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Heavy-Tailed Random Walks on Complexes of Half-Lines

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Abstract We study a random walk on a complex of finitely many half-lines joined at a common origin; jumps are heavy-tailed and of two types, either one-sided (towards the origin) or two-sided (symmetric). Transmission between half-lines via the origin is governed by an irreducible Markov transition matrix, with associated stationary distribution μ_k. If χ_k is 1 for one-sided half-lines k and $1/2$ for two-sided half-lines, and α_k is the tail exponent of the jumps on half-line k, we show that the recurrence classification for the case where all $\alpha_k \chi_k \in (0, 1)$ is determined by the sign of $\sum_k \mu_k \cot(\chi_k \pi \alpha_k)$. In the case of two half-lines, the model fits naturally on $\mathbb{R}$ and is a version of the *oscillating random walk* of Kemperman. In that case, the cotangent criterion for recurrence becomes linear in α_1 and α_2; our general setting exhibits the essential nonlinearity in the cotangent criterion. For the general model, we also show existence and non-existence of polynomial moments of return times. Our moments results are sharp (and new) for several cases of the oscillating random walk; they are apparently even new for the case of a homogeneous random walk on $\mathbb{R}$ with symmetric increments of tail exponent $\alpha \in (1, 2)$.

Keywords Random walk · Heavy tails · Recurrence · Transience · Passage-time moments · Lyapunov functions · Oscillating random walk · Cotangent criterion

Mathematics Subject Classification (2010) 60J05 (Primary); 60J10 · 60G50 (Secondary)

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Springer

1 Introduction

We study Markov processes on a complex of half-lines $\mathbb{R}_+ \times \mathcal{S}$, where $\mathcal{S}$ is finite, and all half-lines are connected at a common origin. On a given half-line, a particle performs a random walk with a heavy-tailed increment distribution, until it would exit the half-line, when it switches (in general, at random) to another half-line to complete its jump.

To motivate the development of the general model, we first discuss informally some examples; we give formal statements later.

The *one-sided oscillating random walk* takes place on two half-lines, which we may map onto $\mathbb{R}$. From the positive half-line, the increments are negative with density proportional to $y^{-1-\alpha}$, and from the negative half-line, the increments are positive with density proportional to $y^{-1-\beta}$, where $\alpha, \beta \in (0, 1)$. The walk is transient if and only if $\alpha + \beta < 1$; this is essentially a result of Kemperman [15].

The oscillating random walk has several variations and has been well studied over the years (see, e.g. [17–19,21]). This previous work, as we describe in more detail below, is restricted to the case of *two* half-lines. We generalize this model to an arbitrary number of half-lines, labelled by a finite set $\mathcal{S}$, by assigning a rule for travelling from half-line to half-line.

First, we describe a deterministic rule. Let T be a positive integer. Define a *routing schedule of length T* to be a sequence $\sigma = (i_1, \ldots, i_T)$ of T elements of $\mathcal{S}$, dictating the sequence in which half-lines are visited, as follows. The walk starts from line i_1, and, on departure from line i_1 jumps over the origin to i_2, and so on, until departing i_T it returns to i_1; on line $k \in \mathcal{S}$, the walk jumps towards the origin with density proportional to $y^{-1-\alpha_k}$ where $\alpha_k \in (0, 1)$. One simple example takes a cyclic schedule in which σ is a permutation of the elements of $\mathcal{S}$. In any case, a consequence of our results is that now the walk is transient if and only if

$$\sum_{k \in \mathcal{S}} \mu_k \cot(\pi \alpha_k) > 0, \tag{1.1}$$

where μ_k is the number of times k appears in the sequence σ. In particular, in the cyclic case the transience criterion is $\sum_{k \in \mathcal{S}} \cot(\pi \alpha_k) > 0$.

It is easy to see that, if $\mathcal{S}$ contains two elements, the *cotangent criterion* (1.1) is equivalent to the previous one for the one-sided oscillating walk ($\alpha_1 + \alpha_2 < 1$). For more than two half-lines, the criterion is nonlinear, and it was necessary to extend the model to more than two lines in order to see the essence of the behaviour.

More generally, we may choose a random routing rule between lines: on departure from half-line $i \in \mathcal{S}$, the walk jumps to half-line $j \in \mathcal{S}$ with probability $p(i, j)$. The deterministic cyclic routing schedule is a special case in which $p(i, i') = 1$ for i' the successor to i in the cycle. In fact, this set-up generalizes the arbitrary deterministic routing schedule described above, as follows. Given the schedule sequence σ of length T, we may convert this to a cyclic schedule on an extended state space consisting of μ_k copies of line k and then reading σ as a permutation. So the deterministic routing model is a special case of the model with Markov routing, which will be the focus of the rest of the paper.

Our result again will say that (1.1) is the criterion for transience, where μ_k is now the stationary distribution associated with the stochastic matrix $p(i, j)$. Our general model also permits *two-sided* increments for the walk from some of the lines, which contribute terms involving $\cot(\pi\alpha_k/2)$ to the cotangent criterion (1.1). These two-sided models also generalize previously studied classical models (see, e.g. [15,17,23]). Again, it is only in our general setting that the essential nature of the cotangent criterion (1.1) becomes apparent.

Rather than $\mathbb{R}_+ \times \mathcal{S}$, one could work on $\mathbb{Z}_+ \times \mathcal{S}$ instead, with mass functions replacing probability densities; the results would be unchanged.

The paper is organized as follows. In Sect. 2, we formally define our model and describe our main results, which as well as a recurrence classification include results on existence of moments of return times in the recurrent cases. In Sect. 3, we explain how our general model relates to the special case of the oscillating random walk when $\mathcal{S}$ has two elements, and state our results for that model; in this context, the recurrence classification results are already known, but the existence-of-moments results are new even here, and are in several important cases sharp. The present work was also motivated by some problems concerning many-dimensional, partially homogeneous random walks similar to models studied in [5,6,13]: we describe this connection in Sect. 4. The main proofs are presented in Sects. 5, 6, and 7, the latter dealing with the critical boundary case which is more delicate and requires additional work. We collect various technical results in the Appendix.

2 Model and Results

Consider $(X_n, \xi_n; n \in \mathbb{Z}_+)$, a discrete-time, time-homogeneous Markov process with state space $\mathbb{R}_+ \times \mathcal{S}$, where $\mathcal{S}$ is a finite non-empty set. The state space is equipped with the appropriate Borel sets, namely sets of the form $B \times A$ where $B \in \mathcal{B}(\mathbb{R}_+)$ is a Borel set in $\mathbb{R}_+$, and $A \subseteq \mathcal{S}$. The process will be described by:

- an irreducible stochastic matrix labelled by $\mathcal{S}$, $P = (p(i, j); i, j \in \mathcal{S})$; and
- a collection $(w_i; i \in \mathcal{S})$ of probability density functions, so $w_i : \mathbb{R} \to \mathbb{R}_+$ is a Borel function with $\int_{\mathbb{R}} w_i(y)\mathrm{d}y = 1$.

We view $\mathbb{R}_+ \times \mathcal{S}$ as a complex of half-lines $\mathbb{R}_+ \times \{k\}$, or *branches*, connected at a central origin $\mathcal{O} := \{0\} \times \mathcal{S}$; at time n, the coordinate ξ_n describes which branch the process is on, and X_n describes the distance along that branch at which the process sits. We will call X_n a *random walk* on this complex of branches.

To simplify notation, throughout we write $\mathbb{P}_{x,i}[\,\cdot\,]$ for $\mathbb{P}[\,\cdot\, \mid (X_0, \xi_0) = (x, i)]$, the conditional probability starting from $(x, i) \in \mathbb{R}_+ \times \mathcal{S}$; similarly, we use $\mathbb{E}_{x,i}$ for the corresponding expectation. The transition kernel of the process is given for $(x, i) \in \mathbb{R}_+ \times \mathcal{S}$, for all Borel sets $B \subseteq \mathbb{R}_+$ and all $j \in \mathcal{S}$, by

$$\mathbb{P}\big[(X_{n+1}, \xi_{n+1}) \in B \times \{j\} \mid (X_n, \xi_n) = (x, i)\big]$$
$$= \mathbb{P}_{x,i}\,[(X_1, \xi_1) \in B \times \{j\}]$$
$$= p(i, j) \int_B w_i(-z - x)\mathrm{d}z + \mathbf{1}\{i = j\} \int_B w_i(z - x)\mathrm{d}z. \qquad (2.1)$$

The dynamics of the process represented by (2.1) can be described algorithmically as follows. Given $(X_n, \xi_n) = (x, i) \in \mathbb{R}_+ \times \mathcal{S}$, generate (independently) a spatial increment φ_{n+1} from the distribution given by w_i and a random index $\eta_{n+1} \in \mathcal{S}$ according to the distribution $p(i, \cdot)$. Then,

- if $x + \varphi_{n+1} \geq 0$, set $(X_{n+1}, \xi_{n+1}) = (x + \varphi_{n+1}, i)$; or
- if $x + \varphi_{n+1} < 0$, set $(X_{n+1}, \xi_{n+1}) = (|x + \varphi_{n+1}|, \eta_{n+1})$.

In words, the walk takes a w_{ξ_n}-distributed step. If this step would bring the walk beyond the origin, it passes through the origin and switches onto branch η_{n+1} (or, if η_{n+1} happens to be equal to ξ_n, it reflects back along the same branch).

The finite irreducible stochastic matrix P is associated with a (unique) positive invariant probability distribution $(\mu_k; k \in \mathcal{S})$ satisfying

$$\sum_{j \in \mathcal{S}} \mu_j p(j, k) - \mu_k = 0, \quad \text{for all } k \in \mathcal{S}. \tag{2.2}$$

For future reference, we state the following.

(A0) Let $P = (p(i, j); i, j \in \mathcal{S})$ be an irreducible stochastic matrix, and let $(\mu_k; k \in \mathcal{S})$ denote the corresponding invariant distribution.

Our interest here is when the w_i are *heavy-tailed*. We allow two classes of distribution for the w_i: *one-sided* or *symmetric*. It is convenient, then, to partition $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}^{\mathrm{one}} \cup \mathcal{S}^{\mathrm{sym}}$ where $\mathcal{S}^{\mathrm{one}}$ and $\mathcal{S}^{\mathrm{sym}}$ are disjoint sets, representing those branches on which the walk takes, respectively, one-sided and symmetric jumps. The w_k are then described by a collection of positive parameters $(\alpha_k; k \in \mathcal{S})$.

For a probability density function $v : \mathbb{R} \to \mathbb{R}_+$, an exponent $\alpha \in (0, \infty)$, and a constant $c \in (0, \infty)$, we write $v \in \mathfrak{D}_{\alpha,c}$ to mean that there exists $c : \mathbb{R}_+ \to (0, \infty)$ with $\sup_y c(y) < \infty$ and $\lim_{y \to \infty} c(y) = c$ for which

$$v(y) = \begin{cases} c(y) y^{-1-\alpha} & \text{if } y > 0 \\ 0 & \text{if } y \leq 0. \end{cases} \tag{2.3}$$

If $v \in \mathfrak{D}_{\alpha,c}$ is such that (2.3) holds and $c(y)$ satisfies the stronger condition $c(y) = c + O(y^{-\delta})$ for some $\delta > 0$, then we write $v \in \mathfrak{D}^+_{\alpha,c}$.

Our assumption on the increment distributions w_i is as follows.

(A1) Suppose that, for each $k \in \mathcal{S}$, we have an exponent $\alpha_k \in (0, \infty)$, a constant $c_k \in (0, \infty)$, and a density function $v_k \in \mathfrak{D}_{\alpha_k, c_k}$. Then suppose that, for all $y \in \mathbb{R}$, w_k is given by

$$w_k(y) = \begin{cases} v_k(-y) & \text{if } k \in \mathcal{S}^{\mathrm{one}} \\ \frac{1}{2} v_k(|y|) & \text{if } k \in \mathcal{S}^{\mathrm{sym}}. \end{cases} \tag{2.4}$$

We say that X_n is *recurrent* if $\liminf_{n \to \infty} X_n = 0$, a.s., and *transient* if $\lim_{n \to \infty} X_n = \infty$, a.s. An irreducibility argument shows that our Markov chain

(X_n, ξ_n) displays the usual recurrence/transience dichotomy and exactly one of these two situations holds; however, our proofs establish this behaviour directly using semi-martingale arguments, and so we may avoid discussion of irreducibility here.

Throughout we define, for $k \in \mathcal{S}$,

$$\chi_k := \frac{1 + \mathbf{1}\{k \in \mathcal{S}^{\text{one}}\}}{2} = \begin{cases} \frac{1}{2} & \text{if } k \in \mathcal{S}^{\text{sym}}; \\ 1 & \text{if } k \in \mathcal{S}^{\text{one}}. \end{cases}$$

Our first main result gives a recurrence classification for the process.

Theorem 1 *Suppose that* (A0) *and* (A1) *hold.*

(a) *Suppose that* $\max_{k \in \mathcal{S}} \chi_k \alpha_k \geq 1$. *Then,* X_n *is recurrent.*
(b) *Suppose that* $\max_{k \in \mathcal{S}} \chi_k \alpha_k < 1$.
   (i) *If* $\sum_{k \in \mathcal{S}} \mu_k \cot(\chi_k \pi \alpha_k) < 0$, *then* X_n *is recurrent.*
   (ii) *If* $\sum_{k \in \mathcal{S}} \mu_k \cot(\chi_k \pi \alpha_k) > 0$, *then* X_n *is transient.*
   (iii) *Suppose in addition that the densities* v_k *of* (A1) *belong to* $\mathfrak{D}^+_{\alpha_k, c_k}$ *for each* k. *Then,* X_n *is recurrent if* $\sum_{k \in \mathcal{S}} \mu_k \cot(\chi_k \pi \alpha_k) = 0$.

In the recurrent cases, it is of interest to quantify recurrence via existence or non-existence of passage-time moments. For $a > 0$, let $\tau_a := \min\{n \geq 0 : X_n \leq a\}$, where throughout the paper we adopt the usual convention that $\min \emptyset := +\infty$. The next result shows that in all the recurrent cases, excluding the boundary case in Theorem 1(b)(iii), the tails of τ_a are *polynomial*.

Theorem 2 *Suppose that* (A0) *and* (A1) *hold. In cases* (a) *and* (b)(i) *of Theorem* 1, *there exist* $0 < q_0 \leq q_1 < \infty$ *such that for all* $x > a$ *and all* $k \in \mathcal{S}$,

$$\mathbb{E}_{x,k}\left[\tau_a^q\right] < \infty, \quad \text{for } q < q_0 \quad \text{and} \quad \mathbb{E}_{x,k}\left[\tau_a^q\right] = \infty, \quad \text{for } q > q_1.$$

Remark 1 Our general results have $q_0 < q_1$ so that Theorem 2 does not give sharp estimates; these remain an open problem.

We do have sharp results in several particular cases for *two* half-lines, in which case our model reduces to the *oscillating random walk* considered by Kemperman [15] and others. We present these sharp moments results (Theorems 4 and 6) in the next section, which discusses in detail the case of the oscillating random walk, and also describes how our recurrence results relate to the known results for this classical model.

3 Oscillating Random Walks and Related Examples

3.1 Two Half-Lines Become One Line

In the case of our general model in which $\mathcal{S}$ consists of two elements, $\mathcal{S} = \{-1, +1\}$, say, it is natural and convenient to represent our random walk on the whole real line $\mathbb{R}$. Namely, if $\omega(x, k) := kx$ for $x \in \mathbb{R}_+$ and $k = \pm 1$, we let $Z_n = \omega(X_n, \xi_n)$.

The simplest case has no reflection at the origin, only transmission, i.e. $p(i, j) = 1\{i \neq j\}$, so that $\mu = (\frac{1}{2}, \frac{1}{2})$. Then, for $B \subseteq \mathbb{R}$ a Borel set,

$$\mathbb{P}\left[Z_{n+1} \in B \mid (X_n, \xi_n) = (x, i)\right] = \mathbb{P}_{x,i}\left[(X_1, \xi_1) \in B^+ \times \{+1\}\right]$$
$$+ \mathbb{P}_{x,i}\left[(X_1, \xi_1) \in B^- \times \{-1\}\right],$$

where $B^+ = B \cap \mathbb{R}_+$ and $B^- = \{-x : x \in B, \ x < 0\}$. In particular, by (2.1), writing w_+ for w_{+1}, for $x \in \mathbb{R}_+$,

$$\mathbb{P}\left[Z_{n+1} \in B \mid (X_n, \xi_n) = (x, +1)\right] = \int_{B^+} w_+(z - x)\mathrm{d}z + \int_{B^-} w_+(-z - x)\mathrm{d}z$$
$$= \int_B w_+(z - x)\mathrm{d}z,$$

and, similarly, writing $w_-(y)$ for $w_{-1}(-y)$, for $x \in \mathbb{R}_+$,

$$\mathbb{P}\left[Z_{n+1} \in B \mid (X_n, \xi_n) = (x, -1)\right] = \int_B w_-(z + x)\mathrm{d}z.$$

For $x \neq 0$, ω is invertible with

$$\omega^{-1}(x) = \begin{cases} (|x|, +1) & \text{if } x > 0 \\ (|x|, -1) & \text{if } x < 0, \end{cases}$$

and hence we have for $x \in \mathbb{R}\backslash\{0\}$ and Borel $B \subseteq \mathbb{R}$,

$$\mathbb{P}\left[Z_{n+1} \in B \mid Z_n = x\right] = \begin{cases} \int_B w_+(z - x)\mathrm{d}z & \text{if } x > 0 \\ \int_B w_-(z - x)\mathrm{d}z & \text{if } x < 0. \end{cases} \tag{3.1}$$

We may make an arbitrary non-trivial choice for the transition law at $Z_n = 0$ without affecting the behaviour of the process, and then, (3.1) shows that Z_n is a time-homogeneous Markov process on $\mathbb{R}$. Now, Z_n is *recurrent* if $\liminf_{n\to\infty} |Z_n| = 0$, a.s., or *transient* if $\lim_{n\to\infty} |Z_n| = \infty$, a.s. The one-dimensional case described at (3.1) has received significant attention over the years. We describe several of the classical models that have been considered.

