Testing Independence Based on Bernstein Empirical Copula and Copula Density

M. Belalia [a], T. Bouezmarni [a*], F.C. Lemyre [a] and A. Taamouti [b]

[a]*Département de mathématiques, Université de Sherbrooke, Canada;* [b]*Durham University Business School, UK*

In this paper we provide three nonparametric tests of independence between continuous random variables based on the Bernstein copula distribution function and the Bernstein copula density function. The first test is constructed based on a Cramér-von Mises divergence-type functional based on the empirical Bernstein copula process. The two other tests are based on the Bernstein copula density and use Cramér-von Mises and Kullback-Leibler divergence-type functionals, respectively. Furthermore, we study the asymptotic null distribution of each of these test statistics. Finally, we consider a Monte Carlo experiment to investigate the performance of our tests. In particular we examine their size and power which we compare with those of the classical nonparametric tests that are based on the empirical distribution function.

Keywords: Bernstein empirical copula; Copula density; Cramér–von Mises statistic; Kullback-Leibler divergence-type; Independence test.

1. Introduction

Testing for independence between random variables is important in statistics, economics, finance and other disciplines. In economics, tests of independence are useful to detect possible economic causal effects that can be of great importance for policy-makers. In finance, identifying the dependence between different asset prices (returns) is essential for risk management and portfolio selection. Standard tests of independence are given by the usual T-test and F-test that are defined in the context of linear regression models. However, these tests are only appropriate for testing independence in Gaussian models, thus they might fail to capture nonlinear dependence. With the recent growing interest in nonlinear dependence, it is not surprising that there has been a search for alternative dependence measures and tests of independence. In this paper we propose three nonparametric tests

*Corresponding author. Email: Taoufik.Bouezmarni@USherbrooke.ca

of independence. These tests are model-free, hence they can be used to detect both linear and nonlinear dependence.

Nonparametric tests of independence have recently gained momentum. In particular, several statistical procedures have been proposed to test for the independence between two continuous random variables X and Y. Most classical tests of independence were initially based on some measures of dependence such as the Pearson linear correlation coefficient ρ, which takes the value 0 under the null hypothesis of no correlation. Other tests of independence have been constructed using other popular measures of dependence that are based on ranks. The rank-based measures of dependence do not depend on the marginal distributions. The most used rank-based measures are Kendall's tau and Spearman's rho. The independence tests that are based on Kendall's tau (resp. Spearman's rho) were investigated by Prokhorov (2001) (resp. Borkowf (2002)). These tests are usually inconsistent, which means that under some alternatives their power functions do not tend to one as the sample size tend to infinity.

To overcome the inconsistency problem of the above tests, Blum, Kiefer, and Rosenblatt (1961) were among the first to use nonparametric test statistics based on comparison of empirical distribution functions. For bivariate random variables X and Y, Blum et al. (1961) use a Cramér–von Mises distance to compare the joint empirical distribution function of (X, Y); say $H_n(x, y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i \leq x, Y_i \leq y)$, with the product of its corresponding marginal empirical distributions, i.e., $F_n(x) = H_n(x, \infty)$ and $G_n(y) = H_n(\infty, y)$. Thereafter, other researchers have considered using empirical characteristic functions; for review see Feuerverger (1993) and Bilodeau and Lafaye de Micheaux (2005), among others.

When the marginal distributions of a random vector $\mathbf{X} = (X_1, ..., X_d)$ are continuous, Sklar (1959) has shown that there exists a unique distribution function C [hereafter copula function] with uniform marginal distributions, such as

$$H(x_1, ..., x_d) = C\left(F_1(x_1), ..., F_d(x_d)\right),$$

for $x_1, ..., x_d \in \mathbb{R}^d$.[1] Under the null hypothesis of independence between the components of $\mathbf{X}$, the copula function is equal to the independent copula C_π, which is defined as $C_\pi(\mathbf{u}) = C_\pi(u_1, ..., u_d) = \prod_{j=1}^{d} u_j$, for $\mathbf{u} \in [0, 1]^d$, i.e.,

$$\mathcal{H}_0 : C(\mathbf{u}) = C_\pi(\mathbf{u}) \equiv \prod_{j=1}^{d} u_j, \text{ for } \mathbf{u} \in [0, 1]^d. \tag{1}$$

The tests based on the distribution and characteristic functions discussed above have inspired Dugué (1975), Deheuvels (1981a,b,c), Ghoudi, Kulperger, and Rémillard (2001), and Genest and Rémillard (2004) to construct tests of mutual independence between the

[1]For more details on copula theory, the readers are referred to an excellent book by Nelsen (2006)

components of $\mathbf{X}$ based on the observations $\mathbf{X}_i = (X_{i,1}, ..., X_{i,d})$, for $i = 1, .., n$, and using the test statistic

$$S_n = \int_{[0,1]^d} n \left\{ C_n(\mathbf{u}) - \prod_{j=1}^{d} u_j \right\}^2 d\mathbf{u}, \tag{2}$$

where $C_n(\mathbf{u})$ is the empirical copula originally proposed by Deheuvels (1979) and defined as

$$C_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \mathbb{I} \{V_{i;j} \leq u_j\}, \text{ for } \mathbf{u} = (u_1, ..., u_d) \in [0,1]^d, \tag{3}$$

where $\mathbb{I}\{.\}$ is an indicator function and $V_{i;j} = F_{j;n}(X_{i,j})$, for $j = 1, ..., d$, with $F_{j;n}(.)$ is the empirical cumulative distribution function of the component $X_{i,j}$, for $i = 1, \ldots, n$. An interesting aspect of the above test statistic is that, under the null of mutual independence, the empirical process $\mathbb{C}_n(\mathbf{u}) = \sqrt{n}\,(C_n(\mathbf{u}) - C_\pi(\mathbf{u}))$ can be decomposed, using the Möbius transform, into $2^d - d - 1$ sub-processes $\sqrt{n}\mathcal{M}_A(C_n)$, for $A \subseteq \{1, ..., d\}$ and $|A| > 1$, that converge jointly to tight centred mutually independent Gaussian processes; see Blum et al. (1961), Rota (1964) and Genest and Rémillard (2004). However, this test fails when the dependence happens only at the tails. For example, as we will see in Section 5, when the data are generated from Student copula with Kendall's tau equal to 0 and degree of freedom equal to 2, the power of the test which is based on the empirical copula is low. This indicates that the empirical copula-based test is not able to detect tail dependence. In general, this test does not perform well in term of power in the presence of weak dependencies.

In this paper, we propose several nonparametric copula-based tests for independence that are easy to implement and provide a better power compared to the empirical copula-based test. The first test is a Cramér-von Mises-type test that we construct using Bernstein empirical copula. Bernstein empirical copula was first studied by Sancetta and Satchell (2004) for i.i.d. data, who showed that, under some regularity conditions, any copula function can be approximated by a Bernstein copula. Recently, Janssen, Swanepoel, and Veraverbeke (2012) have shown that the Bernstein empirical copula outperforms the classical empirical copula estimator. This latter result has motivated us to use the Bernstein copula function, instead of the standard empirical copula, for testing the null hypothesis in (1). For weak dependencies, our results show that the test based on Bernstein empirical copula outperforms the empirical copula-based test. However, the two tests fail in term of power when the null hypothesis is for example a Student copula with zero Kendall's tau and small degree of freedom. The difficulty of distinguishing between the independent copula and Student copula with zero Kendall's tau and small degree of freedom, illustrated in Figure 1 and discussed in Section 3.2, may explain the low power of nonparametric copula distribution-based tests.

To overcome the above problem, we introduce two other nonparametric tests based on Bernstein empirical copula density. Bouezmarni, Rombouts, and Taamouti (2010) have studied the Bernstein copula density estimator and derived its asymptotic properties under dependent data. These properties have recently been reinvestigated in Janssen, Swanepoel, and Veraverbeke (2014). The motivation for using Bernstein copula density in the construction of our tests is illustrated in Figure 1, which shows that the copula density is flexible in terms of detecting the independence between the variables of interest. In particular, the shape of the copula density changes according to the type and degree of dependencies. Thus, our second test is a Cramér-von Mises-type test which is defined in terms of Bernstein copula density estimator. The third test that we propose is based on Kullback-Leibler divergence which is originally defined in terms of probability density functions. This divergence can be rewritten in terms of copula density, see Blumentritt and Schmid (2012). Consequently, the third test is a Kullback-Leibler divergence-type test that we construct based on Bernstein copula density estimator. Our results show that these two tests outperform both the Bernstein copula and empirical copula-based tests, and are able to detect the weak dependencies and the dependence that happens at the extreme regions of the Student copula.

Furthermore, we establish the asymptotic distribution of each of these tests under the null hypothesis of independence, and we show their consistency under a fixed alternative. Finally, we run a Monte Carlo experiment to investigate the performance of these tests. In particular, we examine and compare their empirical size and power to those of nonparametric test which is based on the empirical copula process considered in Deheuvels (1981c), Genest, Quessy, and Rémillard (2006), and Kojadinovic and Holmes (2009).

The remainder of the paper is organized as follows. In Section 2, we provide the definition of Bernstein copula distribution and its properties. Thereafter, we define the process of Bernstein copula $\{\mathbb{B}_{k,n}(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$ that we use to construct our first test of independence. In Section 3, we define the Bernstein copula density that we use to build our second test of independence based on Cramér-von Mises divergence. Section 4 is devoted to our third nonparametric test of independence that we construct based on Kullback-Liebler divergence which we define in terms of Bernstein copula density. We establish the asymptotic distribution of each of these test statistics under the null, and we show their consistency under a fixed alternative. Section 5 reports the results of a Monte Carlo simulation study to illustrate the performance (empirical size and power) of the proposed test statistics. We conclude in Section 6. The proofs of main theoretical results and some technical computations are presented in Appendix A and B, respectively.

2. Test of independence using Bernstein copula

2.1. *Bernstein copula distribution*

In this section, we define the estimator of Bernstein copula distribution and we discuss its asymptotic properties. This estimator will be used to build the first test of independence. Sancetta and Satchell (2004) were the first to apply a Bernstein polynomial for the estimation of copulas. The Bernstein copula estimator is given by

$$C_{k,n}(\mathbf{u}) = \sum_{v_1=0}^{k} \dots \sum_{v_d=0}^{k} C_n\left(\frac{v_1}{k}, \dots, \frac{v_d}{k}\right) \prod_{j=1}^{d} P_{v_j,k}(u_j), \text{ for } \mathbf{u} = (u_1, \dots, u_d) \in [0,1]^d, \tag{4}$$

where $C_n(.)$ is the empirical copula defined in Equation (3), $P_{v_j,k}(.)$ is the binomial probability mass function with parameters v_j and k, and k is an integer that represents a bandwidth parameter and depends on the sample size n. Janssen et al. (2012) have studied the asymptotic properties (almost sure consistency and asymptotic normality) of the estimator in (4). In particular, they provided its asymptotic bias and variance and showed that this estimator outperforms the empirical copula in terms of mean squared error.

We now define the following empirical Bernstein copula process under the null hypothesis of independence:

$$\mathbb{B}_{k,n}(\mathbf{u}) = n^{1/2}\left\{C_{k,n}(\mathbf{u}) - C_\pi(\mathbf{u})\right\}, \text{ for } \mathbf{u} \in [0,1]^d, \tag{5}$$

where $C_\pi(\mathbf{u})$ is the independent copula function defined in Equation (1). The following Lemma from Janssen et al. (2012) states the weak convergence of the process $\mathbb{B}_{k,n}$ under $\mathcal{H}_0$ in (1). It will be used to establish the asymptotic distribution of our first test of independence presented in Section 2.2.

Lemma 1 (Janssen et al. (2012)) *Suppose that $k \to \infty$ as $n \to \infty$. Then, under $\mathcal{H}_0$, the process $\mathbb{B}_{k,n}$ converges weakly to Gaussian process, $\mathbb{C}_\pi(\mathbf{u})$, with mean zero and covariance function given by:*

$$\mathrm{E}\left[\left\{\mathbb{I}(U_1 \le u_1, \dots, U_d \le u_d) - C_\pi(\mathbf{u}) - \sum_{j=1}^{d} C_{\pi j}(\mathbf{u})\{\mathbb{I}(U_j \le u_j) - u_j\}\right\}\right.$$
$$\left. \times \left\{\mathbb{I}(U_1 \le v_1, \dots, U_d \le v_d) - C_\pi(\mathbf{v}) - \sum_{j=1}^{d} C_{\pi j}(\mathbf{v})\{\mathbb{I}(U_j \le v_j) - v_j\}\right\}\right],$$

where U_j, for $j = 1, \dots, d$, are i.i.d. $U[0,1]$, $C_{\pi j}(\mathbf{u}) = \prod_{i \neq j} u_i$, and $\mathbb{I}(.)$ is an indicator function.

2.2. *Test of independence*

The empirical Bernstein copula process in (5) will be used to construct the test statistic of our first nonparametric test of independence. For a given sample $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$, a convenient way for testing $\mathcal{H}_0$ in (1) is by measuring the distance between the Bernstein empirical copula $C_{k,n}(\mathbf{u})$ and the independent copula function C_π in (1). This distance can be measured using a Cramér–von Mises divergence that leads to the following test statistic:

$$T_n = n \int_{[0,1]^d} \left[C_{k,n}(\mathbf{u}) - \prod_{j=1}^{d} u_j \right]^2 d\mathbf{u} = n \int_{[0,1]^d} \mathbb{B}_{k,n}^2(\mathbf{u}) d\mathbf{u}. \tag{6}$$

Other test statistics can be obtained using different criteria such as the one used in Kolmogorov-Smirnov test statistic. We can also consider integrating with respect to the Lebesgue-Steiltjes measure $dC_n(\mathbf{u})$, but under the null hypothesis H_0 this should lead to similar result as the test statistic in Equation (6). The following proposition provides an explicit expression for the test statistic T_n in (6) [see the proof of Proposition 1 in Appendix A].

PROPOSITION 1 *If we note* $\sum_{v_1=0}^{k} \ldots \sum_{v_d=0}^{k} \sum_{s_1=0}^{k} \ldots \sum_{s_d=0}^{k} = \sum_{(v,s)}$, *then*

$$T_n = n \sum_{(v,s)} C_n\left(\frac{v_1}{k}, ..., \frac{v_d}{k}\right) C_n\left(\frac{s_1}{k}, ..., \frac{s_d}{k}\right) \prod_{j=1}^{d} \binom{k}{v_j} \binom{k}{s_j} \beta(v_j + s_j + 1, 2k - v_j - s_j + 1)$$
$$- 2n \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k} C_n\left(\frac{v_1}{k}, ..., \frac{v_d}{k}\right) \prod_{j=1}^{d} \binom{k}{v_j} \beta(v_j + 2, k - v_j + 1) + \frac{n}{3^d},$$

where $\beta(.,.)$ *is the beta function which is defined as* $\beta(w_1, w_2) = \int_0^1 t^{w_1-1} (1-t)^{w_2-1} dt$, *for* w_1 *and* w_2 *two positive integers.*

The next result that follows from Lemma 1 and the continuous mapping theorem establishes the asymptotic distribution of test statistic T_n.

PROPOSITION 2 *Suppose that* $k \to \infty$ *as* $n \to \infty$. *Then, under the null hypothesis of independence* $\mathcal{H}_0$ *in* (1), *the test statistic* T_n *in (6) converges in distribution to the following integral of a Gaussian process:*

$$\int_{[0,1]^d} \mathbb{C}_\pi^2(\mathbf{u}) d\mathbf{u},$$

where the process $\mathbb{C}_\pi(\mathbf{u})$ *is defined in Lemma 1.*

The asymptotic distribution of T_n in Proposition 2 can be used to make a decision about rejecting or failing to reject $\mathcal{H}_0$. A Monte Carlo simulation-based approach can also be used to simulate the distribution of the test statistic T_n under the null hypothesis H_0. The latter approach consists in generating several samples under H_0, i.e., we generate random vectors $[0,1]^d$ under the null hypothesis of independence and for each of these samples we calculate the test statistic T_n. Thereafter and for a given significance level $\alpha \in (0,1)$, we compute the (1-α)-*th* empirical quantile of the simulated distribution of the test statistic T_n. We then reject the null hypothesis of independence if the observed test statistic, computed using the observed data, is greater than the calculated (1-α)-*th* quantile. In finite sample settings, our simulation results suggest that a Monte Carlo-simulation based approach provides a better approximation for the distribution of T_n compared to the asymptotic distribution. This means that it is better to use critical values (p-values) that are calculated using Monte Carlo simulation instead of the ones that come from the asymptotic distribution.

We next establish the consistency of our first test for a fixed alternative [see the proof of Proposition 3 in Appendix A].

