermi level one finds an infinite number of evanescent operators.

## II. THE FREE BOSON IN NONINTEGER $d$

As a first order of business we will have to make the definition of a quantum field theory in noninteger dimensions more precise. We will investigate local operators and their correlation functions, and leave the study of other observables like the S-matrix or nonlocal operators to future work.<sup>4</sup> In this section we will consider the free massless scalar field  $\phi$  in noninteger dimensions in more detail. Unless otherwise mentioned, we will always work in Euclidean signature.

For any *integer*  $d > 2$ , the set of local operators is generated by taking symmetrized normal-ordered products, or “words,” of the elementary “letters” which consist of zero or more derivatives acting on the fundamental field  $\phi$ . Each operator therefore looks like

$$:\partial^{n_1}\phi\partial^{n_2}\phi\dots\partial^{n_k}\phi:(x), \quad (2.1)$$

where  $\partial^n$  stand for various derivatives of order  $n$ , some of whose indices may be contracted (see below). Correlation functions and operator product expansions (OPEs) of these operators can be computed by Wick’s theorem.

In this paper we will mainly consider scalar, parity-even operators.<sup>5</sup> A basis for such operators is produced by taking operators of the form (2.1) and contracting all the indices on the derivatives with the inverse metric  $\delta^{\mu\nu}$ . Contracting two derivatives acting on the same  $\phi$  is considered to give rise to a zero operator, and such operators are not included in the basis. This is justified, because operators proportional to the equation of motion  $\square\phi = 0$ ,  $\square := \partial^\mu\partial_\mu$ , have nonvanishing correlation functions only at coincident points. If one is interested in correlation functions at noncoincident points, as is the case here, one can drop such operators from the start. In the above discussion, this means that we will eliminate operators of the form  $\partial^n\square\phi \times (\text{anything})$ .

Through the operator-state correspondence the set of operators also defines the Hilbert space of the theory on the  $(d-1)$ -sphere  $S^{d-1}$ . As is familiar, the states in this Hilbert space can be decomposed into highest-weight representations of the  $d$ -dimensional conformal algebra  $SO(d+1, 1)$  built on top of conformal primary operators satisfying  $[K_\mu, \mathcal{O}(0)] = 0$ . For our purposes, we will not need to explicitly organize all states in such representations. With some abuse of language, in this work we will sometimes talk about states and operators interchangeably.

Moving to *noninteger*  $d$ , most of the above story extends in a fairly standard and natural manner. The scaling

<sup>4</sup>Results on nonlocal operators in the epsilon expansion can be found, for example, in Refs. [12,13].

<sup>5</sup>See however Sec. II E for a brief discussion of general tensorial operators, the epsilon tensor and the parity operation.

dimension of  $\phi$  is analytically continued from integer  $d$ , i.e.  $\Delta_\phi = (d-2)/2$ . One does not dwell too much on the range of the index  $\mu$  for noninteger  $d$ . One also imposes that  $\partial_\mu x^\nu = \delta_\mu^\nu$  and furthermore that the metric satisfies  $\delta^{\mu\nu} \delta_{\mu\nu} = d$ . This procedure completely defines correlation functions of local operators in flat space, with the exception of those involving the epsilon tensor, which we discuss in more detail below. It basically amounts to the analytic continuation of coefficients multiplying the tensor structures.

We will also need to integrate correlation functions over either all of  $\mathbb{R}^d$  or on  $S^{d-1}$ . We will use the standard dimensional regularization integration rules [4,14]. In particular the volume of a unit sphere in  $\mathbb{R}^d$  is  $S_d := \text{Vol}(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ . As explained in Refs. [4,14], such  $d$ -dimensional integrals can be properly defined if one considers vectors to have infinitely many components in noninteger dimensions; one may show that they truncate to finite-dimensional integrals when  $d$  becomes integer.

Finally, it will be helpful to have a definition of correlation functions on the cylinder  $\mathbb{R} \times S^{d-1}$ . We define these formally through the standard Weyl transformation rules from flat space. Mapping to the cylinder is morally equivalent to the radial quantization of the theory, which will be extensively used in the following sections. Notice that if we can integrate correlation functions then we can also define nonlocal operators, like the generators of the conformal algebra, and hence conformal transformations on local operators are defined unambiguously. We will therefore also be able to group operators into conformal multiplets and distinguish between primary and descendant operators.

Although the above setup defines all local observables needed in this work, some aspects of the continuation to noninteger  $d$  remain somewhat mysterious. These issues relate to the representation theory of the algebras  $so(d)$  and  $so(d+1, 1)$  in noninteger  $d$ , which does not appear to be developed in the mathematical literature.<sup>6</sup> One concrete relevant question concerns the meaning of the analytic continuation of the partition function  $Z$  of the free boson on  $S^1 \times S^{d-1}$  [16]. For integer  $d$  the partition function counts all states on the sphere with weight one (times  $q^\Delta$ , where  $\Delta$  is the dimension and  $q$  is the modular parameter). In analytically continued  $Z$ , the vector  $|\partial_\mu \phi\rangle$  is then counted with weight  $d$ , whereas we expect that this representation is infinite dimensional for noninteger  $d$ . This leads to the conclusion that in noninteger  $d$ , the partition function  $Z$  only counts degrees of freedom in a regulated sense. Is there an intrinsic representation-theoretic meaning to the dimensions of  $so(d)$  representations analytically continued

<sup>6</sup>An analytic continuation of the algebra  $sl(d)$  to general  $d$  has been constructed in Ref. [15].

to noninteger  $d$ ? It would be interesting to better understand this question.<sup>7</sup>

### A. Evanescent operators

Consider the sequence of scalar, parity-even operators of the form

$$R_n(x) := \delta^{\mu_1 \nu_1} \delta^{\mu_2 \nu_2} \dots \delta^{\mu_n \nu_n} : M_{\mu_1 \nu_1}(x) \dots M_{\mu_n \nu_n}(x) :, \quad (2.2)$$

where  $n = 1, 2, \dots$  is a positive integer and we have introduced the shorthand notation

$$M_{\mu\nu}(x) = \partial_\mu \partial_\nu \phi(x). \quad (2.3)$$

The product of deltas has to be antisymmetrized in the second indices as indicated by the notation. Explicitly, the first few of these operators are given by

$n$	$R_n$	
2	$-\text{tr} M^2$	
3	$2\text{tr} M^3$	
4	$-6\text{tr} M^4 + 3(\text{tr} M^2)^2$	
5	$24\text{tr} M^5 - 20(\text{tr} M^2)(\text{tr} M^3)$	(2.4)

where e.g.  $\text{tr} M^3$  stands for the contraction  $M_{\mu\nu} M_{\nu\lambda} M_{\lambda\mu}$  etc. Notice that  $\text{tr} M = \square \phi = 0$ . Clearly, with this definition each of the operators  $R_n$  is defined for any  $d$ , integer or noninteger.

However if  $d < n$  the number of possible values of the  $\nu$  indices on the rhs of Eq. (2.2) is too small: for each choice of  $\nu_1, \dots, \nu_n$  there are necessarily duplicate indices and total antisymmetrization is not possible. This implies that each of the operators  $R_n$  vanishes when  $d = 1, 2, \dots, n-1$ .<sup>8</sup> These are, then, examples of evanescent operators—operators which are identically zero for certain integer dimensions, but nontrivial for noninteger dimensions.

We hasten to add that  $R_n$ 's are not the only evanescents of the theory. For example, multiplying an evanescent  $R_n$  by any other operator gives again an evanescent. More evanescents can be constructed by applying the same basic

<sup>7</sup>Similar issues arise for  $O(N)$  models and  $Q$ -state Potts models with noninteger  $N$  or  $Q$ , relevant for percolation and self-avoiding walks. There is a wide statistical physics literature on the partition function of such models where the coefficients are polynomials of  $N$  or  $\sqrt{Q}$ , interpreted as quantum dimensions of quantum group representations. We just give two references [17,18] as points of entry. We thank Jesper Jacobsen and Hirohiko Shimada for the discussions.

<sup>8</sup>This can also be shown by expressing the  $R_n$  operators in terms of the eigenvalues of the matrix  $M$  [1] or by applying the Cayley-Hamilton theorem [19]. We thank Miguel Paulos for suggesting to us a proof using antisymmetrization.

antisymmetrization idea to other tensors. To give just one example: one can replace some of the  $M$ 's in Eq. (2.2) with  $M'_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi$ . In Sec. II D we will see a systematic approach to count and construct all the evanescent states below some cutoff in dimension, based on the Gram matrix. A natural conjecture, which we checked extensively, is that all the evanescents appear from antisymmetrizations of the above sort. It would be nice to prove this.

Let us finally note that  $R_n$ 's are not primary operators, as they can be represented as total derivatives:

$$R_n = \partial_{\mu_1} \{ \delta^{\mu_1 \nu_1} \delta^{\mu_2 \nu_2} \dots \delta^{\mu_n \nu_n} : \partial_{\nu_1} \phi M_{\mu_2 \nu_2} \dots M_{\mu_n \nu_n} : \}. \quad (2.5)$$

Below we will further analyze the conformal multiplets to which  $R_n$  belong.

### B. Negative-norm states

We will now establish that the Hilbert space of local operators of the free scalar theory is not positive definite, and so the theory is not unitary, away from integer  $d$ .

Whether a Hilbert space is positive definite or not is encoded in the Gram matrix, or the matrix of inner products. In conformal field theory it is customary to work in radial quantization, where the inner products are evaluated by inserting the operators at zero and infinity. The radial quantization Gram matrix is thus defined as

$$G_{ij} \equiv \langle \mathcal{O}_i | \mathcal{O}_j \rangle_{\text{rad}} := \lim_{x \rightarrow 0} \langle [\mathcal{O}_i(x)]^\dagger \mathcal{O}_j(x) \rangle, \quad (2.6)$$

where the conjugate operator  $[\mathcal{O}_i(x)]^\dagger$  is inserted at the point  $Rx = x^\mu/x^2$  and is rescaled appropriately to have a finite limit. We will extensively use the above Gram matrix in the rest of the paper, and it will be discussed in more detail in Sec. II D.

For now, it will however be convenient to introduce a second inner product  $\tilde{G}$ . Here, one of the operators is inserted at the point N with coordinates  $x_d = 1/2, x_1, \dots, x_{d-1} = 0$ . Conjugate operators  $[\mathcal{O}(N)]^+$  are inserted at the point S with coordinates  $x_d = -1/2, x_1, \dots, x_{d-1} = 0$ , which is the reflection of N through the plane  $x_d = 0$ .<sup>9</sup> The Gram matrix  $\tilde{G}_{ij}$  is then defined as

$$\tilde{G}_{ij} \equiv \langle \mathcal{O}_i | \mathcal{O}_j \rangle_{\text{NS}} := \langle [\mathcal{O}_i(N)]^+ \mathcal{O}_j(N) \rangle. \quad (2.7)$$

This inner product is adapted to the so-called North-South (NS) quantization scheme [20,21], where states in the Hilbert space are defined by inserting operators at N.

