

# A Survey on the Computational Complexity of Colouring Graphs with Forbidden Subgraphs <sup>\*</sup>

Petr A. Golovach<sup>1</sup>, Matthew Johnson<sup>2</sup>, Daniël Paulusma<sup>2</sup>, and Jian Song<sup>2</sup>

<sup>1</sup> Department of Informatics, Bergen University,  
PB 7803, 5020 Bergen, Norway  
petr.golovach@ii.uib.no

<sup>2</sup> School of Engineering and Computing Sciences, Durham University,  
Science Laboratories, South Road, Durham DH1 3LE, United Kingdom  
{matthew.johnson2, daniel.paulusma, jian.song}@durham.ac.uk

**Abstract.** For a positive integer  $k$ , a  $k$ -colouring of a graph  $G = (V, E)$  is a mapping  $c : V \rightarrow \{1, 2, \dots, k\}$  such that  $c(u) \neq c(v)$  whenever  $uv \in E$ . The COLOURING problem is to decide, for a given  $G$  and  $k$ , whether a  $k$ -colouring of  $G$  exists. If  $k$  is fixed (that is, it is not part of the input), we have the decision problem  $k$ -COLOURING instead. We survey known results on the computational complexity of COLOURING and  $k$ -COLOURING for graph classes that are characterized by one or two forbidden induced subgraphs. We also consider a number of variants: for example, where the problem is to extend a partial colouring, or where lists of permissible colours are given for each vertex. Finally, we also survey results for graph classes defined by some other forbidden pattern.

## 1 Introduction

To colour a graph is to label its vertices so that no two adjacent vertices have the same label. We call the labels *colours*. In a graph colouring problem one typically seeks to colour a graph using as few colours as possible, or perhaps simply to decide whether a given number of colours is sufficient. Graph colouring problems are central to the study of both structural and algorithmic graph theory and have very many theoretical and practical applications. Many variants and generalizations of the concept have been investigated, and there are some excellent surveys [1, 78, 101, 108] and a book [71] on the subject. We survey *computational complexity* results of graph colouring problems (for a short survey see [18]).

As we will note in the following subsection, the complexity of many graph colouring problems is fully understood when the possible input is any graph, and it is therefore natural to study the complexity of problems where the input is restricted. For example, one well-known result for graph colouring is due to Grötschel, Lovász, and Schrijver [53] who have shown that the problem of whether a *perfect* graph can be coloured with at most  $k$  colours for a given integer  $k$  is polynomial-time solvable; in contrast, the problem for general graphs is NP-complete [74].

Perfect graphs are an example of a graph class that is closed under vertex deletion, and, like all such graph classes, can be characterized by a family of forbidden induced subgraphs (an infinite family in the case of perfect graphs). In recent years, colouring problems for classes with forbidden-induced-subgraph characterizations have been extensively studied, and this survey is a response to the need for these results to be collected together. In fact, such a task is beyond the scope of a single paper and so our aim here is to report on the computational complexity of graph colouring problems for graph classes characterized by the absence of *one or two* forbidden induced subgraphs (for a survey on computational complexity results and open problems for colouring graphs characterized by more than two forbidden induced subgraphs or for which some graph parameter is bounded, see [96]).

<sup>\*</sup> The research leading to these results has received funding from EPSRC (EP/G043434/1) and the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement no. 267959.

## 1.1 Graph Colouring Problems

We consider finite undirected graphs with no multiple edges and no self-loops. That is, a graph  $G$  is an ordered pair  $(V, E)$  that consists of a finite set  $V$  of elements called *vertices* and a finite set  $E$  of unordered pairs of members of  $V$  called *edges*. The sets  $V$  and  $E$  are called the *vertex set* and *edge set* of  $G$ , respectively, and an edge containing  $u$  and  $v$  is denoted  $uv$ . The vertex and edge sets of a graph  $G$  can also always be referred to as  $V(G)$  and  $E(G)$ , and, when there is no possible ambiguity, we shall not always be careful in distinguishing between a graph and its vertex or edge set; that is, for example, we will write that a vertex belongs to a graph (rather than to the vertex set of the graph). A graph  $G' = (V', E')$  is a *subgraph* of  $G$  (and  $G$  is a *supergraph* of  $G'$ ) if  $V' \subseteq V$  and  $E' \subseteq E$ ; we say that  $G'$  is a *proper* subgraph of  $G$  if  $G'$  is a subgraph of  $G$  and  $G' \neq G$ .

A *colouring* of a graph  $G = (V, E)$  is a mapping  $c : V \rightarrow \{1, 2, \dots\}$  such that  $c(u) \neq c(v)$  whenever  $uv \in E$ . We call  $c(u)$  the *colour* of  $u$ . We let  $c(U) = \{c(u) \mid u \in U\}$  for  $U \subseteq V$ . If  $c(V) \subseteq \{1, \dots, k\}$ , then  $c$  is also called a *k-colouring* of  $G$ . For a colour  $c$ , the set of all vertices of  $G$  with colour  $c$  forms a *colour class*. We say that  $G$  is *k-colourable* if a  $k$ -colouring exists, and the *chromatic number* of  $G$  is the smallest integer  $k$  for which  $G$  is  $k$ -colourable and is denoted  $\chi(G)$ . A graph  $G$  is *k-vertex-critical* if  $\chi(G) = k$  and  $\chi(G') \leq k - 1$  for any subgraph  $G'$  of  $G$  obtained by deleting a vertex.