PROPOSITION 3 *Suppose that $k \to \infty$ as $n \to \infty$. Then, the test based on the test statistic T_n in (6) is consistent for any bounded copula density c such that*

$$\int \left(C(\mathbf{u}) - \prod_{j=1}^{d} u_j \right)^2 d\mathbf{u} > 0.$$

3. Test of independence using Bernstein copula density

3.1. *Bernstein copula density*

In this section, we define the estimator of Bernstein copula density that we will use to build our second nonparametric test of independence. Before doing so, let us first recall the definition of copula density using copula distribution. If it exists, the copula density, denoted by c, is defined as follows:

$$c(\mathbf{u}) = \partial^d C(\mathbf{u}) / \partial u_1 ... \partial u_d, \tag{7}$$

where C is the copula distribution.

Now, from Equation (7) and since the Bernstein copula distribution introduced in Section 2 is absolutely continuous, the Bernstein copula density is defined as follows:

$$c_k(\mathbf{u}) = \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k} C\left(\frac{v_1}{k}, ..., \frac{v_d}{k}\right) \prod_{j=1}^{d} P'_{v_j,k}(u_j),$$

where $P'_{v_j,k}(u)$ is the derivative of the binomial probability function $P_{v_j,k}(u)$ with respect to u. Thus, the estimator of Bernstein copula density is given by

$$c_{k,n}(\mathbf{u}) = \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k} C_n\left(\frac{v_1}{k}, ..., \frac{v_d}{k}\right) \prod_{j=1}^{d} P'_{v_j,k}(u_j), \tag{8}$$

where $C_n(.)$ is the empirical copula distribution. From Bouezmarni et al. (2010), the Bernstein copula density estimator can be rewritten as follows:

$$c_{k,n}(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} K_k\left(\mathbf{u}, \mathbf{V}_i\right), \text{ for } \mathbf{u} \in [0,1]^d, \tag{9}$$

with

$$K_k(\mathbf{u}, \mathbf{V}_i) = k^d \sum_{\nu_1=1}^{k-1} ... \sum_{\nu_d=1}^{k-1} \mathbb{I}\{\mathbf{V}_i \in A_k(\nu)\} \prod_{j=1}^{d} P_{\nu_j,k-1}(u_j),$$

where $P_{\nu_j,k-1}(.)$ is the binomial probability mass function with parameters ν_j and $k-1$, $\mathbf{V}_i = (F_{1;n}(X_{i,1}), ..., F_{d;n}(X_{i,d}))$, with $F_{j;n}(.)$, for $j = 1, ..., d$, the empirical distribution based on $X_{i,j}$, for $i = 1, \ldots, n$, and $A_k(\nu) = \left[\frac{\nu_1}{k}, \frac{\nu_1+1}{k}\right] \times ... \times \left[\frac{\nu_d}{k}, \frac{\nu_d+1}{k}\right]$, with k an integer that plays the role of bandwidth parameter.

The Bernstein copula density estimator in (9) is proposed and investigated in Sancetta and Satchell (2004) for i.i.d. data. Later, Bouezmarni et al. (2010) have used a Bernstein polynomial to estimate the copula density for time series data. They provided the asymptotic bias and variance, uniform a.s. convergence, and asymptotic normality of the estimator of Bernstein copula density for α-mixing data. Recently, Janssen et al. (2014) have reinvestigated this estimator by establishing its asymptotic normality under i.i.d. data.

3.2. *Test of independence*

We will now use the estimator of Bernstein copula density in Equation (9) to define the test statistic of our second nonparametric test of independence. Before doing so, observe that testing the null hypothesis of independence is equivalent to testing

$$\mathcal{H}_0 : c(\mathbf{u}) = 1, \qquad \mathbf{u} \in [0,1]^d.$$

To test the above null hypothesis, we consider the following Cramér–von Mises-type test statistic

$$I_n(\mathbf{u}) = \int_{[0,1]^d} (c_{k,n}(\mathbf{u}) - 1)^2 d\mathbf{u}, \tag{10}$$

where $c_{k,n}(\mathbf{u})$ is the Bernstein copula density estimator in Equation (9).

As mentioned in the introduction, building tests of independence based on Bernstein copula *density* instead of Bernstein copula *distribution* is motivated by the fact that the copula density is able to capture the dependence even when the Kendall's tau coefficient is small or equal to zero. For example, it is straightforward to see that when Kendall's tau is equal to zero, one can not distinguish between the Student copula distribution and the independent copula. However, it is easier to distinguish between their corresponding copula density functions. For example, if we consider a Student's probability density function $t_{\nu+1}$with the number of degrees of freedom equal to $\nu = 2$ and Kendall's tau τ, then the lower/upper tail-dependence coefficient of the Student copula density is equal to $\lambda = 2t_{\nu+1}(-\sqrt{1+\nu}\sqrt{1-\tau}/\sqrt{1+\tau})$. Hence, even if we take Kendall's tau equal to zero, the tail-dependence coefficient λ will be equal to 0.1816901, thus different from zero. This situation is illustrated in Figure 1 where Kendall's tau is taken to be equal to zero.

Now, to establish the asymptotic distribution of the test statistic I_n under the null $\mathcal{H}_0$, we need to introduce the following additional term. For any integers v_1 and v_2 such that $0 \leq v_1, v_2 \leq k-1$, we define

$$\begin{aligned}\Gamma_k(v_1, v_2) &= \int_0^1 P_{v_1,k-1}(u) P_{v_2,k-1}(u)\, du \\ &= \binom{k-1}{v_1}\binom{k-1}{v_2}\beta(v_1+v_2+1, 2k-1-v_1-v_2).\end{aligned} \tag{11}$$

The following proposition provides a practical expression for the test statistic I_n in the bivariate case [see the proof of Proposition 4 in Appendix A].

Proposition 4 *Using similar notations to those in Proposition 1, the test statistic in (10) can be rewritten as follows:*

$$I_n(\mathbf{u}) = k^4 \sum_{\substack{v_1, v_1'=0 \\ v_2, v_2'=0}}^{k-1} \Upsilon_k(v_1, v_2)\Upsilon_k(v_1', v_2')\Gamma_k(v_1, v_1')\Gamma_k(v_2, v_2') - 1,$$

where $\Gamma_k(.,.)$ *is defined in Equation (11) and* $\Upsilon_k(v_1, v_2) = C_n\left(\frac{v_1+1}{k}, \frac{v_2+1}{k}\right) - C_n\left(\frac{v_1}{k}, \frac{v_2+1}{k}\right) - C_n\left(\frac{v_1+1}{k}, \frac{v_2}{k}\right) + C_n\left(\frac{v_1}{k}, \frac{v_2}{k}\right)$, *with* $C_n(.,.)$ *denotes the empirical copula.*

The following theorem provides the asymptotic distribution of the test statistic I_n under $\mathcal{H}_0$ [see the proof of Theorem 1 in Appendix A].

Theorem 1 *Suppose that* $k \to \infty$ *together with* $n^{-1/2}k^{3d/4}\log\log^2(n) \to 0$ *when* $n \to \infty$. *Then, under* $\mathcal{H}_0$, *we have*

$$\mathcal{I}_{n,k} := nk^{-d}\left\{\frac{I_n - 2^{-d}\pi^{d/2}n^{-1}k^{d/2}}{2^{1/2}\sqrt{\left\{\sum_{v_1,v_2=0}^{k-1}\Gamma_k(v_1,v_2)^2\right\}^d - k^{-2d}}}\right\} \xrightarrow{d} \mathcal{N}(0,1),$$

where I_n and $\Gamma_k(.,.)$ are defined in Equations (10) and (11), respectively.

The proof of the following Corollary can be found in Appendix A.

COROLLARY 1 *Suppose that the assumptions of Theorem 1 are satisfied. Then, there exists a constant $R > 0$ such that*

$$nk^{-d/4}\left\{\frac{I_n - 2^{-d}\pi^{d/2}n^{-1}k^{d/2}}{R^d\sqrt{2}}\right\} \xrightarrow{d} \mathcal{N}(0,1),$$

where I_n is defined in Equation (10).

As for our first test, our simulation results suggest that it is better to use a Monte Carlo simulation-based approach, instead of the asymptotic distribution, for the calculation of critical values (p-values) of the test statistic I_n. A brief description of Monte Carlo simulation approach can be found at the end of Section 2.2. We next establish the consistency of our second test based on the test statistic I_n [see the proof of Proposition 5 in Appendix A].

PROPOSITION 5 *Assume that $k \to \infty$ together with $n^{-1/2}k^{3d/4}\log\log^2(n) \to 0$ when $n \to \infty$. Then, the test based on the test statistic I_n in (10) is consistent for any bounded copula density c such that*

$$\int (c(\mathbf{u}) - 1)^2 \, d\mathbf{u} > 0.$$

4. Test of independence based on Kullback-Leibler divergence

4.1. *Measure of dependence*

Relative entropy, also known as the Kullback-Leibler divergence, is a measure of multivariate association which is originally defined in terms of probability density functions. Following Blumentritt and Schmid (2012), we rewrite the Kullback-Leibler measure in terms of copula density to disentangle the dependence structure from the marginal distributions. Blumentritt and Schmid (2012) propose an estimator for Kullback-Leibler measure of dependence using the Bernstein copula density estimator. Since the latter is guaranteed to be non-negative, this helps avoid having negative values inside the logarithmic function of the Kullback-Leibler distance. Furthermore, there is no boundary bias problem when we use the Bernstein copula density estimator because by smoothing with beta densities this estimator does not assign weights outside its support.

We now review the theoretical aspects of the above measure. Joe et al. (1987), Joe (1989a), and Joe (1989b) have introduced relative entropy as a measure of multivariate association for the random vector $\mathbf{X}$. The relative entropy is defined as

$$\delta(c) = \int_{\mathbb{R}^d} \log\left[\frac{f(\mathbf{x})}{\prod_{i=1}^d f_i(x_i)}\right] f(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{12}$$

where f is the joint probability density of $\mathbf{X}$ and f_i is the marginal probability density of its component X_i, for $i = 1, ..., d$. According to Sklar (1959), the density function of $\mathbf{X}$ can be expressed as

$$f(x_1, ..., x_d) = c\,(F_1(x_1), ..., F_d(x_d)) \prod_{i=1}^{d} f_i(x_i), \tag{13}$$

where c is the density copula function. Using Equation (13), we can show that the relative entropy in (12) can be rewritten in terms of copula density as

$$\delta(c) \equiv \delta(c) = \int_{[0,1]^d} \log\,[c(\mathbf{u})]c(\mathbf{u})\, \mathrm{d}\mathbf{u}. \tag{14}$$

The measure $\delta(c)$ does not depend on the marginal distributions of $\mathbf{X}$, but only on the copula density c. We will next define a nonparametric estimator of $\delta(c)$ that we will use to construct the test statistic of our third test of independence, and we will establish its asymptotic normality.

4.2. ***Test of independence***

We have shown that the Kullback-Leibler measure of dependence $\delta(c)$ can be expressed in terms of copula density function c. Thus, an estimator of that measure can be obtained by replacing the unknown copula density c by its Bernstein copula density estimator in Equation (9):

$$\delta_n(c) = \int_{[0,1]^d} \log\,[c_{k,n}(\mathbf{u})]\; c_{k,n}(\mathbf{u}) d\mathbf{u}. \tag{15}$$

where $c_{k,n}(\mathbf{u})$ is the Bernstein copula density estimator defined in Equation (9). In practice, we suggest to replace $c_{k,n}(u)du$ in $\delta_n(c)$ by $dC_n(u)$, i.e., to use the following test statistic:

$$\begin{aligned} \tilde{\delta}_n(c) &= \int_{[0,1]^d} \log\,[c_{k,n}(\mathbf{u})]\; dC_n(\mathbf{u}) \\ &= \frac{1}{n}\sum_{i=1}^{n} \log\,[c_{k,n}(\mathbf{V_i})]\,. \end{aligned} \tag{16}$$

Now, observe that the null hypothesis of independence is equivalent to the nullity of the measure $\delta(c)$. Thus, our third nonparametric test of independence is based on $\delta_n(c)$. In other words, we use $\delta_n(c)$ in Equation (15) as a test statistic to test the null hypothesis $\mathcal{H}_0$. The following theorem provides the asymptotic normality of the test statistic $\delta_n(c)$ [see the proof of Theorem 2 in Appendix A].

THEOREM 2 *Suppose that the assumptions of Theorem 1 are satisfied. Then, under $\mathcal{H}_0$, we have*

$$nk^{-d}\left\{\frac{2\delta_n(c)-2^{-d}\pi^{d/2}n^{-1}k^{d/2}}{2^{1/2}\sqrt{\left\{\sum_{v_1,v_2=0}^{k-1}\Gamma_k(v_1,v_2)^2\right\}^d-k^{-2d}}}\right\}\xrightarrow{d}\mathcal{N}(0,1),$$

where $\Gamma_k(.,.)$ and $\delta_n(c)$ are defined in (11) and (15), respectively.

The result in Theorem 2 remains unchanged when we replace $\delta_n(c)$ by $\tilde{\delta}_n(c)$ in (16).

As for the test statistics T_n and I_n, our simulation results suggest that it is better to use a Monte Carlo simulation-based approach, instead of the asymptotic distribution, for the calculation of critical values (p-values) of the test statistic $\delta_n(c)$. A brief description of Monte Carlo simulation approach can be found at the end of Section 2.2. Furthermore, the consistency of the test based on $\delta_n(c)$ can be established under the same conditions as the ones we needed for the consistency of I_n, using similar arguments as in the proof of Proposition 5.

5. Simulation studies

We run a Monte Carlo experiment to investigate the performance of nonparametric tests of independence proposed in the previous sections. In particular, we study the power of the test statistics T_n, I_n and δ_n using different samples sizes: $n = 100$, 200, 400, 500. To calculate the critical values of these test statistics under the null and at 5% significance level, we simulate independent data using the independent copula. Thereafter, we evaluate the empirical power of our tests using different copula functions that generate data under different degrees of dependence following different values of Kendall's tau coefficient $\tau = 0$, 0.1, 0.25. For Kendall's τ coefficient greater than 0.5, all the tests provide good and comparable results. The copulas under consideration are Normal, Student, Clayton and Gumbel copulas. Moreover, we compare the power functions of our tests to the power function of the following classical test which is based on the empirical copula process considered in Deheuvels (1981c), Genest, Quessy, and Rémillard (2006), and Kojadinovic and Holmes (2009):

$$S_n = n \int_{[0,1]^2} \{C_n(u_1,u_2) - C_\pi(u_1,u_2)\}^2 du_1 du_2. \tag{17}$$

The test statistics T_n, I_n and δ_n depend on the bandwidth parameter k which is needed to estimate the copula density (distribution). We take various values of k to investigate the sensitivity of the power functions of our nonparametric tests to the bandwidth parameter. A practical bandwidth can be selected using a similar approach to the one proposed by Omelka, Gijbels, and Veraverbeke (2009) for kernel-based copula estimation, but this is not investigated in this paper and left for future research. Omelka et al. (2009)'s approach involves an Edgeworth expansion of the asymptotic distribution of the test statistics. Finally, we use Monte-Carlo approximations, based on 1000 replications, to compute the critical values and the empirical power of all the tests, S_n, T_n, I_n and δ_n.

In the simulations, we consider two scenarios for the marginal distributions used to compute the test statistics. In the first one, we assume that the marginal distributions are known and given by a uniform distribution. In the second scenario, we consider that the marginal distributions are estimated. In the latter scenario we consider different models for the marginal distributions: uniform, normal and Student. Simulation results for the empirical power of the tests that are based on the statistics T_n, I_n, δ_n, and S_n are reported in Tables 1-3 for the first scenario and in Tables 4-6 for the second scenario. We only provide the results for normal marginals as the results for other distributions are quite similar.

Table 1 compares the power function of our first nonparametric test which is based on the Bernstein copula distribution T_n to the power function of the classical test which is based on the empirical copula S_n. The simulation results for different copulas, samples sizes, and degrees of dependence show that both tests provide good empirical size. The power of the two tests increases with sample size and degree of dependence measured by Kendall's tau. Furthermore, the power functions of both tests are comparable for moderate degree of dependency, but the test based on the Bernstein copula dominates the one based on the empirical copula when Kendall's tau is small. Finally, the two tests fail in terms of power in the case of Student copula with Kendall's tau equal to zero. Recall that in the case of Student copula, Kendall's tau equal to zero does not imply independence, because the dependence may happen in the tail regions.

Tables 2 and 3 provide the empirical size and power of nonparametric tests that are based on the test statistics I_n and δ_n, respectively. From these, we see that the two tests generally control the size. Their powers increase with the sample size and the strength of dependence. Compared to the empirical copula-based test S_n, we find that these tests do much better in terms of power, especially in the case of Student copula with zero Kendall's tau. For example, when $n = 500$ and $k = 25$ the powers of I_n and δ_n tests are equal to 0.823 and 0.434, respectively, whereas the one of S_n test is equal to 0.048.