<sup>9</sup>If  $\mathcal{O}$  has nonzero spin, then  $[\mathcal{O}(N)]^+$  also contains a factor  $\Theta = (-1)^k$ , where  $k$  is the number of Lorentz indices perpendicular to the plane  $x_d = 0$ .

The two Gram matrices  $G$  and  $\tilde{G}$  are related to each other, because there is a global conformal transformation which maps 0 and  $\infty$  to N, S respectively. If there is a negative-norm state with respect to one norm, there will be one with respect to the other. Notice however that descendants transform nonhomogeneously under this transformation, and so it is not true in general that  $\tilde{G}_{ij}$  is proportional to  $G_{ij}$  (it is only true for primaries).

Defining the norm via  $\tilde{G}$  is more intuitive because of the relation to the reflection positivity in flat space, and also because the NS conjugation acts simply in the presence of derivatives. The  $G$ -norm is related to reflection positivity on the cylinder  $S^{d-1} \times \mathbb{R}$ , and its evaluation is slightly more complicated. In compensation it satisfies various nice selection rules which will be described in Sec. II D.

In this section we will exhibit operators in free massless scalar theory which have negative  $\tilde{G}$ -norm. The existence of such operators was first observed by us in Ref. [1], and here we will give a slightly more detailed discussion. In the simplest and most explicit examples, the negative-norm operators will be Lorentz scalars. Recall that the two-point function of a Hermitian scalar operator  $\mathcal{O}(x)$  (not necessarily a conformal primary) takes the form

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = c(\mathcal{O})/|x|^{2\Delta_{\mathcal{O}}}, \quad (2.8)$$

where  $\Delta_{\mathcal{O}}$  is the scaling dimension of  $\mathcal{O}$ . For scalars the coefficient  $c(\mathcal{O})$  is precisely the  $\tilde{G}$ -norm of  $\mathcal{O}$ . This coefficient must be positive in a unitary conformal field theory (CFT), while our operators will have it negative.<sup>10</sup>

If one wants to look for negative-norm states, the evanescent operators discussed in the previous section are good candidates. Indeed, their  $\tilde{G}$ -norms  $c(\mathcal{O})$  will have zeroes at (some) integer  $d$ 's, and so assuming that these zeroes are first order,  $c(\mathcal{O})$  will be negative on one side of each zero.

Consider thus the operators  $R_n$ . Since  $R_n$  is identically zero for  $d = 1, 2, \dots, n-1$ , we know that any correlation function  $\langle R_n(x) \dots \rangle$  will vanish for these  $d$ . In particular, this will be true for the two-point function coefficient  $c(R_n)$ . Let us then write

$$c(R_n) = \hat{c}(d, n) \prod_{i=1}^{n-1} (d-i), \quad (2.9)$$

On general grounds, we can say that  $\hat{c}(d, n) > 0$  for all integer  $d \geq n$ , since the operator is then nontrivial, and the

<sup>10</sup>Since our understanding of field theory in noninteger dimensions is somewhat incomplete, one may wonder what precisely is meant by their ‘‘unitarity.’’ The conclusions of this paper are based on the natural assumption that  $c(\mathcal{O}) > 0$  is a necessary ingredient in any meaningful definition of a unitary theory.

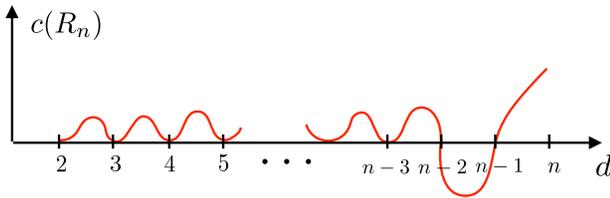


FIG. 1. The schematic behavior of  $c(R_n)$ ,  $n \geq 4$ , as a function of  $d$  in the range  $d \geq 2$ . The norm vanishes for  $d = 2, \dots, n-1$ , is negative in the interval  $n-2 < d < n-1$ , and is positive otherwise.

theory unitary. An explicit computation gives the following formula<sup>11</sup>:

$$\hat{c}(d, n) = n!(n-1)[d(n-1) + 2n]d^n(d-2)^n(d-n+3)_n, \quad (2.10)$$

where  $(x)_n \equiv \Gamma(x+n)/\Gamma(x)$  is the Pochhammer symbol.<sup>12</sup>

We thus see that  $\hat{c}(d, n)$  is positive at  $d = n-1$ , so that this zero of  $c(R_n)$  remains first order. Consequently,  $c(R_n)$  is negative for  $n-2 < d < n-1$ ; see Fig. 1. For example,  $c(R_4)$  is negative for  $2 < d < 3$ ,  $c(R_5)$  is negative for  $3 < d < 4$  etc. We conclude that the free massless scalar boson in any noninteger dimension  $d > 2$  contains negative-norm states and is therefore not unitary.<sup>13</sup>

The somewhat complicated form of the  $R_n$  operators explains why the negative-norm states have never been observed before our work [1]. We stress however that  $R_n$  are nothing but linear combinations of normal-ordered products of the derivatives of  $\phi$ . They can be obtained by taking repeated OPEs of  $M_{\mu\nu}$ . In this sense these operators are not much different from, say,  $:\phi^2:$  or the stress tensor. In particular, there is no way to throw them out from the theory without destroying its consistency.

Above we discussed the two-point function of the  $R_n$  operators. Other correlators involving  $R_n$  can be computed similarly. Just like the two-point function, these correlators will vanish at integer  $d = 1 \dots n-1$  but will be generally nonzero for other  $d$ . Here is one example, for  $n = 5$ , of a nonzero three-point function:

$$\langle \phi^2(x)\phi^3(y)R_5(0) \rangle = -240(d-4)(d-3)(d-2)^6 d \times \frac{(d+2)x^2y^2 - 3d(x \cdot y)^2}{|x|^{2+2d}|y|^{2+3d}}. \quad (2.11)$$

<sup>11</sup>We have verified this formula  $n = 2, 3, 4, 5, 6$  and conjecture that it holds for all  $n$ .

<sup>12</sup>We are working in the normalization in which  $\langle \phi(x)\phi(0) \rangle = 1/|x|^{2\Delta_\phi}$ .

<sup>13</sup>Notice the similarity with the proof that  $O(N)$  models are not unitary for noninteger  $N$  [22].

As expected, the three-point function vanishes at  $d = 1, 2, 3, 4$  when  $R_5$  becomes null.<sup>14</sup> Notice that the three-point function does not have the standard Polyakov form of the correlator of three scalar primaries, because as we observed above  $R_n$  is not a primary.

### C. Primary decomposition and negative norms

In the previous section we saw that the free massless scalar in  $d > 2$  contains negative-norm descendant states. What does this mean in terms of primaries? Clearly, there should be either

Scenario (1): negative-norm primaries above the unitarity bound,

or

Scenario (2): positive-norm primaries below the unitarity bound,

or both. Recall that for primaries above the unitarity bound the norm of the descendants is related to that of the primary by a positive factor, so that if the norm of the primary is negative, so will be the norm of the descendant. For primaries below the unitarity bound there is at least one descendant whose norm has the sign opposite to that of the primary.

To decide between these scenarios, we would like to decompose the operators  $R_n$  into a linear combination of descendants of primaries. In principle, it is a straightforward exercise to find such a decomposition for any given operator  $\mathcal{O}$ . We start by acting on  $\mathcal{O}$  with  $K_\mu$  several times until the result is zero. If, schematically,  $(K_\mu)^N \mathcal{O} = 0$ , then  $(K_\mu)^{N-1} \mathcal{O}$  is a linear combination of primaries, call them  $\mathcal{O}_i$  whose descendants  $\mathcal{D}_i$  contribute to  $\mathcal{O}$ . Subtracting these descendants with appropriate coefficients, we get  $(K_\mu)^{N-1}(\mathcal{O} - c_i \mathcal{D}_i) = 0$ . We can then move one level up and repeat the procedure recursively until the full decomposition is obtained.

In practice however it is a bit tedious to carry this out for the operators  $R_n$ , which contain many derivatives, and one has to act many times with  $K_\mu$  before hitting zero. We performed the decomposition for  $n \leq 4$ , and we will describe here the result for  $R_4$ . Its decomposition in general  $d$  includes seven terms:

$$R_4 = c_1 \square^4 \mathcal{O}^{(1)} + c_2 \square^2 \partial_\mu \partial_\nu \mathcal{O}_{\mu\nu}^{(2)} + c_3 \square^2 \mathcal{O}^{(3)} + c_4 \square \partial_\mu \partial_\nu \mathcal{O}_{\mu\nu}^{(4)} + c_5 \partial_\mu \partial_\nu \partial_\lambda \partial_\sigma \mathcal{O}_{\mu\nu\lambda\sigma}^{(5)} + c_6 \square \mathcal{O}^{(6)} + c_7 \partial_\mu \partial_\nu \mathcal{O}_{\mu\nu}^{(7)}, \quad (2.12)$$

where

$$\begin{aligned} \mathcal{O}^{(1)} &= \phi^4, & \mathcal{O}^{(3)} &= [(\partial\phi)^2]^2 + \dots, \\ \mathcal{O}^{(6)} &= (\partial\phi)^2(\partial\partial\phi)^2 + \dots \end{aligned} \quad (2.13)$$

<sup>14</sup>For  $d = 1$  we have to use the identity  $(x \cdot y)^2 = x^2 y^2$ .

are scalar primaries (the  $\dots$  terms are fixed by the primary condition),

$$\begin{aligned}\mathcal{O}^{(2)} &= \phi^2 T_{\mu\nu}, & \mathcal{O}^{(4)} &= \partial_\mu \phi \partial_\nu \phi (\partial\phi)^2 + \dots, \\ \mathcal{O}^{(7)} &= \partial_\mu \phi \partial_\nu \phi (\partial\partial\phi)^2 + \dots\end{aligned}\quad (2.14)$$

are symmetric traceless spin-two primaries (the first of these is a normal-ordered product of  $\phi^2$  and the stress tensor), and

$$\mathcal{O}^{(5)} = \partial_\mu \phi \partial_\nu \phi \partial_\lambda \phi \partial_\sigma \phi + \dots \quad (2.15)$$

is a symmetric traceless primary of spin four.