We shall define a number of decision problems.

### Colouring Problems

#### COLOURING

*Instance* : A graph  $G$  and a positive integer  $k$ .

*Question* : Is  $G$   $k$ -colourable?

If  $k$  is *fixed*, that is, not part of the input, then we have the following problem.

#### k-COLOURING

*Instance* : A graph  $G$ .

*Question* : Is  $G$   $k$ -colourable?

### Precolouring Extension Problems

A *k-precolouring* of a graph  $G = (V, E)$  is a mapping  $c_W : W \rightarrow \{1, 2, \dots, k\}$  for some subset  $W \subseteq V$ . A  $k$ -colouring  $c$  of  $G$  is an extension of a  $k$ -precolouring  $c_W$  of  $G$  if  $c(v) = c_W(v)$  for each  $v \in W$ .

#### PRECOLOURING EXTENSION

*Instance* : A graph  $G$ , a positive integer  $k$  and a  $k$ -precolouring  $c_W$  of  $G$ .

*Question* : Can  $c_W$  be extended to a  $k$ -colouring of  $G$ ?

#### k-PRECOLOURING EXTENSION

*Instance* : A graph  $G$  and a  $k$ -precolouring  $c_W$  of  $G$ .

*Question* : Can  $c_W$  be extended to a  $k$ -colouring of  $G$ ?

### List Colouring Problems

A *list assignment* of a graph  $G = (V, E)$  is a function  $L$  with domain  $V$  such that for each vertex  $u \in V$ ,  $L(u)$  is a subset of  $\{1, 2, \dots\}$ . We refer to this set as the *list of admissible colours* for  $u$ . If  $L(u) \subseteq \{1, \dots, k\}$  for each  $u \in V$ , then  $L$  is also called a *k-list assignment*. The *size* of a list assignment  $L$  is the maximum list size  $|L(u)|$  over all vertices  $u \in V$ . A colouring  $c$  *respects*  $L$  if

$c(u) \in L(u)$  for all  $u \in V$ . There are three decision problems as we can fix either the number of colours or the size of the list assignment.

**LIST COLOURING**

*Instance*: A graph  $G$  and a list assignment  $L$  for  $G$ .

*Question*: Is there a colouring of  $G$  that respects  $L$ ?

**$\ell$ -LIST COLOURING**

*Instance*: A graph  $G$  and a list assignment  $L$  for  $G$  of size at most  $\ell$ .

*Question*: Is there a colouring of  $G$  that respects  $L$ ?

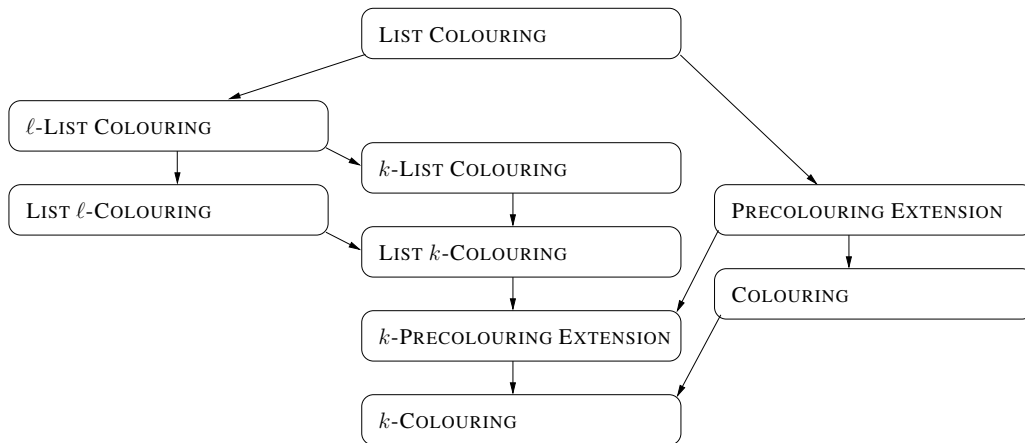
**LIST  $k$ -COLOURING**

*Instance*: A graph  $G$  and a  $k$ -list assignment  $L$  for  $G$ .

*Question*: Is there a colouring of  $G$  that respects  $L$ ?

Note that  $k$ -COLOURING can be viewed as a special case of  $k$ -PRECOLOURING EXTENSION by choosing  $W = \emptyset$ , and that  $k$ -PRECOLOURING EXTENSION can be viewed as a special case of LIST  $k$ -COLOURING by choosing  $L(u) = \{c_W(u)\}$  if  $u \in W$  and  $L(u) = \{1, \dots, k\}$  if  $u \in V \setminus W$ . Also LIST  $k$ -COLOURING can be readily seen to be a special case of  $k$ -LIST COLOURING, since if each list is a subset of  $\{1, \dots, k\}$ , then the size of the list assignment is certainly at most  $k$ . Similarly, from our definitions, we see that it follows that, whenever  $\ell_1 \leq \ell_2$ ,  $\ell_1$ -LIST COLOURING is a special case of  $\ell_2$ -LIST COLOURING, and that whenever  $k_1 \leq k_2$ , LIST  $k_1$ -COLOURING is a special case of LIST  $k_2$ -COLOURING. In Figure 1 we display all these relationships, which are implicitly assumed throughout the survey. Having this figure in mind we can say that NP-completeness results propagate upwards and polynomial time solvability results propagate downwards. Note that the relationships displayed in Figure 1 remain valid even if we restrict our attention to special graph classes — that is, if each of the problems accepts as input only certain graphs.