The same remark applies when the degree of dependencies is small. For example, under Clayton copula and when $\tau = 0.1$, $k = 25$, and $n = 400$, the powers of I_n and δ_n tests are equal to 0.813 and 0.506, respectively, whereas the power of S_n test is equal to 0.294. The difference becomes even more important when we increase the sample seize. Finally, we find that the Cramér-von Mises-type test which is defined in terms of Bernstein copula density generally outperforms the test based on Kullback-Leibler divergence and defined as a function of Bernstein copula density estimator.

Table 4 shows the power of the tests T_n and S_n using estimated marginal distributions. We observe a significant improvement in the power of the test S_n compared to the results in Table 1. But we still find that the test T_n does better than the test S_n in many cases. Tables 5-6 show the power of the tests I_n and δ_n. We see clearly that the tests I_n and δ_n do better than the test S_n for Student copula and very low dependence, especially for $\tau = 0$. However, in many cases the test S_n does better than the tests I_n and δ_n when $\tau = 0.1$. Finally, it seems that the test I_n does better than the other ones (δ_n and T_n).

6. Conclusion

We provided three different nonparametric tests of independence between continuous random variables based on estimators of Bernstein copula distribution and Bernstein copula density. The first two tests were constructed using Cramér-von Mises divergence that we define as a function of the empirical Bernstein copula process and the empirical Bernstein copula density, respectively. The third test is based on Kullback-Leibler divergence originally defined in terms of probability density functions. We first rewrote the Kullback-Leibler divergence in terms of copula density, see also Blumentritt and Schmid (2012). Thereafter, we constructed the third test using an estimator of Kullback-Leibler divergence defined as a logarithmic function of the estimator of Bernstein copula density. Furthermore, we provided the asymptotic distribution of each of these tests under the null, and we established their consistency under a fixed alternative. Finally, we ran a Monte Carlo experiment to investigate the performance of these tests. In particular, we examined and compared their empirical size and power to those of classical nonparametric test which is based on the empirical copula considered in Deheuvels (1981c), Genest et al. (2006), and Kojadinovic and Holmes (2009).

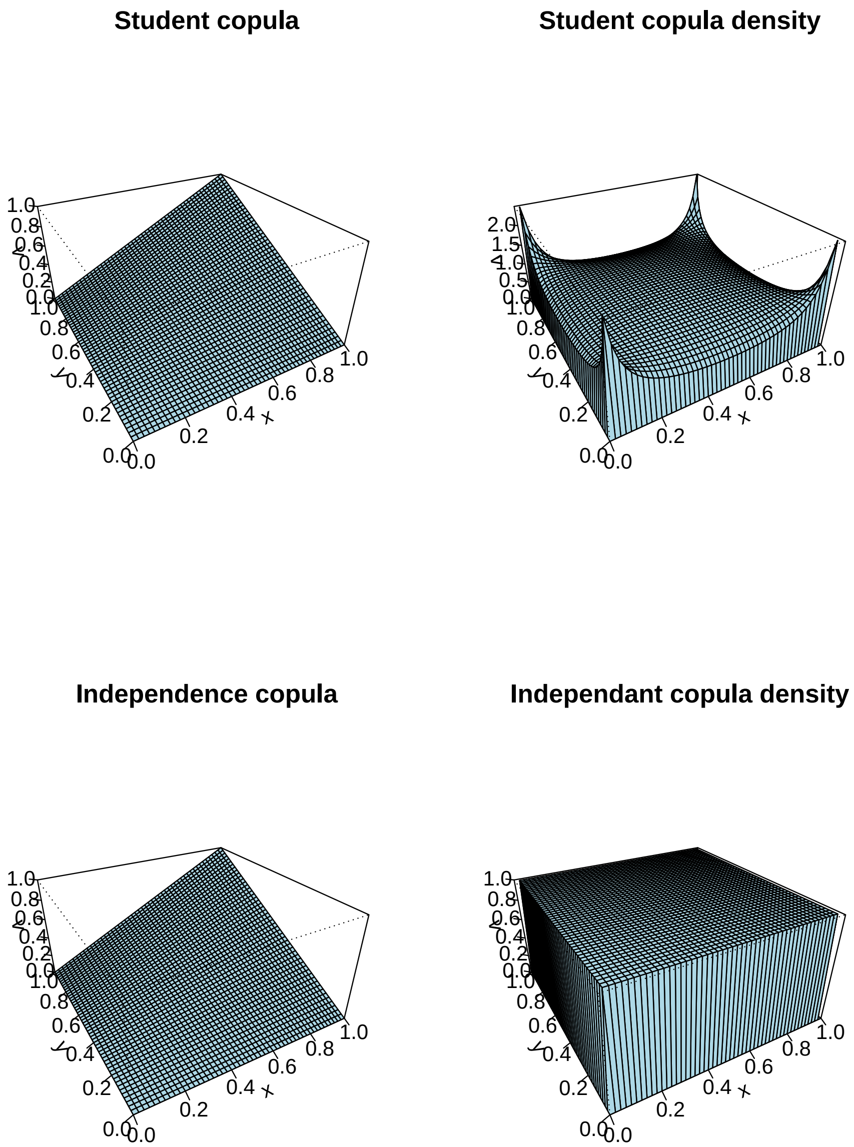

Figure 1. This figure compares the student copula distribution (in the top of the left-hand side panel) and the independent copula distribution (in the bottom of the left-hand side panel) and between the student copula density (in the top of the right-hand side panel) and the independent copula density (in the bottom of the right-hand side panel).

Statistic T_n for Normal copula												
	$n = 100$			$n = 200$			$n = 400$			$n = 500$		
k	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$
5	0.050	0.258	0.932	0.038	0.420	0.996	0.052	0.778	1.000	0.024	0.824	1.000
10	0.064	0.276	0.948	0.044	0.472	0.998	0.064	0.814	1.000	0.026	0.846	1.000
15	0.064	0.252	0.948	0.036	0.484	0.998	0.056	0.806	1.000	0.028	0.860	1.000
20	0.064	0.270	0.948	0.048	0.488	0.998	0.066	0.806	1.000	0.030	0.864	1.000
25	0.062	0.268	0.950	0.044	0.492	0.998	0.056	0.806	1.000	0.026	0.860	1.000
30	0.070	0.274	0.950	0.044	0.478	0.998	0.060	0.794	1.000	0.028	0.860	1.000
$\mathbf{S_n}$	0.074	0.122	0.314	0.048	0.126	0.490	0.060	0.242	0.840	0.030	0.260	0.916
Statistic T_n for Student copula												
5	0.064	0.280	0.912	0.032	0.442	0.994	0.062	0.762	1.000	0.048	0.838	1.000
10	0.062	0.340	0.944	0.044	0.476	0.998	0.100	0.818	1.000	0.042	0.868	1.000
15	0.066	0.332	0.938	0.048	0.502	0.998	0.098	0.814	1.000	0.040	0.890	1.000
20	0.064	0.328	0.938	0.054	0.492	0.998	0.098	0.824	1.000	0.056	0.898	1.000
25	0.064	0.328	0.936	0.064	0.512	0.998	0.094	0.820	1.000	0.048	0.884	1.000
30	0.076	0.360	0.940	0.058	0.504	0.998	0.098	0.816	1.000	0.056	0.896	1.000
$\mathbf{S_n}$	0.054	0.128	0.328	0.044	0.142	0.494	0.054	0.260	0.866	0.048	0.264	0.900
Statistic T_n for Clayton copula												
5	0.052	0.242	0.936	0.034	0.388	0.990	0.044	0.750	1.000	0.036	0.830	1.000
10	0.054	0.270	0.962	0.044	0.454	0.996	0.054	0.838	1.000	0.032	0.880	1.000
15	0.048	0.244	0.956	0.044	0.456	0.996	0.048	0.822	1.000	0.040	0.886	1.000
20	0.054	0.262	0.968	0.044	0.452	0.996	0.050	0.838	1.000	0.044	0.902	1.000
25	0.058	0.256	0.966	0.050	0.480	0.996	0.050	0.826	1.000	0.044	0.884	1.000
30	0.060	0.252	0.966	0.044	0.430	0.996	0.050	0.824	1.000	0.042	0.886	1.000
$\mathbf{S_n}$	0.042	0.114	0.348	0.046	0.146	0.536	0.060	0.294	0.902	0.036	0.248	0.936
Statistic T_n for Gumbel copula												
5	0.038	0.296	0.928	0.020	0.426	0.998	0.044	0.798	1.000	0.040	0.810	1.000
10	0.046	0.358	0.956	0.026	0.460	1.000	0.068	0.844	1.000	0.036	0.842	1.000
15	0.034	0.342	0.952	0.022	0.490	1.000	0.058	0.836	1.000	0.044	0.864	1.000
20	0.040	0.344	0.952	0.026	0.482	1.000	0.062	0.836	1.000	0.046	0.864	1.000
25	0.036	0.332	0.950	0.026	0.490	1.000	0.056	0.828	1.000	0.040	0.860	1.000
30	0.038	0.372	0.966	0.020	0.488	1.000	0.062	0.830	1.000	0.040	0.870	1.000
$\mathbf{S_n}$	0.060	0.106	0.372	0.028	0.138	0.542	0.080	0.272	0.844	0.052	0.224	0.910

Table 1. This table compares the empirical size and power of the test statistics T_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas) with known marginal distributions, different values of Kendall's tau coefficient τ ($\tau = 0, 0.1, 0.25$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth k.

	$n=100$			$n=200$			$n=400$			$n=500$		
Statistic I_n for Normal copula												
k	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$
5	0.068	0.317	0.955	0.056	0.510	0.999	0.037	0.757	1.000	0.053	0.863	1.000
10	0.056	0.330	0.946	0.063	0.494	1.000	0.046	0.756	1.000	0.044	0.842	1.000
15	0.059	0.299	0.896	0.050	0.432	0.997	0.054	0.704	1.000	0.061	0.832	1.000
20	0.056	0.259	0.854	0.040	0.354	0.992	0.041	0.649	1.000	0.059	0.787	1.000
25	0.044	0.219	0.803	0.054	0.339	0.986	0.032	0.553	1.000	0.063	0.749	1.000
30	0.049	0.191	0.762	0.057	0.304	0.981	0.038	0.527	1.000	0.057	0.698	1.000
$\mathbf{S_n}$	0.074	0.122	0.314	0.048	0.126	0.490	0.060	0.242	0.840	0.030	0.260	0.916
Statistic I_n for Student copula												
5	0.096	0.413	0.955	0.149	0.585	1.000	0.146	0.851	1.000	0.178	0.933	1.000
10	0.241	0.507	0.965	0.356	0.725	1.000	0.481	0.935	1.000	0.582	0.975	1.000
15	0.300	0.528	0.957	0.458	0.740	0.999	0.662	0.958	1.000	0.751	0.981	1.000
20	0.322	0.505	0.940	0.491	0.720	0.998	0.695	0.952	1.000	0.817	0.990	1.000
25	0.306	0.485	0.910	0.528	0.720	0.997	0.693	0.942	1.000	0.823	0.989	1.000
30	0.316	0.473	0.898	0.511	0.722	0.998	0.709	0.945	1.000	0.823	0.986	1.000
$\mathbf{S_n}$	0.054	0.128	0.328	0.044	0.142	0.494	0.054	0.260	0.866	0.048	0.264	0.900
Statistic I_n for Clayton copula												
5	0.063	0.351	0.971	0.054	0.614	1.000	0.052	0.817	1.000	0.055	0.917	1.000
10	0.062	0.390	0.976	0.063	0.649	1.000	0.051	0.879	1.000	0.047	0.939	1.000
15	0.059	0.379	0.963	0.057	0.628	1.000	0.054	0.873	1.000	0.052	0.943	1.000
20	0.048	0.347	0.952	0.047	0.579	1.000	0.040	0.856	1.000	0.050	0.930	1.000
25	0.043	0.325	0.927	0.050	0.560	1.000	0.039	0.813	1.000	0.052	0.922	1.000
30	0.045	0.287	0.914	0.055	0.541	0.998	0.039	0.799	1.000	0.043	0.904	1.000
$\mathbf{S_n}$	0.042	0.114	0.348	0.046	0.146	0.536	0.060	0.294	0.902	0.036	0.248	0.936
Statistic I_n for Gumbel copula												
5	0.056	0.380	0.941	0.051	0.595	1.000	0.052	0.802	1.000	0.058	0.901	1.000
10	0.062	0.400	0.952	0.065	0.621	1.000	0.052	0.860	1.000	0.049	0.926	1.000
15	0.065	0.370	0.949	0.065	0.598	1.000	0.050	0.872	1.000	0.057	0.931	1.000
20	0.060	0.348	0.940	0.051	0.571	1.000	0.041	0.841	1.000	0.059	0.929	1.000
25	0.055	0.315	0.909	0.051	0.562	0.999	0.030	0.811	1.000	0.058	0.913	1.000
30	0.065	0.315	0.877	0.050	0.535	0.998	0.035	0.797	1.000	0.044	0.909	1.000
$\mathbf{S_n}$	0.060	0.106	0.372	0.028	0.138	0.542	0.080	0.272	0.844	0.052	0.224	0.910

Table 2. This table compares the empirical size and power of the test statistics I_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas) with known marginal distributions, different values of Kendall's tau coefficient τ ($\tau = 0, 0.1, 0.25$) , different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth k.

Statistic δ_n for Normal copula												
	$n=100$			$n=200$			$n=400$			$n=500$		
k	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$
5	0.044	0.222	0.924	0.066	0.428	0.998	0.050	0.740	1.000	0.046	0.840	1.000
10	0.036	0.172	0.844	0.066	0.330	0.982	0.048	0.606	1.000	0.050	0.670	1.000
15	0.046	0.176	0.754	0.070	0.268	0.964	0.070	0.540	1.000	0.068	0.590	1.000
20	0.050	0.134	0.644	0.058	0.196	0.924	0.070	0.424	1.000	0.062	0.500	1.000
25	0.056	0.144	0.574	0.062	0.172	0.902	0.062	0.370	1.000	0.076	0.440	1.000
30	0.046	0.112	0.490	0.062	0.148	0.848	0.068	0.304	0.998	0.050	0.358	0.998
$\mathbf{S_n}$	0.074	0.122	0.314	0.048	0.126	0.490	0.060	0.242	0.840	0.030	0.260	0.916
Statistic δ_n for Student copula												
5	0.112	0.330	0.934	0.170	0.562	0.996	0.390	0.872	1.000	0.428	0.934	1.000
10	0.154	0.316	0.836	0.226	0.512	0.986	0.526	0.844	1.000	0.536	0.920	1.000
15	0.162	0.302	0.760	0.228	0.466	0.956	0.488	0.788	1.000	0.572	0.892	1.000
20	0.158	0.242	0.674	0.194	0.392	0.928	0.434	0.724	1.000	0.492	0.814	1.000
25	0.154	0.210	0.618	0.194	0.344	0.890	0.406	0.668	1.000	0.434	0.770	1.000
30	0.134	0.180	0.524	0.146	0.304	0.840	0.366	0.586	0.996	0.386	0.672	1.000
$\mathbf{S_n}$	0.054	0.128	0.328	0.044	0.142	0.494	0.054	0.260	0.866	0.048	0.264	0.900
Statistic δ_n for Clayton copula												
5	0.046	0.288	0.952	0.048	0.502	1.000	0.040	0.878	1.000	0.050	0.918	1.000
10	0.056	0.264	0.912	0.050	0.428	1.000	0.040	0.764	1.000	0.042	0.828	1.000
15	0.066	0.240	0.846	0.056	0.342	0.996	0.050	0.668	1.000	0.062	0.752	1.000
20	0.056	0.206	0.788	0.054	0.276	0.982	0.050	0.572	1.000	0.058	0.660	1.000
25	0.058	0.200	0.722	0.050	0.230	0.954	0.050	0.506	1.000	0.056	0.576	1.000
30	0.042	0.150	0.642	0.040	0.188	0.918	0.040	0.434	1.000	0.034	0.478	1.000
$\mathbf{S_n}$	0.042	0.114	0.348	0.046	0.146	0.536	0.060	0.294	0.902	0.036	0.248	0.936
Statistic δ_n for Gumbel copula												
5	0.046	0.296	0.918	0.052	0.494	0.998	0.036	0.794	1.000	0.062	0.854	1.000
10	0.056	0.260	0.852	0.040	0.378	0.992	0.050	0.704	1.000	0.052	0.744	1.000
15	0.078	0.242	0.760	0.036	0.286	0.972	0.062	0.634	1.000	0.060	0.686	1.000
20	0.072	0.188	0.676	0.034	0.248	0.948	0.042	0.548	0.998	0.050	0.600	1.000
25	0.080	0.188	0.622	0.030	0.208	0.924	0.040	0.490	0.998	0.050	0.552	1.000
30	0.072	0.150	0.552	0.024	0.172	0.882	0.046	0.422	0.994	0.042	0.446	0.998
$\mathbf{S_n}$	0.060	0.106	0.372	0.028	0.138	0.542	0.080	0.272	0.844	0.052	0.224	0.910

Table 3. This table compares the empirical size and power of the test statistics δ_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas) with known marginal distributions, different values of Kendall's tau coefficient τ ($\tau = 0, 0.1, 0.25$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth k.