3.2 Examples and Further Results

3.2.1 Homogeneous Symmetric Random Walk

The most classical case is as follows.

(Sym) Let $\alpha \in (0, \infty)$. For $v \in \mathfrak{D}_{\alpha,c}$, suppose that $w_+(y) = w_-(y) = \frac{1}{2}v(|y|)$ for $y \in \mathbb{R}$.

In this case, Z_n describes a random walk with i.i.d. symmetric increments.

Theorem 3 *Suppose that* (Sym) *holds. Then, the symmetric random walk is transient if $\alpha < 1$ and recurrent if $\alpha > 1$. If, in addition, $v \in \mathfrak{D}_{\alpha,c}^{+}$, then the case $\alpha = 1$ is recurrent.*

Theorem 3 follows from our Theorem 1, since in this case

$$\sum_{k \in \mathcal{S}} \mu_k \cot\left(\chi_k \pi \alpha_k\right) = \cot(\pi \alpha / 2).$$

Since it deals with a sum of i.i.d. random variables, Theorem 3 may be deduced from the classical theorem of Chung and Fuchs [7], via, e.g. the formulation of Shepp [23]. The method of the present paper provides an alternative to the classical (Fourier analytic) approach that generalizes beyond the i.i.d. setting (note that Theorem 3 is not, formally, a consequence of Shepp's most accessible result, Theorem 5 of [23], since v does not necessarily correspond to a unimodal distribution in Shepp's sense).

With τ_a as defined previously, in the setting of the present section we have $\tau_a = \min\{n \geq 0 : |Z_n| \leq a\}$. Use $\mathbb{E}_x[\,\cdot\,]$ as shorthand for $\mathbb{E}[\,\cdot\, \mid Z_0 = x]$. We have the following result on existence of passage-time moments, whose proof is in Sect. 6; while part (i) is well known, we could find no reference for part (ii).

Theorem 4 *Suppose that* (Sym) *holds, and that $\alpha > 1$. Let $x \notin [-a, a]$.*

(i) *If $\alpha \geq 2$, then $\mathbb{E}_x\left[\tau_a^q\right] < \infty$ if $q < 1/2$ and $\mathbb{E}_x\left[\tau_a^q\right] = \infty$ if $q > 1/2$.*
(ii) *If $\alpha \in (1, 2)$, then $\mathbb{E}_x\left[\tau_a^q\right] < \infty$ if $q < 1 - \frac{1}{\alpha}$ and $\mathbb{E}_x\left[\tau_a^q\right] = \infty$ if $q > 1 - \frac{1}{\alpha}$.*

Our main interest concerns spatially inhomogeneous models, i.e. in which w_x depends on x, typically only through sgn x, the sign of x. Such models are known as *oscillating random walks* and were studied by Kemperman [15], to whom the model was suggested in 1960 by Anatole Joffe and Peter Ney (see [15, p. 29]).

3.2.2 One-Sided Oscillating Random Walk

The next example, following [15], is a *one-sided oscillating random walk*:

(Osc1) Let $\alpha, \beta \in (0, \infty)$. For $v_+ \in \mathfrak{D}_{\alpha,c_+}$ and $v_- \in \mathfrak{D}_{\beta,c_-}$, suppose that

$$w_+(y) = v_+(-y), \quad \text{and} \quad w_-(y) = v_-(y).$$

In other words, the walk always jumps in the direction of (and possibly over) the origin, with tail exponent α from the positive half-line and exponent β from the negative half-line. The following recurrence classification applies.

Theorem 5 *Suppose that* (Osc1) *holds. Then, the one-sided oscillating random walk is transient if $\alpha + \beta < 1$ and recurrent if $\alpha + \beta > 1$. If, in addition, $v_+ \in \mathfrak{D}_{\alpha,c_+}^{+}$ and $v_- \in \mathfrak{D}_{\beta,c_-}^{+}$, then the case $\alpha + \beta = 1$ is recurrent.*

Theorem 5 was obtained in the discrete-space case by Kemperman [15, p. 21]; it follows from our Theorem 1, since in this case

$$\sum_{k \in \mathcal{S}} \mu_k \cot\left(\chi_k \pi \alpha_k\right) = \frac{1}{2}\cot\left(\pi\alpha\right) + \frac{1}{2}\cot\left(\pi\beta\right) = \frac{\sin\left(\pi\left(\alpha+\beta\right)\right)}{2\sin\left(\pi\alpha\right)\sin\left(\pi\beta\right)}.$$

The special case of (Osc1) in which $\alpha = \beta$ was called *antisymmetric* by Kemperman; here, Theorem 5 shows that the walk is transient for $\alpha < 1/2$ and recurrent for $\alpha > 1/2$. We have the following moments result, proved in Sect. 6.

Theorem 6 *Suppose that* (Osc1) *holds, and that* $\alpha = \beta > 1/2$. *Let* $x \notin [-a, a]$.

(i) *If* $\alpha \geq 1$, *then* $\mathbb{E}_x[\tau_a^q] < \infty$ *if* $q < \alpha$ *and* $\mathbb{E}_x[\tau_a^q] = \infty$ *if* $q > \alpha$.
(ii) *If* $\alpha \in (1/2, 1)$, *then* $\mathbb{E}_x[\tau_a^q] < \infty$ *if* $q < 2 - \frac{1}{\alpha}$ *and* $\mathbb{E}_x[\tau_a^q] = \infty$ *if* $q > 2 - \frac{1}{\alpha}$.

Problem 1 Obtain sharp moments results for general $\alpha, \beta \in (0, 1)$.

3.2.3 Two-Sided Oscillating Random Walk

Another model in the vein of [15] is a *two-sided oscillating random walk*:

(Osc2) Let $\alpha, \beta \in (0, \infty)$. For $v_+ \in \mathfrak{D}_{\alpha,c_+}$ and $v_- \in \mathfrak{D}_{\beta,c_-}$, suppose that

$$w_+(y) = \frac{1}{2}v_+(|y|), \text{ and } w_-(y) = \frac{1}{2}v_-(|y|).$$

Now, the jumps of the walk are symmetric, as under (Sym), but with a tail exponent depending upon which side of the origin the walk is currently on, as under (Osc1).

The most general recurrence classification result for the model (Osc2) is due to Sandrić [21]. A somewhat less general, discrete-space version was obtained by Rogozin and Foss (Theorem 2 of [17, p. 159]), building on [15]. Analogous results in continuous time were given in [4,11]. Here is the result.

Theorem 7 *Suppose that* (Osc2) *holds. Then, the two-sided oscillating random walk is transient if* $\alpha + \beta < 2$ *and recurrent if* $\alpha + \beta > 2$. *If, in addition,* $v_+ \in \mathfrak{D}_{\alpha,c_+}^+$ *and* $v_- \in \mathfrak{D}_{\beta,c_-}^+$, *then the case* $\alpha + \beta = 2$ *is recurrent.*

Theorem 7 also follows from our Theorem 1, since in this case

$$\sum_{k \in \mathcal{S}} \mu_k \cot(\chi_k \pi \alpha_k) = \frac{1}{2}\cot(\pi\alpha/2) + \frac{1}{2}\cot(\pi\beta/2) = \frac{\sin(\pi(\alpha+\beta)/2)}{2\sin(\pi\alpha/2)\sin(\pi\beta/2)}.$$

Problem 2 Obtain sharp moments results for $\alpha, \beta \in (0, 2)$.

3.2.4 Mixed Oscillating Random Walk

A final model is another oscillating walk that mixes the one- and two-sided models:

(Osc3) Let $\alpha, \beta \in (0, \infty)$. For $v_+ \in \mathfrak{D}_{\alpha, c_+}$ and $v_- \in \mathfrak{D}_{\beta, c_-}$, suppose that

$$w_+(y) = \frac{1}{2} v_+(|y|), \quad \text{and} \quad w_-(y) = v_-(y).$$

In the discrete-space case, Theorem 2 of Rogozin and Foss [17, p. 159] gives the recurrence classification.

Theorem 8 *Suppose that* (Osc3) *holds. Then, the mixed oscillating random walk is transient if $\alpha + 2\beta < 2$ and recurrent if $\alpha + 2\beta > 2$. If, in addition, $v_+ \in \mathfrak{D}_{\alpha, c_+}^+$ and $v_- \in \mathfrak{D}_{\beta, c_-}^+$, then the case $\alpha + 2\beta = 2$ is recurrent.*

Theorem 8 also follows from our Theorem 1, since in this case

$$\sum_{k \in \mathcal{S}} \mu_k \cot(\chi_k \pi \alpha_k) = \frac{1}{2} \cot(\pi \alpha / 2) + \frac{1}{2} \cot(\pi \beta) = \frac{\sin(\pi(\alpha + 2\beta)/2)}{2 \sin(\pi \alpha / 2) \sin(\pi \beta)}.$$

3.3 Additional Remarks

It is possible to generalize the model further by permitting the local transition density to vary *within* each half-line. Then, we have the transition kernel

$$\mathbb{P}[Z_{n+1} \in B \mid Z_n = x] = \int_B w_x(z - x) \mathrm{d}z, \tag{3.2}$$

for all Borel sets $B \subseteq \mathbb{R}$. Here, the local transition densities $w_x : \mathbb{R} \to \mathbb{R}_+$ are Borel functions. Variations of the oscillating random walk, within the general setting of (3.2), have also been studied in the literature. Sandrić [19,21] supposes that the w_x satisfy, for each $x \in \mathbb{R}$, $w_x(y) \sim c(x)|y|^{-1-\alpha(x)}$ as $|y| \to \infty$ for some measurable functions c and α; he refers to this as a *stable-like Markov chain*. Under a uniformity condition on the w_x, and other mild technical conditions, Sandrić [19] obtained, via Foster–Lyapunov methods similar in spirit to those of the present paper, sufficient conditions for recurrence and transience: essentially $\liminf_{x \to \infty} \alpha(x) > 1$ is sufficient for recurrence and $\limsup_{x \to \infty} \alpha(x) < 1$ is sufficient for transience. These results can be seen as a generalization of Theorem 3. Some related results for models in continuous-time (Lévy processes) are given in [20,22,24]. Further results and an overview of the literature are provided in Sandrić's Ph.D. thesis [18].

4 Many-Dimensional Random Walks

The next two examples show how versions of the oscillating random walk of Sect. 3 arise as embedded Markov chains in certain two-dimensional random walks.

Example 1 Consider $\xi_n = (\xi_n^{(1)}, \xi_n^{(2)})$, $n \in \mathbb{Z}_+$, a nearest-neighbour random walk on $\mathbb{Z}^2$ with transition probabilities

$$\mathbb{P}\left[\xi_{n+1} = (y_1, y_2) \mid \xi_n = (x_1, x_2)\right] = p(x_1, x_2; y_1, y_2).$$
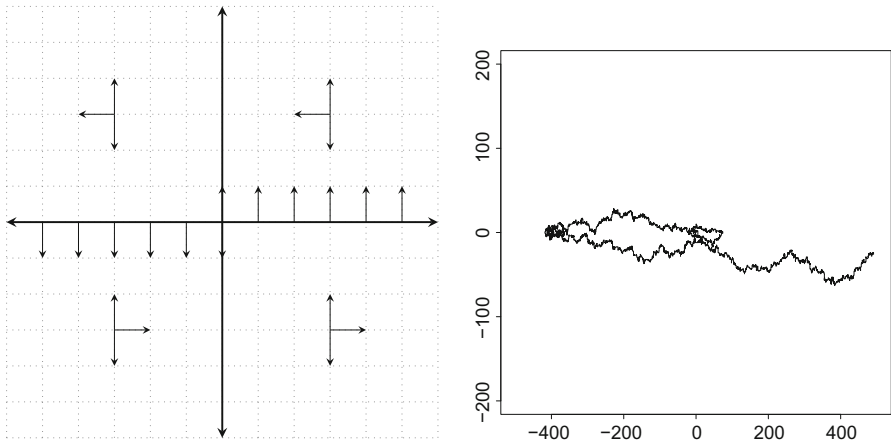
Fig. 1 Pictorial representation of the non-homogeneous nearest-neighbour random walk on $\mathbb{Z}^2$ of Example 1, plus a simulated trajectory of 5000 steps of the walk. We conjecture that the walk is recurrent

Suppose that the probabilities are given for $x_2 \neq 0$ by,

$$p(x_1, x_2; x_1, x_2 + 1) = p(x_1, x_2; x_1, x_2 - 1) = \frac{1}{3};$$

$$p(x_1, x_2; x_1 + 1, x_2) = \frac{1}{3}\mathbf{1}\{x_2 < 0\};$$

$$p(x_1, x_2; x_1 - 1, x_2) = \frac{1}{3}\mathbf{1}\{x_2 > 0\}; \tag{4.1}$$

(the rest being zero) and for $x_2 = 0$ by $p(x_1, 0; x_1, 1) = 1$ for all $x_1 > 0$, $p(x_1, 0; x_1, -1) = 1$ for all $x_1 < 0$, and $p(0, 0; 0, 1) = p(0, 0; 0, -1) = 1/2$. See Fig. 1 for an illustration.

Set $\tau_0 := 0$ and define recursively $\tau_{k+1} = \min\{n > \tau_k : \xi_n^{(2)} = 0\}$ for $k \geq 0$; consider the embedded Markov chain $X_n = \xi_{\tau_n}^{(1)}$. We show that X_n is a discrete version of the oscillating random walk described in Sect. 3. Indeed, $|\xi_n^{(2)}|$ is a reflecting random walk on $\mathbb{Z}_+$ with increments taking values $-1, 0, +1$ each with probability $1/3$. We then (see, e.g. [9, p. 415]) have that for some constant $c \in (0, \infty)$,

$$\mathbb{P}[\tau_1 > r] = (c + o(1)) r^{-1/2}, \quad \text{as} \quad r \to \infty.$$

Suppose that $\xi_0^{(1)} = x > 0$. Since between times τ_0 and τ_1, $\xi_n^{(1)}$ is monotone, we have

$$\mathbb{P}\left[\tau_1 > 3r + r^{3/4}\right] = \mathbb{P}\left[\tau_1 > 3r + r^{3/4}, \xi_{\tau_1} - \xi_{\tau_0} < -r\right]$$
$$+ \mathbb{P}\left[\tau_1 > 3r + r^{3/4}, \xi_{\tau_1} - \xi_{\tau_0} \geq -r\right]$$
$$\leq \mathbb{P}\left[\xi_{\tau_1} - \xi_{\tau_0} < -r\right] + \mathbb{P}\left[\xi_{\lceil 3r + r^{3/4}\rceil} - \xi_{\tau_0} \geq -r\right].$$

In other words,

$$\mathbb{P}\left[\xi_{\tau_1}^{(1)} - \xi_{\tau_0}^{(1)} < -r\right] \geq \mathbb{P}\left[\tau_1 > 3r + r^{3/4}\right] - \mathbb{P}\left[\xi_{\lceil 3r+r^{3/4}\rceil}^{(1)} - \xi_0^{(1)} \geq -r\right].$$

Here, $\xi_n^{(1)} + \frac{n}{3}$ is a martingale with bounded increments, so, by the Azuma–Hoeffding inequality, for some $\varepsilon > 0$ and all $r \geq 1$,

$$\mathbb{P}\left[\xi_{\lceil 3r+r^{3/4}\rceil}^{(1)} - \xi_0^{(1)} \geq -r\right] \leq \mathbb{P}\left[\xi_{\lceil 3r+r^{3/4}\rceil}^{(1)} + \tfrac{1}{3}\lceil 3r + r^{3/4}\rceil - \xi_0^{(1)} \geq \tfrac{1}{3}r^{3/4}\right]$$
$$\leq \exp\left\{-\varepsilon r^{1/2}\right\}.$$

Similarly,

$$\mathbb{P}\left[\xi_{\tau_1}^{(1)} - \xi_{\tau_0}^{(1)} < -r\right] \leq \mathbb{P}\left[\tau_1 > 3r - r^{3/4}\right] + \mathbb{P}\left[\xi_{\lceil 3r-r^{3/4}\rceil}^{(1)} - \xi_0^{(1)} \leq -r\right],$$

where, by Azuma–Hoeffding once more,

$$\mathbb{P}\left[\xi_{\lceil 3r-r^{3/4}\rceil}^{(1)} - \xi_0^{(1)} \leq -r\right] \leq \mathbb{P}\left[\xi_{\lceil 3r-r^{3/4}\rceil}^{(1)} + \tfrac{1}{3}\lceil 3r - r^{3/4}\rceil - \xi_0^{(1)} \leq -\tfrac{1}{3}r^{3/4} + 1\right]$$
$$\leq \exp\left\{-\varepsilon r^{1/2}\right\}.$$

Combining these bounds, and using the symmetric argument for $\{\xi_{\tau_1}^{(1)} > r\}$ when $\xi_0^{(1)} = x < 0$, we see that for $r > 0$,

$$\mathbb{P}\left[X_{n+1} - X_n < -r \mid X_n = x\right] = u(r), \quad \text{if } x > 0, \quad \text{and}$$
$$\mathbb{P}\left[X_{n+1} - X_n > r \mid X_n = x\right] = u(r), \quad \text{if } x < 0, \tag{4.2}$$

where $u(r) = (c + o(1))r^{-1/2}$. Thus, X_n satisfies a discrete-space analogue of (Osc1) with $\alpha = \beta = 1/2$. This is the critical case identified in Theorem 5, but that result does not cover this case due to the rate of convergence estimate for u; a finer analysis is required. We conjecture that the walk is recurrent.