Contrary to the list colouring variants, when  $\ell \geq k$ ,  $k$ -COLOURING is not a special case of  $\ell$ -COLOURING. This is not only clear from its definition (the input consists of the graph only) but can also be illustrated by considering special graph classes. For example, 3-COLOURING is NP-complete for planar graphs [41], whereas 4-COLOURING is polynomial time solvable for these graphs (since, of course, they are all 4-colourable) [2]. Similarly,  $k$ -PRECOLOURING EXTENSION is not a special case of  $\ell$ -PRECOLOURING EXTENSION.



**Figure 1.** Relationships between COLOURING and its variants. An arrow from one problem to another indicates that the latter is a special case of the former;  $k$  and  $\ell$  are any two integers for which  $\ell \geq k$ .

There is one further type of problem.

### Choosability Problems

A graph  $G = (V, E)$  is  $\ell$ -choosable if, for every list assignment  $L$  of  $G$  with  $|L(u)| = \ell$  for all  $u \in V$ , there exists a colouring that respects  $L$ .

#### CHOOSABILITY

*Instance:* A graph  $G = (V, E)$  and a positive integer  $\ell$ .

*Question:* Is  $G$   $\ell$ -choosable?

#### $\ell$ -CHOOSABILITY

*Instance:* A graph  $G = (V, E)$ .

*Question:* Is  $G$   $\ell$ -choosable?

Theorem 1 describes the computational complexity of the problems we have introduced on general graphs. Here,  $\Pi_2^P$  is a complexity class in the polynomial hierarchy containing both NP and coNP; see for example the book of Garey and Johnson [40] for its exact definition.

**Theorem 1.** *The following two statements hold for general graphs.*

- (i) *The problems  $k$ -COLOURING,  $k$ -PRECOLOURING EXTENSION, LIST  $k$ -COLOURING and  $k$ -LIST COLOURING are polynomial-time solvable if  $k \leq 2$  and NP-complete if  $k \geq 3$ .*
- (ii)  *$\ell$ -CHOOSABILITY is polynomial-time solvable if  $\ell \leq 2$  and  $\Pi_2^P$ -complete if  $\ell \geq 3$ .*

*Proof.* Lovász [81] showed that 3-COLOURING is NP-complete; a straightforward reduction from 3-COLOURING shows that  $k$ -COLOURING is NP-complete for all  $k \geq 4$ . Erdős, Rubin and Taylor [32] and Vizing [109] observed that 2-LIST COLOURING is polynomial-time solvable on general graphs. Then (i) follows from the relationships displayed in Figure 1. Erdős, Rubin and Taylor [32] proved (ii).  $\square$

When considering Theorem 1, a natural question to ask is whether further tractable cases can be found if restrictions are placed on the input graphs. This survey reports progress on finding answers to this question.

## 1.2 Notation and Terminology

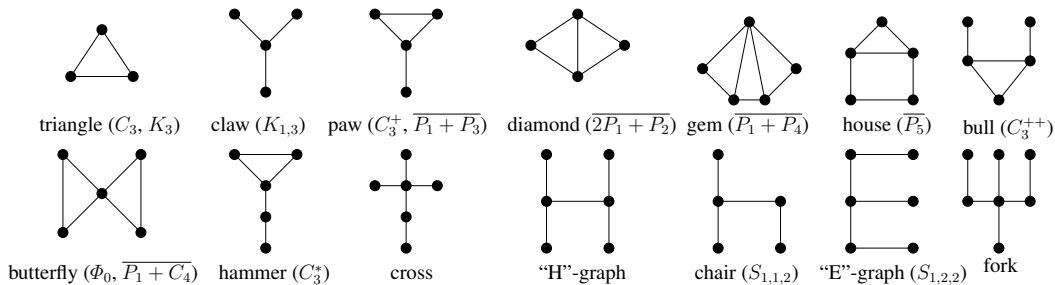
We define the graph classes considered in this survey and other notation and terminology. We refer to the textbook of Diestel [30] for any undefined terms.

Let  $G = (V, E)$  be a graph. For a subset  $S \subseteq V$ , let  $G[S]$  denote the *induced* subgraph of  $G$  that has vertex set  $S$  and edge set  $\{uv \in E(G) \mid u, v \in S\}$ . For a subset  $S \subseteq V$ , we write  $G - S = G[V \setminus S]$ , and for a vertex  $v \in V$ , we use  $G - v = G - \{v\}$ . For a graph  $F$ , we write  $F \subseteq G$  and  $F \subseteq_i G$  to denote that  $F$  is a subgraph or an induced subgraph of  $G$ , respectively. For two graphs  $G$  and  $H$ , a vertex mapping  $f : V(G) \rightarrow V(H)$  is called a (*graph*) *isomorphism* when  $uv \in E(G)$  if and only if  $f(u)f(v) \in E(H)$ , and we say that  $G$  and  $H$  are *isomorphic* whenever such a mapping exists. Let  $G$  be a graph and  $\{H_1, \dots, H_p\}$  be a set of graphs. Then  $G$  is  $(H_1, \dots, H_p)$ -*free* if  $G$  has no *induced* subgraph isomorphic to a graph in  $\{H_1, \dots, H_p\}$ . And  $G$  is *strongly*  $(H_1, \dots, H_p)$ -*free* if  $G$  has no subgraph isomorphic to a graph in  $\{H_1, \dots, H_p\}$ . If  $p = 1$ , we can simply write that  $G$  is (strongly)  $H_1$ -free (rather than (strongly)  $(H_1)$ -free).