	$n=100$			$n=200$			$n=400$			$n=500$		
Statistic T_n for Normal copula												
k	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$
5	0.050	0.284	0.922	0.074	0.566	1.000	0.048	0.786	1.000	0.040	0.862	1.000
10	0.046	0.290	0.940	0.074	0.588	1.000	0.042	0.794	1.000	0.044	0.922	1.000
15	0.046	0.282	0.940	0.072	0.616	1.000	0.040	0.808	1.000	0.042	0.918	1.000
20	0.050	0.306	0.940	0.080	0.608	1.000	0.036	0.794	1.000	0.042	0.924	1.000
25	0.050	0.294	0.942	0.076	0.600	1.000	0.040	0.798	1.000	0.044	0.924	1.000
30	0.044	0.288	0.940	0.076	0.600	1.000	0.038	0.794	1.000	0.034	0.916	1.000
$\mathbf{S_n}$	0.036	0.170	0.890	0.068	0.502	1.000	0.066	0.812	1.000	0.046	0.914	1.000
Statistic T_n for Student copula												
5	0.056	0.272	0.928	0.076	0.576	1.000	0.044	0.728	1.000	0.072	0.848	1.000
10	0.064	0.284	0.930	0.080	0.598	1.000	0.038	0.768	1.000	0.096	0.876	1.000
15	0.066	0.286	0.926	0.088	0.630	1.000	0.048	0.778	1.000	0.100	0.900	1.000
20	0.070	0.306	0.936	0.094	0.620	1.000	0.044	0.768	1.000	0.110	0.908	1.000
25	0.070	0.308	0.936	0.096	0.614	1.000	0.048	0.780	1.000	0.106	0.898	1.000
30	0.068	0.294	0.932	0.086	0.616	1.000	0.050	0.784	1.000	0.096	0.896	1.000
$\mathbf{S_n}$	0.040	0.214	0.874	0.054	0.476	0.998	0.092	0.808	1.000	0.092	0.896	1.000
Statistic T_n for Clayton copula												
5	0.046	0.286	0.914	0.056	0.550	0.998	0.040	0.768	1.000	0.040	0.860	1.000
10	0.042	0.286	0.936	0.046	0.592	0.998	0.044	0.794	1.000	0.056	0.918	1.000
15	0.040	0.268	0.928	0.052	0.596	1.000	0.050	0.810	1.000	0.058	0.918	1.000
20	0.046	0.302	0.944	0.062	0.610	1.000	0.034	0.806	1.000	0.062	0.924	1.000
25	0.044	0.298	0.946	0.056	0.598	1.000	0.044	0.818	1.000	0.056	0.924	1.000
30	0.044	0.278	0.936	0.060	0.576	1.000	0.042	0.808	1.000	0.052	0.918	1.000
$\mathbf{S_n}$	0.018	0.222	0.912	0.042	0.502	0.998	0.066	0.828	1.000	0.058	0.922	1.000
Statistic T_n for Gumbel copula												
5	0.060	0.262	0.916	0.070	0.570	1.000	0.040	0.794	1.000	0.056	0.868	1.000
10	0.050	0.254	0.932	0.064	0.576	0.998	0.048	0.812	1.000	0.074	0.914	1.000
15	0.050	0.254	0.938	0.072	0.634	1.000	0.046	0.836	1.000	0.082	0.924	1.000
20	0.052	0.266	0.934	0.078	0.604	0.998	0.048	0.810	1.000	0.076	0.922	1.000
25	0.054	0.266	0.938	0.076	0.600	0.998	0.048	0.822	1.000	0.076	0.914	1.000
30	0.048	0.272	0.940	0.068	0.618	0.998	0.052	0.828	1.000	0.072	0.918	1.000
$\mathbf{S_n}$	0.026	0.242	0.876	0.060	0.546	1.000	0.060	0.800	1.000	0.068	0.910	1.000

Table 4. This table compares the empirical size and power of the test statistics T_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas) with estimated marginal distributions, different values of Kendall's tau coefficient τ ($\tau = 0, 0.1, 0.25$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth k.

	$n=100$			$n=200$			$n=400$			$n=500$		
Statistic I_n for Normal copula												
k	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$	$\tau=0$	$\tau=0.1$	$\tau=0.25$
5	0.064	0.274	0.940	0.050	0.510	1.000	0.084	0.794	1.000	0.052	0.886	1.000
10	0.046	0.280	0.922	0.054	0.474	0.998	0.068	0.744	1.000	0.064	0.878	1.000
15	0.038	0.242	0.872	0.050	0.446	0.998	0.066	0.698	1.000	0.054	0.820	1.000
20	0.042	0.242	0.830	0.054	0.384	0.992	0.072	0.662	1.000	0.048	0.782	1.000
25	0.042	0.232	0.798	0.048	0.320	0.986	0.062	0.632	1.000	0.046	0.758	1.000
30	0.036	0.204	0.738	0.038	0.286	0.978	0.054	0.590	1.000	0.040	0.698	1.000
$\mathbf{S_n}$	0.036	0.170	0.890	0.068	0.502	1.000	0.066	0.812	1.000	0.046	0.914	1.000
Statistic I_n for Student copula												
5	0.094	0.312	0.926	0.094	0.628	1.000	0.230	0.892	1.000	0.192	0.930	1.000
10	0.168	0.416	0.952	0.308	0.734	1.000	0.588	0.956	1.000	0.668	0.990	1.000
15	0.254	0.438	0.930	0.442	0.776	1.000	0.702	0.956	1.000	0.746	0.988	1.000
20	0.270	0.450	0.926	0.466	0.772	1.000	0.762	0.958	1.000	0.806	0.986	1.000
25	0.284	0.430	0.918	0.472	0.750	0.998	0.772	0.960	1.000	0.824	0.988	1.000
30	0.294	0.438	0.888	0.458	0.716	0.998	0.780	0.946	1.000	0.824	0.984	1.000
$\mathbf{S_n}$	0.040	0.214	0.874	0.054	0.476	0.998	0.092	0.808	1.000	0.092	0.896	1.000
Statistic I_n for Clayton copula												
5	0.056	0.334	0.958	0.066	0.512	1.000	0.072	0.862	1.000	0.064	0.920	1.000
10	0.048	0.386	0.970	0.060	0.562	1.000	0.046	0.878	1.000	0.054	0.950	1.000
15	0.052	0.356	0.956	0.072	0.540	1.000	0.038	0.854	1.000	0.062	0.962	1.000
20	0.050	0.376	0.954	0.072	0.546	0.998	0.032	0.850	1.000	0.042	0.950	1.000
25	0.042	0.352	0.950	0.058	0.460	0.998	0.036	0.828	1.000	0.046	0.946	1.000
30	0.048	0.332	0.928	0.044	0.448	0.998	0.030	0.814	1.000	0.040	0.922	1.000
$\mathbf{S_n}$	0.018	0.222	0.912	0.042	0.502	0.998	0.066	0.828	1.000	0.058	0.922	1.000
Statistic T_n for Gumbel copula												
5	0.038	0.316	0.934	0.038	0.538	1.000	0.068	0.842	1.000	0.062	0.914	1.000
10	0.048	0.350	0.942	0.040	0.578	1.000	0.066	0.882	1.000	0.060	0.944	1.000
15	0.056	0.348	0.938	0.038	0.600	1.000	0.062	0.870	1.000	0.058	0.936	1.000
20	0.058	0.356	0.918	0.044	0.554	1.000	0.068	0.860	1.000	0.060	0.936	1.000
25	0.042	0.348	0.898	0.044	0.512	1.000	0.062	0.846	1.000	0.060	0.934	1.000
30	0.038	0.348	0.894	0.040	0.478	0.998	0.058	0.806	1.000	0.058	0.908	1.000
$\mathbf{S_n}$	0.026	0.242	0.876	0.060	0.546	1.000	0.060	0.800	1.000	0.068	0.910	1.000

Table 5. This table compares the empirical size and power of the test statistics I_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas) with estimated marginal distributions , different values of Kendall's tau coefficient τ ($\tau = 0, 0.1, 0.25$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth k.

Statistic δ_n for Normal copula												
	$n = 100$			$n = 200$			$n = 400$			$n = 500$		
k	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.25$
5	0.070	0.244	0.906	0.062	0.480	0.996	0.048	0.724	1.000	0.072	0.822	1.000
10	0.048	0.166	0.820	0.064	0.364	0.988	0.048	0.592	1.000	0.044	0.662	1.000
15	0.068	0.170	0.742	0.060	0.278	0.966	0.046	0.494	1.000	0.060	0.578	1.000
20	0.060	0.140	0.612	0.052	0.238	0.930	0.044	0.394	1.000	0.050	0.502	1.000
25	0.042	0.108	0.512	0.044	0.188	0.906	0.038	0.336	1.000	0.050	0.438	0.998
30	0.046	0.098	0.472	0.042	0.166	0.842	0.040	0.300	1.000	0.052	0.368	0.998
$\mathbf{S_n}$	0.036	0.170	0.890	0.068	0.502	1.000	0.066	0.812	1.000	0.046	0.914	1.000
Statistic δ_n for Student copula												
5	0.114	0.282	0.906	0.176	0.524	0.996	0.366	0.864	1.000	0.398	0.938	1.000
10	0.130	0.260	0.842	0.222	0.506	0.992	0.492	0.836	1.000	0.542	0.916	1.000
15	0.166	0.256	0.780	0.226	0.398	0.960	0.472	0.782	1.000	0.552	0.898	1.000
20	0.132	0.188	0.692	0.210	0.344	0.930	0.432	0.702	1.000	0.506	0.852	1.000
25	0.102	0.148	0.592	0.176	0.284	0.880	0.382	0.626	0.998	0.476	0.804	1.000
30	0.114	0.136	0.548	0.148	0.262	0.846	0.354	0.590	0.996	0.458	0.740	1.000
$\mathbf{S_n}$	0.040	0.214	0.874	0.054	0.476	0.998	0.092	0.808	1.000	0.092	0.896	1.000
Statistic δ_n for Clayton copula												
5	0.040	0.292	0.948	0.048	0.566	1.000	0.054	0.830	1.000	0.060	0.922	1.000
10	0.040	0.208	0.900	0.056	0.448	0.996	0.056	0.720	1.000	0.042	0.808	1.000
15	0.072	0.196	0.844	0.030	0.366	0.984	0.062	0.632	1.000	0.052	0.746	1.000
20	0.050	0.132	0.750	0.036	0.334	0.974	0.058	0.540	1.000	0.046	0.638	1.000
25	0.046	0.114	0.666	0.030	0.288	0.948	0.048	0.450	1.000	0.064	0.600	1.000
30	0.052	0.110	0.608	0.030	0.250	0.930	0.052	0.414	1.000	0.062	0.514	1.000
$\mathbf{S_n}$	0.018	0.222	0.912	0.042	0.502	0.998	0.066	0.828	1.000	0.058	0.922	1.000
Statistic δ_n for Gumbel copula												
5	0.038	0.240	0.924	0.038	0.490	0.994	0.054	0.758	1.000	0.050	0.896	1.000
10	0.036	0.188	0.850	0.060	0.398	0.988	0.064	0.638	1.000	0.038	0.786	1.000
15	0.050	0.192	0.782	0.054	0.318	0.966	0.044	0.556	1.000	0.060	0.714	1.000
20	0.038	0.138	0.686	0.048	0.272	0.948	0.048	0.480	0.998	0.054	0.648	1.000
25	0.036	0.118	0.600	0.042	0.218	0.898	0.050	0.426	0.998	0.062	0.580	1.000
30	0.044	0.110	0.514	0.050	0.212	0.860	0.052	0.388	0.994	0.048	0.510	1.000
$\mathbf{S_n}$	0.026	0.242	0.876	0.060	0.546	1.000	0.060	0.800	1.000	0.068	0.910	1.000

Table 6. This table compares the empirical size and power of the test statistics δ_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas) with estimated marginal distributions, different values of Kendall's tau coefficient τ ($\tau = 0, 0.1, 0.25$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth k.

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Appendix A. Proofs of Propositions 1, 3, 4, 5, and of Theorem 2

Proof of Proposition 1. First of all, we decompose the test statistic T_n in the following way:

$$\begin{aligned} T_n &= n \int_{[0,1]^d} \left(C_{k,n}(\mathbf{u}) - \prod_{j=1}^{d} u_j \right)^2 du_1 ... du_d \\ &= n \int_{[0,1]^d} (C_{k,n}(\mathbf{u}))^2 du_1 ... du_d - 2n \int_{[0,1]^d} C_{k,n}(\mathbf{u}) \prod_{j=1}^{d} u_j \, du_1 ... du_d \\ &+ n \int_{[0,1]^d} \left(\prod_{j=1}^{d} u_j \right)^2 du_1 ... du_d \\ &= T_{1n} - T_{2n} + T_{3n}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} T_{1n} &= n \int_{[0,1]^d} (C_{k,n}(\mathbf{u}))^2 \, du_1 ... du_d \\ &= n \int_{[0,1]^d} \sum_{(v,s)} C_n \left(\frac{v_1}{k}, ..., \frac{v_d}{k} \right) C_n \left(\frac{s_1}{k}, ..., \frac{s_d}{k} \right) \\ &\times \prod_{j=1}^{d} P_{v_j,k}(u_j) P_{s_j,k}(u_j) du_1 ... du_d. \end{aligned}$$

Using the definition of binomial distribution, we obtain

$$\begin{aligned} T_{1n} &= n \sum_{(v,s)} \binom{k}{v_1} ... \binom{k}{v_d} \binom{k}{s_1} ... \binom{k}{s_d} C_n \left(\frac{v_1}{k}, ..., \frac{v_d}{k} \right) C_n \left(\frac{s_1}{k}, ..., \frac{s_d}{k} \right) \\ &\times \int_{[0,1]^d} \prod_{j=1}^{d} u^{v_j + s_j} (1-u)^{2k - v_j - s_j} du_1 ... du_d \\ &= n \sum_{(v,s)} C_n \left(\frac{v_1}{k}, ..., \frac{v_d}{k} \right) C_n \left(\frac{s_1}{k}, ..., \frac{s_d}{k} \right) \\ &\times \prod_{j=1}^{d} \binom{k}{v_j} \binom{k}{s_j} \beta(v_j + s_j + 1, 2k - v_j - s_j + 1). \end{aligned}$$

In a similar way, we can show that

$$T_{2n} = 2n \sum_{v_1=0}^{k} ... \sum_{v_d=1}^{k} C_n \left(\frac{v_1}{k}, ..., \frac{v_d}{k} \right) \prod_{j=1}^{d} \binom{k}{v_j} \beta(v_j + 2, k - v_j + 1).$$

This concludes the proof of Proposition 1.

■

Proof of Proposition 3. We provide the proof for $d = 2$. The generalization to $d > 2$ is straightforward. For a two-dimensional vector $\mathbf{u} = (u_1, u_2)$, we start by the following decomposition:

$$\begin{aligned}\int \left(\hat{C}_{n,k}(\mathbf{u}) - u_1u_2\right)^2 d\mathbf{u} &= \int \left(\hat{C}_{n,k}(\mathbf{u}) - C(\mathbf{u})\right)^2 d\mathbf{u} + \int (C(\mathbf{u}) - u_1u_2)^2 d\mathbf{u} \\ &\quad + 2\int \left(\hat{C}_{n,k}(\mathbf{u}) - C(\mathbf{u})\right)(C(\mathbf{u}) - u_1u_2)\, d\mathbf{u} \\ &= T_{1,n} + T_{2,n} + T_{3,n}.\end{aligned}$$

From Janssen et al. (2012) and the continuous mapping theorem we have

$$n\int \left(\hat{C}_{n,k}(\mathbf{u}) - C(\mathbf{u})\right)^2 d\mathbf{u} = O_p(1). \tag{A1}$$

Furthermore, from Janssen et al. (2012) , we can show that

$$n\int \left(\hat{C}_{n,k}(\mathbf{u}) - C(\mathbf{u})\right)(C(\mathbf{u}) - u_1u_2)\, d\mathbf{u} = o_p(n). \tag{A2}$$

Therefore, using the fact that $\int(C(u,v) - u_1u_2)^2 d\mathbf{u} > 0$ and from (A1) and (A2), we deduce the consistency of T_n.