The  $\tilde{G}$ -norm of  $R_4$  has a first-order zero at  $d = 3$ , and one may ask how this agrees with its decomposition. The above primaries are normalized so that they do not contain any explicit  $d$ -dependent factors which vanish or blow up at  $d = 3$ . In this normalization, the coefficients  $c_1, \dots, c_6$  turn out to be  $O(d-3)$ , and hence the operators  $\mathcal{O}^{(1)} \dots \mathcal{O}^{(6)}$  only give a subleading  $O(d-3)^2$  contribution to the norm of  $R_4$ . On the other hand  $c_7$  remains finite as  $d \rightarrow 3$ . Therefore, at leading order in  $d-3$ , the norm of  $R_4$  is proportional to that of  $\mathcal{O}^{(7)}$ . The proportionality coefficient is expected<sup>15</sup> to be positive, since  $\mathcal{O}^{(7)}$  is well above the spin-two unitarity bound. We therefore expect that the norm of  $\mathcal{O}^{(7)}$ , as that of  $R_4$ , has a first-order zero at  $d = 3$  and is negative at  $d$  just below 3. This corresponds to the above scenario (1). We computed the norm of  $\mathcal{O}^{(7)}$  explicitly and checked that this interpretation is indeed correct. We do not present the full details, except for the fact that  $\mathcal{O}^{(7)}$  can be written in the form

$$\begin{aligned}\mathcal{O}_{\mu\nu}^{(7)} &= \text{const } \delta^{\mu_1[\nu_1} \delta^{\mu_2|\nu_2} \delta^{\mu_3|\nu_3} \delta^{\mu_4|\nu_4]} \\ &\times (\phi M_{\mu_2, \nu_2} - 9 \partial_{\mu_2} \phi \partial_{\nu_2} \phi) M_{\mu_3, \nu_3} M_{\mu_4, \nu_4} \\ &+ O(d-3),\end{aligned}\quad (2.16)$$

and we can use the same antisymmetrization argument as before to conclude that  $\mathcal{O}^{(7)}$  vanishes in  $d = 3$ .

#### D. The radial quantization Gram matrix

In the previous sections we produced a few examples of negative-norm states in the free scalar theory. We would now like to give a more systematic survey, and in particular count how many states have negative norm. From now on we will switch to using the radial quantization Gram matrix

<sup>15</sup>We say ‘‘expected’’ since strictly speaking the standard unitarity bounds have been derived for the integer dimensions. We rely on the plausible assumption that nothing unexpected happens with the unitarity bounds analytically continued to noninteger  $d$ . At any fixed level of the conformal multiplet, one could check this by an explicit computation, although we have not done this.

$G$  defined in Eq. (2.6). For the systematic analysis  $G$  is more convenient than  $\tilde{G}$  because it satisfies various nice selection rules (see below). Let us briefly review how  $G$  is constructed in practice [1, 19]. For us, the operators  $\mathcal{O}_i, \mathcal{O}_j$  in Eq. (2.6) will be normal-ordered products of  $\phi$  and its derivatives, or linear combinations thereof. The action of conjugation on  $\phi$  is defined by

$$[\phi(x)]^\dagger := |x|^{-2\Delta_\phi} \phi(Rx). \quad (2.17)$$

It is extended to derivatives by (anti)linearity:

$$[\partial_x^n \phi(x)]^\dagger := \partial_x^n \{|x|^{-2\Delta_\phi} \phi(Rx)\}. \quad (2.18)$$

It is also extended to the normal-ordered products of  $\phi$  and its derivatives in an obvious way. With these definitions the Gram matrix will be Hermitian (in fact real and symmetric if the basis  $\mathcal{O}_i$  involves only linear combinations with real coefficients). Computing it is a problem of symbolic algebra, which we realized in MATHEMATICA. We will keep using the normalization from footnote 12, in which  $\langle \phi | \phi \rangle = 1$ .

As mentioned, the radial quantization Gram matrix respects various selection rules, which give it a block structure. One basic rule is that only operators with the same  $SO(d)$  symmetry and the same scaling dimensions have nonzero inner product. This is in fact true for any theory.

Our specific theory will have another, very powerful, selection rule. To describe it, we define the  $\phi$ -type of an operator as the list of the number of derivatives for each factor  $\phi$  in the operator, sorted in increasing order. This list thus has length  $n_\phi$  defined as the number of  $\phi$ 's in the operator. When determining the  $\phi$ -type, contractions are unimportant. For example, the  $\phi$ -type of  $\phi^2$  is  $\{0, 0\}$  and the  $\phi$ -type of  $\phi \partial^2 \phi \partial \phi \partial^3 \phi$ , no matter how the indices are contracted, is  $\{0, 1, 2, 3\}$ . Clearly, if two operators have the same  $\phi$ -type they also have the same scaling dimension, although the converse is not necessarily true.

With this definition, in the free massless scalar theory, a Gram matrix element vanishes unless the two operators are of the same  $\phi$ -type. A very simple partial case of this rule is that the matrix element vanishes unless  $n_\phi(\mathcal{O}) = n_\phi(\mathcal{O}')$ . This is clear since otherwise one cannot form any contraction between  $\mathcal{O}$  and  $\mathcal{O}'$ , while the full rule may not appear totally obvious starting from the definition of  $G$  in terms of two-point functions.<sup>16</sup> This rule however directly follows from the oscillator picture of the free scalar Hilbert space in the canonical quantization, where each factor  $\partial^l \phi$  corresponds to adding an oscillator of angular momentum  $l$ ; see Ref. [1], Eq. (3.13).

<sup>16</sup>See however the discussion in Ref. [19], section C.3.

The above discussion was for general  $d$ . Let us now describe how the Gram matrix can be used to enumerate the null states in 4D, which correspond to evanescent operators when we move slightly away from 4D. This will be also relevant for the subsequent discussion of the WF fixed point.

Using our MATHEMATICA code we evaluated all matrix elements for scalar parity-even operators up to  $\Delta = 23$  in  $d = 4$ . This will be enough for the applications we have in mind. The code becomes slower for longer operators but with additional optimization one could go higher up in dimension if needed.

More precisely, we split the operators into groups having the same  $\phi$ -type. Within each group we computed the Gram matrix keeping the spacetime dimension  $d$  as a free parameter. Setting  $d$  to 4 and counting the number of zero eigenvalues we can then determine the number of null states. We then set  $d$  to  $4 - \epsilon$  and determine the eigenvalues of the Gram matrix to first order in  $\epsilon > 0$ . We found that up to the dimension that we are working all eigenvalues are either  $O(1)$  or  $O(\epsilon)$  as  $\epsilon \rightarrow 0$ . In particular there are no  $O(\epsilon^2)$  eigenvalues up to  $\Delta = 23$ .<sup>17</sup> This fact will play some role later in the WF fixed point discussion. At still higher dimensions there might well be  $O(\epsilon^2)$  eigenvalues.

Our results are summarized in Table I. The first null state occurs at  $\Delta = 15$ ; this is the operator  $R_5$ . At  $\Delta = 16$  we have two null states: the operator  $R'_5$  obtained by replacing one of the  $M$ 's by  $M'$  in  $R_5$ , and the product  $:\phi R_5:$ . In fact the normal-ordered product of any state with a null state is always a null state. The three null states at  $\Delta = 17$  are  $:\phi^2 R_5:$ ,  $:\phi R'_5:$  and  $R''_5$  obtained by two  $M \rightarrow M'$  replacements in  $R_5$ . A similar analysis can be done for  $\Delta > 17$ . We see from the table that the majority (although not all) of the null states acquire a negative norm in  $d = 4 - \epsilon$ .

As an example, let us consider the block of scalar operators of  $\phi$ -type  $\{2, 2, 2, 2, 2\}$ . The space of operators of this  $\phi$ -type is two dimensional, with the basis naturally chosen as

$$\mathcal{O}_1 = (\text{tr}M^2)(\text{tr}M^3) \quad \text{and} \quad \mathcal{O}_2 = \text{tr}M^5 \quad (2.19)$$

where we recall that  $M_{\mu\nu} = \partial_\mu \partial_\nu \phi$ . The Gram matrix restricted to this subsector is

<sup>17</sup>There is no contradiction between this statement and the fact that the  $\tilde{G}$ -norm of  $R_7$  vanishes as  $O(\epsilon^2)$  near  $d = 4$ , see Fig. 1. First of all, the  $G$ -norm discussed in this section is not proportional to the  $\tilde{G}$ -norm except for the primaries, and in fact the  $G$ -norm of  $R_7$  is  $O(\epsilon)$ . Additionally, there is another evanescent operator of the same  $\phi$ -type as  $R_7$ , namely  $:R_2 R_5:$ . These two evanescent mix at  $O(\epsilon)$ , and the resulting eigenvalues are both  $O(\epsilon)$ .

TABLE I.  $N_0$  is the number of scalar parity-even null states in  $d = 4$  with scaling dimension  $\Delta$ .  $N_-$  denotes how many of these null states become negative-norm states slightly below four dimensions (the rest of the null states acquire positive norm).

$\Delta$	15	16	17	18	19	20	21	22	23
$N_0$	1	2	3	6	12	24	46	93	181
$N_-$	1	2	3	5	10	21	41	83	160

$$\begin{aligned} \mathbf{G}_{\{2,2,2,2,2\}} &= 2(d-2)^6(d-1)d^2(d+2)(d+4) \\ &\times \begin{pmatrix} 12d^2(d^2+d+10) & 60d(d^2+2d-4) \\ 60d(d^2+2d-4) & 5(d^4+7d^3+12d^2-48d+96) \end{pmatrix}. \end{aligned} \quad (2.20)$$

Diagonalizing, we find two positive eigenvalues for  $d > 4$ . One of them crosses zero and becomes negative for  $d < 4$ . The corresponding eigenvector is in  $d = 4$  precisely the operator  $R_5$ .

The number of physical states of dimension  $\Delta$  in the massless scalar theory for integer  $d$  can be counted using the oscillator representation [16]; it grows as  $\exp(C\Delta^{1-1/d})$ .<sup>18</sup> The number of null states will grow at least that fast, since we can form high-dimension null states by forming products of low-lying nulls with high-dimension normal states. It would be interesting to obtain a more systematic counting of the null states in any integer  $d$ .

### E. General tensorial operators<sup>19</sup>

Our discussion so far has focused on scalar operators, with a brief mention of symmetric traceless tensor operators in Sec. II C. This was clearly sufficient to demonstrate the nonunitarity of the free boson in noninteger dimensions, but for completeness let us now consider more general tensorial operators.

As we mentioned above, the set of nontrivial local operators for the free boson in  $d$  dimensions is generated by symmetrized normal-ordered products of the form

$$:(\partial_{\mu_1} \dots \partial_{\mu_{n_1}} \phi)(\partial_{\nu_1} \dots \partial_{\nu_{n_2}} \phi) \dots (\partial_{\rho_1} \dots \partial_{\rho_{n_k}} \phi): \quad (2.21)$$

with  $k \geq 1$  and  $n_1 \dots n_k \geq 0$ , where some indices may be contracted modulo the sole constraint that  $\square \phi = 0$ . We would like to find a way to classify these operators into ‘‘irreducible representations of the Euclidean rotation group in  $d$  dimensions.’’ In order to do so we can (1) contract the indices with invariant tensors, and (2) symmetrize and/or antisymmetrize the uncontracted indices.

<sup>18</sup>The same growth rate is expected to hold in any CFT [23,24].

<sup>19</sup>This section is somewhat outside the main line of development and may be skipped on the first reading, but it is helpful for the discussion in Sec. V B.

Let us first consider the invariant tensors. For any *integer*  $d$  there are precisely two fundamental invariants: the inverse metric  $\delta^{\mu\nu}$  and the totally antisymmetric tensor  $\epsilon^{\mu_1 \dots \mu_d}$ . It is clear that our formalism for noninteger  $d$  can include the former but not the latter. Indeed, the theory for noninteger  $d$  should smoothly continue to *any* integer  $d$ . By taking this integer  $d$  as large as we want we can banish the epsilon tensor from appearing in any operator with finitely many  $\phi$ 's. We are therefore left with  $\delta^{\mu\nu}$  as the sole fundamental invariant.