**Observation 1** *If a graph  $H'$  is an induced subgraph of a graph  $H$ , then every  $H'$ -free graph is  $H$ -free. If  $H'$  is a subgraph of  $H$ , then every strongly  $H'$ -free graph is strongly  $H$ -free.*

The *complement* of a graph  $G$  is denoted  $\overline{G}$  and has the same vertex set as  $G$  and an edge between two distinct vertices if and only if these vertices are not adjacent in  $G$ . The *union* of two graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . If  $V(G) \cap V(H) = \emptyset$ , then we call the union of  $G$  and  $H$  the *disjoint union* of  $G$  and  $H$  and denote it  $G + H$ . We denote the disjoint union of  $r$  copies of  $G$  by  $rG$ .

For a graph  $G$ , the *degree*  $\deg_G(u)$  of a vertex  $u$  in  $G$  is the number of edges incident with it, or equivalently the size of its *neighbourhood*  $N_G(u) = \{v \in V \mid uv\}$ . A vertex  $u$  that is adjacent to all other vertices of  $G$  is called a *dominating* vertex of  $G$ . The *minimum degree* of  $G$  is the smallest degree of a vertex in  $G$ , and the *maximum degree* of  $G$ , denoted by  $\Delta(G)$ , is the largest degree of a vertex in  $G$ . If every vertex in  $G$  has degree  $p$ , then  $G$  is said to be *p-regular* (or sometimes just regular).



**Figure 2.** A number of small graphs with special names that we use throughout the survey. Also indicated are notations that will be defined in later sections.

For  $n \geq 1$ , the *complete graph*  $K_n$  is a graph on  $n$  vertices in which each pair of distinct vertices is joined by an edge. For a graph  $G$ , a subgraph isomorphic to a complete graph is called a *clique*, and the *clique number* of  $G$  is the size of its largest clique and is denoted  $\omega(G)$ .

For  $n \geq 1$ , the graph with vertices  $\{u_1, \dots, u_n\}$  and edges  $\{u_1u_2, u_2u_3, \dots, u_{n-1}u_n\}$  is called a *path* and is denoted  $P_n$ . For  $n \geq 3$ , the graph obtained from  $P_n$  by adding the edge  $u_1u_n$ , is called a *cycle* and is denoted  $C_n$ . The *length* of a path or cycle is its number of edges. The *end-vertices* of a path are the vertices of degree 1 (we will also refer to the vertices that comprise an edge as its end-vertices). The graph  $C_3 = K_3$  is also called a *triangle* (see Figure 2), and a  $C_3$ -free graph is also called *triangle-free*. A  $P_4$ -free graph is also called a *cograph*. Notice that  $rP_1$  denotes an *independent set* on  $r$  vertices.

Let  $G = (V, E)$  be a graph. The *girth* of  $G$  is the length of a shortest cycle in  $G$  or infinite if  $G$  has no cycle. Note that a graph has girth at least  $g$  for some integer  $g \geq 4$  if and only if it is  $(C_3, \dots, C_{g-1})$ -free. We say that  $G$  is *connected* if there is a path between every pair of distinct vertices; otherwise it is called *disconnected*. A vertex  $u \in V$  is a *cut vertex* if  $G$  is connected and  $G - u$  is disconnected. If  $G$  is connected and has no cut vertices, it is *2-connected*. A maximal connected subgraph of  $G$  is called a *connected component*. A graph is a *tree* if it is connected and  $(C_3, C_4, \dots)$ -free. A graph is a *forest* if each of its connected components is a tree. A graph is a *linear forest* if each of its connected components is a path.

A graph is *bipartite* if its vertex set can be partitioned into two sets such that every edge has one end-vertex in each set. For  $r \geq 1, s \geq 1$ , the *complete bipartite graph*  $K_{r,s}$  is a bipartite graph whose vertex set can be partitioned into two sets of sizes  $r$  and  $s$  such that there is an edge joining each pair of vertices from distinct sets. For  $r \geq 1$ , the graph  $K_{1,r}$  is also called a *star*. The graph  $K_{1,3}$  is also called a *claw* (see Figure 2), and a  $K_{1,3}$ -free graph is called *claw-free*. A graph is a *complete multipartite* graph if the vertex set can be partitioned so that there is an edge joining every pair of vertices from distinct sets of the partition and no edge joining vertices in the same set.

A graph  $G$  is *perfect* if, for every induced subgraph  $H \subseteq_i G$ , the chromatic number of  $H$  equals its clique number. The *line graph* of a graph  $G = (V, E)$  has vertex set  $E$  and  $x, y \in E$  are adjacent as vertices in the line graph if and only if they are adjacent as edges in  $G$ ; that is, if they share an end-vertex in  $G$ . A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their end-vertices. A graph is a *split* graph if its vertices can be partitioned into two sets that induce a clique and an independent set; if every vertex in the independent set is adjacent to every vertex in the clique, then it is a *complete* split graph. A number of small graphs that have special names are shown in Figure 2.