■

Proof of Proposition 4. Expanding the squared term in the test statistic (10) leads to the following decomposition:

$$I_n = I_n^{(1)} + I_n^{(2)} + 1,$$

with

$$I_n^{(1)} = \int_{[0,1]^2} c_{k,n}^2(\mathbf{u})\mathrm{d}\mathbf{u} \qquad \text{and} \qquad I_n^{(2)} = -2\int_{[0,1]^2} c_{k,n}(\mathbf{u})\mathrm{d}\mathbf{u}.$$

First, by writing

$$c_{k,n}^2(\mathbf{u}) = k^4 \sum_{\substack{v_1,v_1'=0 \\ v_2,v_2'=0}}^{k-1} \Upsilon_k(v_1,v_2)\Upsilon_k(v_1',v_2')P_{v_1,k-1}(u_1)P_{v_1',k-1}(u_1)P_{v_2,k-1}(u_2)P_{v_2',k-1}(u_2),$$

we deduce that

$$\begin{aligned}
I_n^{(1)}(\mathbf{u}) &= k^4 \sum_{\substack{v_1,v_1'=0\\ v_2,v_2'=0}}^{k-1} \Upsilon_k(v_1,v_2)\Upsilon_k(v_1',v_2') \\
&\quad \times \int_0^1 \int_0^1 P_{v_1,k-1}(u_1)P_{v_1',k-1}(u_1)P_{v_2,k-1}(u_2)P_{v_2',k-1}(u_2)\mathrm{d}u_1\mathrm{d}u_1 \\
&= k^4 \sum_{\substack{v_1,v_1'=0\\ v_2,v_2'=0}}^{k-1} \Upsilon_k(v_1,v_2)\Upsilon_k(v_1',v_2')\Gamma_k(v_1,v_1')\Gamma_k(v_2,v_2').
\end{aligned}$$

Second, from the definition of $c_{n,k}\left(.\right)$ in Equation (8), we have

$$\int_{[0,1]^2} c_{k,n}(\mathbf{u})\mathrm{d}\mathbf{u} = \sum_{v_1,v_2=0}^{k} C_n\left(\frac{v_1}{k},\frac{v_2}{k}\right)\int_0^1\int_0^1 P'_{v_1,k}(u_1)P'_{v_2,k}(u_2)\mathrm{d}u_1\mathrm{d}u_2.$$

As $\int_{[0,1]} P'_{v_1,k}(u) = P_{v_1,k}(u)|_0^1 = \mathbb{I}\{v_1=0\}+\mathbb{I}\{v_1=k\}$, the last integral is equal to 1.

■

Proof of Theorem 1 for $d=2$. The following proof corresponds to the bivariate case. For the more general case $d>2$, the proof can be obtained in a similar way. For the bivariate case ($d=2$), we will show that the random variable

$$\mathcal{I}_{n,k} := nk^{-2}\left\{\frac{I_n - 2^{-2}\pi n^{-1}k}{2^{1/2}\sqrt{\left\{\sum_{v_1,v_2=0}^{k-1}\Gamma_k^2(v_1,v_2)\right\}^2 - k^{-4}}}\right\} \tag{A3}$$

is asymptotically normally distributed. First, observe that dealing with term I_n in (A3) is quite tricky since it involves the *pseudo-observations* $\mathbf{V}_1,\ldots,\mathbf{V}_n$. Thus, we consider $\widetilde{I}_n = \int_{[0,1]^2}\{\widetilde{c}_{k,n}(\mathbf{u})-1\}^2\mathrm{d}\mathbf{u}$, where for $\widetilde{\mathbf{V}}_i = (F_1(X_{i,1}),F_2(X_{i,2}))$,

$$\widetilde{c}_{k,n}(\mathbf{u}) := \sum_{v_1,v_2=0}^{k}\widetilde{C}_n\left(\frac{v_1}{k},\frac{v_2}{k}\right)P'_{v_1,k}(u_1)P'_{v_2,k}(u_2) \quad \text{and} \quad \widetilde{C}_n(\mathbf{u}) = n^{-1}\sum_{i=1}^{n}\mathbb{I}\{\widetilde{\mathbf{V}}_i \le \mathbf{u}\}.$$

The new term $\widetilde{I}_n$ is just a version of I_n in which the pseudo-observations $\mathbf{V}_1,\ldots,\mathbf{V}_n$ have been replaced with *"uniformized" observations* $\widetilde{\mathbf{V}}_1,\ldots,\widetilde{\mathbf{V}}_n$. Under the null hypothesis, $\widetilde{\mathbf{V}}_i = (\widetilde{V}_{i,1},\widetilde{V}_{i,2})$ are independent and uniformly distributed random variables.

We now define a new term $\widetilde{\mathcal{I}}_{n,k}$ which is equal to the term in the right hand side of Equation (A3) after replacing I_n by $\widetilde{I}_n$. In the following, the proof of Theorem 1 will be obtained in two steps. In a first step, we show that $\widetilde{\mathcal{I}}_{n,k}$ is asymptotically normally distributed and in a second step we show that the difference $\mathcal{I}_{n,k}-\widetilde{\mathcal{I}}_{n,k}$ is negligible.

A.1. *Asymptotic normality of* $\widetilde{\mathcal{I}}_{n,k}$

Using the decomposition in the proof of Proposition 4, we can obtain the following decomposition:

$$\widetilde{I}_n = \widetilde{I}_{1n} + \widetilde{I}_{2n} - 1,$$

where

$$\begin{aligned}
\widetilde{I}_{1n} &= k^4 n^{-2} \sum_{i=1}^{n} \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1, v_2)\} \Gamma_k(v_1, v_1) \Gamma_k(v_2, v_2) \\
\widetilde{I}_{2n} &= 2k^4 n^{-2} \sum_{i<j}^{n} P_n(\widetilde{\mathbf{V}}_i, \widetilde{\mathbf{V}}_j),
\end{aligned}$$

with $P_n(\widetilde{\mathbf{V}}_i, \widetilde{\mathbf{V}}_j) = \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1, v_2)\} \mathbb{I}\{\widetilde{V}_j \in A_k(v_1', v_2')\} \Gamma_k(v_1, v_1') \Gamma_k(v_2, v_2')$.

We start by studying the first term $\widetilde{I}_{1n}$. As $\mathrm{E}\left(\mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1, v_2)\}\right) = k^{-2}$, we get

$$\mathrm{E}\left(\widetilde{I}_{1n}\right) = k^2 n^{-1} \left\{ \sum_{v_1=0}^{k-1} \Gamma_k^2(v_1, v_1) \right\}.$$

Next, using Lemma 2 in Janssen et al. (2014), we have

$$\sum_{v_1=0}^{k-1} P_{v_1,k-1}^2(u) = \frac{k^{-1/2}}{\sqrt{4\pi u(1-u)}} + o(k^{-1/2}).$$

Then,

$$\mathrm{E}\left(\widetilde{I}_{1n}\right) = \frac{\pi}{4} k n^{-1} + o(k n^{-1}).$$

Thereafter, as $\mathrm{Var}(\ \mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1, v_2)\}) = k^{-2} - k^{-4}$, we deduce that

$$\mathrm{Var}\left(\widetilde{I}_{1n}\right) = k^8 n^{-3} (k^{-2} - k^{-4}) \left\{ \sum_{v_1=0}^{k-1} \Gamma_k^2(v_1, v_1) \right\}^2.$$

Then, from Lemma 5, we can conclude that

$$\widetilde{I}_{1n} = \mathrm{E}\left(\widetilde{I}_{1n}\right) + \left\{ \widetilde{I}_{1n} - \mathrm{E}\left(\widetilde{I}_{1n}\right) \right\} = \frac{\pi}{4} k n^{-1} + O_{\mathrm{P}}(n^{-3/2} k^{3/2}).$$

We now turn our attention to the second term $\widetilde{I}_{2n}$. Observe that

$$\begin{aligned}
\mathrm{E}\left(P_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j)\right) &= \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathrm{E}\left\{\mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1,v_2)\}\mathbb{I}\{\widetilde{\mathbf{V}}_j \in A_k(v_1',v_2')\}\right\}\Gamma_k(v_1,v_1')\Gamma_k(v_2,v_2') \\
&= k^{-4}\left\{\sum_{v_1,v_1'=0}^{k-1}\Gamma_k(v_1,v_1')\right\}^2, \qquad \text{(A4)}
\end{aligned}$$

where the last equality follows from the independence between $\widetilde{\mathbf{V}}_i$ and $\widetilde{\mathbf{V}}_j$ when $i \neq j$. Using Lemma 4, we obtain $\mathrm{E}\left(\widetilde{I}_{2n}\right) = \frac{n-1}{n}$.

In the following, denote $\widetilde{P}_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j) = P_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j) - k^{-4}$. Hence, we can write $\widetilde{I}_{2n} - \frac{n-1}{n} = 2k^4 n^{-1} U_n$, where $U_n = n^{-1}\sum_{i<j}^{n} \widetilde{P}_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j)$.

Let us now show that the random variable U_n is a U-statistic. First, by construction, $\widetilde{P}_n(\cdot,\cdot)$ is centred and symmetric. Second, $\widetilde{P}_n(\cdot,\cdot)$ is degenerated, i.e. for any $\mathbf{v} \in (0,1)^2$, $\mathrm{E}\left(\widetilde{P}_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j) \mid \widetilde{\mathbf{V}}_i = \mathbf{v}\right) = 0$. Indeed, denote by $(v_1^\star, v_2^\star)$ the unique pair of integers (v_1,v_2) such that $\mathbf{v} \in A_k(v_1,v_2)$. Then, as $\mathrm{E}\left(\mathbb{I}\{\widetilde{\mathbf{V}}_j \in A_k(v_1',v_2')\} \mid \widetilde{\mathbf{V}}_i = \mathbf{v}\right) = k^{-2}$, we have:

$$\begin{aligned}
\mathrm{E}\left(P_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j) \mid \widetilde{\mathbf{V}}_i = \mathbf{v}\right) = & \quad k^{-2}\sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\left\{\mathbf{v} \in A_k(v_1,v_2)\right\}\Gamma_k(v_1,v_1')\Gamma_k(v_2,v_2') \\
&= k^{-2}\sum_{\mathbf{v}'=0}^{k-1}\Gamma_k(v_1^\star,v_1')\Gamma_k(v_2^\star,v_2') = k^{-2}\left\{\sum_{v_1'=0}^{k-1}\Gamma_k(v_1^\star,v_1')\right\}^2.
\end{aligned}$$

The latter is equal to k^{-4} using Lemma 4. Hence, $\mathrm{E}\left(\widetilde{P}_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j) \mid \widetilde{\mathbf{V}}_i = \mathbf{v}\right) = 0$.

To show the asymptotic normality of U_n, we use the following lemma that establishes the central limit theorem for the U-statistics.

LEMMA 2 (Hall (1984)) *Let $\{\widetilde{\mathbf{V}}_i : i = 1,\ldots,n\}$ be an i.i.d. sequence. Consider the U-statistic $U_n \equiv \frac{1}{n}\sum_{1\le i<j\le n}\widetilde{P}_n(\widetilde{\mathbf{V}}_i,\widetilde{\mathbf{V}}_j)$, where the symmetric variable function $\widetilde{P}_n$ is centered (i.e., $\mathrm{E}[\widetilde{P}_n(\widetilde{\mathbf{V}}_1,\widetilde{\mathbf{V}}_2)] = 0$) and degenerated. Let*

$$\sigma_n^2 \equiv \mathrm{E}[\widetilde{P}_n^2(\widetilde{\mathbf{V}}_1,\widetilde{\mathbf{V}}_2)], \quad \widetilde{\Pi}_n(\mathbf{v}_1,\mathbf{v}_2) \equiv \mathrm{E}[\widetilde{P}_n(\widetilde{\mathbf{V}}_1,\mathbf{v}_1)\widetilde{P}_n(\widetilde{\mathbf{V}}_1,\mathbf{v}_2)].$$

Then, if

$$\lim_{n\to\infty}\frac{\mathrm{E}[\widetilde{\Pi}_n^2(\widetilde{\mathbf{V}}_1,\widetilde{\mathbf{V}}_2)] + n^{-1}\mathrm{E}[\widetilde{P}_n^4(\widetilde{\mathbf{V}}_1,\widetilde{\mathbf{V}}_2)]}{\sigma_n^4} = 0, \qquad \text{(A5)}$$

the random variable $\sqrt{2}\sigma_n^{-1}U_n$ converges in distribution to a standard normal.

Now, in order to apply Lemma 2 we need to check if Equation (A5) is satisfied. Hence, we need to calculate the three quantities involved in that equation. We start with σ_n^2.

First, recall the definition of P_n and observe that

$$
\begin{aligned}
P_n^2(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2) &= \sum_{\substack{\mathbf{v},\mathbf{v}'=0 \\ \mathbf{w},\mathbf{w}'=0}}^{k-1} \mathbb{I}\left\{\widetilde{\mathbf{V}}_1 \in A_k(v_1, v_2)\right\} \mathbb{I}\left\{\widetilde{\mathbf{V}}_2 \in A_k(v_1', v_2')\right\} \mathbb{I}\left\{\widetilde{\mathbf{V}}_1 \in A_k(w_1, w_2)\right\} \\
&\quad \times \mathbb{I}\left\{\widetilde{\mathbf{V}}_2 \in A_k(w_1', w_2')\right\} \Gamma_k(v_1, v_1')\Gamma_k(v_2, v_2')\Gamma_k(w_1, w_1')\Gamma_k(w_2, w_2').
\end{aligned}
$$

As $\mathbb{I}\{\widetilde{\mathbf{V}}_1 \in A_k(v_1, v_2)\}\mathbb{I}\{\widetilde{\mathbf{V}}_1 \in A_k(w_1, w_2)\} = 0$ unless $(v_1, v_2) = (w_1, w_2)$, we obtain

$$
P_n^2(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2) = \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\left\{\widetilde{\mathbf{V}}_1 \in A_k(v_1, v_2)\right\} \mathbb{I}\left\{\widetilde{\mathbf{V}}_2 \in A_k(v_1', v_2')\right\} \Gamma_k^2(v_1, v_1')\Gamma_k^2(v_2, v_2').
$$

Since $\widetilde{\mathbf{V}}_1$ and $\widetilde{\mathbf{V}}_2$ are independent and uniformly distributed random variables, it follows that

$$
\begin{aligned}
\mathrm{E}\{P_n^2(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)\} &= \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathrm{E}\left(\mathbb{I}\left\{\widetilde{\mathbf{V}}_1 \in A_k(v_1, v_2)\right\} \mathbb{I}\{\widetilde{\mathbf{V}}_2 \in A_k(v_1', v_2')\}\right) \Gamma_k^2(v_1, v_1')\Gamma_k^2(v_2, v_2') \\
&= k^{-4}\left\{\sum_{\mathbf{v}=0}^{k-1} \Gamma_k^2(v_1, v_2)\right\}^2. \qquad \text{(A6)}
\end{aligned}
$$

Then, from Lemma 5

$$
\begin{aligned}
\sigma_n^2 &= \mathrm{E}[P_n^2(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)] - [\mathrm{E}(P_n^2(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2))]^2 \\
&= k^{-4}\left\{\sum_{\mathbf{v}=0}^{k-1} \Gamma_k^2(v_1, v_2)\right\}^2 - k^{-8} \\
&= O(k^{-7}). \qquad \text{(A7)}
\end{aligned}
$$

Now, focusing on the second term of the numerator in Equation (A5), we show that

$$
\begin{aligned}
\mathrm{E}[\widetilde{P}_n^4(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)] &= \sum_{\ell=0}^{4} \binom{4}{\ell} \mathrm{E}[P_n^\ell(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)](-\mathrm{E}[P_n(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)])^{4-\ell} \\
&= k^{-16} \sum_{\ell=0}^{4} \binom{4}{\ell} \mathrm{E}[P_n^\ell(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)](k)^{4\ell}(-1)^{4-\ell}. \qquad \text{(A8)}
\end{aligned}
$$

Similar calculations for computing the term σ_n^2 leads to the following:

$$
\begin{aligned}
\mathrm{E}\{P_n^3(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)\} &= k^{-4}\left\{\textstyle\sum_{\mathbf{v}=0}^{k-1} \Gamma_k^3(v_1, v_2)\right\}^2, \\
\mathrm{E}\{P_n^4(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)\} &= k^{-4}\{\textstyle\sum_{\mathbf{v}=0}^{k-1} \Gamma_k^4(v_1, v_2)\}^2.
\end{aligned}
$$

Plug-in the above results into the Equation (A8) and using Equations (A4) and (A6) and

Lemma 5 allows us to conclude that

$$\begin{aligned}
\mathrm{E}[\widetilde{P}_n^4(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)] &= -3k^{-16} + 6k^{-12}\{\sum_{\mathbf{v}=0}^{k-1} \Gamma_k^2(v_1, v_2)\}^2 \\
&\quad - 4k^{-8}\{\sum_{\mathbf{v}=0}^{k-1} \Gamma_k^3(v_1, v_2)\}^2 + k^{-4}\{\sum_{\mathbf{v}=0}^{k-1} \Gamma_k^4(v_1, v_2)\}^2 \\
&= O(k^{-13}). \qquad \text{(A9)}
\end{aligned}$$