The prescription to classify the operators is then the following. We first remove all the traces from a general tensor, which can be done in arbitrary  $d$  using  $\delta^{\mu\nu} \delta_{\mu\nu} = d$ . The indices on the resulting traceless operators are then symmetrized or antisymmetrized according to Young tableaux.<sup>20</sup> This again can be done in a  $d$ -independent manner, with one important peculiarity: the absence of an epsilon tensor implies that there is no restriction on the number of rows in the tableau. In other words, all the different Young tableaux correspond to *different* irreducible tensor structures in  $d$  dimensions. It should be clear that antisymmetrization over  $k$  indices gives an evanescent operator for any integer  $d < k$ , but a perfectly valid operator otherwise.

Finally we discuss the nontrivial reflection operator  $P$  in the  $O(d)$  group with determinant  $-1$ , which is the generalization of parity to arbitrary  $d$ . For our purposes we can define it to be a reflection in one of the  $d$  coordinates, say coordinate number 1. Under this transformation

$$P\phi(x)P = \phi(Px) \quad (2.22)$$

and therefore

$$P\partial_\mu\phi(x)P = \partial_\mu(\phi(Px)) = (-)^\mu(\partial_\mu\phi)(Px) \quad (2.23)$$

where  $(-)^mu$  is shorthand for a sign that equals  $-1$  if  $\mu$  is the reflected index and  $+1$  otherwise. This clearly extends to higher derivatives, and for any tensorial operator constructed in general  $d$  as described above we have

$$PT_{\mu_1 \dots \mu_k}(x)P = (-1)^{\mu_1} \dots (-1)^{\mu_k} T_{\mu_1 \dots \mu_k}(Px), \quad (2.24)$$

irrespective of how this tensorial operator is constructed from derivatives and  $\phi$ 's, and irrespective of the symmetrization and trace properties of the indices. Operators with the above transformation properties are canonically defined as “even” under the generalized parity operation.

We conclude that in a formalism that can be defined for arbitrary  $d$  there is no room for parity-odd operators.

<sup>20</sup>To obtain a nontrivial operator these tableaux must of course respect the permutation symmetries of the  $\phi$ 's as well as those of the derivatives on a single  $\phi$ . For example, the antisymmetric parts of both  $\partial_\mu\phi\partial_\nu\phi$  and of  $\partial_\mu\partial_\nu\phi$  clearly vanish.

Indeed, for the free scalar theory in integer dimensions a local operator is even/odd under parity precisely if we need an even/odd number of epsilon tensors to define the operator. Therefore, in our formalism the pseudoscalars are replaced with  $d$ -index antisymmetric tensors, the pseudovectors with  $(d-1)$ -index antisymmetric tensors, and so on: for every parity-odd operator in a given integer dimension there exists a parity-even operator that extends to arbitrary dimensions.<sup>21</sup>

Let us finally remark that the selection rules coming from parity conservation in integer  $d$  will now have to follow from ordinary Lorentz symmetry rules. As an example, the fact that the OPE of two scalars does not contain antisymmetric tensors also implies that no pseudoscalars appear.

### III. REVIEW OF THE WILSON-FISHER THEORY AT ONE LOOP

In the previous section, we have shown that the free massless scalar in noninteger  $d$  is not unitary. We will now turn our attention to the WF fixed point in  $d = 4 - \epsilon$  dimensions with  $\epsilon \ll 1$ . It is pretty clear that this theory will also not be unitary. Indeed, its dynamics happens in the same non-positive-definite Hilbert space as that of the free theory. Barring unlikely coincidences and decouplings, negative-norm states found in the free theory will persist in the interacting theory. This fact and its consequences will be studied in detail in Sec. IV.

In this section, we will do some preparatory work, by reviewing the lowest-order computations of the WF critical exponents. This material is completely standard. Unitarity considerations will not play any role here. The main result is that the leading anomalous dimensions are computed by diagonalizing the matrix of the OPE coefficients of the operator  $\phi^4$ , evaluated in the free 4D theory, Eq. (3.18). The reader familiar with this can proceed straight to Sec. IV.

#### A. Anomalous dimension generalities

We will recall here some well-known facts about how one computes in perturbation theory scaling dimensions of local operators at the WF fixed point (or at any other weakly coupled IR fixed point). We will work in the minimal subtraction scheme. This is all standard, see e.g. Refs. [25,26].

One starts with bare operators  $\mathcal{O}_i$  which are the free theory operators from Sec. II. When one computes perturbative corrections to correlation functions of these

<sup>21</sup>The precise recipe is the following: start with the definition of the operator using  $\phi$ 's, derivatives and invariant tensors. Now replace pairs of epsilon tensors with products of the metric, for example in two dimensions  $\epsilon^{ab}\epsilon^{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}$ . Repeat until there is a single epsilon tensor. Simply cross that one out from the definition of the operator, to obtain a parity-even operator with some newly freed indices.

operators, one encounters poles in  $\epsilon$  which are subtracted order by order to define the finite operators denoted  $[\mathcal{O}_i]$ . The bare and finite operators are related by the mixing matrix  $Z$  as follows:

$$\mathcal{O}_i = Z_i^j [\mathcal{O}_j]. \quad (3.1)$$

Then  $Z$  can be expanded order by order in  $g$ :

$$Z = \mathbb{1} + gZ_1 + \dots, \quad (3.2)$$

where the omitted terms describe higher-order contributions in  $g$ . In minimal subtraction, the matrix elements  $Z_i^j$  are fixed to be poles in  $\epsilon$ .

Of course, the action (1.1) also requires field and coupling constant renormalizations of order  $g^2$  and higher. These effects ensure that the theory reaches a fixed point at  $g = g_*$  which at leading order is given in Eq. (1.2). Apart from this, they will not play a role in our discussion.

The above renormalization prescription renders correlation functions of the operators  $[\mathcal{O}_i]$  finite for any value of  $g$ . In particular, correlators of these operators at the IR fixed point can be computed by setting  $g \rightarrow g_*$ . Notice however that the operators  $[\mathcal{O}_i]$  as defined above will usually not have well-defined IR scaling dimensions, i.e. they will be mixtures of operators with different dimensions. To get operators with well-defined IR dimensions, an additional change of basis is needed. Namely, consider the dilatation operator acting in the space of operators:

$$D = D_0 + \Gamma(g_*). \quad (3.3)$$

We split  $D$  into the classical term  $D_0$  and a matrix of anomalous dimensions  $\Gamma$ . We choose a basis of bare operators  $\mathcal{O}_i$  that have a well-defined classical dimension  $\Delta_0$  in  $d$  dimensions. In this basis  $D_0$  is diagonal<sup>22</sup>:

$$(D_0)_i^j = \Delta_{0,i} \delta_i^j. \quad (3.4)$$

By a standard renormalization group argument, the anomalous dimension matrix  $\Gamma(g)$  is related to the mixing matrix  $Z(g)$  in Eq. (3.1) as follows:

$$\frac{d}{d \log \mu} Z = Z \cdot \Gamma(g) \quad (3.5)$$

where the derivative is taken holding the bare coupling  $g_B \equiv g\mu^\epsilon$  fixed. The renormalizability of the  $\phi^4$  theory implies that  $\Gamma$  depends only on  $g$  and not on  $\epsilon$ .

In terms of the dilatation matrix  $D$ , the operators which have well-defined scaling dimensions at the fixed point are linear combinations of finite operators of the form

<sup>22</sup>This simply means that each of the basis operators have to be made of a fixed number of  $\phi$ 's and derivatives. E.g.  $\phi^4$  and  $(\partial\phi)^2$  are good basis elements. However, since the dimensions of these operators are different at  $O(\epsilon)$ , a basis containing their linear combinations would not be a good basis for our purposes.

$$c^i [\mathcal{O}_i] \quad (3.6)$$

where  $c^i$  is a left eigenvector of the matrix  $D$ ,

$$c^i D_i^j = \Delta c^j. \quad (3.7)$$

The scaling dimensions themselves are the eigenvalues  $\Delta$  corresponding to these eigenvectors.

## B. $\Gamma$ at one loop from the OPE

To compute the scaling dimensions, we need the anomalous dimension matrix. At higher orders, the only currently known systematic way of finding it is the full-blown Feynman-diagrammatic perturbation theory. In this paper we will only need  $\Gamma$  at one loop. There is then a well-known simple alternative (see e.g. Ref. [27]) which instead of Feynman diagrams uses the position space OPE, and which we will now review. For us this method is more convenient, since we have a well-developed machinery to compute the OPEs efficiently, described in detail in Ref. [1].

To avoid possible misunderstanding, we would like to stress the difference between the approach adopted here and the recent work [28–32] on how to compute the dimensions of operators at the WF fixed point in the  $\epsilon$ -expansion without any reference to the Lagrangian but just from the constraints of conformal symmetry. That work is interesting and radical although still in its infancy. Here we will be not at all radical: we will be relying as usual on the UV Lagrangian and the path integral to define the theory and do the computation, just phrasing it in the language of the OPE rather than Feynman diagrams.

Let us then compute the one-loop matrix  $Z$  using the OPE method. We will determine the matrix  $Z_1$  in Eq. (3.2); the matrix  $\Gamma$  will then follow. As is well known, in minimal subtraction operators can mix only if they have the same 4D dimension. The matrix  $Z$  is therefore block diagonal, with each block corresponding to operators with the same 4D dimension. We call these “big blocks” to distinguish them from smaller blocks that will be introduced below. Within the big block, the classical dimensions in  $d = 4 - \epsilon$  will differ by  $O(\epsilon)$ , and it will be convenient to introduce an antisymmetric matrix  $\kappa_{ij}$  measuring this difference:

$$\Delta_{0,i} - \Delta_{0,j} = \kappa_{ij} \epsilon. \quad (3.8)$$

Within a big block, we write the one-loop matrix  $Z_1$  as

$$(Z_1)_i^j = \frac{1}{\epsilon} \mu^{\kappa_{ij} \epsilon} z_i^j, \quad (3.9)$$

where the power of  $\mu$  is dictated by dimensional analysis, and  $z$  is an order-one numerical matrix which we have to compute.

Suppose that we want to determine the one-loop counterterm for a generic scalar operator  $\mathcal{O}_i$ . Consider the free theory OPE of  $\phi^4$  with this operator<sup>23</sup>:

$$\begin{aligned} \phi^4(x)\mathcal{O}_i(0) &= \sum_j C_i^j |x|^{-(\Delta_0(\phi^4)+\Delta_{0,i}-\Delta_{0,j})} \mathcal{O}_j(0) \\ &+ \text{nonscalar operators.} \end{aligned} \quad (3.10)$$

Here  $\Delta_0(\phi^4) = d - \epsilon$  is the classical dimension of the operator  $\phi^4$ . The  $C_i^j$  is the free theory OPE coefficient matrix. For simple operators it is easy to compute by counting Wick contractions. For more complicated operators it makes sense to use symbolic algebra, as we will do here following Ref. [1].