Example 2 We present two variations on the previous example, which are superficially similar but turn out to be less delicate. First, modify the random walk of the previous example by supposing that (4.1) holds but replacing the behaviour at $x_2 = 0$ by $p(x_1, 0; x_1, 1) = p(x_1, 0; x_1, -1) = 1/2$ for all $x_1 \in \mathbb{Z}$. See the left-hand part of Fig. 2 for an illustration.

The embedded process X_n now has, for all $x \in \mathbb{Z}$ and for $r \geq 0$,

$$\mathbb{P}\left[X_{n+1} - X_n < -r \mid X_n = x\right] = \mathbb{P}\left[X_{n+1} - X_n > r \mid X_n = x\right] = u(r), \tag{4.3}$$

where $u(r) = (c/2)(1 + o(1))r^{-1/2}$. Thus, X_n is a random walk with symmetric increments, and the discrete version of our Theorem 3 (and also a result of [23])
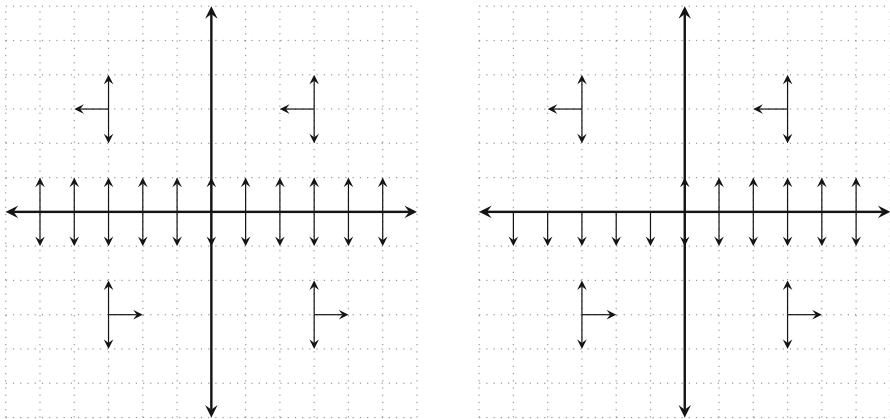
Fig. 2 Pictorial representation of the two non-homogeneous nearest-neighbour random walks on $\mathbb{Z}^2$ of Example 2. Each of these walks is transient

implies that the walk is transient. This walk was studied by Campanino and Petritis [5,6], who proved transience via different methods.

Next, modify the random walk of Example 1 by supposing that (4.1) holds but replacing the behaviour at $x_2 = 0$ by $p(x_1, 0; x_1, 1) = p(x_1, 0; x_1, -1) = 1/2$ if $x_1 \geq 0$, and $p(x_1, 0; x_1, -1) = 1$ for $x_1 < 0$. See the right-hand part of Fig. 2 for an illustration. This time the walk takes a symmetric increment as at (4.3) when $x \geq 0$ but a one-sided increment as at (4.2) when $x < 0$. In this case, the discrete version of our Theorem 8 (and also a result of [17]) shows that the walk is transient.

One may obtain the general model on $\mathbb{Z}_+ \times \mathcal{S}$ as an embedded process for a random walk on complexes of half-spaces, generalizing the examples considered here; we leave this to the interested reader.

5 Recurrence Classification in the Non-critical Cases

5.1 Lyapunov Functions

Our proofs are based on demonstrating appropriate Lyapunov functions; that is, for suitable $\varphi : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ we study $Y_n = \varphi(X_n, \xi_n)$ such that Y_n has appropriate local supermartingale or submartingale properties for the one-step mean increments

$$D\varphi(x, i) := \mathbb{E}\left[\varphi\left(X_{n+1}, \xi_{n+1}\right) - \varphi\left(X_n, \xi_n\right) \mid (X_n, \xi_n) = (x, i)\right]$$
$$= \mathbb{E}_{x,i}\left[\varphi(X_1, \xi_1) - \varphi\left(X_0, \xi_0\right)\right].$$

First, we note some consequences of the transition law (2.1). Let $\varphi : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ be measurable. Then, we have from (2.1) that, for $(x, i) \in \mathbb{R}_+ \times \mathcal{S}$,

$$D\varphi(x, i) = \sum_{j \in \mathcal{S}} p(i, j) \int_{-\infty}^{-x} (\varphi(-x - y, j) - \varphi(x, i)) \, w_i(y) \mathrm{d}y$$

$$+ \int_{-x}^{\infty} (\varphi(x + y, i) - \varphi(x, i)) \, w_i(y) \mathrm{d}y. \qquad (5.1)$$

Our primary Lyapunov function is roughly of the form $x \mapsto |x|^\nu$, $\nu \in \mathbb{R}$, but weighted according to an $\mathcal{S}$-dependent component (realized by a collection of multiplicative weights λ_k); these weights provide a crucial technical tool.

For $\nu \in \mathbb{R}$ and $x \in \mathbb{R}$, we write

$$f_\nu(x) := (1 + |x|)^\nu.$$

Then, for parameters $\lambda_k > 0$ for each $k \in \mathcal{S}$, define for $x \in \mathbb{R}_+$ and $k \in \mathcal{S}$,

$$f_\nu(x, k) := \lambda_k f_\nu(x) = \lambda_k (1 + x)^\nu. \qquad (5.2)$$

Now, for this Lyapunov function, (5.1) gives

$$Df_\nu(x, i) = \sum_{j \in \mathcal{S}} p(i, j) \int_{-\infty}^{-x} (\lambda_j f_\nu(x + y) - \lambda_i f_\nu(x)) \, w_i(y) \mathrm{d}y$$

$$+ \lambda_i \int_{-x}^{\infty} (f_\nu(x + y) - f_\nu(x)) \, w_i(y) \mathrm{d}y. \qquad (5.3)$$

Depending on whether $i \in \mathcal{S}^{\mathrm{sym}}$ or $i \in \mathcal{S}^{\mathrm{one}}$, the above integrals can be expressed in terms of v_i as follows. For $i \in \mathcal{S}^{\mathrm{sym}}$,

$$Df_\nu(x, i) = \sum_{j \in \mathcal{S}} p(i, j) \frac{\lambda_j}{2} \int_x^{\infty} f_\nu(y - x) v_i(y) \mathrm{d}y - \frac{\lambda_i}{2} \int_x^{\infty} f_\nu(x) v_i(y) \mathrm{d}y$$

$$+ \frac{\lambda_i}{2} \int_0^x (f_\nu(x + y) + f_\nu(x - y) - 2f_\nu(x)) v_i(y) \mathrm{d}y$$

$$+ \frac{\lambda_i}{2} \int_x^{\infty} (f_\nu(x + y) - f_\nu(x)) v_i(y) \mathrm{d}y. \qquad (5.4)$$

For $i \in \mathcal{S}^{\mathrm{one}}$,

$$Df_\nu(x, i) = \sum_{j \in \mathcal{S}} p(i, j) \lambda_j \int_x^{\infty} f_\nu(y - x) v_i(y) \mathrm{d}y - \lambda_i \int_x^{\infty} f_\nu(x) v_i(y) \mathrm{d}y$$

$$+ \lambda_i \int_0^x (f_\nu(x - y) - f_\nu(x)) v_i(y) \mathrm{d}y. \qquad (5.5)$$

5.2 Estimates of Functional Increments

In the course of our proofs, we need various integral estimates that can be expressed in terms of classical transcendental functions. For the convenience of the reader, we gather all necessary integrals in Lemmas 1 and 2; the proofs of these results are deferred until the Appendix. Recall that the Euler gamma function Γ satisfies the functional equation $z\Gamma(z) = \Gamma(z + 1)$, and the hypergeometric function $_mF_n$ is defined via a power series (see [1]).

Lemma 1 *Suppose that $\alpha > 0$ and $-1 < \nu < \alpha$. Then,*

$$i_0^{\nu,\alpha} := \int_1^\infty \frac{(1+u)^\nu - 1}{u^{1+\alpha}} du = \frac{1}{\alpha - \nu} \, _2F_1(-\nu, \alpha - \nu; \alpha - \nu + 1; -1) - \frac{1}{\alpha};$$

$$i_{2,0}^\alpha := \int_1^\infty \frac{1}{u^{1+\alpha}} du = \frac{1}{\alpha};$$

$$i_{2,1}^{\nu,\alpha} := \int_1^\infty \frac{(u-1)^\nu}{u^{1+\alpha}} du = \frac{\Gamma(1+\nu)\Gamma(\alpha - \nu)}{\Gamma(1+\alpha)}.$$

Suppose that $\alpha \in (0, 2)$ and $\nu > -1$. Then,

$$i_1^{\nu,\alpha} := \int_0^1 \frac{(1+u)^\nu + (1-u)^\nu - 2}{u^{1+\alpha}} du$$

$$= \frac{\nu(\nu - 1)}{2 - \alpha} \, _4F_3\left(1, 1 - \tfrac{\nu}{2}, 1 - \tfrac{\alpha}{2}, \tfrac{3-\nu}{2}; \tfrac{3}{2}, 2, 2 - \tfrac{\alpha}{2}; 1\right).$$

Suppose that $\alpha \in (0, 1)$ and $\nu > -1$. Then,

$$\tilde{i}_1^{\nu,\alpha} := \int_0^1 \frac{(1-u)^\nu - 1}{u^{1+\alpha}} du = \frac{1}{\alpha}\left(1 - \frac{\Gamma(1+\nu)\Gamma(1-\alpha)}{\Gamma(1-\alpha+\nu)}\right).$$

Recall that the digamma function is $\psi(z) = \frac{d}{dz} \log \Gamma(z) = \Gamma'(z)/\Gamma(z)$, which has $\psi(1) = -\gamma$ where $\gamma \approx 0.5772$ is Euler's constant.

Lemma 2 *Suppose that $\alpha > 0$. Then,*

$$j_0^\alpha := \int_1^\infty \frac{\log(1+u)}{u^{1+\alpha}} du = \frac{1}{\alpha}\left(\psi(\alpha) - \psi\left(\tfrac{\alpha}{2}\right)\right);$$

$$j_2^\alpha := \int_1^\infty \frac{\log(u-1)}{u^{1+\alpha}} du = -\frac{1}{\alpha}(\gamma + \psi(\alpha)).$$

Suppose that $\alpha \in (0, 2)$. Then,

$$j_1^\alpha := \int_0^1 \frac{\log(1-u^2)}{u^{1+\alpha}} du = \frac{1}{\alpha}\left(\gamma + \psi\left(1 - \tfrac{\alpha}{2}\right)\right).$$

Suppose that $\alpha \in (0, 1)$. Then,

$$\tilde{j}_1^\alpha := \int_0^1 \frac{\log(1-u)}{u^{1+\alpha}} du = \frac{1}{\alpha} \left(\gamma + \psi(1-\alpha) \right).$$

Remark 2 The j integrals can be obtained as derivatives with respect to ν of the i integrals, evaluated at $\nu = 0$.

The next result collects estimates for our integrals in the expected functional increments (5.4) and (5.5) in terms of the integrals in Lemma 1.

Lemma 3 *Suppose that $\upsilon \in \mathfrak{D}_{\alpha,c}$. For $\alpha > 0$ and $-1 < \nu < \alpha$ we have*

$$\int_x^\infty f_\nu(y-x)\upsilon(y)dy = cx^{\nu-\alpha} i_{2,1}^{\nu,\alpha} + o\left(x^{\nu-\alpha}\right); \tag{5.6}$$

$$\int_x^\infty f_\nu(x)\upsilon(y)dy = cx^{\nu-\alpha} i_{2,0}^{\nu,\alpha} + o\left(x^{\nu-\alpha}\right); \tag{5.7}$$

$$\int_x^\infty \left(f_\nu(x+y) - f_\nu(x)\right)\upsilon(y)dy = cx^{\nu-\alpha} i_0^{\nu,\alpha} + o\left(x^{\nu-\alpha}\right). \tag{5.8}$$

For $\alpha \in (0, 2)$ and $\nu > -1$ we have

$$\int_0^x \left(f_\nu(x+y) + f_\nu(x-y) - 2f_\nu(x)\right)\upsilon(y)dy = cx^{\nu-\alpha} i_1^{\nu,\alpha} + o\left(x^{\nu-\alpha}\right). \tag{5.9}$$

For $\alpha \in (0, 1)$ and $\nu > -1$ we have

$$\int_0^x \left(f_\nu(x-y) - f_\nu(x)\right)\upsilon(y)dy = cx^{\nu-\alpha} \tilde{i}_1^{\nu,\alpha} + o\left(x^{\nu-\alpha}\right). \tag{5.10}$$

Moreover, if $\upsilon \in \mathfrak{D}_{\alpha,c}^+$ then stronger versions of all of the above estimates hold with $o(x^{\nu-\alpha})$ replaced by $O(x^{\nu-\alpha-\delta})$ for some $\delta > 0$.

Proof These estimates are mostly quite straightforward, so we do not give all the details. We spell out the estimate in (5.6); the others are similar. We have

$$\int_x^\infty f_\nu(y-x)\upsilon(y)dy = x^\nu \int_x^\infty \left(\tfrac{1+y}{x} - 1\right)^\nu c(y)y^{-1-\alpha}dy.$$

With the substitution $u = \frac{1+y}{x}$, this last expression becomes

$$x^{1+\nu} \int_{\frac{1+x}{x}}^\infty (u-1)^\nu u^{-1-\alpha}(x-u^{-1})^{-1-\alpha} c(ux-1)du.$$

Let $\varepsilon \in (0, c)$. Then, there exists $y_0 \in \mathbb{R}_+$ such that $|c(y) - c| < \varepsilon$ for all $y \geq y_0$, so that $|c(ux - 1) - c| < \varepsilon$ for all u in the range of integration, provided $x \geq y_0$. Writing

$$f(u) = (u - 1)^\nu u^{-1-\alpha}, \quad \text{and} \quad g(u) = \left(x - u^{-1}\right)^{-1-\alpha} c(ux - 1),$$

for the duration of the proof, we have that

$$\int_x^\infty f_\nu(y - x) v(y) dy = x^{1+\nu} \int_{\frac{1+x}{x}}^\infty f(u) g(u) du. \tag{5.11}$$

For $u \geq \frac{1+x}{x}$ and $x \geq y_0$, we have

$$g_- := (c - \varepsilon) x^{-1-\alpha} \leq g(u) \leq (c + \varepsilon)(x - 1)^{-1-\alpha} =: g_+,$$

so that $g_+ - g_- \leq 2\varepsilon(x - 1)^{-1-\alpha} + C_1 x^{-2-\alpha}$ for a constant $C_1 < \infty$ not depending on $x \geq y_0$ or ε. Moreover, it is easy to see that $\int_1^\infty |f(u)| du \leq C_2$ for a constant C_2 depending only on ν and α, provided $\nu \in (-1, \alpha)$. Hence, Lemma 18 shows that

$$\left| \int_{\frac{1+x}{x}}^\infty f(u) g(u) du - (c - \varepsilon) x^{-1-\alpha} \int_{\frac{1+x}{x}}^\infty f(u) du \right| \leq 2 C_2 \varepsilon(x - 1)^{-1-\alpha} + C_1 C_2 x^{-2-\alpha},$$

for all $x \geq y_0$. Since also

$$\left| \int_{\frac{1+x}{x}}^\infty f(u) du - i_{2,1}^{\nu,\alpha} \right| \leq \int_1^{\frac{1+x}{x}} |f(u)| du \to 0,$$

as $x \to \infty$, it follows that for any $\varepsilon > 0$ we may choose x sufficiently large so that

$$\left| \int_{\frac{1+x}{x}}^\infty f(u) g(u) du - c x^{-1-\alpha} i_{2,1}^{\nu,\alpha} \right| \leq \varepsilon x^{-1-\alpha},$$

and since $\varepsilon > 0$ was arbitrary, we obtain (5.6) from (5.11). □

We also need the following simple estimates for ranges of α when the asymptotics for the final two integrals in Lemma 3 are not valid.