The first term of the denominator in Equation (A5) requires more attention. For any $\mathbf{z}, \mathbf{z}' \in (0,1)$, we expand the product and show that:

$$\widetilde{\Pi}_n(\mathbf{z}, \mathbf{z}') = \mathrm{E}\{\widetilde{P}_n(\widetilde{\mathbf{V}}_1, \mathbf{z})\widetilde{P}_n(\widetilde{\mathbf{V}}_1, \mathbf{z}')\} = \mathrm{E}\{P_n(\widetilde{\mathbf{V}}_1, \mathbf{z})P_n(\widetilde{\mathbf{V}}_1, \mathbf{z}')\} - k^{-8}.$$

If we denote $S_k(a,b) = \sum_{v=0}^{k-1} \Gamma_k(v,a)\Gamma_k(v,b)$, we have

$$\begin{aligned}
&\mathrm{E}\{P_n(\widetilde{\mathbf{V}}_1, \mathbf{z})P_n(\widetilde{\mathbf{V}}_1, \mathbf{z}')\} \\
&= k^{-2} \sum_{\mathbf{v},\mathbf{v}',\mathbf{v}''=0}^{k-1} \mathbb{I}\{\mathbf{z} \in A_k(v_1', v_2')\}\mathbb{I}\{\mathbf{z}' \in A_k(v_1'', v_2'')\}\Gamma_k(v_1, v_1')\Gamma_k(v_2, v_2')\Gamma_k(v_1, v_1'')\Gamma_k(v_2, v_2'') \\
&= k^{-2} \sum_{\mathbf{v}',\mathbf{v}''=0}^{k-1} \mathbb{I}\{\mathbf{z} \in A_k(v_1', v_2')\}\mathbb{I}\{\mathbf{z}' \in A_k(v_1'', v_2'')\}S_k(v_1', v_1'')S_k(v_2', v_2''),
\end{aligned}$$

Then,

$$\mathrm{E}^2\{P_n(\widetilde{\mathbf{V}}_1, \mathbf{z})P_n(\widetilde{\mathbf{V}}_1, \mathbf{z}')\} = k^{-4} \sum_{\mathbf{v}',\mathbf{v}''=0}^{k-1} \mathbb{I}\{\mathbf{z} \in A_k(v_1', v_2')\}\mathbb{I}\{\mathbf{z}' \in A_k(v_1'', v_2'')\}S_k^2(v_1', v_1'')S_k^2(v_2', v_2'').$$

Finally, as $\mathrm{E}\{\mathbb{I}\{\widetilde{\mathbf{V}}_1 \in A_k(v_1', v_2')\}\mathbb{I}\{\widetilde{\mathbf{V}}_2 \in A_k(v_1', v_2')\}\} = k^{-4}$, we deduce that

$$\begin{aligned}
\mathrm{E}\,(\widetilde{\Pi}_n^2(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)) &= k^{-8}\left\{\sum_{v_1,v_2=0}^{k-1} S_k^2(v_1, v_2)\right\}^2 - k^{-12} \\
&= O(k^{-12}) \qquad \text{from Lemma 5.} \qquad \text{(A10)}
\end{aligned}$$

Now, from Equations (A7), (A9) and (A10), we have

$$\frac{\mathrm{E}\left[\widetilde{\Pi}_n^2(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)\right] + n^{-1}\mathrm{E}\left[\widetilde{P}_n^4(\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2)\right]}{\sigma_n^4} \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Thus, Lemma 2 applies and we conclude that the term $\sqrt{2}\sigma_n^{-1}U_n$ converges in distribution to a standard normal. Hence, we conclude the asymptotic normality of $\widetilde{\mathcal{I}}_{n,k}$.

A.2. *Asymptotic negligibility of $\mathcal{I}_{n,k} - \widetilde{\mathcal{I}}_{n,k}$*

From Stute (1984) and under $\mathcal{H}_0$, we have

$$\begin{aligned} C_n(u,v) - uv &= \widetilde{C}_n(u,v) - uv - v\{\widetilde{C}_n(u,1) - u\} \\ &\quad - u\{\widetilde{C}_n(1,v) - v\} + \xi_n(u,v), \end{aligned} \tag{A11}$$

where $\sup_{u,v} |\xi_n(u,v)| = O_{\mathrm{P}}(n^{-3/4} \log\log(n))$.

Write

$$\bar{\xi}_n = \sup_{u,v} \left| \sum_{\mathbf{v}=0}^{k} \xi_n(u,v) P'_{v_1,k}(u) P'_{v_2,k}(v) \right|$$

and denote $\widehat{I}_n = \int_{[0,1]^2} J^2_{n,k}(u,v) \mathrm{d}u\mathrm{d}v$, where

$$J_{n,k}(u,v) := n^{-1}k^2 \sum_{i=1}^{n} \sum_{\mathbf{v}=0}^{k-1} \widetilde{\lambda}_i(v_1,v_2) P_{v_1,k-1}(u) P_{v_2,k-1}(v) + 1,$$

with $\widetilde{\lambda}_i(v_1,v_2) = \mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1,v_2)\} - k^{-1}\mathbb{I}\{\widetilde{V}_{1i} \in A_k(v_1)\} - k^{-1}\mathbb{I}\{\widetilde{V}_{2i} \in A_k(v_2)\}$. Hence, from Equation (A11), we obtain

$$-\bar{\xi}_n^2 + \widehat{I}_n \le \int_{[0,1]^2} \{c_{k,n}(u,v) - 1\}^2 \le \bar{\xi}_n^2 + \widehat{I}_n.$$

If we denote $A_k = \int_{[0,1]^2} \sum_{\mathbf{v}=0}^{k} |P'_{v_1,k}(u_1) P'_{v_2,k}(u_2)| \mathrm{d}u_1 \mathrm{d}u_2$, which is at most $O(k)$, then $I_n - \widehat{I}_n = O_{\mathrm{P}}(n^{-3/2} \log\log^2(n) k^2)$ and therefore $nk^{-1/2}(I_n - \widehat{I}_n) = o_p(1)$.

Now, we need to show that $nk^{-1/2}(\widehat{I}_n - \widetilde{I}_n)$ is negligible. Since $\int_{[0,1]^2} J_{n,k}(u,v)dudv = 0$ and using similar arguments as in the proof of Proposition 4, we obtain

$$\widehat{I}_n = n^{-2}k^4 \sum_{i,j=1}^{n} \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \widetilde{\lambda}_i(v_1,v_2) \widetilde{\lambda}_j(v'_1,v'_2) \Gamma_k(v_1,v'_1) \Gamma_k(v_2,v'_2) - 1.$$

Hence, expanding the product $\widetilde{\lambda}_i(v_1,v_2)\widetilde{\lambda}_j(v'_1,v'_2)$ leads to the following decomposition: $\widehat{I}_n = \widetilde{I}_n + \sum_{k=1}^{5} \widehat{I}_n^{(j)}$, where the five terms $\widehat{I}_n^{(j)}$ $(j = 1, \ldots, 5)$ are computed below. From Lemma 4, we have

$$\begin{aligned} \widehat{I}_n^{(1)} &= n^{-2}k^2 \sum_{i,j=1}^{n} \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\{\widetilde{V}_{1i} \in A_k(v_1)\} \mathbb{I}\{\widetilde{V}_{1j} \in A_k(v'_1)\} \Gamma_k(v_1,v'_1) \Gamma_k(v_2,v'_2) \\ &= n^{-2} \sum_{i,j=1}^{n} \sum_{v_1,v'_1=0}^{k-1} \mathbb{I}\{\widetilde{V}_{1i} \in A_k(v_1)\} \mathbb{I}\{\widetilde{V}_{1j} \in A_k(v'_1)\}, \\ &= 1 \end{aligned}$$

Similarly,

$$\begin{aligned}\widehat{I}_n^{(2)} &= n^{-2}k^2 \sum_{i,j=1}^{n} \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\{\widetilde{V}_{2i} \in A_k(v_2)\}\mathbb{I}\{\widetilde{V}_{2j} \in A_k(v_2')\}\Gamma_k(v_1, v_1')\Gamma_k(v_2, v_2') \\ &= 1.\end{aligned}$$

Furthermore,

$$\begin{aligned}\widehat{I}_n^{(3)} &= 2n^{-2}k^2 \sum_{i,j=1}^{n} \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\{\widetilde{V}_{1i} \in A_k(v_1)\}\mathbb{I}\{\widetilde{V}_{2j} \in A_k(v_2')\}\Gamma_k(v_1, v_1')\Gamma_k(v_2, v_2') \\ &= 2,\end{aligned}$$

and

$$\begin{aligned}\widehat{I}_n^{(4)} &= -2n^{-2}k^3 \sum_{i,j=1}^{n} \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1, v_2)\}\mathbb{I}\{\widetilde{V}_{1j} \in A_k(v_1')\}\Gamma_k(v_1, v_1')\Gamma_k(v_2, v_2') \\ &= -2n^{-2}k^2 \sum_{i,j=1}^{n} \sum_{\mathbf{v},v_1'=0}^{k-1} \mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1, v_2)\}\mathbb{I}\{\widetilde{V}_{1j} \in A_k(v_1')\}\Gamma_k(v_1, v_1') \\ &= -2n^{-2}k^2 \sum_{i,j=1}^{n} \sum_{v_1,v_1'=0}^{k-1} \mathbb{I}\{\widetilde{V}_{1i} \in A_k(v_1)\}\mathbb{I}\{\widetilde{V}_{1j} \in A_k(v_1')\}\Gamma_k(v_1, v_1').\end{aligned}$$

Write $\widehat{I}_n^{(4)} = \widehat{I}_{1n}^{(4)} + \widehat{I}_{2n}^{(4)}$, where

$$\begin{aligned}\widehat{I}_{1n}^{(4)} &= -2n^{-2}k^2 \sum_{i=0}^{n} \sum_{v_1,v_1'=0}^{k-1} \mathbb{I}\{\widetilde{V}_{1i} \in A_k(v_1)\}\Gamma_k(v_1, v_1') = -2n^{-2}k^2 \\ \widehat{I}_{2n}^{(4)} &= -4n^{-2}k^2 \sum_{i<j}^{n} \sum_{v_1,v_1'=0}^{k-1} \mathbb{I}\{\widetilde{V}_{1i} \in A_k(v_1)\}\mathbb{I}\{\widetilde{V}_{1j} \in A_k(v_1')\}\Gamma_k(v_1, v_1').\end{aligned}$$

Using similar arguments as in Section A.1, we can show that $\widehat{I}_n^{(4)} + 2 = O_{\mathrm{P}}(n^{-1}k^{-3/4} + n^{-2}k^2)$ and

$$\begin{aligned}\widehat{I}_n^{(5)} &= -2n^{-2}k^3 \sum_{i,j=1}^{n} \sum_{\mathbf{v},\mathbf{v}'=0}^{k-1} \mathbb{I}\{\widetilde{\mathbf{V}}_i \in A_k(v_1, v_2)\}\mathbb{I}\{\widetilde{V}_{2j} \in A_k(v_2')\}\Gamma_k(v_1, v_1')\Gamma_k(v_2, v_2') \\ &= -2n^{-2}k^2 \sum_{i,j=1}^{n} \sum_{v_2,v_2'=0}^{k-1} \mathbb{I}\{\widetilde{V}_{2i} \in A_k(v_2)\}\mathbb{I}\{\widetilde{V}_{2j} \in A_k(v_2')\}\Gamma_k(v_2, v_2') \\ &= -2 + O_{\mathrm{P}}(n^{-1}k^{-3/4} + n^{-2}k^2).\end{aligned}$$

We conclude that $nk^{-1/2} \sum_{j=1}^{5} \widehat{I}_n^{(j)} = o_p(1)$. Hence, $nk^{-1/2}(\widehat{I}_n - \widetilde{I}_n)$ is negligible.

■

Proof of Corollary 1. In Lemma 5 of Appendix B, it is shown that there exists a constant $R > 0$ such that

$$k^{3/2} \sum_{v_1,v_2=0}^{k-1} \Gamma_k(v_1,v_2)^2 \longrightarrow R^2, \text{ as } k \to \infty.$$

An application of Theorem 1 together with Slutky's Lemma yields to the result.

■

Proof of Proposition 5. We start with the following decomposition:

$$\begin{aligned}\int (\hat{c}_{n,k}(\mathbf{u}) - 1)^2 \, d\mathbf{u} &= \int (\hat{c}_{n,k}(\mathbf{u}) - c(\mathbf{u}))^2 \, d\mathbf{u} + \int (c(\mathbf{u}) - 1)^2 \, d\mathbf{u} \\ &\quad + 2 \int (\hat{c}_{n,k}(\mathbf{u}) - c(\mathbf{u}))\,(c(\mathbf{u}) - 1) \, d\mathbf{u}, \\ &= I_{1,n} + I_{2,n} + I_{3,n}. \end{aligned} \tag{A12}$$

First, using similar arguments as in the proof of Theorem 1, we can show that

$$nk^{-d/4} \left(\frac{I_{1,n} - n^{-1}k^{d/2}2^{-d}\pi^{d/2}}{\sqrt{2}R^d} \right) = O_p(1). \tag{A13}$$

Second, it was shown in Bouezmarni et al. (2010) that $\|c_{k,n}(\mathbf{u}) - c(\mathbf{u})\|_\infty = o_p(1)$. We can then deduce that

$$nk^{-d/4} I_{3,n} = o_p(nk^{-d/4}). \tag{A14}$$

Finally, from (A12), (A13), (A14) and under a fixed alternative, we obtain

$$nk^{-d/4} \left(\frac{I_n - n^{-1}k^{d/2}B}{\sqrt{2}R^d} \right) \xrightarrow{P} \infty.$$

■

Proof of Theorem 2. Using a Taylor expansion of the function $g(x) = x\log(x)$ around $x^* = 1$, we obtain

$$\begin{aligned}\delta_n(c) &= \frac{1}{2} \int_{[0,1]^d} (c_{k,n}(\mathbf{u}) - 1)^2 \, \mathrm{d}\mathbf{u} + O_p\left(\int_{[0,1]^d} (c_{k,n}(\mathbf{u}) - 1)^3 \, \mathrm{d}\mathbf{u} \right) \\ &= \frac{1}{2} I_n + O_p\left(\int_{[0,1]^d} (c_{k,n}(\mathbf{u}) - 1)^3 \, \mathrm{d}\mathbf{u} \right).\end{aligned}$$

Using Proposition 3 in Bouezmarni et al. (2010) and the fact that

$$\left| \int_{[0,1]^d} (c_{k,n}(\mathbf{u}) - 1)^3 \, \mathrm{d}\mathbf{u} \right| \le I_n \times \sup_{\mathbf{u} \in [0,1]^d} |c_{k,n}(\mathbf{u}) - 1|$$

we conclude that the asymptotic normality of $\delta_n(c)$ is similar to that of $\frac{1}{2}I_n$, which concludes the proof of Theorem 2.

■

Appendix B. Technical computations

B.1. *Some preliminaries*

We begin this section by establishing some properties of the function $\Gamma_k(\cdot,\cdot)$. Before, observe that:

$$\begin{aligned}
\Gamma_k(v_1,v_2) &= \int_0^1 P_{v_1,k-1}(u)P_{v_2,k-1}(u)\mathrm{d}u \\
&= \binom{k-1}{v_1}\binom{k-1}{v_2}\beta(v_1+v_2+1,2k-1-v_1-v_2) \\
&= \frac{\{(k-1)!\}^2}{(2k-1)!}\left\{\frac{(v_1+v_2)!(2k-2-v_1-v_2)!}{(v_1)!(v_2)!(k-1-v_1)!(k-1-v_2)!}\right\}. \qquad \text{(B1)}
\end{aligned}$$

Lemma 3 *The function $\Gamma_k(\cdot,\cdot)$ satisfies:*

(1) $\max_{0\le v_2\le k-1}\Gamma_k(v_1,v_2)=\Gamma_k(v_1,v_1)$, *for fixed* $v_1 \quad (0\le v_1\le k-1)$*;*
(2) $\max_{0\le v_1,v_2\le k-1}\Gamma_k(v_1,v_2)=(2k-1)^{-1}$.