A *tree decomposition* of a graph  $G$  is a tree  $T$  where the elements of  $V(T)$  (called *nodes*) are subsets of  $V(G)$  such that the following three conditions are satisfied:

- for each vertex  $v \in V(G)$ , there is at least one node  $X \in V(T)$  with  $v \in X$ ;
- for each edge  $uv \in E(G)$ , there is a node  $X \in V(T)$  with  $\{u, v\} \subseteq X$ ;
- for each vertex  $v \in V(G)$ , the set of nodes  $\{X \mid v \in X\}$  induces a connected subtree of  $T$ .

If  $X$  is the largest node in a tree decomposition, then the *width* of the decomposition is  $|X| - 1$ . The *treewidth* of  $G$  is the minimum width over all possible tree decompositions of  $G$ . If a tree decomposition  $T$  is a path, then it is a *path decomposition*. The *pathwidth* of  $G$  is the minimum width over all possible path decompositions of  $G$ .

The graph parameter *clique-width* is defined by considering how to construct graphs in which each vertex has a label. Four operations are permitted:

- create a graph with one (labelled) vertex;
- combine two labelled graphs by taking their disjoint union;
- in a labelled graph, for two labels  $i$  and  $j$  with  $i \neq j$ , join by an edge each vertex with label  $i$  to each vertex with label  $j$ ;
- in a labelled graph, for two labels  $i$  and  $j$ , change every instance of label  $i$  to  $j$ .

The *clique-width* of  $G$  is the minimum number of labels needed to construct  $G$  (with some labelling) using these operations. A description of how  $G$  is constructed using these operations is called a  *$q$ -expression* if  $q$  is the number of labels used (so the clique-width of  $G$  is the minimum  $q$  for which  $G$  has a  $q$ -expression). We say that a class of graphs  $\mathcal{G}$  has *bounded* clique-width (or bounded treewidth) if there is a constant  $p$  such that the clique-width (or treewidth) of every graph in  $\mathcal{G}$  is at most  $p$ .

Let  $G = (V, E)$  be a graph. The *contraction* of an edge  $uv \in E$  removes  $u$  and  $v$  from  $G$ , and adds a new vertex  $w$  and edges such that the neighbourhood of  $w$  is the union of the neighbourhoods of  $u$  and  $v$ . Note that, by definition, edge contractions create neither self-loops nor multiple edges. Let  $u \in V$  be a vertex of degree 2 whose neighbours  $v$  and  $w$  are not adjacent. The *vertex dissolution* of  $u$  removes  $u$  and adds the edge  $vw$ . The “dual” operation of a vertex dissolution is *edge subdivision*, which replaces an edge  $vw$  by a new vertex  $u$  and edges  $uv$  and  $uw$ . We say that  $G$  contains another graph  $H$  as a *minor* if  $G$  can be modified into  $H$  by a sequence that consists of edge contractions, edge deletions and vertex deletions. And  $G$  contains  $H$  as a *topological minor* if  $G$  can be modified into  $H$  by a sequence that consists of vertex dissolutions, edge deletions and vertex deletions.

## 2 Results and Open Problems for $H$ -Free Graphs

In this section we consider graph classes characterized by one forbidden induced subgraph; we refer to the collection of all such graph classes as  $H$ -free graphs. In Section 2.1 we consider Colouring, Precolouring Extension and List Colouring Problems, and in Section 2.2 we consider Choosability Problems.

### 2.1 Colouring, Precolouring Extension and List Colouring Problems

Theorem 3 below describes what is known about the complexity of problems where the number of colours is not fixed. We first briefly describe the origin of these results.

Král', Kratochvíl, Tuza, and Woeginger [76] completely classified the computational complexity of COLOURING by showing that it is polynomial-time solvable for  $H$ -free graphs if  $H$  is an induced subgraph of  $P_4$  or of  $P_1 + P_3$ , and NP-complete otherwise. Both Hujter and Tuza [68] and Jansen and Scheffler [70] showed that PRECOLOURING EXTENSION is polynomial-time solvable for  $P_4$ -free graphs. This result was used by Golovach, Paulusma and Song [48] in order to obtain a dichotomy for PRECOLOURING EXTENSION analogous to the one of Král' et al. Jansen and Scheffler [70] also showed the following result which we state as a Theorem as we will use it later in the paper.

**Theorem 2.** *3-LIST COLOURING is NP-complete for complete bipartite graphs.*

As a consequence, 3-LIST COLOURING is NP-complete for  $(P_1 + P_2)$ -free graphs. Jansen [69] implicitly showed that 3-LIST COLOURING is NP-complete for (not necessarily vertex-disjoint) unions of two complete graphs, and thus for  $3P_1$ -free graphs. By combining these results, together with Theorem 1 (i), Golovach et al. [48] obtained dichotomies for LIST COLOURING and  $\ell$ -LIST COLOURING. We summarize all these results:

**Theorem 3.** *Let  $H$  be a graph. Then the following four statements hold for  $H$ -free graphs.*

- (i) COLOURING is polynomial-time solvable if  $H$  is an induced subgraph of  $P_4$  or of  $P_1 + P_3$ ; otherwise it is NP-complete.
- (ii) PRECOLOURING EXTENSION is polynomial-time solvable if  $H$  is an induced subgraph of  $P_4$  or of  $P_1 + P_3$ ; otherwise it is NP-complete.
- (iii) LIST COLOURING is polynomial-time solvable if  $H$  is an induced subgraph of  $P_3$ ; otherwise it is NP-complete.
- (iv) For  $\ell \geq 3$ ,  $\ell$ -LIST COLOURING is polynomial-time solvable if  $H$  is an induced subgraph of  $P_3$ ; otherwise it is NP-complete. [Recall that for  $\ell \leq 2$ ,  $\ell$ -LIST COLOURING is polynomial-time solvable on general graphs.]