Lemma 4 *Suppose that $v \in \mathfrak{D}_{\alpha,c}$.*

(i) For $\alpha \geq 2$ and any $\nu \in (0, 1)$, there exist $\varepsilon > 0$ and $x_0 \in \mathbb{R}_+$ such that, for all $x \geq x_0$,

$$\int_0^x (f_\nu(x + y) + f_\nu(x - y) - 2f_\nu(x)) v(y) dy \leq \begin{cases} -\varepsilon x^{\nu-2} \log x & \text{if } \alpha = 2, \\ -\varepsilon x^{\nu-2} & \text{if } \alpha > 2. \end{cases}$$

(ii) For $\alpha \geq 1$ and any $v > 0$, there exist $\varepsilon > 0$ and $x_0 \in \mathbb{R}_+$ such that, for all $x \geq x_0$,

$$\int_0^x (f_v(x - y) - f_v(x)) \, v(y) \mathrm{d}y \leq \begin{cases} -\varepsilon x^{v-1} \log x & \text{if } \alpha = 1, \\ -\varepsilon x^{v-1} & \text{if } \alpha > 1. \end{cases}$$

Proof For part (i), set $a_v(z) = (1+z)^v + (1-z)^v - 2$, so that

$$\int_0^x (f_v(x+y) + f_v(x-y) - 2f_v(x)) \, v(y) \mathrm{d}y$$
$$= (1+x)^v \int_0^x a_v\left(\tfrac{y}{1+x}\right) c(y) y^{-1-\alpha} \mathrm{d}y.$$

Suppose that $\alpha \geq 2$ and $v \in (0, 1)$. For $v \in (0, 1)$, calculus shows that $a_v(z)$ has a single local maximum at $z = 0$, so that $a_v(z) \leq 0$ for all z. Moreover, Taylor's theorem shows that for any $v \in (0, 1)$ there exists $\delta_v \in (0, 1)$ such that $a_v(z) \leq -(v/2)(1-v)z^2$ for all $z \in [0, \delta_v]$. Also, $c(y) \geq c/2 > 0$ for all $y \geq y_0$ sufficiently large. Hence, for all $x \geq y_0/\delta_v$,

$$\int_0^x a_v\left(\tfrac{y}{1+x}\right) c(y) y^{-1-\alpha} \mathrm{d}y \leq \int_{y_0}^{\delta_v x} a_v\left(\tfrac{y}{1+x}\right) c(y) y^{-1-\alpha} \mathrm{d}y$$
$$\leq -\frac{cv(1-v)}{4(1+x)^2} \int_{y_0}^{\delta_v x} y^{1-\alpha} \mathrm{d}y,$$

which yields part (i) of the lemma.

For part (ii), suppose that $\alpha \geq 1$ and $v > 0$. For any $v > 0$, there exists $\delta_v \in (0, 1)$ such that $(1-z)^v - 1 \leq -(v/2)z$ for all $z \in [0, \delta_v]$. Moreover, $c(y) \geq c/2 > 0$ for all $y \geq y_0$ sufficiently large. Hence, since the integrand is non-positive, for $x > y_0/\delta_v$,

$$(1+x)^v \int_0^x \left(\left(1 - \tfrac{y}{1+x}\right)^v - 1\right) c(y) y^{-1-\alpha} \mathrm{d}y$$
$$\leq \frac{c}{2}(1+x)^v \int_{y_0}^{\delta_v x} \left(\left(1 - \tfrac{y}{1+x}\right)^v - 1\right) y^{-1-\alpha} \mathrm{d}y$$
$$\leq -\frac{cv(1+x)^v}{4(1+x)} \int_{y_0}^{\delta_v x} y^{-\alpha} \mathrm{d}y,$$

and part (ii) follows. $\qquad\square$

Lemma 5 *Suppose that (A1) holds and $\chi_i \alpha_i < 1$. Then for $v \in (-1, 1 \wedge \alpha_i)$, $\flat \in \{\text{one, sym}\}$, and $i \in \mathcal{S}^\flat$, as $x \to \infty$,*

$$Df_v(x, i) = \chi_i \lambda_i c_i x^{v-\alpha_i} i_{2,1}^{v,\alpha_i} \left(\frac{(P\lambda)_i}{\lambda_i} + R^\flat(\alpha_i, v)\right) + o(x^{v-\alpha_i}),$$

where $\boldsymbol{\lambda} = (\lambda_k; k \in \mathcal{S})$,

$$R^{\text{sym}}(\alpha, \nu) = \frac{i_0^{\nu,\alpha} + i_1^{\nu,\alpha} - i_{2,0}^{\alpha}}{i_{2,1}^{\nu,\alpha}}, \quad \text{and} \quad R^{\text{one}}(\alpha, \nu) = \frac{\tilde{i}_1^{\nu,\alpha} - i_{2,0}^{\alpha}}{i_{2,1}^{\nu,\alpha}}.$$

Moreover, if $\nu_i \in \mathfrak{D}_{\alpha_i,c_i}^+$ *then, for some* $\delta > 0$,

$$Df_\nu(x, i) = \chi_i \lambda_i c_i x^{\nu - \alpha_i} i_{2,1}^{\nu,\alpha_i} \left(\frac{(P\boldsymbol{\lambda})_i}{\lambda_i} + R^{\flat}(\alpha_i, \nu) \right) + O(x^{\nu - \alpha_i - \delta}).$$

Finally, as $\nu \to 0$,

$$R^{\flat}(\alpha, \nu) = \begin{cases} R^{\text{sym}}(\alpha, \nu) = -1 + \nu\pi \cot(\pi\alpha_i/2) + o(\nu) & \text{if } \flat = \text{sym} \\ R^{\text{one}}(\alpha, \nu) = -1 + \nu\pi \cot(\pi\alpha_i) + o(\nu) & \text{if } \flat = \text{one}. \end{cases} \quad (5.12)$$

Proof The above expressions for $Df_\nu(x, i)$ follow from (5.4) and (5.5) with Lemma 3.

Additionally, we compute that $R^{\text{sym}}(\alpha, 0) = R^{\text{one}}(\alpha, 0) = -1$. For ν in a neighbourhood of 0, uniformity of convergence of the integrals over $(1, \infty)$ enables us to differentiate with respect to ν under the integral sign to get

$$\begin{aligned} \frac{\partial}{\partial \nu} R^{\text{sym}}(\alpha, \nu)|_{\nu=0} &= \frac{j_0^\alpha + j_1^\alpha}{i_{2,1}^{0,\alpha}} - R^{\text{sym}}(\alpha, 0) \frac{j_2^\alpha}{i_{2,1}^{0,\alpha}} \\ &= \psi\left(1 - \frac{\alpha}{2}\right) - \psi\left(\frac{\alpha}{2}\right) \\ &= \pi \cot\left(\frac{\pi\alpha}{2}\right), \end{aligned}$$

using Lemma 2 and the digamma reflection formula (equation 6.3.7 from [1, p. 259]), and then, the first formula in (5.12) follows by Taylor's theorem. Similarly, for the second formula in (5.12),

$$\begin{aligned} \frac{\partial}{\partial \nu} R^{\text{one}}(\alpha, \nu)|_{\nu=0} &= \frac{\tilde{j}_1^\alpha}{i_{2,1}^{0,\alpha}} - R^{\text{one}}(\alpha, 0) \frac{j_2^\alpha}{i_{2,1}^{0,\alpha}} \\ &= \psi(1 - \alpha) - \psi(\alpha) \\ &= \pi \cot(\pi\alpha). \end{aligned}$$

This completes the proof. $\qquad\square$

We conclude this subsection with two algebraic results.

Lemma 6 *Suppose that* (A0) *holds. Given* $(b_k; k \in \mathcal{S})$ *with* $b_k \in \mathbb{R}$ *for all* k, *there exists a solution* $(\theta_k; k \in \mathcal{S})$ *with* $\theta_k \in \mathbb{R}$ *for all* k *to the system of equations*

$$\sum_{j \in \mathcal{S}} p(k, j)\theta_j - \theta_k = b_k, \quad (k \in \mathcal{S}), \quad (5.13)$$

if and only if $\sum_{k \in \mathcal{S}} \mu_k b_k = 0$. Moreover, if a solution to (5.13) exists, we may take $\theta_k > 0$ for all $k \in \mathcal{S}$.

Proof As column vectors, we write $\boldsymbol{\mu} = (\mu_k; k \in \mathcal{S})$ for the stationary probabilities as given in (A0), $\mathbf{b} = (b_k; k \in \mathcal{S})$, and $\boldsymbol{\theta} = (\theta_k; k \in \mathcal{S})$. Then, in matrix-vector form, (5.13) reads $(P - I)\boldsymbol{\theta} = \mathbf{b}$, while $\boldsymbol{\mu}$ satisfies (2.2), which reads $(P - I)^\top \boldsymbol{\mu} = \mathbf{0}$, the homogeneous system adjoint to (5.13) (here I is the identity matrix and $\mathbf{0}$ is the vector of all 0s).

A standard result from linear algebra (a version of the Fredholm alternative) says that $(P - I)\boldsymbol{\theta} = \mathbf{b}$ admits a solution $\boldsymbol{\theta}$ if and only if the vector $\mathbf{b}$ is orthogonal to any solution $\mathbf{x}$ to $(P - I)^\top \mathbf{x} = \mathbf{0}$; but, by (A0), any such $\mathbf{x}$ is a scalar multiple of $\boldsymbol{\mu}$. In other words, a solution $\boldsymbol{\theta}$ to (5.13) exists if and only if $\boldsymbol{\mu}^\top \mathbf{b} = 0$, as claimed.

Finally, since P is a stochastic matrix, $(P - I)\mathbf{1} = \mathbf{0}$, where $\mathbf{1}$ is the column vector of all 1s; hence, if $\boldsymbol{\theta}$ solves $(P - I)\boldsymbol{\theta} = \mathbf{b}$, then so does $\boldsymbol{\theta} + \gamma \mathbf{1}$ for any $\gamma \in \mathbb{R}$. This implies the final statement in the lemma. $\qquad\square$

Lemma 7 *Let $U = (U_{k,\ell}; k, \ell = 0, \ldots, M)$ be a given upper triangular matrix having all its upper triangular elements non-negative ($U_{k,\ell} \geq 0$ for $0 \leq k < \ell \leq M$ and vanishing all other elements) and $A = (A_k; k = 1, \ldots, M)$ a vector with positive components. Then, there exists a unique lower triangular matrix $L = (L_{k,\ell}; k, \ell = 0, \ldots, M)$ (so diagonal and upper triangular elements vanish) satisfying*

(i) $L_{m,m-1} = (UL)_{m,m} + A_m$ for $m = 1, \ldots, M$;
(ii) $L_{k,\ell} = \sum_{r=0}^{k-\ell-1} L_{\ell+r+1,\ell+r} = L_{\ell+1,\ell} + \cdots + L_{k,k-1}$ for $0 \leq \ell < k \leq M$.

Also, all lower triangular elements of L are positive, i.e. $L_{k,\ell} > 0$ for $0 \leq \ell < k \leq M$.

Proof We construct L inductively. Item (i) demands

$$L_{m,m-1} = \sum_{\ell=0}^{M} U_{m,\ell} L_{\ell,m} + A_m = \sum_{\ell=m+1}^{M} U_{m,\ell} L_{\ell,m} + A_m. \qquad (5.14)$$

In the case $m = M$, with the usual convention that an empty sum is 0, the demand (5.14) is simply $L_{M,M-1} = A_M$. So we can start our construction taking $L_{M,M-1} = A_M$, which is positive by assumption [item (ii) makes no demands in the case $k = M$, $\ell = M - 1$].

Suppose now that all matrix elements $L_{k,\ell}$ have been computed in the lower-right corner Λ_m ($1 \leq m \leq M$):

$$\Lambda_m = \begin{matrix} L_{m,m-1} & & \\ L_{m+1,m-1} & L_{m+1,m} & \\ \vdots & \vdots & \ddots \\ L_{M,m-1} & L_{M,m} & \ldots & L_{M,M-1} \end{matrix}$$

The elements of L involved in statement (i) (for given m) and in statement (ii) for $\ell = m - 1$ are all in Λ_m; thus, as part of our inductive hypothesis we may suppose

that the elements of Λ_m are such that (i) holds for the given m, and (ii) holds with $\ell = m - 1$ and all $m \le k \le M$. We have shown that we can achieve this for Λ_M.

The inductive step is to construct from Λ_m ($2 \le m \le M$) elements $L_{k,m-2}$ for $m - 1 \le k \le M$ and hence complete the array Λ_{m-1} in such a way that (i) holds for $m - 1$ replacing m, that (ii) holds for $\ell = m - 2$, and that all elements are positive. Now, (5.14) reveals the demand of item (i) as

$$L_{m-1,m-2} = \sum_{\ell=m}^{M} U_{m-1,\ell} L_{\ell,m-1} + A_{m-1},$$

which we can achieve since the elements of L on the right-hand side are all in Λ_m, and since $A_{m-1} > 0$ we get $L_{m-1,m-2} > 0$.

The $\ell = m - 2$ case of (ii) demands that for $m \le k \le M$ we have

$$L_{k,m-2} = \sum_{r=0}^{k-m+1} L_{m-1+r,m-2+r} = L_{m-1,m-2} + \cdots + L_{k,k-1}$$

which involves only elements of Λ_m in addition to $L_{m-1,m-2}$, which we have already defined, and positivity of all the $L_{k,m-2}$ follows by hypothesis. This gives us the construction of Λ_{m-1} and establishes the inductive step.

This algorithm can be continued down to Λ_1. But then the lower triangular matrix L is totally determined. The diagonal and upper triangular elements of L do not influence the construction, and may be set to zero. $\qquad\square$

Corollary 1 *Let the matrix U and the vector A be as in Lemma 7. Let $\mathcal{L}$ be the set of lower triangular matrices $\tilde{L}$ satisfying*

(i) $\tilde{L}_{m,m-1} > (U\tilde{L})_{m,m} + A_m$ *for $m = 1, \ldots, M$;*
(ii) $\tilde{L}_{k,\ell} = \sum_{r=0}^{k-\ell-1} \tilde{L}_{\ell+r+1,\ell+r} = \tilde{L}_{\ell+1,\ell} + \cdots + \tilde{L}_{k,k-1}$ *for $0 \le \ell < k \le M$,*

viewed as subset of the positive cone $\mathcal{V} = (0, \infty)^{\frac{M(M-1)}{2}}$. Then, $\mathcal{L}$ is a non-empty, open subset of $\mathcal{V}$.

5.3 Supermartingale Conditions and Recurrence

We use the notation

$$\underline{\alpha} := \min_{k \in \mathcal{S}} \alpha_k; \quad \bar{\alpha} := \max_{k \in \mathcal{S}} \alpha_k; \quad \alpha_\star := \min_{k \in \mathcal{S}} \{\alpha_k \wedge (1/\chi_k)\} \quad \text{and}$$
$$\alpha^\star := \max_{k \in \mathcal{S}} \{\alpha_k \wedge (1/\chi_k)\}.$$

We start with the case $\max_{k \in \mathcal{S}} \chi_k \alpha_k < 1$. We will obtain a local supermartingale by choosing the λ_k carefully. Lemma 6, which shows how the stationary probabilities μ_k enter, is crucial; a similar idea was used for random walks on strips in Section 3.1 of [10]. Next is our key local supermartingale result in this case.

Proposition 1 *Suppose that* (A0) *and* (A1) *hold, and that* $\max_{k \in \mathcal{S}} \chi_k \alpha_k < 1$.

(i) *If* $\sum_{k \in \mathcal{S}} \mu_k \cot(\chi_k \pi \alpha_k) < 0$, *then there exist* $\nu \in (0, \underline{\alpha})$, $\lambda_k > 0$ $(k \in \mathcal{S})$, $\varepsilon > 0$, *and* $x_0 \in \mathbb{R}_+$ *such that for all* $x \geq x_0$ *and all* $i \in \mathcal{S}$,

$$Df_\nu(x, i) = \mathbb{E}_{x,i}\left[f_\nu(X_1, \xi_1) - f_\nu(X_0, \xi_0)\right] \leq -\varepsilon x^{\nu - \bar{\alpha}},$$

(ii) *If* $\sum_{k \in \mathcal{S}} \mu_k \cot(\chi_k \pi \alpha_k) > 0$, *then there exist* $\nu \in (-1, 0)$, $\lambda_k > 0$ $(k \in \mathcal{S})$, $\varepsilon > 0$, *and* $x_0 \in \mathbb{R}_+$ *such that for all* $x \geq x_0$ *and all* $i \in \mathcal{S}$,

$$Df_\nu(x, i) = \mathbb{E}_{x,i}[f_\nu(X_1, \xi_1) - f_\nu(X_0, \xi_0)] \leq -\varepsilon x^{\nu - \bar{\alpha}},$$

Proof Write $a_k = \pi \cot(\chi_k \pi \alpha_k)$, $k \in \mathcal{S}$. First, we prove part (i). By hypothesis, $\sum_{k \in \mathcal{S}} \mu_k a_k = -\delta$ for some $\delta > 0$. Set $b_k = -a_k - \delta$ for all $k \in \mathcal{S}$, so that $\sum_{k \in \mathcal{S}} \mu_k b_k = 0$. Lemma 6 shows that for these b_k, we can find a collection of $\theta_k \in (0, \infty)$ so that

$$\sum_{j \in \mathcal{S}} p(i, j)\theta_j - \theta_i + a_i = -\delta, \quad \text{for all } i \in \mathcal{S}; \tag{5.15}$$

fix these θ_k for the rest of the proof. We then choose $\boldsymbol{\lambda} = (\lambda_k; k \in \mathcal{S})$ of the form $\lambda_k := \lambda_k(\nu) = 1 + \theta_k \nu$, for some $\nu \in (0, 1 \wedge \underline{\alpha})$.