Proof. To show Item (1), we calculate the following ratio

$$\begin{aligned}
\frac{\Gamma_k(v_1,v_2)}{\Gamma_k(v_1,v_2+1)} &= \frac{(v_2+1)}{(v_1+v_2+1)}\frac{(2k-2-v_1-v_2)}{(k-1-v_2)} \\
&= \frac{-v_2^2+v_2(2k-3-v_1)+2k-2-v1}{-v_2^2+v_2(k-2-v_1)+(k-1)(v_1+1)}.
\end{aligned}$$

Hence, the ratio is less than one, which means that $\Gamma_k(v_1,v_2)\le\Gamma_k(v_1,v_2+1)$, if and only if $(k-1)v_2\le kv_1-(k-1)$. In other words, if and only if $v_2\le\frac{k}{k-1}v_1-1$. Item (1) is therefore proven since v_1 and v_2 are between 0 and $k-1$. We now show Item (2). Similarly,

$$\begin{aligned}
\frac{\Gamma_k(v_1,v_1)}{\Gamma_k(v_1+1,v_1+1)} &= \frac{(v_1+1)^2}{(2v_1+2)(2v_1+1)}\frac{(2k-2-2v_1)(2k-3-2v_1)}{(k-1-v_1)^2} \\
&= \frac{(v_1+1)}{(2v_1+1)}\frac{(2k-3-2v_1)}{(k-1-v_1)} \\
&= \frac{-2v_1^2+v_1(2k-5)+2k-3}{-2v_1^2+v_1(2k-3)+k-1}.
\end{aligned}$$

Again, the latter is less than one provided $2v_1\ge k-2$. It follows that if $v_1\ge\frac{k-2}{2}$, then the maximum is achieved at $\Gamma_k(k-1,k-1)=(2k-1)^{-1}$. Otherwise, the maximum is at $\Gamma_k(0,0)=(2k-1)^{-1}$. ■

B.2. *Computation of* $\sum_{v=0}^{k-1}\Gamma_k(v_1,v_2)$ *and* $\sum_{v_1=0}^{k-1}\Gamma_k(v_1,v_2)$

The results of this section are given in the following lemma.

LEMMA 4 *The function* $\Gamma_k(\cdot,\cdot)$ *satisfies:*

(1) $\sum_{v_1=0}^{k-1}\Gamma_k(v_1,v_2)=k^{-1}$, *for any fixed* v_2*;*
(2) $\sum_{\mathbf{v}=0}^{k-1}\Gamma_k(v_1,v_2)=1$.

Proof. Recall that $\Gamma_k(v_1,v_2)=\int_{[0,1]}P_{v_1,k-1}(u)P_{v_2,k-1}(u)\mathrm{d}u$. Using the fact the sum of the binomial probabilities is equal to 1 and because $\int_{[0,1]}P_{v_1,k-1}(u)\mathrm{d}u=k^{-1}$, we deduce that

$$\begin{aligned}\sum_{v_1=0}^{k-1}\Gamma_k(v_1,v_2) &= \int_{[0,1]}\left\{\sum_{v_1=0}^{k-1}P_{v_1,k-1}(u)\right\}P_{v_2,k-1}(u)\mathrm{d}u\\ &= \int_{[0,1]}P_{v_2,k-1}(u)\mathrm{d}u=k^{-1}.\end{aligned}$$

Then Item (1) is proved. Item (2) is a direct result from (1).

■

B.3. *Computation of* $\sum_{\mathbf{v}=0}^{k-1}\Gamma_k^j(v_1,v_2)$, $j=2,3,4$ *and* $\sum_{\mathbf{v}=0}^{k-1}S_k^2(v_1,v_2)$

The following lemma provides the orders of sums that involve either $\Gamma_k(\cdot,\cdot)$ or $S_k(\cdot,\cdot)$.

LEMMA 5 *The functions* $\Gamma_k(\cdot,\cdot)$ *and* $S_k(\cdot,\cdot)$ *satisfy:*

(1) *There exists a constant* $R>0$ *such that* $k^{3/2}\sum_{\mathbf{v}=0}^{k-1}\Gamma_k^2(v_1,v_2)\to R^2$*;*
(2) $\sum_{\mathbf{v}=0}^{k-1}\Gamma_k^3(v_1,v_2)=O(k^{-3}\log(k))$*;*
(3) $\sum_{\mathbf{v}=0}^{k-1}\Gamma_k^4(v_1,v_2)=O(k^{-9/2})$*;*
(4) $\sum_{\mathbf{v}=0}^{k-1}S_k^2(v_1,v_2)=O(k^{-7/2})$.

As the proof of lemma is rather long and technical, it is divided into subsections.

B.3.1. Proof of Lemma 5-(1)

The proof is done in two steps. First, in Part I we will show that $\sum_{\mathbf{v}=0}^{k-1}\Gamma_k^2(v_1,v_2)=O(k^{-3/2})$. In Part II, we will demonstrate that there exists a constant $C>0$ such that $\sum_{\mathbf{v}=0}^{k-1}\Gamma_k^2(v_1,v_2)\geq Ck^{-3/2}$. In a final part, the proof of Item (1) will follow from an application of the monotone convergence theorem.

Part I: First, from the symmetry of $\Gamma_k(\cdot,\cdot)$, we have

$$\sum_{\mathbf{v}=0}^{k-1}\Gamma_k^2(v_1,v_2)\leq 2\sum_{v_1\leq v_2}^{k-1}\Gamma_k^2(v_1,v_2)\leq 4\sum_{v_1\leq v_2}^{[\frac{k-1}{2}]}\Gamma_k^2(v_1,v_2),$$

where [.] denotes the integer part.

As a starting point, take L as the smallest integer such that $2^L > \sqrt{k-1}$. Write $q_j = 2^j$, $A_j = \{(v_1, v_2) \in [0, 2^L] : q_j \leq v_2 < q_{j+1}$ and $v_1 \leq v_2\}$. Hence, from Lemma 3, we have $\max_{(v_1,v_2)\in A_j} \Gamma_k(v_1, v_2) = \Gamma_k(q_j, q_j) = \nu_k(\frac{q_j}{k-1})$, where $\nu_k(\cdot)$ is defined in Lemma 6. Using Lemma 6, we have

$$\nu_k(\frac{q_j}{k-1}) \leq \frac{e^{\frac{1}{12(k-1)}}\sqrt{k-1}}{(2k-1)\sqrt{\pi q_j(k-1-q_j)}} \leq \frac{e^{\frac{1}{12(k-1)}}\sqrt{2}}{(2k-1)\sqrt{\pi}} \times 2^{-j/2},$$

where the last inequality follows from the fact that $k-1-q_j \geq \frac{k-1}{2}$ and the definition of q_j. Since the number of elements in A_j is bounded by $(q_{j+1} - q_j) \times q_{j+1} = 2 * 2^{2j}$, we have

$$\sum_{v_1 \leq v_2}^{2^L} \Gamma_k^2(v_1, v_2) \leq \frac{4e^{\frac{1}{6(k-1)}}}{(2k-1)^2\pi} \sum_{j=1}^{L} 2^{2j} \times 2^{-j} = \frac{4e^{\frac{1}{6(k-1)}}}{(2k-1)^2\pi}\{2^{L+1} - 1\}.$$

As $2^{L+1} - 1 \leq 4 \times 2^L = 4\sqrt{k-1}$, we deduce that

$$\sum_{v_1 \leq v_2}^{2^L} \Gamma_k^2(v_1, v_2) \leq \frac{16e^{\frac{1}{6(k-1)}}\sqrt{k-1}}{(2k-1)^2\pi}.$$

Next, take L_2 as the smallest integer such that $2^{L_2} > \frac{k-1}{2}$. Write again $q_j = 2^j$, $A_j = \{(v_1, v_2) \in [0, 2^{L_2}] : q_j \leq v_1 < q_{j+1}, v_1 \leq v_2$ and $\frac{v2-v1}{2} \leq \sqrt{q_j}\}$. In a similar way and from Lemma 6, we have

$$\begin{aligned}
\sum_{j=L}^{L_2} \sum_{(v_1,v_2)\in A_j} \Gamma_k^2(v_1, v_2) &\leq \frac{4e^{\frac{1}{6(k-1)}}}{(2k-1)^2\pi} \sum_{j=1}^{L_2} \sqrt{q_j}(q_{j+1} - q_j) \times q_j^{-1} \\
&\leq \frac{8e^{\frac{1}{6(k-1)}}}{(2k-1)^2\pi} \sum_{j=1}^{L_2/2} 2^j \\
&= \frac{8e^{\frac{1}{6(k-1)}}}{(2k-1)^2\pi}\{2^{L_2/2+1} - 1\}.
\end{aligned}$$

Hence,

$$\sum_{j=L}^{L_2} \sum_{(v_1,v_2)\in A_j} \Gamma_k^2(v_1, v_2) \leq \frac{16e^{\frac{1}{6(k-1)}}\sqrt{k-1}}{(2k-1)^2\pi}.$$

Now, let $\bar{L}_j$ be the smallest integer such that $2^{\bar{L}_j} > 2q_j^{1/2}$, and denotes $\alpha_j^{(\ell)} = 2^\ell q_j^{1/2}$. We consider $A_j^{(\ell)} = \Big\{(v_1, v_2) \in [0, 2^{L_2}] : q_j \leq v_1 < q_{j+1}, v_1 \leq v_2$ and $\alpha_j^{(\ell)} \leq \frac{v2-v1}{2} \leq \alpha_j^{(\ell+1)}\Big\}$. Then, for any $(v_1, v_2) \in A_j^{(\ell)}$, we have

$$\Gamma_k(v_1, v_2) \leq \Gamma_k(q_j - \alpha_j^\ell, q_j + \alpha_j^\ell).$$

Hence, from Lemma 10 with $a = \frac{q_j}{(k-1)}$ and $\alpha = \frac{\alpha_j^\ell}{a(k-1)} = \frac{2^\ell}{\sqrt{q_j}}$, we obtain

$$\Gamma_k(v_1, v_2) \leq \frac{1}{2k-1} \frac{e^{\frac{1}{12(k-1)}}}{2\sqrt{\pi}} q_j^{-1/2} e^{-\frac{2^\ell}{6}}, \quad \text{for} \quad v_1, v_2 \text{ in } A_j^{(\ell)}.$$

Finally, we have

$\sum_{j=L}^{L_2} \sum_{\ell=0}^{\bar{L}_j} \sum_{(v_1,v_2)\in A_j^{(\ell)}} \Gamma_k^2(v_1, v_2)$

$$\begin{aligned}
&\leq \frac{1}{(2k-1)^2} \frac{e^{\frac{1}{6(k-1)}}}{4\pi} \sum_{j=L}^{L_2} \sum_{\ell=0}^{\bar{L}_j} (q_{j+1} - q_j)(\alpha_j^{(\ell+1)} - \alpha_j^{(\ell)}) q_j^{-1} e^{-\frac{2^\ell}{3}} \\
&= \frac{1}{(2k-1)^2} \frac{e^{\frac{1}{6(k-1)}}}{4\pi} \sum_{j=L}^{L_2} \sum_{\ell=0}^{\bar{L}_j} 2^\ell q_j^{1/2} e^{-\frac{2^\ell}{3}} \\
&\leq \frac{1}{(2k-1)^2} \frac{e^{\frac{1}{6(k-1)}}}{4\pi} \times \left\{ 2 \sum_{j=1}^{L_2/2} 2^j \right\} \times \left\{ \sum_{\ell \geq 0} 2^\ell e^{-\frac{2^\ell}{3}} \right\} \\
&= \frac{1}{(2k-1)^2} \frac{e^{\frac{1}{6(k-1)}}}{4\pi} \times \left\{ 2(2^{L_2/2+1} - 1) \right\} \times \left\{ \sum_{\ell \geq 0} 2^\ell e^{-\frac{2^\ell}{3}} \right\}
\end{aligned}$$

Since $\sum_{\ell\geq 0} 2^\ell e^{-\frac{2^\ell}{3}}$ converges, there exists a constant S such that

$$\sum_{j=L}^{L_2} \sum_{\ell=0}^{\bar{L}_j} \sum_{(v_1,v_2)\in A_j^{(\ell)}} \Gamma_k^2(v_1, v_2) \leq \frac{1}{(2k-1)^2} \frac{e^{\frac{1}{6(k-1)}} \sqrt{k-1}}{\pi} \times S.$$

This concludes Part I.

Part II: Following similar argument as the one used in Part I, take L as the greatest integer such that $2^L \leq \frac{k-1}{2}$. From Lemma 3, for any $q_j \leq v_1 < q_{j+1}$, we have $\Gamma_k(v_1, v_1) \geq \Gamma_k(q_{j+1}, q_{j+1})$. We consider $A_j = \left\{(v_1, v_2) \in [0, 2^L] : q_j \leq v_1 < q_{j+1}, v_1 \leq v_2 \text{ and } \frac{v2-v1}{2} \leq \sqrt{q_j}\right\}$, where $q_j = 2^j$. Then, from Lemma 11 with $\rho = 1$, for any $(v_1, v_2) \in A_j$, we obtain

$$\Gamma_k(v_1, v_2) \geq \frac{\left(\frac{3}{4}\right)^{\frac{2}{3}} e^{-\frac{10}{8}}}{2k-1} \frac{e^{\frac{-1}{12(k-1)}}}{2\sqrt{\pi}} q_{j+1}^{-1/2}.$$

Since A_j contains at least $(q_{j+1} - q_j) \times q_j^{1/2} = 2^{3j/2}$ elements, we get

$$\begin{aligned}
\sum_{j=1}^{L} \sum_{(v_1,v_2)\in A_j} \Gamma_k^2(v_1, v_2) \quad &\geq \quad \frac{\left(\frac{3}{4}\right)^{\frac{4}{3}} e^{-\frac{5}{2}}}{(2k-1)^2} \frac{e^{\frac{-1}{6(k-1)}}}{4\pi} \sum_{j=1}^{L} 2^{\frac{3j}{2}} 2^{-(j+1)} \\
&\geq \quad \frac{1}{2} \frac{\left(\frac{3}{4}\right)^{\frac{4}{3}} e^{-\frac{5}{2}}}{(2k-1)^2} \frac{e^{\frac{-1}{6(k-1)}}}{4\pi} \{2^{L/2+1} - 1\}.
\end{aligned}$$

This completes the proof because $2^{L/2+1} \geq \sqrt{k-1}$.

B.4. *Proof of Lemma 5-(2)-(4)*

First, from the symmetry of $\Gamma_k(\cdot,\cdot)$, we have

$$\sum_{v_1,v_2=0}^{k-1} \Gamma_k^3(v_1,v_2) \leq 2 \sum_{v_1\leq v_2}^{k-1} \Gamma_k^3(v_1,v_2) \leq 4 \sum_{v_1\leq v_2}^{[\frac{k-1}{2}]} \Gamma_k^3(v_1,v_2).$$

As a starting point, take L as the smallest integer such that $2^L > \frac{k-1}{2}$. Write $q_j = 2^j$ and consider the set $A_j = \{(v_1,v_2) \in [0,2^L] : q_j \leq v_2 < q_{j+1}, v_1 \leq v_2 \text{ and } \frac{v_2-v_1}{2} \leq \sqrt{q_j}\}$. Hence, from the proof of Lemma 3, we obtain $\max_{(v_1,v_2)\in A_j} \Gamma_k(v_1,v_2) = \Gamma_k(q_j,q_j) = \nu_k(\frac{q_j}{k-1})$. With the help of Lemma 6, we have

$$\nu_k\left(\frac{q_j}{k-1}\right) \leq \frac{e^{\frac{1}{12(k-1)}}\sqrt{k-1}}{(2k-1)\sqrt{\pi q_j(k-1-q_j)}} \leq \frac{e^{\frac{1}{12(k-1)}}\sqrt{2}}{(2k-1)\sqrt{\pi}} \times 2^{-j/2},$$

where the last inequality follows from the fact that $k-1-q_j \geq \frac{k-1}{2}$ and the definition of q_j. Since the number of elements in A_j is bounded by $(q_{j+1}-q_j) \times \sqrt{q_{j+1}} = 2 * 2^{\frac{3j}{2}}$, we have

$$\sum_{j=1}^{L} \sum_{(v_1,v_2)\in A_j} \Gamma_k^3(v_1,v_2) \leq \frac{8e^{\frac{1}{4(k-1)}}}{(2k-1)^3\pi^{3/2}} \sum_{j=1}^{L} 2^{\frac{3j}{2}} \times 2^{-\frac{3j}{2}} = \frac{8e^{\frac{1}{4(k-1)}}}{(2k-1)^3\pi^{3/2}} \times L.$$

As $L \leq \log(k-1)+1$, we deduce that

$$\sum_{j=1}^{L} \sum_{(v_1,v_2)\in A_j} \Gamma_k^3(v_1,v_2) \leq \frac{8e^{\frac{1}{4(k-1)}}}{(2k-1)^3\pi^{3/2}} \{\log(k-1)+1\}.$$

We are now ready to use the result developed in Lemma 10. Let $\bar{L}_j$ be the smallest integer such that $2^{\bar{L}_j} > 2q_j^{1/2}$, and write $\alpha_j^{(\ell)} = 2^\ell q_j^{1/2}$, $A_j^{(\ell)} = \{(v_1,v_2) \in [0,2^{L_2}] : q_j \leq v_1 < q_{j+1}, v_1 \leq v_2 \text{ and } \alpha_j^{(\ell)} \leq \frac{v2-v1}{2} \leq \alpha_j^{(\ell+1)}\}$. Hence, for any $(v_1,v_2) \in A_j^{(\ell)}$, we have

$$\Gamma_k(v_1,v_2) \leq \Gamma_k(q_j-\alpha_j^\ell, q_j+\alpha_j^\ell).$$

Hence, we can use Lemma 10 with $a = \frac{q_j}{(k-1)}$ and $\alpha = \frac{\alpha_j^\ell}{a(k-1)} = \frac{2^\ell}{\sqrt{q_j}}$ to obtain

$$\Gamma_k(v_1,v_2) \leq \frac{1}{2k-1} \frac{e^{\frac{1}{12(k-1)}}}{2\sqrt{\pi}} q_j^{-1/2} e^{-\frac{2^\ell}{6}}.$$

Finally, we have

$\sum_{j=1}^{L}\sum_{\ell=0}^{\bar{L}_j}\sum_{(v_1,v_2)\in A_j^{(\ell)}}\Gamma_k^3(v_1,v_2)$

$$\begin{aligned}
&\le \frac{1}{(2k-1)^3}\frac{e^{\frac{1}{4(k-1)}}}{2^3\pi^{3/2}}\sum_{j=1}^{L}\sum_{\ell=0}^{\bar{L}_j}(q_{j+1}-q_j)(\alpha_j^{(\ell+1)}-\alpha_j^{(\ell)})q_j^{-3/2}e^{-\frac{2^\ell}{2}}\\
&= \frac{1}{(2k-1)^3}\frac{e^{\frac{1}{4(k-1)}}}{2^3\pi^{3/2}}\times L\times\sum_{\ell=0}^{\bar{L}_j}2^\ell e^{-\frac{2^\ell}{2}}.
\end{aligned}$$

Since $\sum_{\ell\ge 0}2^\ell e^{-\frac{2^\ell}{3}}$ converges, there exists a constant S such that

$$\sum_{j=1}^{L}\sum_{\ell=0}^{\bar{L}_j}\sum_{(v_1,v_2)\in A_j^{(\ell)}}\Gamma_k(v_1,v_2)^3\le\frac{1}{(2k-1)^3}\frac{e^{\frac{1}{4(k-1)}}}{2^3\pi^{3/2}}\{\log(k-1)+1\}\times S.$$

This concludes the proof of Item (2). In order to show Items (3) and (4), we use very similar techniques as in the proof of Item (2).