Now, the matrices  $C$  and  $z$  can be related by the following path integral argument. The leading correction to any free theory correlation function  $\langle \mathcal{O}_i(0) \dots \rangle_{\text{free}}$  is given by

$$-\frac{g\mu^\epsilon}{4!} \int d^d x \langle \mathcal{O}_i(0) \phi^4(x) \dots \rangle_{\text{free}}. \quad (3.11)$$

The pole in  $\epsilon$  that we need to subtract will come from the short-distance part of the integration region,  $x \rightarrow 0$ . In this region the OPE is valid, and we can replace  $\mathcal{O}_i(0)\phi^4(x)$  by the rhs of Eq. (3.10). Since the  $x$  integral is spherically symmetric, it picks out only the scalars, and gives rise to a factor of  $S_d = 2\pi^2 + O(\epsilon)$ . Next, we integrate over the radial direction. If the result is order  $1/\epsilon$ , then it will have to be compensated by counterterms. This occurs only if the operator  $\mathcal{O}_j$  is within the same ‘‘big block’’ as  $\mathcal{O}_i$ —a manifestation of the mentioned fact that only operators with the same 4D dimensions mix. Doing the radial integral, the pole part is then given by

$$-\frac{g}{4!} \frac{2\pi^2}{(1 - \kappa_{ij})\epsilon} C_i^j \langle \mathcal{O}_j(0) \dots \rangle_{\text{free}}, \quad (3.12)$$

where on the rhs we have a sum over all  $j$  in the big block of  $i$ .

To be a bit more precise, the OPE coefficients  $C_i^j$  depend on  $\epsilon$ , and the actual residue of the  $1/\epsilon$  pole in Eq. (3.12) is obtained by replacing  $C_i^j$  with

$$\bar{C}_i^j \equiv C_i^j|_{d=4}. \quad (3.13)$$

We cancel the pole by adding to  $\mathcal{O}_i$  the corresponding counterterm, defining

$$[\mathcal{O}_i] = \mathcal{O}_i + g\mu^{\kappa_{ij}\epsilon} \frac{\pi^2/12}{(1 - \kappa_{ij})\epsilon} \bar{C}_i^j \mathcal{O}_j. \quad (3.14)$$

<sup>23</sup>We do not distinguish between conformal primaries and descendants here.

Notice that the power of  $\mu$  on the rhs of this expression is not given by the above computation but is reconstructed by dimensional analysis. Comparing to Eq. (3.1), we obtain our final result:

$$z_i^j = \begin{cases} -\frac{\pi^2/12}{1 - \kappa_{ij}} \bar{C}_i^j, & i, j \in \text{same big block,} \\ 0, & \text{otherwise.} \end{cases} \quad (3.15)$$

To summarize: to compute the one-loop mixing matrix  $Z_1$ , it is enough to evaluate the OPE with the operator  $\phi^4$ . Moreover, we only need a small part of this OPE, namely the part which involves scalars with the same 4D dimension.

Notice that in deriving Eq. (3.12) we needed to assume that  $\kappa_{ij} \neq 1$ , because otherwise the integral is logarithmic and is not rendered finite in dimensional regularization. If this occurred it would be paradoxical, signaling a breakdown of the minimal subtraction scheme. However, this does not happen, because it can be shown that  $\bar{C}_i^j = 0$  for all such pairs of operators.

In fact, an even stronger statement holds (see Appendix A for the proof): for operators with the same 4D dimension,  $\bar{C}_i^j$  is zero unless  $\kappa_{ij} = 0$ . An equivalent form of the latter condition is that  $\mathcal{O}_j$  and  $\mathcal{O}_i$  contain the same number of  $\phi$ 's. The set of all operators which have the same 4D dimension and the same number of  $\phi$ 's will be called the ‘‘small block’’ of  $\mathcal{O}_i$ . In other words, we are saying that at one loop operators mix only within the small blocks. This fact was already noticed in Ref. [9]. Examples show that this rule does not extend to higher orders in perturbation theory.<sup>24</sup>

Using Eqs. (3.5), (3.9), and (3.15), the one-loop anomalous dimension matrix within a given big block is given by

$$\begin{aligned} \Gamma_i^j(g) &= g(-1 + \kappa_{ij})z_i^j \\ &= g \frac{\pi^2}{12} \bar{C}_i^j \quad (\text{canonical normalization}). \end{aligned} \quad (3.16)$$

The specification in parentheses reflects that in the above discussion we used the OPE coefficients computed using the canonically normalized scalar field as in Eq. (1.1). We will now pass to the CFT normalization of footnote 12 used elsewhere in the paper. The two normalizations are related by

$$\phi_{\text{can}} = k_d^{1/2} \phi_{\text{CFT}}, \quad k_d = 1/[(d-2)S_d]. \quad (3.17)$$

For the matrix of OPE coefficients of the  $\phi^4$  operator this introduces an extra factor  $k_d^2$ , which in 4D becomes  $k_4^2 = 16\pi^4$ . Taking into account this factor and substituting the critical coupling (1.2), we find

<sup>24</sup>For example  $\phi^4$  mixes with  $\partial^2(\phi^2)$  at  $O(g^2)$ .

$$\Gamma_i^j(g_*) = \frac{\epsilon}{36} \bar{C}_i^j \quad (\text{CFT normalization}), \quad (3.18)$$

which is our final master formula for the one-loop anomalous dimension matrix. To avoid any possible confusion, we emphasize that this formula is valid when operators  $i, j$  are within the same big block. In addition, within a given big block  $\bar{C}_i^j = 0$  unless the two operators are in the same small block. For  $i, j$  not in the same big block we have  $\Gamma_i^j = 0$  no matter what  $\bar{C}_i^j$  is (it may be nonzero in this case).

### C. Examples

Here are a few examples of anomalous dimensions computed in this formalism. Let us consider scalar operators that are even under the  $\mathbb{Z}_2$  symmetry  $\phi \rightarrow -\phi$ . The OPEs will be written schematically, indicating the coefficient  $\bar{C}_i^j$  in the CFT normalization but leaving the dependence on  $x$  implicit. Nonscalar operators and operators from different big blocks will be omitted. With  $\gamma_{\mathcal{O}}$  we denote the eigenvalue of the anomalous dimension matrix  $\Gamma(g_*)$  corresponding to the operator  $[\mathcal{O}]$ .

For the  $\phi^2$  operator we have

$$\phi^4 \times \phi^2 = 12\phi^2. \quad (3.19)$$

We recover the known result for the leading anomalous dimension  $\gamma_{\phi^2} = \epsilon/3$ .

We next consider the big block of operators of 4D dimension four, whose basis is formed by  $\phi^4$  and  $\partial^2\phi^2$ . The OPE turns out to be diagonal:

$$\phi^4 \times \phi^4 = 72\phi^4, \quad \phi^4 \times \partial^2\phi^2 = 12\partial^2\phi^2. \quad (3.20)$$

This is in agreement with the small block selection rule. The leading anomalous dimension  $\gamma_{\phi^4} = 2\epsilon$  agrees with the well-known result, while for  $\partial^2\phi^2$  we find  $\gamma = \gamma_{\phi^2}$  as it should be.

The big block of dimension six consists of  $\phi^6$ ,  $\partial^2\phi^4$  and  $\partial^2\partial^2\phi^2$ . The  $\phi^4$  OPE coefficients are again diagonal: 180, 72, 12, and there is only one new anomalous dimension.

The situation at dimension eight is more interesting. The big block has six operators. Three operators,  $\phi^8$ ,  $\partial^2\phi^6$  and  $\partial^2\partial^2\partial^2\phi^2$ , form three small blocks by themselves. The  $\phi^4$  OPE matrix in this sector is diagonal: 336, 180, 12. We get one new eigenvalue as expected. The three remaining operators have four derivatives and four  $\phi$ 's each, and thus form a separate small block. In the basis

$$\begin{aligned} A_1 &= \phi^2(\partial_\mu\partial_\nu\phi)^2, \\ A_2 &= \phi(\partial_\mu\partial_\nu\phi)(\partial_\mu\phi)(\partial_\nu\phi), \\ A_3 &= (\partial\phi)^2(\partial\phi)^2, \end{aligned} \quad (3.21)$$

the  $\phi^4$  OPE matrix in this sector is

$$\begin{pmatrix} 56 & 32 & 0 \\ 4 & 60 & 6 \\ 0 & 16 & 48 \end{pmatrix}. \quad (3.22)$$

The eigenvalues are 72, 52, 40, and the corresponding left eigenvectors are

$$\begin{aligned} c_1 &= (1, 4, 1), & c_2 &= (-2, 2, 3), \\ c_3 &= (1, -4, 3). \end{aligned} \quad (3.23)$$

In fact, the operator  $c_1 \cdot \mathcal{A}$  is proportional to  $\partial^2\partial^2\phi^4$  and indeed has the same anomalous dimension. The other two linear combinations,  $c_2 \cdot \mathcal{A}$  and  $c_3 \cdot \mathcal{A}$ , are not derivatives of previously considered operators, and give two new anomalous dimensions. The two eigenvalues and eigenstates we report are in agreement with Ref. [10].

## IV. NONUNITARITY AT THE WILSON-FISHER FIXED POINT

In the previous section we reviewed the standard facts about the leading anomalous dimensions at the WF fixed point. The considered examples concerned low-dimension operators. We know from Sec. II that once we move to sufficiently high dimension we will encounter evanescent operators and the related issue of nonunitarity. This raises several questions, such as:

- (i) Do the negative-norm states persist when we pass from the free theory to the WF fixed point?
- (ii) Should the above recipe for computing anomalous dimensions be modified in the presence of evanescent operators/negative norms?
- (iii) Is there any smoking-gun consequence of the non-unitarity at the level of the anomalous dimensions?

These and related questions will be discussed in this section.

### A. Robustness of negative-norm states

One potentially confusing issue is as follows. If we write evanescent operators in the most natural parametrization in terms of  $\phi$  and its derivatives, without any  $d$ -dependent factors, like in the examples considered in Sec. II, their norm vanishes for  $\epsilon \rightarrow 0$ . This may create a feeling that the evanescent operators are ‘‘fragile,’’ in the sense that even small perturbative corrections could dramatically change their properties, e.g. change the sign of their norm, or perhaps decouple them completely from the theory. For the same reason one could worry that one has to be especially careful when computing their anomalous dimensions, and that the recipe from Sec. III would require modifications. As we will now explain, this is not the case.

Basically what happens is that although correlators involving evanescent operators are small (order  $\epsilon$ ), corrections to them are still one power of  $\epsilon$  smaller. Indeed, these

corrections involve correlators of the same evanescent operator with insertions of  $\phi^4$ , which are again order  $\epsilon$ , times the coupling  $g$  which provides an extra suppression; see Eq. (3.11).