Since $i_{2,1}^{\nu,\alpha} > 0$ for all $\alpha > 0$ and all $\nu \in (0, \alpha)$, Lemma 5 shows that $Df_\nu(x, i)$ will be negative for all i and all x sufficiently large provided that we can find ν such that $\frac{(P\boldsymbol{\lambda})_i}{\lambda_i} + R^\flat(\alpha_i, \nu) < 0$ for all i. By (5.12), writing $\boldsymbol{\theta} = (\theta_k; k \in \mathcal{S})$,

$$\begin{aligned}
\frac{(P\boldsymbol{\lambda})_i}{\lambda_i} + R^\flat(\alpha_i, \nu) &= \frac{(P\boldsymbol{\lambda})_i}{\lambda_i} - 1 + \nu a_i + o(\nu) \\
&= \frac{1 + \nu(P\boldsymbol{\theta})_i}{1 + \nu\theta_i} - 1 + \nu a_i + o(\nu) \\
&= \frac{\nu\left((P\boldsymbol{\theta})_i - \theta_i\right)}{1 + \nu\theta_i} + \nu a_i + o(\nu) \\
&= \nu\left((P\boldsymbol{\theta})_i - \theta_i\right) + \nu a_i + o(\nu) \\
&= -\nu\delta + o(\nu),
\end{aligned}$$

by (5.15). Therefore, by Lemma 5, we can always find sufficiently small $\nu > 0$ and a vector $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\nu)$ with strictly positive elements for which $Df_\nu(x, i) \leq -\varepsilon x^{\nu - \alpha_i}$ for some $\varepsilon > 0$, all i, and all x sufficiently large. Maximizing over i gives part (i).

The argument for part (ii) is similar. Suppose $\nu \in (-1, 0)$. This time, $\sum_{k \in \mathcal{S}} \mu_k a_k = \delta$ for some $\delta > 0$, and we set $b_k = -a_k + \delta$, so that $\sum_{k \in \mathcal{S}} \mu_k b_k = 0$ once more. Lemma 6 now shows that we can find θ_k so that

$$\sum_{j \in \mathcal{S}} p(i, j)\theta_j - \theta_i + a_i = \delta, \quad \text{for all} \quad i \in \mathcal{S}. \tag{5.16}$$

With this choice of θ_k, we again set $\lambda_k = 1 + \theta_k \nu$; note we may assume $\lambda_k > 0$ for all k for ν sufficiently small.

Again, $Df_\nu(x, i)$ will be non-positive for all i and all x sufficiently large provided that we can find λ and ν such that $\frac{(P\lambda)_i}{\lambda_i} + R^\flat(\alpha_i, \nu) < 0$ for all i. Following a similar argument to before, we obtain with (5.16) that

$$\frac{(P\lambda)_i}{\lambda_i} + R^\flat(\alpha_i, \nu) = \nu \left((P\theta)_i - \theta_i\right) + \nu a_i + o(\nu) = \nu \delta + o(\nu).$$

Thus, we can find for $\nu < 0$ close enough to 0 a vector $\lambda = \lambda(\nu)$ with strictly positive elements for which $Df_\nu(x, i) \leq -\varepsilon x^{\nu - \alpha_i}$ for all i and all x sufficiently large. $\qquad \square$

Now we examine the case $\max_{k \in \mathcal{S}} \chi_k \alpha_k \geq 1$.

Proposition 2 *Suppose that* (A0) *and* (A1) *hold, and* $\max_{k \in \mathcal{S}} \chi_k \alpha_k \geq 1$. *Then, there exist* $\nu \in (0, \alpha_\star)$, $\lambda_k > 0$ $(k \in \mathcal{S})$, $\varepsilon > 0$, *and* $x_0 \in \mathbb{R}_+$ *such that for all* $i \in \mathcal{S}$,

$$Df_\nu(x, i) = \mathbb{E}_{x,i} \left[f_\nu(X_1, \xi_1) - f_\nu(X_0, \xi_0) \right] \leq -\varepsilon x^{\nu - \alpha^\star}, \quad \text{for all} \quad x \geq x_0. \tag{5.17}$$

Before starting the proof of this proposition, we introduce the following notation. For $k \in \mathcal{S}$, denote by $a_k = \pi \cot(\pi \chi_k \alpha_k)$ and define the vector $\mathbf{a} = (a_k; k \in \mathcal{S})$. For $i \in \mathcal{S}$ and $A \subseteq \mathcal{S}$, write $P(i, A) = \sum_{j \in A} p(i, j)$. Define $\mathcal{S}_0 = \{i \in \mathcal{S} : \chi_i \alpha_i \geq 1\}$ and recursively, for $m \geq 1$,

$$\mathcal{S}_m := \left\{ i \in \mathcal{S} \setminus \cup_{\ell=0}^{m-1} \mathcal{S}_\ell : P(i, \mathcal{S}_{m-1}) > 0 \right\}.$$

Denote by $M := \max\{m \geq 0 : \mathcal{S}_m \neq \emptyset\}$. Since P is irreducible, the collection $(\mathcal{S}_m; m = 0, \ldots, M)$ is a partition of $\mathcal{S}$.

Proof of Proposition 2 It suffices to find $x_0 \in \mathbb{R}_+$, $\varepsilon > 0$, $\nu \in (0, \alpha_\star)$, and an open, non-empty subset G of the positive cone $\mathcal{C} := (0, \infty)^{|\mathcal{S}|}$ such that

$$G \subseteq \cap_{i \in \mathcal{S}} \{\lambda \in \mathcal{C} : Df_\nu(x, i) \text{ satisfies condition (5.17)}\}.$$

Now, for $i \in \mathcal{S}_0$, inequality (5.17) is satisfied thanks to (5.4), (5.5), and Lemmas 3 and 4 for every choice of λ (with positive components) and $\nu \in (0, \underline{\alpha})$. Hence, the previous condition reduces to the requirement

$$G \subseteq \cap_{i \in \mathcal{S} \setminus \mathcal{S}_0} \{\lambda \in \mathcal{C} : Df_\nu(x, i) \text{ satisfies condition (5.17)}\}. \tag{5.18}$$

The rest of the proof is devoted into establishing this fact.

Suppose that $i \in \mathcal{S}_m$ with $m = 1, \ldots, M$. Then, $\chi_i \alpha_i < 1$ by construction, and Lemma 5 shows that condition (5.17) will be satisfied if the system of inequalities

$$\frac{(P\lambda)_i}{\lambda_i} + R^\flat(\alpha_i, \nu) < 0, \ i = 1, \ldots, M \tag{5.19}$$

has non-trivial solutions λ for sufficiently small ν. Thanks to Lemma 5, we have $R^\flat(\alpha_i, \nu) = -1 + \nu a_i + o(\nu)$. We will obtain (5.17) if, for ν sufficiently small,

$$\mathcal{R} = \mathcal{R}(P, \nu, \mathbf{a}) := \bigcap_{m=1}^{M} \left\{ \lambda \in \mathcal{C} : \frac{(P\lambda)_i}{\lambda_i} < 1 - \nu a_i, i \in \mathcal{S}_m \right\} \neq \emptyset. \tag{5.20}$$

We seek a solution $\lambda \in \mathcal{R}$ (for sufficiently small ν), under the Ansatz that the λ_j are constant on every $\mathcal{S}_\ell$, i.e. the vector λ has the form $\hat{\lambda}$ with $\hat{\lambda}_j = \lambda^{(\ell)}$ for all $j \in \mathcal{S}_\ell$. Suppose that $i \in \mathcal{S}_m$. Then, $p(i, j) = 0$ for $j \in \mathcal{S}_\ell$ with $\ell < m - 1$, so that

$$\frac{(P\hat{\lambda})_i}{\hat{\lambda}_i} = \sum_{j \in \mathcal{S}} p(i, j) \frac{\hat{\lambda}_j}{\hat{\lambda}_i} = \sum_{\ell=0}^{M} P(i, \mathcal{S}_\ell) \frac{\lambda^{(\ell)}}{\lambda^{(m)}} = \sum_{\ell=m-1}^{M} P(i, \mathcal{S}_\ell) \frac{\lambda^{(\ell)}}{\lambda^{(m)}}. \tag{5.21}$$

We introduce the auxiliary matrix $\rho = (\rho_{k,\ell}; k, \ell \in \{0, \ldots, M\})$ defined by $\rho_{k,\ell} := \lambda^{(k)} / \lambda^{(\ell)}$. By construction, $\rho_{k,k} = 1$ and $\rho_{k,\ell} = 1/\rho_{\ell,k}$. Let

$$L_{k,\ell} = \frac{1}{\nu} \log \rho_{k,\ell} = -\frac{1}{\nu} \log \rho_{\ell,k} = \frac{1}{\nu} \left(\log \lambda^{(k)} - \log \lambda^{(\ell)} \right). \tag{5.22}$$

It suffices to determine the upper triangular part of ρ, or, equivalently, the lower triangular array $(L_{k,\ell}; 0 \leq \ell < k \leq M)$. We do so recursively, starting with $L_{M,M-1}$. In the case $i \in \mathcal{S}_M$, the condition in (5.20) reads, by (5.21),

$$\rho_{M-1,M} < \frac{1 - P(i, \mathcal{S}_M) - \nu a_i}{P(i, \mathcal{S}_{M-1})} = \frac{P\left(i, \mathcal{S}_M^c\right)}{P(i, \mathcal{S}_{M-1})} - \nu \frac{a_i}{P(i, \mathcal{S}_{M-1})}$$
$$= 1 - \nu \frac{a_i}{P(i, \mathcal{S}_{M-1})}.$$

On introducing the constant $A_M = \max_{i \in \mathcal{S}_M} \frac{a_i}{P(i, \mathcal{S}_M)}$, it is enough to choose $\rho_{M-1,M} < 1 - \nu A_M = \exp(-\nu A_M) + o(\nu)$. In other words, with $\rho_{\ell,k} = \exp(-\nu L_{k,\ell})$, we see that the choice $L_{M,M-1} > A_M$ satisfies the condition in (5.20) for $i \in \mathcal{S}_M$.

Suppose now that we have determined the condition in (5.20) for $i \in \cup_{\ell=m}^{M} \mathcal{S}_\ell$. Then, for $i \in \mathcal{S}_{m-1}$ the condition amounts, via (5.21), to

$$\rho_{m-1,m} P(i, \mathcal{S}_{m-1}) < 1 - \nu a_i - P(i, \mathcal{S}_m) - \sum_{\ell=m+1}^{M} \rho_{\ell,m} P(i, \mathcal{S}_\ell).$$

Using the fact that for $\ell < m$ we have $\rho_{\ell,m} = \exp(-\nu L_{m,\ell})$ and for $\ell > m$ we have $\rho_{\ell,m} = \exp(\nu L_{\ell,m})$, the above expression becomes, up to $o(\nu)$ terms

$$1 - \nu L_{m,m-1} < \frac{P\left(i, \left(\cup_{\ell=m}^{M} \mathcal{S}_\ell\right)^c\right)}{P(i, \mathcal{S}_{m-1})} - \nu \sum_{\ell=m+1}^{M} \frac{P(i, \mathcal{S}_l)}{P(i, \mathcal{S}_{m-1})} L_{\ell,m} - \nu \frac{a_i}{P(i, \mathcal{S}_{m-1})}$$

$$= 1 - \nu \sum_{\ell=m+1}^{M} \frac{P(i, \mathcal{S}_\ell)}{P(i, \mathcal{S}_{m-1})} L_{\ell,m} - \nu \frac{a_i}{P(i, \mathcal{S}_{m-1})}.$$

Introducing the upper triangular matrix $U = (U_{m,n}; 0 \leq m < n \leq M)$ defined by $U_{m,n} = \max_{i \in \mathcal{S}_m} \frac{P(i, \mathcal{S}_n)}{P(i, \mathcal{S}_{m-1})}$ for $m \geq 1$, and the vector $A_m = \max_{i \in \mathcal{S}_m} \frac{a_i}{P(i, \mathcal{S}_{m-1})}$ for $m = 1, \ldots, M$, the condition in (5.20) is satisfied if we solve the recursion

$$L_{m,m-1} > (UL)_{m,m} + A_m, \quad \text{for} \quad m = M, M-1, \ldots, 1,$$

with initial condition $L_{M,M-1} > A_M$ and condition $L_{\ell,m} > 0$ for $0 \leq m < \ell \leq M$. Additionally, we have from (5.22) that

$$L_{k,\ell} = L_{k,k-1} + L_{k-1,k-2} + \cdots + L_{\ell+1,\ell}, \quad 0 \leq \ell < k \leq M.$$

Hence, by Corollary 1, there exist non-trivial solutions for the lower triangular matrix L within an algorithmically determined region $\mathcal{L}$. The positivity of the lower triangular part of L implies that the components of $\boldsymbol{\lambda}$ are ordered: $\lambda^{(m)} < \lambda^{(m+1)}$ for $0 \leq m < M-1$. Given L, the ratios of the $\lambda^{(k)}$ are determined, and by construction, the $\lambda^{(k)}$ are positive. $\qquad \square$

We are almost ready to complete the proof of Theorem 1, excluding part (b)(iii); first, we need one more technical result concerning non-confinement.

Lemma 8 *Suppose that* (A0) *and* (A1) *hold. Then,* $\limsup_{n \to \infty} X_n = \infty$, *a.s.*

Proof We claim that for each $x \in \mathbb{R}_+$, there exists $\varepsilon_x > 0$ such that

$$\mathbb{P}\left[X_{n+1} - X_n \geq 1 \mid (X_n, \xi_n) = (y, i)\right] \geq \varepsilon_x, \text{ for all } y \in [0, x] \text{ and all } i \in \mathcal{S}.$$
$$(5.23)$$

Indeed, given $(x, i) \in \mathbb{R}_+ \times \mathcal{S}$, we may choose $j \in \mathcal{S}$ so that $p(i, j) > 0$ and we may choose $z_0 \geq x$ sufficiently large so that, for some $\varepsilon > 0$, $v_i(z) \geq \varepsilon z^{-1-\alpha_i}$ for all $z \geq z_0$. Then if $y \in [0, x]$,

$$\mathbb{P}\left[X_{n+1} \geq y + 1 \mid (X_n, \xi_n) = (y, i)\right] \geq p(i, j) \int_{2z_0+1}^{\infty} \varepsilon z^{-1-\alpha_i} \, dz = \varepsilon_{x,i} > 0,$$

which gives (5.23). The local escape property (5.23) implies the lim sup result by a standard argument: see, e.g. [16, Proposition 3.3.4]. $\qquad \square$

Proof of Theorem 1 We are not yet ready to prove part (b)(iii): we defer that part of the proof until Sect. 7.

The other parts of the theorem follow from the supermartingale estimates in this section together with the technical results from the Appendix. Indeed, under the conditions of part (a) or (b)(i) of the theorem, we have from Propositions 2 or 1(i), respectively, that for suitable $\nu > 0$ and λ_k,

$$\mathbb{E}\left[f_\nu(X_{n+1}, \xi_n) - f_\nu(X_n, \xi_n) \mid X_n, \xi_n\right] \le 0, \quad \text{on } \{X_n \ge x_0\}.$$

Thus, we may apply Lemma 16, which together with Lemma 8 shows that $\liminf_{n\to\infty} X_n \le x_0$, a.s. Thus, there exists an interval $I \subseteq [0, x_0 + 1]$ such that $(X_n, \eta_n) \in I \times \{i\}$ i.o., where i is some fixed element of $\mathcal{S}$. Let $\tau_0 := 0$ and for $k \in \mathbb{N}$ define $\tau_k = \min\{n > \tau_{k-1} + 1 : (X_n, \eta_n) \in I \times \{i\}\}$. Given $i \in \mathcal{S}$, we may choose $j, k \in \mathcal{S}$ such that $p(i, j) > \delta_1$ and $p(j, k) > \delta_1$ for some $\delta_1 > 0$; let $\gamma = \alpha_i \vee \alpha_j$. Then, we may choose $\delta_2 \in (0, 1)$ and $z_0 \in \mathbb{R}_+$ such that $v_i(z) > \delta_2 z^{-1-\gamma}$ and $v_j(z) > \delta_2 z^{-1-\gamma}$ for all $z \ge z_0$. Then, for any $\varepsilon \in (0, 1)$,

$$\mathbb{P}\left[X_{\tau_k+2} < \varepsilon \mid X_{\tau_k}\right] \ge \mathbb{P}\left[X_{\tau_k+2} < \varepsilon, X_{\tau_k+1} \in [z_0 + 1, z_0 + 2], \eta_{\tau_k+1} = j \mid X_{\tau_k}\right]$$
$$\ge \delta_1^2 \delta_2^2 \varepsilon (z_0 + 3)^{-1-\gamma} (x_0 + z_0 + 3)^{-1-\gamma},$$

uniformly in k. Thus, Lévy's extension of the Borel–Cantelli lemma shows $X_n < \varepsilon$ infinitely often. Thus, since $\varepsilon \in (0, 1)$ was arbitrary, $\liminf_{n\to\infty} X_n = 0$, a.s.