Theorem 3 gives a complete complexity classification for problems where the number of colours is not fixed; that is, it is part of the input. Once such a classification was found, the natural direction for further research was to impose an upper bound on the number of available colours, and there is now an extensive literature on such problems. We survey the known results. We start, in Theorems 4 and 5, with more general results; we will soon see why they are useful.

Král' et al. [76] showed (in order to prove that COLORING is NP-complete for  $H$ -free graphs whenever  $H$  has a cycle) that 3-COLOURING is NP-complete for graphs of girth at least  $g$  for any fixed  $g \geq 3$ . Using a similar reduction, Kamiński and Lozin [72] extended this result to all  $k \geq 3$  though in fact a stronger result had been previously obtained by Emden-Weinert, Hougardy and Kreuter [31]:

**Theorem 4.** *For all  $k \geq 3$  and all  $g \geq 3$ ,  $k$ -COLOURING is NP-complete for graphs with girth at least  $g$  and with maximum degree at most  $6k^{13}$ .*

Theorem 4 implies that for any  $k \geq 3$ ,  $k$ -COLOURING is NP-complete for the class of  $H$ -free graphs whenever  $H$  contains a cycle. Let us remind the reader once more that Figure 1 tells us that NP-completeness results propagate upwards, which, combined with Theorems 3 and 4, allows us to say that the complexity of Colouring, Precolouring Extension and List Colouring problems for  $H$ -free graphs is classified except when  $H$  is a forest.

The following theorem is due to Holyer [64], who settled the case  $k = 3$ , and Leven and Galil [80] who settled the case  $k \geq 4$ .

**Theorem 5.** *For all  $k \geq 3$ ,  $k$ -COLOURING is NP-complete for line graphs of  $k$ -regular graphs.*

Because line graphs are easily seen to be claw-free, Theorem 5 implies that for all  $k \geq 3$ ,  $k$ -COLOURING is NP-complete on  $H$ -free graphs whenever  $H$  is a forest with a vertex of degree at least 3. This leaves only the case in which  $H$  is a linear forest.

Combining a result from Balas and Yu [4] on the number of maximal independent sets in an  $sP_2$ -free graph and a result from Tsukiyama, Ide, Ariyoshi and Shirakawa [107] on the enumeration of such sets leads to the result that  $k$ -COLOURING is polynomial-time solvable on  $sP_2$ -free graphs for any two integers  $k$  and  $s$ ; see, for example, the paper of Dabrowski, Lozin, Raman and Ries [27] for a proof of this result. By a few additional arguments, it is possible to obtain the following new result, which is stronger (notice that polynomial-time results propagate downwards in Figure 1).

**Theorem 6.** *For all  $k \geq 1$ ,  $s \geq 1$ , LIST  $k$ -COLOURING is polynomial-time solvable on  $sP_2$ -free graphs.*

*Proof.* Let  $k \geq 1$  and  $s \geq 1$ . Let  $G$  be an  $sP_2$ -free graph with a  $k$ -list assignment  $L$ . By the results of Balas and Yu [4] and Tsukiyama et al. [107], we can enumerate all maximal independent sets of  $G$  in polynomial time. For each maximal independent set  $I$  and each colour  $i \in \{1, \dots, k\}$ , we colour each vertex of  $W = \{u \in I : i \in L(u)\}$  with  $i$ , and then, recursively, attempt to colour  $G - W$  with the remaining colours. The running time of this algorithm is  $(kn)^{O(k)}$ . The algorithm can fail: it might not colour every vertex. However, if it succeeds then the resulting colouring will respect  $L$ .

It remains to show that the algorithm will find a colouring if one exists. Consider the set of vertices  $W$  coloured  $i$  in some colouring. They belong to a maximal independent set  $I$ , and we can assume that  $W = \{u \in I : i \in L(u)\}$  (by changing the colours of some vertices if necessary; the colouring will still be proper). So at some point the algorithm will consider  $i$  and  $I$  and colour  $W$  with  $i$ . By applying the same argument to  $G - W$  (which we know can be coloured with the remaining colours), we can see that the algorithm will obtain a colouring.  $\square$

The following theorem summarizes what is known for colouring problems on  $H$ -free graphs when the number of colours is fixed.

**Theorem 7.** *Let  $H$  be a graph. Then the following five statements hold:*

(i)  $k$ -COLOURING is NP-complete for  $H$ -free graphs if

1.  $k \geq 3$  and  $H \supseteq_i C_r$  for  $r \geq 3$
2.  $k \geq 3$  and  $H \supseteq_i K_{1,3}$
3.  $k \geq 4$  and  $H \supseteq_i P_7$
4.  $k \geq 5$  and  $H \supseteq_i P_6$ .