A more systematic way to explain the same is to observe that the magnitude of the norm of an operator is unphysical, and once it is nonzero we can always rescale the operators to have order-one positive or negative norm. This would require multiplying evanescent operators by a factor  $1/\sqrt{\epsilon}$ . After the rescaling all Gram matrix eigenvalues will be  $O(1)$ . In detail, the Gram matrix before the rescaling has the schematic form

$$G_{ij} \sim \begin{pmatrix} 1 & \epsilon \\ \epsilon & \epsilon \end{pmatrix}, \quad (4.1)$$

where we indicate the order of magnitude of the matrix elements; the first column stays for the normal states, whose norm remains  $O(1)$  as  $\epsilon \rightarrow 0$ , and the second for the evanescents. After the rescaling this becomes

$$G_{ij}^R \sim \begin{pmatrix} 1 & \sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 \end{pmatrix}. \quad (4.2)$$

We see that at leading order in  $\epsilon$ , the mixing between the two groups of states, described by the off-diagonal matrix elements, is a subleading effect. The orthonormal basis of the Hilbert space can be determined by ignoring this mixing and diagonalizing the Gram matrix separately in the normal and evanescent sectors. It is important for this last statement that all eigenvalues in the evanescent sector be  $O(\epsilon)$  and not higher order. As mentioned in Sec. II D, this is true up to the dimension that we are working.

Let us now apply the same rescaling to the OPE matrix  $C_i^j$ . Before rescaling we have

$$C_i^j \sim \begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix}. \quad (4.3)$$

Here we took into account that if  $\mathcal{O}_i$  is an evanescent, all operators appearing on the rhs of Eq. (3.10) are necessarily evanescents. This is because the three-point functions  $\langle (\text{evanescent})\phi^4(\text{any}) \rangle$  have to vanish as  $d \rightarrow 4$ . Thus we have a selection rule  $C_{\text{evanescent}}^{\text{normal}} = O(\epsilon)$ , reflected in Eq. (4.3). After rescaling, paying attention to the lower and upper indices, we have

$$C_i^R{}^j \sim \begin{pmatrix} 1 & \sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 \end{pmatrix}. \quad (4.4)$$

This has the same structure as the rescaled Gram matrix (4.2). We conclude that at leading order the anomalous dimensions in the evanescent and the normal operator sectors can be computed independently. Mixing between

the two sectors is a higher-order effect. Leading anomalous dimensions in the evanescent sector are generically  $O(\epsilon)$ , just as in the normal sector. They are described by the order- $\epsilon$  master formula (3.18), where  $\bar{C}_i^j$  is the OPE matrix in the evanescent sector with  $\epsilon$  set to zero.

When we work in the rescaled frame, the distinction between the normal and evanescent operators fades away (except that some evanescent operators have negative norm). For the normal operators, the most important effect of turning on the interaction is to induce anomalous dimensions which change the long-distance scaling of the correlation functions. On the other hand the numerical coefficient of the two-point correlation function obtains only finite corrections of relative size  $O(\epsilon)$  between its UV and IR values. This is manifest in the standard solution of the Callan-Symanzik equation in terms of the running coupling. This argument does not depend on the sign of the coefficient and applies also to the negative-norm states. We conclude that the states which had negative norm in the UV will remain negative-norm at the WF fixed point for  $\epsilon \ll 1$ . The negative-norm states are robust.

## B. Complex anomalous dimensions

Unitarity puts well-known constraints on the spectrum of scaling dimensions in CFTs [33,34]. In particular, the spectrum of scaling dimensions in a unitary CFT is always real and bounded from below. One expects that nonunitary interacting theories should generically violate these properties, although sometimes they are preserved even without unitarity.<sup>25</sup>

In this section we will show that, as a dramatic repercussion of nonunitarity, the WF fixed point contains states whose dimensions become complex at order  $\epsilon$ .

First of all let us see how the complex eigenvalues can appear in our language. As explained above, the order- $\epsilon$  dimensions are computed by solving the eigenvalue problem

$$v^i \bar{C}_i^j = \lambda v^j. \quad (4.5)$$

This can be solved in each small block separately, as different small blocks do not mix. We can also multiply both sides by the Gram matrix and obtain the generalized eigenvalue problem

$$v^i \bar{C}_{ij} = \lambda v^i G_{ij}. \quad (4.6)$$

Here the matrix appearing on the lhs is the matrix of three-point functions:

<sup>25</sup>One example is the  $i\phi^3$  Lee-Yang critical point in  $2 \leq d < 6$  which is  $PT$  symmetric and the spectrum is real, although the unitarity bounds are violated. For  $d = 2$  this is the  $\mathcal{M}_{2,5}$  minimal model.

TABLE II. Distribution of the evanescent states from Table I into small blocks characterized by a given  $n_\phi$ . Complex eigenvalues first occur in the small block shown in boldface.

$\Delta$	15	16	17	17	18	18	19	19	19	20	20	20
$n_\phi$	5	6	5	7	6	8	5	7	9	6	8	10
$N_0$	1	2	1	2	4	2	3	7	2	14	8	2
$N_+$	0	0	0	0	1	0	0	2	0	1	2	0
$N_-$	1	2	1	2	3	2	3	5	2	13	6	2
$\Delta$	21	21	21	21	22	22	22	22	23	23	23	23
$n_\phi$	5	7	9	11	6	8	10	12	5	7	9	11
$N_0$	7	29	8	2	43	40	8	2	17	110	44	8
$N_+$	0	3	2	0	3	5	2	0	0	13	6	2
$N_-$	7	26	6	2	40	35	6	2	17	97	38	6

$$\bar{C}_{ij} \equiv \bar{C}_i^k G_{kj} = \frac{1}{S_d} \int_{|x|=1} d^d x \langle i | \phi^4(x) | j \rangle. \quad (4.7)$$

As such it is a real and symmetric matrix in our basis, while  $\bar{C}_i^j$  did not have any symmetry properties. Notice that the Gram matrix  $\phi$ -type selection rule is stronger than the small block selection rule, and so the small blocks remain decoupled even in the generalized eigenvalue problem formulation.

Now, if the theory is unitary then the Gram matrix is positive definite. It can then be orthonormalized, reducing the generalized eigenvalue problem to a standard one which can have only real eigenvalues. Thus the anomalous dimensions will be real in unitary theories.

In our problem, the normal and evanescent sector are decoupled at order  $\epsilon$ , and the normal sector is unitary. Thus the anomalous dimensions in the normal sector are all real.

Turning to the evanescent sector, it contains negative-norm states, so there is potential for complex eigenvalues. Notice however that if a decoupled sector contains *only* negative-norm states, the eigenvalues will be again real by the same argument as above.

On the other hand generalized eigenvalue problems with both negative- and positive-norm states can lead to complex eigenvalues, provided the off-diagonal mixing is sufficiently large. As the simplest  $2 \times 2$  example, consider

$$\bar{C}_{ij} = \begin{pmatrix} 1 & \sigma \\ \sigma & 1 \end{pmatrix}, \quad G = \text{diag}\{1, -1\}, \quad (4.8)$$

which has two eigenvalues  $\pm(1 - \sigma^2)^{1/2}$ , that are complex for  $\sigma > 1$ . Complex eigenvalues, if they exist, will always form complex-conjugate pairs as in this example.

In Table II, which is a more detailed version of Table I, we show the distribution of evanescent states in small blocks. Recall that for a given  $\Delta$  the small blocks are characterized by  $n_\phi$ —the number of  $\phi$ 's inside the operator. We see that the first small block with both negative- and positive-norm evanescents is the one at  $\Delta = 18$  with

$n_\phi = 6$ . However the off-diagonal mixing turns out to be insufficient, so the anomalous dimensions are still real. This situation persists until  $\Delta = 22$ .<sup>26</sup> Finally, at  $\Delta = 23$  we find complex anomalous dimensions, which appear in the small block with  $n_\phi = 7$ .

This finding is important and we will document it here in detail. Here is how the computation proceeds:

1. Compute the Gram matrix of dimension  $\Delta = 23$  parity-even scalar operators in  $d = 4$ , and identify all the null states. The null states are zero-eigenvalue eigenvectors  $v_{(a)}^i$  of the 4D Gram matrix

$$\bar{G}_{ij} = G_{ij}|_{d=4}. \quad (4.9)$$

Here  $i$  runs over all  $\Delta = 23$  basis states, and  $a$  numbers the null states. This step is simplified by the fact that the Gram matrix satisfies the  $\phi$ -type selection rule, and is block diagonal. There are 2814 operators at  $\Delta = 23$ , and the largest Gram matrix block has size 97, corresponding to the  $\phi$ -type  $\{1, 1, 2, 2, 3, 3, 4\}$ .<sup>27</sup>

2. The previous step results in 181 null states for  $\Delta = 23$ , which split into five small blocks (see Table II). The  $n_\phi = 7$  small block contains 110 states. Although this is not strictly speaking necessary, let us check that it has both positive- and negative-norm states when we move away from 4D,

<sup>26</sup>Although this is not essential, there is another circumstance which contributes to the rarity of complex anomalous dimension. We will demonstrate it by an example. Take the small block with  $\Delta = 20$ ,  $n_\phi = 6$ , which contains one positive- and 13 negative-norm evanescents, so one could hope to find complex eigenvalues here. However in fact four of these states will be of the form  $\partial^2$  acting on the evanescents in the  $\Delta = 18$ ,  $n_\phi = 6$  small block, and thus have the same anomalous dimensions, which as we said are real because the off-diagonal mixing is small. Thus we only have ten really new evanescents in the  $\Delta = 20$ ,  $n_\phi = 6$  small block, and they are all negative-norm, so their anomalous dimensions are bound to be real.

<sup>27</sup>This 97 is not the same as the boldface 97 in Table II—in fact this block contains only 13 null states.

which is the necessary condition to find complex dimensions.

The  $(4 - \epsilon)$ -dimensional Gram matrix restricted to the evanescent sector is

$$\mathcal{G}_{ab} = v_{(a)}^i G_{ij} v_{(b)}^j. \quad (4.10)$$

Here on the rhs we have the  $(4 - \epsilon)$ -dimensional  $G$  but the 4D null state eigenvectors. I.e. we just use the same 4D null vectors  $v_{(a)}^i$  as a basis of evanescent states in  $(4 - \epsilon)$ -dimensions.<sup>28</sup> The so-defined matrix  $\mathcal{G}_{ab}$  is the lower-right corner in the schematic Gram matrix (4.1). We can then diagonalize  $\mathcal{G}_{ab}$  expanded to first order in  $\epsilon$ , and count how many of its eigenstates have positive and negative norm. In the  $n_\phi = 7$  small block we find 13 positive- and 97 negative-norm states below 4D, as reported in the table.

3. To find the anomalous dimensions, we have to diagonalize the matrix  $\bar{C}_i^j$  within each small block. As explained above, the normal and evanescent operators decouple at order  $\epsilon$ . Anomalous dimensions of the evanescent operators are computed by diagonalizing the matrix  $\bar{C}_i^j$  restricted to the corresponding subspace. This is the lower-right corner matrix in the schematic equation (4.3). This matrix can be isolated in several ways; one that we used is as follows. Take a 4D null operator described by the vector  $v_{(a)}^i$  and act on it with the 4D OPE matrix  $\bar{C}_i^j$  (which we know). We know that we should get a linear combination of evanescent. Thus we simply solve the linear equation

$$v_{(a)}^i \bar{C}_i^j = C_a^b v_{(b)}^j \quad (4.11)$$

for an unknown matrix  $C_a^b$ .