On the other hand, under the conditions of part (b)(ii) of the theorem, we have from Proposition 1(ii) that for suitable $\nu < 0$ and λ_k,

$$\mathbb{E}\left[f_\nu(X_{n+1}, \xi_n) - f_\nu(X_n, \xi_n) \mid X_n, \xi_n\right] \le 0, \quad \text{on } \{X_n \ge x_1\},$$

for any x_1 sufficiently large. Thus, we may apply Lemma 17, which shows that for any $\varepsilon > 0$ there exists $x \in (x_1, \infty)$ for which, for all $n \ge 0$,

$$\mathbb{P}\left[\inf_{m \ge n} X_m \ge x_1 \mid \mathcal{F}_n\right] \ge 1 - \varepsilon, \quad \text{on } \{X_n \ge x\}.$$

Set $\sigma_x = \min\{n \ge 0 : X_n \ge x\}$. Then, on $\{\sigma_x < \infty\}$,

$$\mathbb{P}\left[\inf_{m \ge \sigma_x} X_m > x_1 \mid \mathcal{F}_{\sigma_x}\right] \ge 1 - \varepsilon, \quad \text{a.s.}$$

But on $\{\sigma_x < \infty\} \cap \{\inf_{m \ge \sigma_x} X_m > x_1\}$, we have $\liminf_{m\to\infty} X_m \ge x_1$, so

$$\mathbb{P}\left[\liminf_{m\to\infty} X_m \ge x_1\right] \ge \mathbb{E}\left[\mathbb{P}\left[\inf_{m \ge \sigma_x} X_m > x_1 \mid \mathcal{F}_{\sigma_x}\right]\mathbf{1}\{\sigma_x < \infty\}\right]$$
$$\ge (1 - \varepsilon)\mathbb{P}[\sigma_x < \infty] = (1 - \varepsilon),$$

by Lemma 8. Since $\varepsilon > 0$ was arbitrary, we get $\liminf_{m\to\infty} X_m \ge x_1$, a.s., and since x_1 was arbitrary we get $\lim_{m\to\infty} X_m = \infty$, a.s. □

6 Existence or Non-existence of Moments

6.1 Technical Tools

The following result is a straightforward reformulation of Theorem 1 of [2].

Lemma 9 *Let Y_n be an integrable $\mathcal{F}_n$-adapted stochastic process, taking values in an unbounded subset of $\mathbb{R}_+$, with $Y_0 = x_0$ fixed. For $x > 0$, let $\sigma_x := \inf\{n \geq 0 : Y_n \leq x\}$. Suppose that there exist $\delta > 0$, $x > 0$, and $\gamma < 1$ such that for any $n \geq 0$,*

$$\mathbb{E}\left[Y_{n+1} - Y_n \mid \mathcal{F}_n\right] \leq -\delta Y_n^\gamma, \quad \text{on } \{n < \sigma_x\}. \tag{6.1}$$

Then, for any $p \in [0, 1/(1 - \gamma))$, $\mathbb{E}[\sigma_x^p] < \infty$.

The following companion result on non-existence of moments is a reformulation of Corollary 1 of [2].

Lemma 10 *Let Y_n be an integrable $\mathcal{F}_n$-adapted stochastic process, taking values in an unbounded subset of $\mathbb{R}_+$, with $Y_0 = x_0$ fixed. For $x > 0$, let $\sigma_x := \inf\{n \geq 0 : Y_n \leq x\}$. Suppose that there exist $C_1, C_2 > 0$, $x > 0$, $p > 0$, and $r > 1$ such that for any $n \geq 0$, on $\{n < \sigma_x\}$ the following hold:*

$$\mathbb{E}\left[Y_{n+1} - Y_n \mid \mathcal{F}_n\right] \geq -C_1; \tag{6.2}$$

$$\mathbb{E}\left[Y_{n+1}^r - Y_n^r \mid \mathcal{F}_n\right] \leq C_2 Y_n^{r-1}; \tag{6.3}$$

$$\mathbb{E}\left[Y_{n+1}^p - Y_n^p \mid \mathcal{F}_n\right] \geq 0. \tag{6.4}$$

Then, for any $q > p$, $\mathbb{E}[\sigma_x^q] = \infty$ for $x_0 > x$.

6.2 Proof of Theorem 2

Proof of Theorem 2 Under conditions (a) or (b)(i) of Theorem 1, we have from Propositions 2 or 1, respectively, that there exist positive λ_k and constants $\varepsilon > 0$, $\beta > 0$ and $\nu \in (0, \beta)$ such that,

$$Df_\nu(x, i) \leq -\varepsilon x^{\nu - \beta}, \quad \text{for all} \quad x \geq x_0 \quad \text{and all} \quad i.$$

Let $Y_n = f_\nu(X_n, \xi_n)$. Then, Y_n is bounded above and below by positive constants times $(1 + X_n)^\nu$, so we have that (6.1) holds for x sufficiently large with $\gamma = 1 - (\beta/\nu)$. It follows from Lemma 9 that $\mathbb{E}[\sigma_x^p] < \infty$ for $p \in (0, \nu/\beta)$, which gives the claimed existence of moments result.

It is not hard to see that *some* moments of the return time fail to exist, due to the heavy-tailed nature of the model, and an argument is easily constructed using the 'one big jump' idea: a similar idea is used in [14]. We sketch the argument. For any x, i, for all y sufficiently large we have $\mathbb{P}_{x,i}[X_1 \geq y - x] \geq \varepsilon y^{-\tilde{\alpha}}$. Given such a first jump, with uniformly positive probability the process takes time at least of order y^β

to return to a neighbourhood of zero (where β can be bounded in terms of α); this can be proved using a suitable maximal inequality as in the proof of Theorem 2.10 of [14]. Combining these two facts shows that with probability of order $y^{-\bar{\alpha}}$ the return time to a neighbourhood of the origin exceeds order y^{β}. This polynomial tail bound yields non-existence of sufficiently high moments. □

6.3 Explicit Cases: Theorems 4 and 6

We now restrict attention to the case $\mathcal{S} = \{1, 2\}$ with $\alpha_1 = \alpha_2 = \alpha$ and $\chi_1 = \chi_2 = \chi$, so both half-lines are of the same type. Take $\lambda_1 = \lambda_2 = 1$ and $\nu \in (0, \alpha)$, so that $f_\nu(x, i) = (1 + x)^\nu$. Then, Lemma 5 shows that, for $i \in \mathcal{S}^\flat$, $\flat \in \{\text{sym}, \text{one}\}$,

$$Df_\nu(x, i) = \chi c_i x^{\nu - \alpha} C^\flat(\nu, \alpha) + o(x^{\nu - \alpha}), \tag{6.5}$$

where

$$C^{\text{sym}}(\nu, \alpha) = i_{2,1}^{\nu,\alpha} + i_0^{\nu,\alpha} + i_1^{\nu,\alpha} - i_{2,0}^\alpha;$$
$$C^{\text{one}}(\nu, \alpha) = i_{2,1}^{\nu,\alpha} + \tilde{i}_1^{\nu,\alpha} - i_{2,0}^\alpha.$$

The two cases we are interested in are the recurrent two-sided symmetric case, where $\chi = \frac{1}{2}$ (i.e. $\mathcal{S} = \mathcal{S}^{\text{sym}}$) with $\alpha > 1$, and the recurrent one-sided antisymmetric case, where $\chi = 1$ (i.e. $\mathcal{S} = \mathcal{S}^{\text{one}}$) with $\alpha > \frac{1}{2}$.

Lemma 11 *Let $\flat \in \{\text{sym}, \text{one}\}$ and $\chi\alpha \in (\frac{1}{2}, 1)$. The function $\nu \mapsto C^\flat(\nu, \alpha)$ is continuous for $\nu \in [0, \alpha)$ with $C(0, \alpha) = 0$ and $\lim_{\nu \uparrow \alpha} C^\flat(\nu, \alpha) = \infty$. There exists $\nu_0^\flat = \nu_0^\flat(\alpha) \in (0, \alpha)$ such that $C^\flat(\nu, \alpha) < 0$ for $\alpha \in (0, \nu_0^\flat)$, $C^\flat(\nu_0^\flat, \alpha) = 0$, and $C^\flat(\nu, \alpha) > 0$ for $\alpha \in (\nu_0^\flat, \alpha)$*

Proof We give the proof only in the case $\flat = \text{sym}$; the other case is very similar. Thus, $\chi = \frac{1}{2}$, and for ease of notation, we write just C instead of $C^\flat$.

Clearly $C(0, \alpha) = 0$. For $\nu \geq 1$, convexity of the function $z \mapsto z^\nu$ on $\mathbb{R}_+$ shows that $(1 + u)^\nu + (1 - u)^\nu - 2 \geq 0$ for all $u \in [0, 1]$, so that $i_1^{\nu,\alpha} \geq 0$; clearly, $i_{2,1}^{\nu,\alpha}$ and $i_0^{\nu,\alpha}$ are also non-negative. Hence, by the expression for $i_{2,1}^{\nu,\alpha}$ in Lemma 1,

$$\liminf_{\nu \uparrow \alpha} C(\nu, \alpha) \geq -\frac{1}{\alpha} + \liminf_{\nu \uparrow \alpha} \frac{\Gamma(\nu + 1)\Gamma(\alpha - \nu)}{\Gamma(1 + \alpha)},$$

which is $+\infty$. Moreover, by Lemma 2 and the subsequent remark,

$$\frac{\partial}{\partial \nu} C(\nu, \alpha) \bigg|_{\nu = 0} = \frac{1}{\alpha} \left(\psi \left(1 - \frac{\alpha}{2} \right) - \psi \left(\frac{\alpha}{2} \right) \right) = \frac{\pi}{\alpha} \cot \left(\frac{\pi\alpha}{2} \right),$$

which is negative for $\alpha \in (1, 2)$. Hence, $C(\nu, \alpha) < 0$ for $\nu > 0$ small enough.

Since $\nu \mapsto C(\nu, \alpha)$ is a non-constant analytic function on $[0, \alpha)$, its zeros can accumulate only at α, but this is ruled out by the fact that $C(\nu, \alpha) \to \infty$ as $\nu \to \alpha$.

Hence, $C(\cdot, \alpha)$ has only finitely many zeros in $[0, \alpha)$; one is at 0, and there must be at least one zero in $(0, \alpha)$, by Rolle's theorem. Define $v_- := v_-(\alpha)$ and $v_+ := v_+(\alpha)$ to be the smallest and largest such zeros, respectively.

Suppose $0 < v_1 \leq v_2 < \alpha$. By Jensen's inequality, $\mathbb{E}_{x,i}[(1 + X_1)^{v_2}] \geq \left(\mathbb{E}_{x,i}[(1 + X_1)^{v_1}]\right)^{v_2/v_1}$. Hence

$$
\begin{aligned}
(1 + x)^{v_2} + Df_{v_2}(x, i) &\geq \left((1 + x)^{v_1} + Df_{v_1}(x, i)\right)^{v_2/v_1} \\
&= (1 + x)^{v_2} \left(1 + (1 + x)^{-v_1} Df_{v_1}(x, i)\right)^{v_2/v_1} \\
&= (1 + x)^{v_2} + \frac{v_2}{v_1} x^{v_2 - v_1} Df_{v_1}(x, i) + o(x^{v_2 - \alpha}),
\end{aligned}
$$

using Taylor's theorem and the fact that $Df_{v_1}(x, i) = O(x^{v_1 - \alpha})$, by (6.5). By another application of (6.5), it follows that

$$
\chi c_i x^{v_2 - \alpha} C(v_2, \alpha) + o(x^{v_2 - \alpha}) \geq \frac{v_2}{v_1} \chi c_i x^{v_2 - \alpha} C(v_1, \alpha) + o(x^{v_2 - \alpha}).
$$

Multiplying by $x^{\alpha - v_2}$ and taking $x \to \infty$, we obtain

$$
C(v_2, \alpha) \geq \frac{v_2}{v_1} C(v_1, \alpha), \quad \text{for } 0 < v_1 \leq v_2 < \alpha.
$$

In particular, (i) if $C(v_1, \alpha) \geq 0$ then $C(v, \alpha) \geq 0$ for all $v \geq v_1 > 0$; and (ii) if $C(v_1, \alpha) > 0$ and $v_2 > v_1$, we have $C(v_2, \alpha) > C(v_1, \alpha)$. It follows from these two observations that $C(v, \alpha) = 0$ for $v \in [v_-, v_+]$, which is not possible unless $v_- = v_+$. Hence, there is exactly one zero of $C(\cdot, \alpha)$ in $(0, \alpha)$; call it $v_0(\alpha)$. □

Lemma 12 *The positive zero of $C^\flat(\cdot, \alpha)$ described in Lemma 11 is given by $v_0^{\text{one}}(\alpha) = 2\alpha - 1$ or $v_0^{\text{sym}}(\alpha) = \alpha - 1$.*

Proof First suppose $\flat = $ one. Then, from Lemma 1 we verify that for $\alpha \in (\frac{1}{2}, 1)$,

$$
C^{\text{one}}(2\alpha - 1, \alpha) = \frac{\Gamma(2\alpha)\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)} - \frac{\Gamma(2\alpha)\Gamma(1 - \alpha)}{\alpha \Gamma(\alpha)} = 0,
$$

since $\alpha \Gamma(\alpha) = \Gamma(1 + \alpha)$.

Now, suppose that $\flat = $ sym and $\alpha \in (1, 2)$. To verify $C^{\text{sym}}(\alpha - 1, \alpha) = 0$, it is simpler to work with the integral representations directly, rather than the hypergeometric functions. After the substitution $z = 1/u$, we have

$$
i_0^{\alpha - 1, \alpha} = \int_0^1 \left((1 + z)^{\alpha - 1} - z^{\alpha - 1}\right) dz = \frac{2^\alpha - 2}{\alpha}.
$$

Similarly, after the same substitution,

$$
i_1^{\alpha - 1, \alpha} = \int_1^\infty \left((z + 1)^{\alpha - 1} + (z - 1)^{\alpha - 1} - 2z^{\alpha - 1}\right) dz,
$$

which we may evaluate as

$$
\begin{aligned}
i_1^{\alpha-1,\alpha} &= \lim_{y\to\infty} \int_1^y \left((z+1)^{\alpha-1} + (z-1)^{\alpha-1} - 2z^{\alpha-1} \right) \mathrm{d}z \\
&= \frac{2-2^\alpha}{\alpha} + \frac{1}{\alpha} \lim_{y\to\infty} y^\alpha \left((1+y^{-1})^\alpha + (1-y^{-1})^\alpha - 2 \right) \\
&= \frac{2-2^\alpha}{\alpha}.
\end{aligned}
$$

Finally, we have that $i_{2,1}^{\alpha-1,\alpha} - i_{2,0}^\alpha = \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} - \frac{1}{\alpha} = 0$, so altogether we verify that $C^{\mathrm{sym}}(\alpha-1,\alpha) = 0$. $\qquad\square$

We can now complete the proofs of Theorems 4 and 6.

Proof of Theorem 4 Let $Y_n = f_\nu(X_n, \xi_n)$. First suppose that $\alpha \in (1,2)$. Then, we have from (6.5) together with Lemmas 11 and 12 that, for any $\nu \in (0, \alpha-1)$,

$$
\mathbb{E}[Y_{n+1} - Y_n \mid \mathcal{F}_n] \leq -\varepsilon Y_n^{1-(\alpha/\nu)}, \quad \text{on} \quad \{Y_n \geq y_0\},
$$

for some $\varepsilon > 0$ and $y_0 \in \mathbb{R}_+$. It follows from Lemma 9 that $\mathbb{E}[\sigma^p] < \infty$ for $p < \nu/\alpha$ and since $\nu < \alpha-1$ was arbitrary we get $\mathbb{E}[\sigma^p] < \infty$ for $p < 1 - (1/\alpha)$.

For the non-existence of moments when $\alpha \in (1,2)$, we will apply Lemma 10 with $Y_n = f_\nu(X_n, \xi_n) = (1+X_n)^\nu$ for some $\nu \in (0, \alpha)$. Then, condition (6.2) follows from (6.5), which also shows that for $r \in (1, \alpha/\nu)$,

$$
\mathbb{E}\left[Y_{n+1}^r - Y_n^r \mid (X_n, \xi_n) = (x,i) \right] = \frac{c_i}{2} x^{r\nu-\alpha} C^{\mathrm{sym}}(r\nu, \alpha) + o(x^{r\nu-\alpha}),
$$

so that $\mathbb{E}[Y_{n+1}^r - Y_n^r \mid \mathcal{F}_n] \leq c_i C^{\mathrm{sym}}(r\nu, \alpha) Y_n^{r-(\alpha/\nu)}$, for all Y_n sufficiently large. Since $\alpha/\nu > 1$ condition (6.3) follows. Finally, we may choose $\nu < \alpha$ close enough to α and then take $\gamma \in (\alpha-1, \nu)$ so that from (6.5), with Lemmas 11 and 12, $\mathbb{E}[Y_{n+1}^{\gamma/\nu} - Y_n^{\gamma/\nu} \mid \mathcal{F}_n] \geq 0$, for all Y_n sufficiently large. Thus, we may apply Lemma 10 to obtain $\mathbb{E}[\sigma^p] = \infty$ for $p > \gamma/\nu$, and taking γ close to $\alpha-1$ and ν close to α we can achieve any $p > 1 - (1/\alpha)$, as claimed.