(ii) LIST  $k$ -COLOURING is NP-complete for  $H$ -free graphs if

1.  $k \geq 4$  and  $H \supseteq_i P_6$
2.  $k \geq 5$  and  $H \supseteq_i P_2 + P_4$ .

(iii) LIST  $k$ -COLOURING is polynomial-time solvable for  $H$ -free graphs if  $k \leq 2$  or

1.  $k \leq 3$  and  $H \subseteq_i sP_1 + P_7$  for  $s \geq 0$
2.  $k \leq 3$  and  $H \subseteq_i sP_3$  for  $s \geq 1$
3.  $k \geq 1$  and  $H \subseteq_i sP_1 + P_5$  for  $s \geq 0$
4.  $k \geq 1$  and  $H \subseteq_i sP_2$  for  $s \geq 1$ .

(iv) 4-PRECOLOURING EXTENSION is polynomial-time solvable for  $H$ -free graphs if  $H \subseteq_i P_2 + P_3$ .

*Proof.* For each case, we refer to the literature or to a result stated above. In some cases we will make additional comments referring to earlier (weaker) results that provided techniques or suggested approaches that were important in obtaining the final result.

(i) We first consider the NP-completeness results for  $k$ -COLOURING.

1. This follows immediately from Theorem 4.
2. This is a direct consequence of Theorem 5 and the fact that every line graph is claw-free.



3. Woeginger and Sgall [112] showed that 4-COLOURING is NP-complete for  $P_{12}$ -free graphs. This bound was improved in a number of other papers. First, Le, Randerath and Schiermeyer [79] showed that 4-COLOURING is NP-complete for  $P_9$ -free graphs. Then, Broersma, Golovach, Paulusma and Song [16] showed that 4-COLOURING is NP-complete for  $P_8$ -free graphs. Finally, the strongest NP-completeness result for 4-COLOURING is due to Huang [65], who showed that it is NP-complete for  $P_7$ -free graphs (we note also that Broersma et al. [16] had already shown that 4-PRECOLOURING EXTENSION is NP-complete for  $P_7$ -free graphs).
  4. Broersma et al. [14] had shown that 5-PRECOLOURING EXTENSION is NP-complete for  $P_6$ -free graphs. Huang [65] improved this (and also a result of Woeginger and Sgall [112] who showed that 5-COLOURING is NP-complete for  $P_8$ -free graphs) by proving that 5-COLOURING is NP-complete for  $P_6$ -free graphs.
- (ii) Next we look at the NP-completeness results for LIST- $k$ -COLOURING.
1. This is a result of Golovach, Paulusma and Song [48].
  2. Couturier, Golovach, Kratsch and Paulusma [22] showed that LIST  $k$ -COLOURING is NP-complete for some integer  $k$  on  $H$ -free graphs, whenever  $H$  is a supergraph of  $P_1 + P_5$  with at least five edges. In particular, they proved that LIST 5-COLOURING is NP-complete on  $(P_2 + P_4)$ -free graphs.
- (iii) We now turn to the polynomial-time results for LIST- $k$ -COLOURING.<sup>3</sup> Before we consider the individual cases, we discuss an observation of Broersma et al. [16] that we will use twice. They noticed that 3-PRECOLOURING EXTENSION is polynomial-time solvable for  $(P_1 + H)$ -free graphs whenever it is polynomial-time solvable for  $H$ -free graphs (and by repeated application the problem is, in fact, solvable for  $(sP_1 + H)$ -free graphs for any  $s \geq 0$ ). We note that analogous statements can be made about 3-COLOURING and LIST 3-COLOURING.
1. Randerath and Schiermeyer [100] showed that 3-COLOURING is polynomial-time solvable on  $P_6$ -free graphs. This was generalized by Broersma, Fomin, Golovach and Paulusma [14] who showed that 3-PRECOLOURING EXTENSION is polynomial-time solvable for  $P_6$ -free graphs. In fact, their proof shows polynomial-time solvability of LIST 3-COLOURING for  $P_6$ -free graphs. Broersma et al. [16] showed that 3-PRECOLOURING EXTENSION can be solved in polynomial time on  $(P_2 + P_4)$ -free graphs. Their proof can be used to show that LIST 3-COLOURING is polynomial-time solvable on  $(P_2 + P_4)$ -free graphs. Recently, Bonomo, Chudnovsky, Maceli, Schaudt, Stein and Zhong [6] gave an  $O(|V|^{23})$  time algorithm for LIST 3-COLOURING on  $P_7$ -free graphs (thereby solving Problem 17 in [102] and Problem 56 in [101]). The same authors showed that 3-COLOURING can be solved in  $O(|V|^7)$  time on  $(C_3, P_7)$ -free graphs.
  2. A further result of Broersma et al. [16] showed that 3-PRECOLOURING EXTENSION is polynomial-time solvable on  $sP_3$ -free graphs for all  $s \geq 1$ . In fact, though they did not state it explicitly, the result holds for LIST 3-COLOURING on  $sP_3$ -free graphs.
  3. This is a result of Couturier et al. [22]. It generalizes an earlier result of Hoàng, Kamiński, Lozin, Sawada, and Shu [60] who proved that for every integer  $k \geq 1$ , LIST  $k$ -COLOURING is polynomial-time solvable on  $P_5$ -free graphs. Previously, (different) proofs for the case  $k \leq 3$  were given by Woeginger and Sgall [112] and Randerath, Schiermeyer and Tewes [102].
  4. This is Theorem 6.
- (iv) This is a result of Golovach, Paulusma and Song [49]. □

As a consequence of Theorem 7 we obtain dichotomies for  $k$ -COLOURING,  $k$ -PRECOLOURING EXTENSION and LIST  $k$ -COLOURING when  $H$  is small. These are stated in Theorem 8.