The order- $\epsilon$  anomalous dimensions of the evanescent are the eigenvalues of  $C_a^b$ . For the  $\Delta = 23$ ,  $n_\phi = 7$  sector it is a  $110 \times 110$  matrix with rational coefficients which we do not report here. The matrix can be diagonalized numerically and one sees two complex-conjugate eigenvalue pairs:

$$\lambda \approx 16.93372103 \pm 5.59469106i, 42.88540243 \pm 1.07557547i. \quad (4.12)$$

As a matter of fact the characteristic polynomial of  $C_a^b$  has a degree 26 factor which is irreducible over  $\mathbb{Q}$  and has the above complex eigenvalues as its roots. The expression for this factor is not particularly illuminating; it is included as a comment in the TeX source file of the arXiv version of this paper.

We conclude that there are two pairs of scalar operators in the Wilson-Fisher fixed point in  $4 - \epsilon$  dimensions

<sup>28</sup>There is arbitrariness in how to extend the null state from 4D, since we can always add any other state with an explicit  $\epsilon$  factor, and it will still be an evanescent. It is easy to see that this arbitrariness does not affect the results.

with complex-conjugate IR scaling dimensions. By the master formula (3.18), their dimensions are related to the above  $\lambda$ 's by

$$\Delta = 23 - \frac{7}{2}\epsilon + \gamma, \quad \gamma = \frac{\lambda}{36}\epsilon + O(\epsilon^2). \quad (4.13)$$

We have checked that the eigenoperators corresponding to these complex dimensions can be represented as total derivatives and thus are not primaries. It should be possible to determine the primaries from which they originate acting along the lines of Sec. II C, but we have not done this exercise.

### C. Comparison to the work of Kehrein *et al.*

The spectrum of Wilson-Fisher fixed points in  $4 - \epsilon$  dimensions has been previously studied in a series of remarkable papers by Kehrein *et al.* [9–11] Here we would like to review their work and to compare it with our results. In particular we would like to explain why they have not observed any complex dimensions or negative-norm states.

They considered the general  $O(N)$  case, but we will specialize to  $N = 1$  in this comparison. We will focus on the first two papers [9,10] which were devoted to the one-loop anomalous dimensions.<sup>29</sup>

Kehrein *et al.* built their composite operators by multiplying the elementary “letters”

$$\Phi^{(l,m_1,m_2)} = h_{\alpha_1 \dots \alpha_l}^{(m_1,m_2)} \partial_{\alpha_1} \dots \partial_{\alpha_l} \phi, \quad (4.14)$$

obtained by contracting the derivatives of  $\phi$  with constant symmetric traceless rank- $l$  tensors. These tensors  $h_{\alpha_1 \dots \alpha_l}^{(m_1,m_2)}$ ,  $m_1, m_2 = -\frac{1}{2}l \dots \frac{1}{2}l$  were chosen to form a basis for the  $(\frac{1}{2}l, \frac{1}{2}l)$  representation of  $so(4) = so(3) \oplus so(3)$ . This is in 4D, and when they move to  $d = 4 - \epsilon$ , they continue using the same 4D operator basis. In fact,  $\epsilon$  enters in their computation only to produce the  $1/\epsilon$  pole in the one-loop integral, and to supply the value of the fixed point coupling.

Since Kehrein *et al.* worked only with 4D tensors, it is not surprising that they have not observed the evanescent operators whose very existence is due to peculiarities of tensor algebra in noninteger dimensions. *A fortiori* they could not observe nonunitarity and complex dimensions. Their way of proceeding amounts to restricting from the start to the subspace of normal operators, and neglecting the evanescent. Within this subspace, their results about order- $\epsilon$  anomalous dimensions are correct and in agreement with ours (see e.g. examples in Sec. III C).<sup>30</sup>

<sup>29</sup>The third paper [11] studied two-loop anomalous dimensions for  $N > 1$  and for a particular subclass of operators, transforming as symmetric traceless tensors in all internal and derivative indices [i.e. without any  $O(N)$  or Lorentz index contractions].

<sup>30</sup>They computed their anomalous dimension matrix in a rather different-looking, but in fact equivalent, way.

### D. Higher-order effects

Given that at order  $\epsilon$  the evanescent operators decouple from the normal ones, it may be tempting to dismiss them as a curiosity which will never affect anything physically relevant. This would be wrong. In fact, already at order  $\epsilon^2$  the effects of evanescent operators are expected to change the anomalous dimensions of physical operators. To make this precise, we will work in a basis where we separate normal operators and evanescents and write the anomalous dimension matrix as follows:

$$\Gamma(g) = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}. \quad (4.15)$$

The blocks  $\gamma_{11}$ ,  $\gamma_{12}$  and  $\gamma_{22}$  are nonzero at order  $g$ , whereas we explained above that the order- $g$  contribution to  $\gamma_{21}$  vanishes. There is however no selection rule that forbids contributions of order  $g^2$  and higher to  $\gamma_{21}$ . This means that starting at two loops, scaling operators in the Wilson-Fisher theory are generically a mixture of normal operators and evanescents in the 4D free theory.

In order to distinguish between normal operators and evanescents at higher loops, we can apply a field redefinition to the evanescents. This means that we are modifying the renormalization scheme. The mixing matrix in this new scheme [5] is schematically given by

$$Z = \mathbb{1} + g \left( Z_{1,0} + \frac{1}{\epsilon} Z_{1,1} \right) + g^2 \left( Z_{2,0} + \frac{1}{\epsilon} Z_{2,1} + \frac{1}{\epsilon^2} Z_{2,2} \right) + O(g^3). \quad (4.16)$$

The  $1/\epsilon^n$  counterterms cancel UV divergences as before, whereas the finite counterterms  $Z_{n,0}$  are added to ensure that  $\gamma_{21}$  vanishes order by order in  $g$ . At this stage, the matrix  $\Gamma$  is block triangular, so it can be diagonalized separately for normal operators and evanescents. However, the matrix  $\gamma_{11}$  now gets an explicit contribution at order  $g^2$  from the evanescents, via the finite counterterms  $Z_{1,0}$ . An example of this phenomenon applied to four-fermion operators is described in Refs. [5,35]. It would be interesting to exhibit such two-loop contributions coming from evanescent operators in the Wilson-Fisher theory as well. We stress that in the scalar sector of WF, such contributions can appear only for operators of classical dimension  $\Delta \geq 15$ . In particular, they will not affect the dimensions of  $\phi$ ,  $\phi^2$  and  $\phi^4$  at any order in  $\epsilon$ .

However, since high-dimension operators appear in the OPE of low-dimension ones, the four-point functions of low-dimension physical operators will be affected. In Appendix B we discuss these four-point functions, and

in particular how the effects of unitarity violation could in principle be observed in them.

## V. FINAL COMMENTS

### A. Some open problems

It would be worthwhile to extend our analysis to other theories, and to more systematically enumerate all the causes of unitarity violation in noninteger dimensions. In particular, we would like to know if nonunitarity is always related to evanescent operators or whether other mechanisms exist that make the continuation of a theory to noninteger  $d$  violate unitarity. It would also be interesting to understand the implications of nonunitarity for other observables in the theory, in particular the recent work [36–40] on the extension of the  $a$  central charge and the free energy  $F$  to theories in noninteger dimensions.

### B. Spectrum continuity

The standard lore says that the Wilson-Fisher fixed point in  $d$  dimensions provides an “analytic continuation” which interpolates between the free theory in 4D and the Ising model critical point in 3D and 2D.<sup>31</sup> The nature of this analytic continuation is still not fully understood. We will now describe how one might think about it, and how the results of this paper fit into the picture.

The WF fixed points in  $d = 2, 3, 4$  are bona fide unitary CFTs, and are nonperturbatively defined.<sup>32</sup> The interpolating fixed points in noninteger dimensions were initially defined only perturbatively, by analytically continuing Feynman diagrams. It is not *a priori* clear if they make sense nonperturbatively. However in much of the literature it is tacitly assumed that they do.

For example it is assumed that the dimensions of the most important scalar operators  $\phi$ ,  $\phi^2$  are analytic functions of  $d$  which reduce to the free theory values in  $d = 4$  and to the dimensions of the leading  $\mathbb{Z}_2$ -odd and -even scalar primaries  $\sigma$ ,  $\epsilon$  in 3D and 2D.

Furthermore, it seems reasonable to assume that this is true not just for  $\phi$ ,  $\phi^2$  but that the whole CFT spectrum is well defined for noninteger  $d$  and interpolates continuously the spectra for  $d = 2, 3, 4$  (“spectrum continuity”). E.g. for the low-dimension scalar operators we expect to see curves as in Fig. 2.

The spectrum continuity hypothesis seems reasonable both for primaries and for their descendants, i.e. if a primary interpolates then its whole conformal multiplet can be assumed to interpolate. Notice that the “number of states” in a multiplet, to the extent this concept makes sense in noninteger dimensions, has to change continuously with

<sup>31</sup>Or even down to  $d = 1$ ; see footnote 1.

<sup>32</sup>For  $d = 3$  this has not yet been proved, although the evidence, recently reviewed in section IV of Ref. [41], is overwhelming.

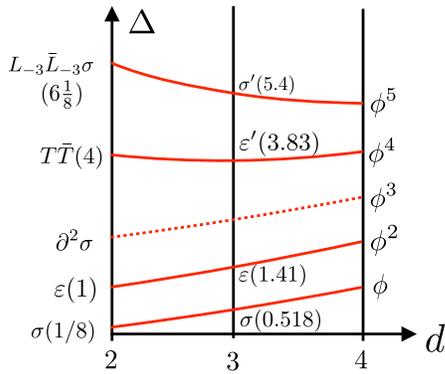


FIG. 2. Expected interpolating behavior for the low-dimension scalar operators. The exact 2D dimensions and the approximate 3D dimensions are given in parentheses. The operator  $\phi^3$  is a primary in 4D but is the descendant  $\partial^2\sigma$  below 4D. Other descendant states are not shown.

$d$ . E.g. if  $\mathcal{O}$  is a scalar primary then on the first descendant level  $\partial_\mu\mathcal{O}$  we have  $d$  states.<sup>33</sup>

There are some states which are primaries in 4D but become descendants below 4D. One such state is  $\phi^3$  which becomes a descendant of  $\phi$  below 4D; see Fig. 2. In perturbation theory this is a consequence of the equation of motion, but more abstractly it can be seen as the phenomenon of conformal multiplet recombination, which takes place whenever a field which saturates the unitarity bound in 4D moves above the bound below 4D. The same happens for the spin- $l$  currents  $\vec{\phi}\partial^l\phi - \text{traces}$ ,  $l = 4, 6, \dots$ , which are conserved in 4D but not in  $d < 4$ . In  $d = 2$  one acquires again infinitely many conserved currents as a consequence of the Virasoro algebra. How this algebra gets broken in  $d = 2 + \epsilon$  is also an interesting open question.