Next, suppose that $\alpha \geq 2$. A similar argument to before but this time using Lemmas 3 and 4 shows that for any $\nu \in (0,1)$,

$$
\mathbb{E}[Y_{n+1} - Y_n \mid \mathcal{F}_n] \leq -\varepsilon Y_n^{1-(2/\nu)}, \quad \text{on} \quad \{Y_n \geq y_0\},
$$

for some $\varepsilon > 0$ and $y_0 \in \mathbb{R}_+$. Lemma 9 then shows that $\mathbb{E}[\sigma^p] < \infty$ for $p < \nu/2$ and since $\nu \in (0,1)$ was arbitrary we get $\mathbb{E}[\sigma^p] < \infty$ for $p < 1/2$.

We sketch the argument for $\mathbb{E}[\sigma^p] = \infty$ when $p > 1/2$. For $\nu \in (1, \alpha)$, it is not hard to show that $Df_\nu(x,i) \geq 0$ for all x sufficiently large, and we may verify the other conditions of Lemma 10 to show that $\mathbb{E}[\sigma^p] = \infty$ for $p > 1/2$. $\qquad\square$

Proof of Theorem 6 Most of the proof is similar to that of Theorem 4, so we omit the details. The case where a different argument is required is the non-existence part of the case $\alpha \geq 1$. We have that for some $\varepsilon > 0$ and all y sufficiently large, $\mathbb{P}_{x,i}[X_1 \geq y] \geq \varepsilon(x + y)^{-\alpha}$. A similar argument to Lemma 4 shows that for any $\nu \in (0, 1)$, for some $C \in \mathbb{R}_+$, $\mathbb{E}[X_{n+1}^{\nu} - X_n^{\nu} \mid X_n = x] \geq -C$. Then, a suitable maximal inequality implies that with probability at least $1/2$ started from $X_1 \geq y$ it takes at least cy^{ν} steps for X_n to return to a neighbourhood of 0, for some $c > 0$. Combining the two estimates gives

$$\mathbb{P}_{x,i}\left[\sigma \geq cy^{\nu}\right] \geq \frac{1}{2}(x + y)^{-\alpha},$$

which implies $\mathbb{E}[\sigma^p] = \infty$ for $p \geq \alpha/\nu$, and since $\nu \in (0, 1)$ was arbitrary, we can achieve any $p > \alpha$. □

7 Recurrence Classification in the Critical Cases

7.1 Logarithmic Lyapunov Functions

In this section, we prove Theorem 1(b)(iii). Throughout this section, we write $a_k := \cot(\chi_k \pi \alpha_k)$ and suppose that $\max_{k \in \mathcal{S}} \chi_k \alpha_k < 1$, that $\sum_{k \in \mathcal{S}} \mu_k a_k = 0$, and that $v_i \in \mathfrak{D}_{\alpha_i, c_i}^+$ for all $i \in \mathcal{S}$, that is, for $y > 0$, $v_i(y) = c_i(y)y^{-1-\alpha_i}$, with $\alpha_i \in (0, \infty)$ and $c_i(y) = c_i + O(y^{-\delta})$, where $\delta > 0$ may be chosen so as not to depend upon i.

To prove recurrence in the critical cases, we need a function that grows more slowly than any power; now, the weights λ_k are additive rather than multiplicative. For $x \in \mathbb{R}$, write $g(x) := \log(1 + |x|)$. Then, for $x \in \mathbb{R}_+$ and $k \in \mathcal{S}$, define

$$g(x, k) := g(x) + \lambda_k = \log(1 + |x|) + \lambda_k, \tag{7.1}$$

where $\lambda_k > 0$ for all $k \in \mathcal{S}$. Also write

$$h(x, k) := (g(x, k))^{1/2} = (\log(1 + |x|) + \lambda_k)^{1/2}.$$

Lemma 6 shows that there exist $\lambda_k > 0$ ($k \in \mathcal{S}$) such that

$$a_k + \sum_{j \in \mathcal{S}} p(i, j)(\lambda_j - \lambda_i) = 0; \tag{7.2}$$

we fix such a choice of the λ_k from now on.

We prove recurrence by establishing the following result.

Lemma 13 *Suppose that the conditions of Theorem 1(b)(iii) hold, and that $(\lambda_k; k \in \mathcal{S})$ are such that (7.2) holds. Then, there exists $x_0 \in \mathbb{R}_+$ such that*

$$\mathbb{E}\left[h(X_{n+1}, \xi_{n+1}) - h(X_n, \xi_n) \mid X_n = x, \xi_n = i\right] \leq 0, \text{ for } x \geq x_0 \quad \text{and all} \quad i \in \mathcal{S}.$$

It is not easy to perform the integrals required to estimate $Dh(x, i)$ (and hence establish Lemma 13) directly. However, the integrals for $Dg(x, i)$ are computable (they appear in Lemma 2), and we can use some analysis to relate $Dh(x, i)$ to $Dg(x, i)$. Thus, the first step in our proof of Lemma 13 is to estimate $Dg(x, i)$.

For the Lyapunov function g, (5.1) gives for $x \in \mathbb{R}_+$ and $i \in \mathcal{S}$,

$$
\begin{aligned}
Dg(x, i) &= \mathbb{E}\left[g(X_{n+1}, \xi_{n+1}) - g(X_n, \xi_n) \mid (X_n, \xi_n) = (x, i)\right] \\
&= \sum_{j \in \mathcal{S}} p(i, j)(\lambda_j - \lambda_i) \int_{-\infty}^{-x} w_i(y)dy + \int_{-\infty}^{\infty} (g(x + y) - g(x))\, w_i(y)dy \\
&= \chi_i \sum_{j \in \mathcal{S}} p(i, j)(\lambda_j - \lambda_i)T_i(x) + G_i^\flat(x),
\end{aligned}
\tag{7.3}
$$

where we have used the fact that $g(x)$ is defined for all $x \in \mathbb{R}$ and symmetric about 0, and we have introduced the notation

$$
T_i(x) := \int_x^\infty v_i(y)dy = \frac{c_i}{\alpha_i}x^{-\alpha_i} + O(x^{-\alpha_i - \delta}),
$$

and

$$
G_i^\flat(x) := \begin{cases} \frac{1}{2} \int_{-\infty}^{\infty} (g(x + y) - g(x))\, v_i(|y|)dy & \text{if } \flat = \text{sym} \\ \int_0^\infty (g(x - y) - g(x))\, v_i(y)dy & \text{if } \flat = \text{one.} \end{cases}
\tag{7.4}
$$

The next lemma, proved in the next subsection, estimates the integrals in (7.4).

Lemma 14 *Suppose that* $v_i \in \mathfrak{D}_{\alpha_i, c_i}^+$. *Then, for* $\flat \in \{\text{one, sym}\}$, *for* $\alpha \in (0, 1/\chi_i)$, *for some* $\eta > 0$, *as* $x \to \infty$,

$$
G_i^\flat(x) = \chi_i c_i x^{-\alpha_i} \frac{\pi}{\alpha_i} \cot(\chi_i \pi \alpha_i) + O(x^{-\alpha_i - \eta}).
$$

Note that Lemma 14 together with (7.3) shows that

$$
Dg(x, i) = \frac{\chi_i c_i}{\alpha_i}\left(a_i + \sum_{j \in \mathcal{S}} p(i, j)(\lambda_j - \lambda_i)\right) x^{-\alpha_i} + O(x^{-\alpha_i - \eta}) = O(x^{-\alpha_i - \eta}),
\tag{7.5}
$$

by (7.2). This is not enough by itself to establish recurrence, since the sign of the $O(x^{-\alpha_i - \eta})$ term is unknown. This is why we need the function $h(x, i)$.

7.2 Proof of Recurrence in the Critical Case

Proof of Lemma 14 To ease notation, we drop the subscripts i everywhere for the duration of this proof. From (7.4) we obtain, for $x > 0$,

$$
\begin{aligned}
G^{\text{sym}}(x) = &\frac{1}{2} \int_x^\infty \log\left(\frac{1+y-x}{1+x}\right) v(y)dy \\
&+ \frac{1}{2} \int_x^\infty \log\left(\frac{1+x+y}{1+x}\right) v(y)dy \\
&+ \frac{1}{2} \int_0^x \left[\log\left(\frac{1+x+y}{1+x}\right) + \log\left(\frac{1+x-y}{1+x}\right)\right] v(y)dy.
\end{aligned}
\tag{7.6}
$$

Let $\alpha \in (0, 2)$. The claim in the lemma for $\flat = \text{sym}$ will follow from the estimates

$$
\int_x^\infty \log\left(\frac{1+x+y}{1+x}\right) v(y)dy = cx^{-\alpha} j_0^\alpha + O(x^{-\alpha-\eta});
$$
$$
\int_x^\infty \log\left(\frac{1+y-x}{1+x}\right) v(y)dy = cx^{-\alpha} j_2^\alpha + O(x^{-\alpha-\eta});
$$
$$
\int_0^x \left[\log\left(\frac{1+x+y}{1+x}\right) + \log\left(\frac{1+x-y}{1+x}\right)\right] v(y)dy = cx^{-\alpha} j_1^\alpha + O(x^{-\alpha-\eta});
\tag{7.7}
$$

since then we obtain from (7.6) with (7.7) and Lemma 2 that

$$
G^{\text{sym}}(x) = \frac{c}{2}x^{-\alpha}\frac{1}{\alpha}\left(\psi\left(1-\tfrac{\alpha}{2}\right) - \psi\left(\tfrac{\alpha}{2}\right)\right) + O(x^{-\alpha-\eta}),
$$

which yields the stated result via the digamma reflection formula (equation 6.3.7 from [1, p. 259]). We present here in detail the proof of only the final estimate in (7.7); the others are similar. Some algebra followed by the substitution $u = y/(1+x)$ shows that the third integral in (7.7) is

$$
\int_0^x \log\left(1 - \frac{y^2}{(1+x)^2}\right) c(y)y^{-1-\alpha}dy
$$
$$
= (1+x)^{-\alpha} \int_0^{\frac{x}{1+x}} \log(1-u^2)c(u(1+x))u^{-1-\alpha}du.
$$

There is a constant $C \in \mathbb{R}_+$ such that for all x sufficiently large,

$$
\int_0^{\frac{\sqrt{x}}{1+x}} \left|\log(1-u^2)c(u(1+x))u^{-1-\alpha}\right| du \leq C \int_0^{\frac{\sqrt{x}}{1+x}} u^{1-\alpha}du = O\left(x^{(\alpha/2)-1}\right),
$$

using Taylor's theorem for log and the fact that $c(y)$ is uniformly bounded. On the other hand, for $u > \frac{\sqrt{x}}{1+x}$ we have $c(u(1+x)) = c + O(x^{-\delta/2})$, so that

$$\int_{\frac{\sqrt{x}}{1+x}}^{\frac{x}{1+x}} \log(1 - u^2) c(u(1+x)) u^{-1-\alpha} du = c \int_{\frac{\sqrt{x}}{1+x}}^{\frac{x}{1+x}} \log(1 - u^2) u^{-1-\alpha} du + O\left(x^{-\delta/2}\right).$$

Here we have that

$$\int_{\frac{\sqrt{x}}{1+x}}^{\frac{x}{1+x}} \log(1 - u^2) u^{-1-\alpha} du = j_1^\alpha + O\left(x^{(\alpha/2)-1}\right) + O\left(x^{-1} \log x\right).$$

Combining these estimates and using the fact that $\alpha \in (0, 2)$, we obtain the final estimate in (7.7). The claim in the lemma for $\flat$ = one follows after some analogous computations, which we omit. $\qquad\square$

Now, we relate $Dh(x, i)$ to $Dg(x, i)$, by comparing the individual integral terms.

Lemma 15 *Suppose that* $v_i \in \mathfrak{D}_{\alpha_i, c_i}^+$. *Then, for all x and all i,*

$$\int_x^\infty \big(h(x + y, i) - h(x, i)\big) v_i(y) dy \leq \frac{1}{2h(x, i)} \int_x^\infty (g(x + y) - g(x)) v_i(y) dy;$$

$$\int_0^x \big(h(x - y, i) - h(x, i)\big) v_i(y) dy \leq \frac{1}{2h(x, i)} \int_0^x (g(x - y) - g(x)) v_i(y) dy;$$

$$and \int_0^x \big(h(x + y, i) + h(x - y, i) - 2h(x, i)\big) v_i(y) dy$$

$$\leq \frac{1}{2h(x, i)} \int_0^x \big(g(x + y) + g(x - y) - 2g(x)\big) v_i(y) dy.$$

Finally, there exists $\varepsilon > 0$ such that, for all $i, j \in \mathcal{S}$ and all x sufficiently large,

$$\int_x^\infty (h(y - x, j) - h(x, i)) v_i(y) dy$$

$$\leq \frac{1}{2h(x, i)} \int_x^\infty (g(y - x, j) - g(x, i)) v_i(y) dy - \varepsilon \frac{x^{-\alpha_i}}{\log^{3/2} x}.$$

Proof The proof is based on the observation that, since $(h(x, i))^2 = g(x, i)$,

$$h(z, j) - h(x, i) = \frac{g(z, j) - g(x, i)}{h(z, j) + h(x, i)}.$$

Thus, for $y \geq 0$,

$$h(x + y, i) - h(x, i) = \frac{g(x + y) - g(x)}{h(x + y, i) + h(x, i)} \leq \frac{g(x + y) - g(x)}{2h(x, i)}, \qquad (7.8)$$

since $g(x + y) - g(x) \geq 0$ and $h(x + y, i) \geq h(x, i)$. This gives the first inequality in the lemma. Similarly, for $y \in [0, x]$,

$$h(x - y, i) - h(x, i) = \frac{g(x - y) - g(x)}{h(x - y, i) + h(x, i)} \leq \frac{g(x - y) - g(x)}{2h(x, i)},$$

since $g(x - y) - g(x) \leq 0$ and $h(x - y, i) \leq h(x, i)$. This gives the second inequality and also yields the third inequality once combined with the $y \in [0, x]$ case of (7.8).

Finally, for $y \geq x$ note that

$$h(y - x, j) - h(x, i) = \frac{g(y - x, j) - g(x, i)}{h(y - x, j) + h(x, i)}. \tag{7.9}$$

Also note that, for $y \geq x > 0$, $g(x, i) = g(x) + \lambda_i$ and $g(y - x, j) = g(y - x) + \lambda_j$, so

$$g(y - x, j) - g(x, i) = \log\left(e^{\lambda_j - \lambda_i} \frac{1 + y - x}{1 + x}\right).$$

So the sign of the expression in (7.9) is non-positive for $y \leq \psi(x) := x - 1 + (1 + x)e^{\lambda_i - \lambda_j}$ and non-negative for $y \geq \psi(x)$, and

$$g(\psi(x) - x, j) - g(x, i) = 0. \tag{7.10}$$

By the monotonicity in y of the denominator, the expression in (7.9) satisfies

$$\frac{g(y - x, j) - g(x, i)}{h(y - x, j) + h(x, i)} \leq \frac{g(y - x, j) - g(x, i)}{h(\psi(x) - x, j) + h(x, i)}$$

both for $y \in [0, \psi(x)]$ and for $y \in [\psi(x), \infty)$. Here $h(\psi(x) - x, j) = h(x, i)$, by (7.10). Hence, we obtain the bound

$$\int_x^\infty (h(y - x, j) - h(x, i)) \, v_i(y) \mathrm{d}y \leq \int_x^\infty \left(\frac{g(y - x, j) - g(x, i)}{2h(x, i)}\right) v_i(y) \mathrm{d}y.$$

To improve on this estimate, suppose that $y \geq Kx$ where $K \in \mathbb{N}$ is such that $Kx > \psi(x)$. Then, using the fact that the numerator in (7.9) is positive, we may choose $K \in \mathbb{N}$ such that for all j and all $y \geq Kx$,

$$h(y - x, j) - h(x, i) \leq \frac{g(y - x, j) - g(x, i)}{h((K - 1)x, j) + h(x, i)}$$

$$\leq \frac{g(y - x, j) - g(x, i)}{(\log(1 + |x|) + \lambda_i + 1)^{1/2} + h(x, i)},$$

for K sufficiently large (depending on $\max_j \lambda_j$) and and all x sufficiently large. Here

$$(\log(1 + |x|) + \lambda_i + 1)^{1/2} = h(x, i) \left(1 + \frac{1}{g(x, i)}\right)^{1/2} = h(x, i) + \frac{1 + o(1)}{2 \log^{1/2} x}.$$

It follows that, for some $\varepsilon > 0$, for all x sufficiently large, for $y \geq Kx$,

$$h(y - x, j) - h(x, i) \leq \frac{g(y - x, j) - g(x, i)}{2h(x, i)} - \varepsilon \left(\frac{g(y - x, j) - g(x, i)}{\log^{3/2} x}\right).$$

The final inequality in the lemma now follows since, for all x sufficiently large,

$$\begin{aligned}
\int_{Kx}^{\infty} &(g(y - x, j) - g(x, i))\, v_i(y) \mathrm{d}y \\
&\geq \frac{c_i}{2} \int_{Kx}^{\infty} \log\left(e^{\lambda_j - \lambda_i}\left(\frac{2 + y}{1 + x} - 1\right)\right) y^{-1 - \alpha_i} \mathrm{d}y \\
&\geq \frac{c_i}{2} x^{-\alpha_i} \int_{K/2}^{\infty} \log\left(e^{\lambda_j - \lambda_i}(u - 1)\right) u^{-1 - \alpha_i} \mathrm{d}u,
\end{aligned}$$

where we used the substitution $u = \frac{2+y}{1+x}$. For K sufficiently large, the term inside the logarithm is uniformly positive, and the claimed bound follows. $\qquad\square$

Now, we may complete the proofs of Lemma 13 and then Theorem 1(b)(iii).