B.5. *Technical Lemmas used in the proof of Lemma 5*

In the following, we use the well-know inequality for k factorial:

$$\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\le k!\le\sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k}}. \tag{B2}$$

Lemma 6 *Let $0<a<1$ such that $a(k-1)\in\mathbb{N}$. Write $\nu_k(a)=\Gamma_k(a(k-1),a(k-1))$. Then, we have*

$$\frac{e^{\frac{-1}{12(k-1)}}}{(2k-1)\sqrt{\pi a(1-a)(k-1)}}\le\nu_k(a)\le\frac{e^{\frac{1}{12(k-1)}}}{(2k-1)\sqrt{\pi a(1-a)(k-1)}}.$$

Proof. First, using Equation (B2), we deduce that:

$$\frac{2^{2k}e^{\frac{-1}{24k}}}{\sqrt{\pi k}}\le\binom{2k}{k}\le\frac{2^{2k}e^{\frac{1}{24k}}}{\sqrt{\pi k}}. \tag{B3}$$

Next, notice that $\nu_k(a)=\frac{\binom{2a(k-1)}{a(k-1)}\binom{2(1-a)(k-1)}{(1-a)(k-1)}}{(2k-1)\binom{2(k-1)}{k-1}}$. Hence, from Equation (B3), we have

$$\frac{2^{2a(k-1)}e^{\frac{-1}{24a(k-1)}}}{\sqrt{\pi a(k-1)}}\le\binom{2a(k-1)}{a(k-1)}\le\frac{2^{2a(k-1)}e^{\frac{1}{24a(k-1)}}}{\sqrt{\pi a(k-1)}},$$

$$\frac{2^{2(1-a)(k-1)}e^{\frac{-1}{24(1-a)(k-1)}}}{\sqrt{\pi(1-a)(k-1)}}\le\binom{2(1-a)(k-1)}{(1-a)(k-1)}\le\frac{2^{2(1-a)(k-1)}e^{\frac{1}{24(1-a)(k-1)}}}{\sqrt{\pi(1-a)(k-1)}},$$

and

$$\sqrt{\pi(k-1)}2^{-2(k-1)}e^{\frac{-1}{24(k-1)}} \leq \binom{2(k-1)}{k-1}^{-1} \leq \sqrt{\pi(k-1)}2^{-2(k-1)}e^{\frac{1}{24(k-1)}}. \tag{B4}$$

From the three last equations we obtain

$$\frac{e^{\frac{-1}{12(k-1)}}}{\sqrt{\pi a(1-a)(k-1)}} \leq \frac{\binom{2a(k-1)}{a(k-1)}\binom{2(1-a)(k-1)}{(1-a)(k-1)}}{\binom{2(k-1)}{k-1}} \leq \frac{e^{\frac{1}{12(k-1)}}}{\sqrt{\pi a(1-a)(k-1)}}.$$

Hence,

$$\frac{e^{\frac{-1}{12(k-1)}}}{(2k-1)\sqrt{\pi a(1-a)(k-1)}} \leq \nu_k(a) \leq \frac{e^{\frac{1}{12(k-1)}}}{(2k-1)\sqrt{\pi a(1-a)(k-1)}}.$$

This concludes the proof.

■

The next Lemmas, 7, 8 and 9 will be useful to prove Lemmas 10 and 11.

LEMMA 7 *Let $0 < a, b < 1$ such that $a(k-1) \in \mathbb{N}$ together with $b(k-1) \in \mathbb{N}$. Write $\nu_k(a,b) = \Gamma_k(a(k-1), b(k-1))$. Then, we have*

$$\frac{1}{2k-1}\frac{e^{\frac{-1}{12(k-1)}}}{2\sqrt{\pi(k-1)}}\sqrt{\frac{(a+b)(2-a-b)}{a(1-a)b(1-b)}}\upsilon_k(a,b) \leq \nu_k(a,b),$$

and

$$\nu_k(a,b) \leq \frac{1}{2k-1}\frac{e^{\frac{1}{12(k-1)}}}{2\sqrt{\pi(k-1)}}\sqrt{\frac{(a+b)(2-a-b)}{a(1-a)b(1-b)}}\upsilon_k(a,b),$$

where

$$\upsilon_k(a,b) = \left\{\frac{2^{-2}(2-a-b)^2}{(1-a)(1-b)}\right\}^{k-1}\left\{\frac{(\frac{a+b}{2-a-b})^{a+b}}{(\frac{a}{1-a})^a(\frac{b}{1-b})^b}\right\}^{k-1}.$$

Proof. Recall that

$$\Gamma_k(a(k-1), b(k-1)) = \frac{\binom{(a+b)(k-1)}{a(k-1)}\binom{(2-a-b)(k-1)}{(1-a)(k-1)}}{(2k-1)\binom{2(k-1)}{k-1}}.$$

Using Equation (B2), we deduce that

$$\frac{e^{\frac{-1}{12(a+b)(k-1)}}}{\sqrt{2\pi\frac{ab}{a+b}(k-1)}}\left\{\frac{(a+b)^{a+b}}{a^ab^b}\right\}^{k-1} \leq \binom{(a+b)(k-1)}{a(k-1)} \leq \frac{e^{\frac{1}{12(a+b)(k-1)}}}{\sqrt{2\pi\frac{ab}{a+b}(k-1)}}\left\{\frac{(a+b)^{a+b}}{a^ab^b}\right\}^{k-1}.$$

Similarly, we have

$$\binom{(2-a-b)(k-1)}{(1-a)(k-1)} \geq \frac{e^{\frac{-1}{12(2-a-b)(k-1)}}}{\sqrt{2\pi\frac{(1-a)(1-b)}{2-a-b}(k-1)}} \left\{\frac{(2-a-b)^{2-a-b}}{(1-a)^{(1-a)}(1-b)^{1-b}}\right\}^{k-1}$$

and

$$\binom{(2-a-b)(k-1)}{(1-a)(k-1)} \leq \frac{e^{\frac{1}{12(2-a-b)(k-1)}}}{\sqrt{2\pi\frac{(1-a)(1-b)}{2-a-b}(k-1)}} \left\{\frac{(2-a-b)^{2-a-b}}{(1-a)^{(1-a)}(1-b)^{1-b}}\right\}^{k-1}.$$

Putting the two last equations together with Equation (B4) leads to

$$\nu_k(a,b) \geq \frac{1}{2k-1}\frac{e^{\frac{-1}{12(k-1)}}}{2\sqrt{\pi(k-1)}}\sqrt{\frac{(a+b)(2-a-b)}{a(1-a)b(1-b)}}\left\{\frac{2^{-2}(2-a-b)^2}{(1-a)(1-b)}\right\}^{k-1}\left\{\frac{(\frac{a+b}{2-a-b})^{a+b}}{(\frac{a}{1-a})^a(\frac{b}{1-b})^b}\right\}^{k-1},$$

and

$$\nu_k(a,b) \leq \frac{1}{2k-1}\frac{e^{\frac{1}{12(k-1)}}}{2\sqrt{\pi(k-1)}}\sqrt{\frac{(a+b)(2-a-b)}{a(1-a)b(1-b)}}\left\{\frac{2^{-2}(2-a-b)^2}{(1-a)(1-b)}\right\}^{k-1}\left\{\frac{(\frac{a+b}{2-a-b})^{a+b}}{(\frac{a}{1-a})^a(\frac{b}{1-b})^b}\right\}^{k-1},$$

which is the desired result.

■

Lemma 8 *Let* $\alpha_a = a(1-a)\alpha$, *with* $\alpha \in (0,1)$. *Then, we have*

$$\upsilon_k(a+\alpha_a, a-\alpha_a) = \left\{\frac{1}{1-\alpha^2 a^2}\right\}^{1-a}\left\{\frac{1}{1-\alpha^2(1-a)^2}\right\}^{a}\left\{\frac{1-\alpha a}{1+\alpha a}\times\frac{1-(1-a)\alpha}{1+(1-a)\alpha}\right\}^{\alpha a(1-a)}.$$

Proof. The lemma can be proved by some algebra calculations. ■

For the next lemma, we need the following notations:

$$p(a,\alpha) = \frac{\alpha^2 a^2(1-a)\{1-2\alpha a+\alpha^2 a^2\}}{(1+2\alpha a)(1-\alpha^2 a^2)}$$

and $T(a,\alpha) = T_1(a,\alpha)T_1(1-a,\alpha)T_2(a,\alpha)T_2(1-a,\alpha)$ where

$$T_1(a,\alpha) = (1-\alpha^2 a^2)^{\frac{\alpha^2 a^2}{1-\alpha^2 v^2}} \text{ and } T_2(a,\alpha) = (\frac{1-\alpha a}{1+\alpha a})^{\frac{-2\alpha^2 a^2(1-a)}{1+2\alpha a}}.$$

Lemma 9 *Let* $\alpha_a = a(1-a)\alpha$, *with* $\alpha \in (0,1)$. *Then, we have*

$$T(a,\alpha)e^{-\{p(a,\alpha)+p(1-a,\alpha)\}} \leq \upsilon_k(a+\alpha_a, a-\alpha_a) \leq e^{-\{p(a,\alpha)+p(1-a,\alpha)\}}.$$

Proof. In the following we use the well-know identity

$$\frac{e}{1+\frac{1}{n}} \leq \left(1+\frac{1}{n}\right)^n \leq e. \tag{B5}$$

Similarly, we deduce

$$e^{-1}(1-\frac{1}{n}) \leq \left(1-\frac{1}{n}\right)^n \leq e^{-1}. \tag{B6}$$

Indeed, taking any real number $t \in (1-\frac{1}{n}, 1)$, we have

$$1 \leq t^{-1} \leq \frac{n}{n-1}.$$

Hence,

$$\int_{1-\frac{1}{n}}^{1} 1\mathrm{d}t \leq \int_{1-\frac{1}{n}}^{1} t^{-1}\mathrm{d}t \leq \int_{1-\frac{1}{n}}^{1} \frac{n}{n-1}\mathrm{d}t,$$

which leads to

$$\frac{1}{n} \leq -\log\left(1-\frac{1}{n}\right) \leq \frac{1}{n-1} \quad \text{and} \quad e^{\frac{-1}{n}} \geq \left(1-\frac{1}{n}\right) \geq e^{\frac{-1}{n-1}}.$$

On one hand,

$$\left(1-\frac{1}{n}\right)^n \leq e^{-1}.$$

On the other hand,

$$\left(e^{\frac{-1}{n-1}}\right)^{n-1} \leq \left(1-\frac{1}{n}\right)^{n-1}.$$

Multiplying the latter equation by $(1-\frac{1}{n})$ entails Equation (B4). Next, write

$$\left\{\frac{1}{1-\alpha^2a^2}\right\} = \left(\left\{1+\frac{1}{\frac{1}{\alpha^2a^2}-1}\right\}^{\frac{1}{\alpha^2a^2}-1}\right)^{\frac{\alpha^2a^2}{1-\alpha^2a^2}}.$$

Using Equation (B5), we have

$$T_1(a,\alpha) \times e^{\frac{\alpha^2a^2(1-a)}{1-\alpha^2a^2}} \leq \left\{\frac{1}{1-\alpha^2a^2}\right\}^{(1-a)} \leq e^{\frac{\alpha^2a^2(1-a)}{1-\alpha^2a^2}}.$$

Similarly, we obtain

$$T_1(1-a,\alpha) \times e^{\frac{\alpha^2a(1-a)^2}{1-\alpha^2(1-a)^2}} \leq \left\{\frac{1}{1-\alpha^2(1-a)^2}\right\}^a \leq e^{\frac{\alpha^2a(1-a)^2}{1-\alpha^2(1-a)^2}}.$$

Finally using Equation (B6), we get

$$\begin{aligned} T_2(a,\alpha)\times e^{\frac{-2\alpha^2a^2(1-a)}{1+2\alpha a}} &\le \left\{\frac{1-\alpha a}{1+\alpha a}\right\}^{\alpha a(1-a)} \le e^{\frac{-2\alpha^2a^2(1-a)}{1+2\alpha a}} \\ T_2(1-a,\alpha)\times e^{\frac{-2\alpha^2a(1-a)^2}{1+2\alpha(1-a)}} &\le \left\{\frac{1-\alpha(1-a)}{1+\alpha(1-a)}\right\}^{\alpha a(1-a)} \le e^{\frac{-2\alpha^2a(1-a)^2}{1+2\alpha(1-a)}}. \end{aligned}$$

Hence, multiplying the previous equations and using Lemma 8 yields to

$$T(a,\alpha)e^{-\{p(a,\alpha)+p(1-a,\alpha)\}} \le \upsilon_k(a+\alpha_a, a-\alpha_a) \le e^{-\{p(a,\alpha)+p(1-a,\alpha)\}},$$

where $p(a,\alpha) = \frac{\alpha^2a^2(1-a)\{1-2\alpha a+\alpha^2a^2\}}{(1+2\alpha a)(1-\alpha^2a^2)}$ and $T(a,\alpha) = T_1(a,\alpha)T_1(1-a,\alpha)T_2(a,\alpha)T_2(1-a,\alpha)$. This completes the proof.

■

The next two lemmas are useful to prove Lemma 5

Lemma 10 *For any $a \in (0,\frac{1}{2})$ and $\alpha \in (0,1)$, write $v^{\pm}(a) = [a \pm \alpha a(1-a))(k-1)]$. Then, we have*

$$\Gamma_k(v^+(a), v^-(a)) \le \frac{1}{2k-1}\frac{e^{\frac{1}{12(k-1)}}}{2\sqrt{\pi(k-1)}}a^{-1/2}e^{-\frac{a\alpha^2}{6}(k-1)}.$$

Proof. Lemma 10 is a consequence of Lemmas 7 and 9 together with the fact that $p(1-a,\alpha) \ge \frac{1}{6}\alpha^2 a$ and $p(a,\alpha) \ge 0$. ■

Lemma 11 *For any $a \in (0,\frac{1}{2})$ and $\alpha \in (0,1)$, write $v^{\pm}(a) = [a \pm \alpha a(1-a))(k-1)]$. If there exists a constant $\rho > 0$ independent of k such that $(k-1)\alpha^2 a < \rho$, then*

$$\Gamma_k(v^+(a), v^-(a)) \ge \frac{\{\frac{3}{4}\}^{\frac{2\rho}{3}}e^{-\frac{10\rho}{8}}}{2k-1}\frac{e^{\frac{-1}{12(k-1)}}}{2\sqrt{\pi(k-1)}}a^{-1/2}.$$

Proof. When $(k-1)\alpha^2 a < \rho$ and $a \in (0,1/2)$, one has $p(a,\alpha) < \frac{5}{8}\rho$ together with $p(1-a,\alpha) < \frac{5}{8}\rho$. Moreover, we obtain $T_1(a,\alpha) \ge \{\frac{3}{4}\}^{\rho/3}$, $T_1((1-a),\alpha) \ge \{\frac{3}{4}\}^{\rho/3}$, $T_2(a,\alpha) \ge 1$ and $T_2(1-a,\alpha) \ge 1$. The result follows from an application of Lemmas 7 and 9. ■