The results in this paper add the following ingredients to this story. First of all we have proven the existence of evanescent operators. Pictorially speaking, these appear in Fig. 2 as new “evanescent” lines that correspond to additional states beyond those in the physical theories in integer dimensions. We do not see any consistent way which would allow one to exclude these states for noninteger  $d$ . Second, the anomalous dimensions are no longer guaranteed to be real; in Fig. 2 there will be scaling dimensions of high-dimension evanescent operators with a nonzero imaginary part. These operators have vanishing norms in integer dimensions, so then (and only then) we can consistently remove these states and recover a unitary theory. The hypothesis of spectrum continuity gets modified accordingly: the most natural assumption is that spectrum

<sup>33</sup>Conceptually speaking evanescent operators naturally emerge from the framework discussed e.g. in Ref. [4], where a vector like  $x^d$  has infinitely many components in generic  $d$ , but in integer  $d$  all but the first  $d$  components decouple. For our purposes it has not been necessary to adopt this viewpoint.

continuity is still true in the enlarged Hilbert space which includes the evanescent operators with their potentially complex dimensions.

### C. Numerical bootstrap in noninteger dimensions

Let us finally discuss the effects of nonunitarity for the analysis in Ref. [42] of the Wilson-Fisher fixed point using conformal bootstrap methods [43] in  $2 < d < 4$ . Such an analysis requires only a definition of the conformal blocks in noninteger  $d$ , which is relatively straightforward: one defines the blocks as a solution of the conformal Casimir equation [44], where  $d$  enters simply as a parameter. In Refs. [45–47] it was found that scaling dimension bounds in  $d = 2$  and  $d = 3$  exhibit kinks that essentially coincide with the location of the Ising CFT. In Ref. [42] it was shown that these kinks continue to exist in noninteger  $d$  and that their location agrees with resummed perturbative results obtained from the  $4 - \epsilon$  expansions.

Reference [42] assumed that the WF theories in  $4 - \epsilon$  dimensions are unitary. This assumption was crucial for the bootstrap analysis, as it implied that all squared OPE coefficients were positive, and all scaling dimensions were real and consistent with unitarity bounds. We have now shown that these assumptions are in fact untrue. In spite of this fact, the results of Ref. [42] produced entirely reasonable-looking results for the dimensions of  $\phi$  and  $\phi^2$ . This may seem at first paradoxical. The explanation of the paradox, already given in Ref. [1], must have to do with the fact that the unitarity violation effects occur only at relatively high dimension. In a four-point function the high-dimension operators decouple exponentially fast [24] and consequently the numerical bootstrap methods are rather insensitive to their behavior. Also, there are many positive-norm operators which live at about the same dimension as the negative-norm ones, which leads to further suppression of unitarity-violating effects (see also Appendix B).

There are other cases when the numerical bootstrap applied to nonunitary theories gave reasonable results, probably for the same reason of the mildness of the unitarity-violating effects [e.g. theories with four supercharges in noninteger dimensions [48,49], and the  $O(N)$  model at  $d = 5.95$  [50]]. Surprisingly, even the 3D  $O(N)$  model with noninteger  $0 < N < 2$  seems to be in this class [51].

In other nonunitary theories the unitarity violation strength is order one and the numerical bootstrap assuming positive OPE coefficients is inadequate. This is what may have been observed for  $1 < d < 2$  in Ref. [3]. The severe truncation methods of Gliozzi and collaborators [52–54] are currently the only ones applicable in such situations.

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*Note added.*—A similar prescription is also sometimes used in multiloop QFT computations involving  $\gamma^5$  matrices [55,56]. We thank John Gracey for pointing this out to us.

## APPENDIX A: SMALL BLOCK SELECTION RULE

Let us recall the small block selection rule stated in Sec. III B: suppose that scalars  $\mathcal{A}$  and  $\mathcal{B}$  have the same 4D dimension, and that  $\phi^4(x) \times \mathcal{A}(0) \supset \mathcal{B}(0)$ . Then  $\mathcal{A}$  and  $\mathcal{B}$  have the same number of  $\phi$ 's.

We first observed this rule empirically, and then found the following proof. The same result was obtained by different means in Ref. [9].

*Proof.*—The OPE is obtained by Wick contracting  $k$  out of four  $\phi$ 's inside  $\phi^4(x)$  with  $\phi$ 's inside  $\mathcal{A}$ , expanding the remaining part of  $\phi^4(x)$  in Taylor series around zero, and contracting indices to get a scalar. The number of  $\phi$ 's inside  $\mathcal{B}$  will be the same if  $k = 2$ .

We thus have to show that any other  $k$  cannot give a scalar  $\mathcal{B}$  of the same 4D dimension as  $\mathcal{A}$ . Notice that every Wick contraction strikes out fields out of  $\mathcal{A}$ , lowering the remaining dimension by the number of contracted fields plus the number of derivatives they carried. Once Wick contractions are finished, the dimension is raised by multiplying with what remains of  $\phi^4(x)$  and by all derivatives we put on it when Taylor expanding. The challenge is to show that these two procedures cannot compensate each other.

The simplest cases are then  $k = 0$  and  $k = 4$ . For  $k = 0$  the dimension is not lowered as there are no Wick contractions, while in the second step it is raised by at least four units. Analogously, for  $k = 4$  the dimension is only lowered by Wick contractions, by at least four units, but cannot be subsequently raised since nothing is left of  $\phi^4(x)$ .

The cases  $k = 1$  and  $k = 3$  require more work. We start with  $k = 1$ . Let us write  $\mathcal{A}$  in the schematic form  $\mathcal{A}'\partial^l\phi_1$ , where  $\phi_1$  is the  $\phi$  inside  $\mathcal{A}$  which gets Wick contracted, and the notation shows that it carries  $l$  derivatives. When these derivatives fall on the propagator  $\langle\phi(x)\phi_1(0)\rangle$ , they produce a rank- $l$  traceless symmetric tensor made out of  $x$ . To generate a scalar operator, the  $\phi^3(x)$  which remained after the Wick contraction must be Taylor expanded to the same order  $l$ , or higher. The generated  $\mathcal{B}$  state is of the schematic form  $(\partial^2)^n\partial^l(\phi^3)\mathcal{A}'$ . Its 4D dimension is thus larger than that of  $\mathcal{A}$  by at least two units. The crux of this argument is that the extra dimension lost in the first step due to derivatives carried by  $\phi_1$  is necessarily recovered in the second step if we want to get a scalar state.

Finally let us consider  $k = 3$ . We write  $\mathcal{A}$  in the form  $\mathcal{A}'\partial^l(\phi_1\phi_2\phi_3)$ , where  $\phi_i$  get Wick contracted. The notation means that  $\phi_i$ 's carry  $l$  derivatives in total, but they can be arbitrarily distributed among these fields. Let  $l' \leq l$  denote the number of these derivatives which are contracted with  $\phi$ 's inside  $\mathcal{A}'$ . The rest of them are contracted among themselves. After Wick contractions, we are left with  $\phi(x)\mathcal{A}'(0)$  where  $\mathcal{A}'$  has  $l'$  open indices. To get a scalar  $\mathcal{B}$ , we must Taylor expand  $\phi(x)$  to order exactly  $l'$ . We thus get  $\mathcal{B}$  of the form  $\partial^{l'}\phi\mathcal{A}'$ . The resulting 4D dimension is lower than that of  $\mathcal{A}$  by at least two units. The equation of motion of  $\phi$  was important here, preventing us from considering the states of the form  $(\partial^2)^n\partial^l\phi\mathcal{A}'$ .

## APPENDIX B: NONUNITARITY AND FOUR-POINT FUNCTIONS

The nonunitarity of the theory in noninteger  $d$  is intrinsically linked to operators of high dimension. However, because of the OPE, we should be able to observe the nonunitarity also by considering higher-point correlation functions of “simple” low-dimension operators like  $\phi^2(x)$  and  $\phi^3(x)$ . In this appendix we briefly discuss the consequences of nonunitarity in the specific example of a four-point function of identical Hermitian scalar operators  $\mathcal{O}(x)$ .

Conformal invariance dictates that

$$\langle\mathcal{O}(x_1)\dots\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_{\mathcal{O}}}x_{34}^{2\Delta_{\mathcal{O}}}}G(u, v) \quad (\text{B1})$$

where  $u$  and  $v$  are the familiar cross ratios. The appearance of a primary operator  $\mathcal{K}(x)$  of dimension  $\Delta$  and spin  $\ell$  in the  $\mathcal{O}(x_1)\mathcal{O}(x_2)$  OPE leads to a term in  $G(u, v)$  of the form

$$G(u, v) = \dots + au^{(\Delta-\ell)/2}(1-v)^\ell + \dots \quad (\text{B2})$$

which is the leading term corresponding to this multiplet as  $u \rightarrow 0$  and (subsequently)  $v \rightarrow 1$ . If  $\mathcal{K}(x)$  is Hermitian with a positive/negative norm then  $a$  is positive/negative. If  $\mathcal{K}(x)$  is not Hermitian then its conjugate also appears in the OPE, with conjugate coefficient, leading to

$$\begin{aligned} G(u, v) &= \dots + (au^{(\Delta-\ell)/2} + \bar{a}u^{(\bar{\Delta}-\ell)/2})(1-v)^\ell + \dots \\ &= \dots + 2|a|u^{(\Delta_1-\ell)/2} \cos(\Delta_2 \log(u)/2 + \theta) \\ &\quad \times (1-v)^\ell + \dots \end{aligned} \quad (\text{B3})$$

with  $\Delta = \Delta_1 + i\Delta_2$  and  $a = |a|\exp(i\theta)$ . Clearly, both negative-norm states as well as complex dimensions lead to very distinct behavior in the OPE limit.

The nonunitarity might be harder to detect in perturbation theory. In that case both the coefficient  $a$  and the dimension  $\Delta$  will have perturbative expansions. We write

$$\begin{aligned} a &= a^0 + \delta a, \\ \Delta &= \Delta^0 + \gamma. \end{aligned} \quad (\text{B4})$$

Unfortunately there are in general other primary operators with the same spin and  $\Delta^0$  that appear in the conformal block decomposition. Let us assume that there are  $K$  of them, and denote them with a subscript  $i$ . We then find

$$G(u, v) = \cdots + u^{(\Delta^0 - \ell)/2} (1 - v)^\ell \sum_{n=0}^{\infty} X_n \log(u)^n + \cdots \quad (\text{B5})$$

with

$$X_n = \sum_{i=1}^K (a_i^0 + \delta a_i) (\gamma_i)^n. \quad (\text{B6})$$

In the epsilon expansion the  $\gamma_i$  are  $O(\epsilon)$  and therefore the coefficients  $X_n$  are  $O(\epsilon^n)$ . On the right-hand side of Eq. (B6) we have  $2K$  independent parameters, so in general the  $X_{n \geq 2K}$  can be expressed in terms of the  $X_{0 \leq n < 2K}$ . Conversely, in order to resolve even the one-loop corrections to  $\delta a_i$  and  $\gamma_i$  we need to know the leading-order behavior of  $X_{0 \leq n < 2K}$ , which requires knowledge of the correction at order  $\epsilon^{2K-1}$ . We conclude that the nonunitarity might show up only at relatively high loop order in the four-point functions.

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