Proof of Lemma 13 Lemma 15 together with (7.5) shows that,

$$Dh(x, i) \leq \frac{Dg(x, i)}{2h(x, i)} - \varepsilon \frac{x^{-\alpha_i}}{\log^{3/2} x} \leq -\varepsilon \frac{x^{-\alpha_i}}{\log^{3/2} x} + O(x^{-\alpha_i - \eta}) \leq 0,$$

for all x sufficiently large. $\qquad\square$

Proof of Theorem 1(b)(iii) Lemma 13 with Lemma 16 shows that $\liminf_{n \to \infty} X_n \leq x_0$, a.s., and then, a similar argument to that in the proof of parts (a) and (b)(i) of Theorem 1 shows that $\liminf_{n \to \infty} X_n = 0$, a.s. $\qquad\square$

Appendix: Technical Results

Semimartingale Results

Lemma 16 *Let (X_n, ξ_n) be an $\mathcal{F}_n$-adapted process taking values in $\mathbb{R}_+ \times \mathcal{S}$. Let $f : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ be such that $\lim_{x \to \infty} f(x, i) = \infty$ for all $i \in \mathcal{S}$, and $\mathbb{E}\, f(X_0, \xi_0) < \infty$. Suppose that there exist $x_0 \in \mathbb{R}_+$ and $C < \infty$ for which, for all $n \geq 0$,*

$$\mathbb{E}\left[f\left(X_{n+1}, \xi_{n+1}\right) - f\left(X_n, \xi_n\right) \mid \mathcal{F}_n\right] \leq 0, \quad \text{on } \{X_n > x_0\}, \ a.s.;$$
$$\mathbb{E}\left[f\left(X_{n+1}, \xi_{n+1}\right) - f\left(X_n, \xi_n\right) \mid \mathcal{F}_n\right] \leq C, \quad \text{on } \{X_n \leq x_0\}, \ a.s.$$

Then,

$$\mathbb{P}\left[\left\{\limsup_{n \to \infty} X_n < \infty\right\} \cup \left\{\liminf_{n \to \infty} X_n \leq x_0\right\}\right] = 1.$$

Proof First note that, by hypothesis, $\mathbb{E} f(X_1, \xi_1) \leq \mathbb{E} f(X_0, \xi_0) + C < \infty$, and iterating this argument, it follows that $\mathbb{E} f(X_n, \xi_n) < \infty$ for all $n \geq 0$.

Fix $n \in \mathbb{Z}_+$. For $x_0 \in \mathbb{R}_+$ in the hypothesis of the lemma, write $\lambda = \min\{m \geq n : X_m \leq x_0\}$. Let $Y_m = f(X_{m \wedge \lambda}, \xi_{m \wedge \lambda})$. Then, $(Y_m, m \geq n)$ is an $(\mathcal{F}_m, m \geq n)$-adapted non-negative supermartingale. Hence, by the supermartingale convergence theorem, there exists $Y_\infty \in \mathbb{R}_+$ such that $\lim_{m \to \infty} Y_m = Y_\infty$, a.s. In particular,

$$\limsup_{m \to \infty} f(X_m, \xi_m) \leq Y_\infty, \quad \text{on } \{\lambda = \infty\}.$$

Set $\zeta_i = \sup\{x \geq 0 : f(x, i) \leq 1 + Y_\infty\}$, which has $\zeta_i < \infty$ a.s. since $\lim_{x \to \infty} f(x, i) = \infty$. Then, $\limsup_{m \to \infty} X_m \leq \max_i \zeta_i < \infty$ on $\{\lambda = \infty\}$. Hence

$$\mathbb{P}\left[\left\{\limsup_{n \to \infty} X_n < \infty\right\} \cup \left\{\inf_{m \geq n} X_m \leq x_0\right\}\right] = 1.$$

Since $n \in \mathbb{Z}_+$ was arbitrary, the result follows:

$$\mathbb{P}\left[\left\{\limsup_{n \to \infty} X_n < \infty\right\} \cup \bigcap_{n \geq 0}\left\{\inf_{m \geq n} X_m \leq x_0\right\}\right] = 1.$$

This completes the proof. □

Lemma 17 *Let (X_n, ξ_n) be an $\mathcal{F}_n$-adapted process taking values in $\mathbb{R}_+ \times \mathcal{S}$. Let $f : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ be such that $\sup_{x,i} f(x, i) < \infty$ and $\lim_{x \to \infty} f(x, i) = 0$ for all $i \in \mathcal{S}$. Suppose that there exists $x_1 \in \mathbb{R}_+$ for which $\inf_{y \leq x_1} f(y, i) > 0$ for all i and*

$$\mathbb{E}[f(X_{n+1}, \xi_{n+1}) - f(X_n, \xi_n) \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n \geq x_1\}, \quad \text{for all } n \geq 0.$$

Then, for any $\varepsilon > 0$ there exists $x \in (x_1, \infty)$ for which, for all $n \geq 0$,

$$\mathbb{P}\left[\inf_{m \geq n} X_m \geq x_1 \ \Big| \ \mathcal{F}_n\right] \geq 1 - \varepsilon, \quad \text{on } \{X_n \geq x\}.$$

Proof Fix $n \in \mathbb{Z}_+$. For $x_1 \in \mathbb{R}_+$ in the hypothesis of the lemma, write $\lambda = \min\{m \geq n : X_m \leq x_1\}$ and set $Y_m = f(X_{m \wedge \lambda}, \xi_{m \wedge \lambda})$. Then, $(Y_m, m \geq n)$ is an $(\mathcal{F}_m, m \geq n)$-adapted non-negative supermartingale, and so converges a.s. as $m \to \infty$ to some $Y_\infty \in \mathbb{R}_+$. Moreover, by the optional stopping theorem for supermartingales,

$$Y_n \geq \mathbb{E}[Y_\infty \mid \mathcal{F}_n] \geq \mathbb{E}[Y_\infty \mathbf{1}\{\lambda < \infty\} \mid \mathcal{F}_n], \ a.s.$$

Here, we have that, a.s.,

$$Y_\infty \mathbf{1}\{\lambda < \infty\} = \lim_{m \to \infty} Y_m \mathbf{1}\{\lambda < \infty\} = f(X_\lambda, \xi_\lambda) \mathbf{1}\{\lambda < \infty\}$$
$$\geq \min_i \inf_{y \leq x_1} f(y, i) \mathbf{1}\{\lambda < \infty\}.$$

Combining these inequalities we obtain

$$\min_i \inf_{y \leq x_1} f(y, i) \mathbb{P}[\lambda < \infty \mid \mathcal{F}_n] \leq Y_n, \text{ a.s.}$$

In particular, on $\{X_n \geq x > x_1\}$, we have $Y_n = f(X_n, \xi_n)$ and so

$$\min_i \inf_{y \leq x_1} f(y) \mathbb{P}[\lambda < \infty \mid \mathcal{F}_n] \leq f(X_n, \xi_n) \leq \max_i \sup_{y \geq x} f(y, i).$$

Since $\lim_{y \to \infty} f(y, i) = 0$ and $\inf_{y \leq x_1} f(y, i) > 0$, given $\varepsilon > 0$ we can choose $x > x_1$ large enough so that

$$\frac{\max_i \sup_{y \geq x} f(y, i)}{\min_i \inf_{y \leq x_1} f(y, i)} < \varepsilon;$$

the choice of x depends only on f, x_1, and ε, and, in particular, does not depend on n. Then, on $\{X_n \geq x\}$, $\mathbb{P}[\lambda < \infty \mid \mathcal{F}_n] < \varepsilon$, as claimed.

Proofs of Integral Computations

Proof of Lemma 1 Consider $i_0^{\nu,\alpha}$. With the change of variable $v = 1/u$, we get

$$\int_0^1 u^{-1-\alpha}(1+u)^\nu \mathrm{d}u = \int_0^1 v^{\alpha-\nu-1}(1+v)^\nu \mathrm{d}v$$
$$= \frac{\Gamma(\alpha - \nu)\Gamma(1)}{\Gamma(\alpha - \nu + 1)} {}_2F_1(-\nu, \alpha - \nu; \alpha - \nu + 1; -1),$$

by the integral representation for the Gauss hypergeometric function (see equation 15.3.1 of [1, p. 558]). The given expression for $i_0^{\nu,\alpha}$ follows.

The integral $i_{2,0}^\alpha$ is trivial. Consider $i_{2,1}^{\nu,\alpha}$. The substitution $v = 1/u$ gives

$$\int_1^\infty u^{-1-\alpha}(u-1)^\nu \mathrm{d}u = \int_0^1 (1-v)^\nu v^{\alpha-\nu-1} \mathrm{d}v = \frac{\Gamma(1+\nu)\Gamma(\alpha - \nu)}{\Gamma(1+\alpha)},$$

by the integral formula for the Beta function (see equation 6.2.1 of [1, p. 258]), provided $\nu > -1$ and $\alpha - \nu > 0$. Hence, we obtain the given expression for $i_{2,1}^{\nu,\alpha}$.

Next, consider $i_1^{\nu,\alpha}$. By considering separately the asymptotics of the integrand as $u \downarrow 0$ and $u \uparrow 1$, we see that $i_1^{\nu,\alpha}$ is finite provided $\alpha \in (0, 2)$ and $\nu > -1$. For

$u \in [-1, 1]$, we have the Taylor series expansion $(1 + u)^{\nu} = \sum_{n \geq 0} \frac{u^n}{n!} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-n)}$. Hence,

$$\left((1 + u)^{\nu} + (1 - u)^{\nu} - 2\right) u^{-1-\alpha} = 2 \sum_{n \geq 1} \frac{u^{2n-1-\alpha}}{(2n)!} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1 - 2n)}.$$

Here, the power series for $n \geq 2$ converges normally (hence uniformly) over $|u| \leq 1$. This remark allows interchanging summation and integration to obtain $i_1^{\nu,\alpha} = W^{\nu,\alpha}(1)$, where for $|z| \leq 1$ we define

$$W^{\nu,\alpha}(z) := 2 \sum_{n \geq 1} \frac{z^n}{2n - \alpha} \frac{1}{(2n)!} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1 - 2n)} \qquad (7.11)$$

$$= \frac{1}{2 - \alpha} \nu(\nu - 1)z \left[1 + \frac{2 - \alpha}{4 - \alpha} \frac{1}{4!}(\nu - 2)(\nu - 3)z \right.$$

$$\left. + \frac{2 - \alpha}{6 - \alpha} \frac{1}{6!}(\nu - 2)(\nu - 3)(\nu - 4)(\nu - 5)z^2 + \cdots\right]$$

$$= \frac{1}{2 - \alpha} \nu(\nu - 1)z \left[\sum_{n \geq 0} c_n z^n\right],$$

where $c_n = \frac{2-\alpha}{2n+2-\alpha} \frac{1}{(2(n+1))!}(\nu - 2)(\nu - 3) \cdots (\nu - 2n - 1)$, for $n \geq 1$ and $c_0 = 1$. An elementary computation yields

$$\frac{c_{n+1}}{c_n} = \frac{(n + 1 - \alpha/2)(n + 1 - \nu/2)(n + 3/2 - \nu/2)(n + 1)}{(n + 2 - \alpha/2)(n + 2)(n + 1/2)} \frac{1}{n + 1}.$$

Therefore (see [3, p. 10] or [12, equation 5.81, p. 207] for a more easily accessible reference), $\sum_{n \geq 0} c_n z^n = {}_4F_3(1, 1 - \frac{\nu}{2}, 1 - \frac{\alpha}{2}, \frac{3-\nu}{2}, 1; 2, \frac{3}{2}, 2 - \frac{\alpha}{2}; z)$. Now, the series defining the generalized hypergeometric function ${}_pF_q(\beta_1, \ldots, \beta_p; \gamma_1, \ldots, \gamma_q; z)$ for $p = q + 1$ converges for all z with $|z| < 1$; for $z = 1$, the series converges for $\sum_{i=1}^q \gamma_i > \sum_{j=1}^p \beta_j$, a condition that reduces to $\nu > -1$ in the present case. Hence,

$$i_1^{\nu,\alpha} = \frac{\nu(\nu - 1)}{2 - \alpha} {}_4F_3\left(1, 1 - \frac{\nu}{2}, 1 - \frac{\alpha}{2}, \frac{3-\nu}{2}, 1; 2, \frac{3}{2}, 2 - \frac{\alpha}{2}; 1\right).$$

Finally, consider $\tilde{i}_1^{\nu,\alpha}$. Due to the singularity at $u = 0$, we compute

$$\int_0^1 \left((1 - u)^{\nu} - 1\right) u^{-1-\alpha} du = \lim_{t \downarrow 0} \int_t^1 \left((1 - u)^{\nu} - 1\right) u^{-1-\alpha} du.$$

For the last integral, we get, for $\alpha > 0$ and $t > 0$,

$$\int_t^1 \left((1 - u)^{\nu} - 1\right) u^{-1-\alpha} du = \int_t^1 (1 - u)^{\nu} u^{-1-\alpha} du + \frac{1 - t^{-\alpha}}{\alpha}.$$

With the substitution $v = \frac{1-u}{1-t}$, the last integral becomes

$$\int_t^1 (1-u)^v u^{-1-\alpha} du = (1-t)^{1+v} \int_0^1 v^v (1 - (1-t)v)^{-\alpha-1} dv$$

$$= \frac{(1-t)^{1+v}}{1+v} {}_2F_1 (1+\alpha, 1+v; 2+v; 1-t),$$

provided $v > -1$, by the integral representation for the Gauss hypergeometric function (equation 15.3.1 of [1, p. 558]). Now, by equation 15.3.6 from [1, p. 559],

$$\begin{aligned}
{}_2F_1(1+\alpha, 1+v; 2+v; 1-t) &= \frac{\Gamma(2+v)\Gamma(-\alpha)}{\Gamma(1-\alpha+v)} {}_2F_1(1+\alpha, 1+v; 1+\alpha; t) \\
&\quad + t^{-\alpha} \frac{1+v}{\alpha} {}_2F_1(1-\alpha+v, 1; 1-\alpha; t) \\
&= \frac{\Gamma(2+v)\Gamma(-\alpha)}{\Gamma(1-\alpha+v)} + \frac{1+v}{\alpha} t^{-\alpha} + O\left(t^{1-\alpha}\right),
\end{aligned}$$

as $t \downarrow 0$, provided $\alpha \in (0, 1)$. Combining these results, we get

$$\int_t^1 ((1-u)^v - 1)u^{-1-\alpha} du = \frac{1}{\alpha} + \frac{\Gamma(1+v)\Gamma(-\alpha)}{\Gamma(1-\alpha+v)} + O\left(t^{1-\alpha}\right).$$

Letting $t \downarrow 0$ and using the fact that $-\alpha\Gamma(-\alpha) = \Gamma(1-\alpha)$ we obtain the given expression for $\tilde{i}_1^{v,\alpha}$. □

Proof of Lemma 2 We appeal to tables of standard integrals (Mellin transforms) from Section 6.4 of [8]. In particular, the given formulae for j_0^α, j_2^α, and $\tilde{j}_1^\alpha$ follow from, respectively, equations 6.4.17, 6.4.20, and 6.4.19 of [8, pp. 315–316]; also used are formulas 6.3.7 and 6.3.8 from [1, p. 259] and the fact that $\psi(1) = -\gamma$. Lastly, for j_1^α we use the substitution $u^2 = s$ to obtain

$$j_1^\alpha = \frac{1}{2} \int_0^1 \frac{\log(1-s)}{s^{1+(\alpha/2)}} ds = \frac{1}{2} \tilde{j}_1^{\alpha/2}.$$

This completes the proof. □

Finally, we need the following elementary fact.

Lemma 18 *Let $A \subseteq \mathbb{R}_+$ be a Borel set, and let f and g be measurable functions from A to $\mathbb{R}$. Suppose that there exist constants g_- and g_+ with $0 < g_- < g_+ < \infty$ such that $g_- \leq g(u) \leq g_+$ for all $u \in A$. Then,*

$$\left| \int_A f(u)g(u) du - g_- \int_A f(u) du \right| \leq (g^+ - g^-) \int_A |f(u)| du.$$

Proof It suffices to suppose that $\int_A |f(u)|\mathrm{d}u < \infty$. Then, $\int_A |f(u)g(u)|\mathrm{d}u \leq g_+ \int_A |f(u)|\mathrm{d}u < \infty$, and

$$\left| \int_A f(u)g(u)\mathrm{d}u - g_- \int_A f(u)\mathrm{d}u \right| = \left| \int_A f(u)(g(u) - g_-)\mathrm{d}u \right|$$
$$\leq \int_A |f(u)|(g(u) - g_-)\mathrm{d}u.$$

This completes the proof. □

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