**Theorem 8.** *Let  $H$  be a graph and  $k$  an integer. Then the following three statements hold:*

<sup>3</sup> Two of these results were only formulated in the literature for  $k$ -PRECOLOURING EXTENSION instead of for LIST  $k$ -COLOURING. In the Appendix we give proofs for LIST  $k$ -COLOURING or explain how the known proofs for  $k$ -PRECOLOURING EXTENSION can be modified accordingly.

- (i) If  $|V(H)| \leq 6$ , then 3-COLOURING, 3-PRECOLOURING EXTENSION, LIST 3-COLOURING are polynomial-time solvable on  $H$ -free graphs if  $H$  is a linear forest, and NP-complete otherwise.
- (ii) If  $|V(H)| \leq 5$ , then 4-COLOURING, 4-PRECOLOURING EXTENSION are polynomial-time solvable on  $H$ -free graphs if  $H$  is a linear forest, and NP-complete otherwise.
- (iii) If  $|V(H)| \leq 4$  and  $k \geq 5$ , then  $k$ -COLOURING,  $k$ -PRECOLOURING EXTENSION, LIST  $k$ -COLOURING are polynomial-time solvable on  $H$ -free graphs if  $H$  is a linear forest, and NP-complete otherwise.

Note that statement (ii) of Theorem 8 cannot be stated also for LIST 4-COLOURING due to exactly one missing case, which is the complexity of LIST 4-COLOURING for  $(P_2 + P_3)$ -free graphs.

Theorem 7 also implies that for  $H$ -free graphs, 3-COLOURING is classified for all graphs  $H$  on seven vertices except when  $H \in \{P_2 + P_5, P_3 + P_4\}$ , that 4-COLOURING is classified for all graphs  $H$  on six vertices, except when  $H \in \{P_1 + P_2 + P_3, P_2 + P_4, 2P_3, P_6\}$ , and that 5-COLOURING is classified for all graphs  $H$  on five vertices, except when  $H = P_2 + P_3$ .

Table 1 shows a summary of the existing results for  $P_r$ -free graphs obtained from Theorem 7. We include this table, because  $k$ -COLOURING restricted to graphs characterized by forbidden induced subgraphs was most actively studied for forbidden induced paths. By comparing Table 1 with similar tables that can be found in several earlier papers [16, 48, 60, 79, 100, 101, 112] one can see the gradual progress that has been made over the years.

$r$	$k$ -COLOURING				$k$ -PRECOLOURING EXTENSION				LIST $k$ -COLOURING			
	$k = 3$	$k = 4$	$k = 5$	$k \geq 6$	$k = 3$	$k = 4$	$k = 5$	$k \geq 6$	$k = 3$	$k = 4$	$k = 5$	$k \geq 6$
$r \leq 5$	P	P	P	P	P	P	P	P	P	P	P	P
$r = 6$	P	?	NP-c	NP-c	P	?	NP-c	NP-c	P	NP-c	NP-c	NP-c
$r = 7$	P	NP-c	NP-c	NP-c	P	NP-c	NP-c	NP-c	P	NP-c	NP-c	NP-c
$r \geq 8$	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c

**Table 1.** The complexity of  $k$ -COLOURING,  $k$ -PRECOLOURING EXTENSION and LIST  $k$ -COLOURING on  $P_r$ -free graphs for fixed  $k$  and  $r$ .

**Open Problem 1** Complete the classification of the complexity of  $k$ -COLOURING,  $k$ -PRECOLOURING EXTENSION and LIST  $k$ -COLOURING for  $H$ -free graphs.

**Two Important Subproblems** First, as noted, the complexity status of 4-COLOURING for  $P_6$ -free graphs is still open. One of the key ingredients in the proofs of the two aforementioned hardness results of 4-COLOURING for  $P_7$ -free graphs and 5-COLOURING for  $P_6$ -free graphs by Huang [65] are the so-called *nice  $k$ -critical* graphs. A graph  $G = (V, E)$  is nice  $k$ -critical for some integer  $k$  if it is  $k$ -vertex-critical, and if moreover,  $G$  contains three independent vertices  $v_1, v_2, v_3$  such that  $\omega(G - \{v_1, v_2, v_3\}) = \omega(G) = k - 1$ . In his hardness reductions, Huang [65] uses the existence of  $P_7$ -free nice 3-critical graphs and  $P_6$ -free nice 4-critical graphs. He also proved that  $P_6$ -free nice 3-critical graphs do not exist. Hence, new techniques are required to determine the computational complexity of 4-COLOURING for  $P_6$ -free graphs.

The second intriguing open question (Problem 18 in [102] and Problem 57 in [101]) that must be answered when solving Open Problem 1 is whether there exists an integer  $r \geq 8$  such that 3-COLOURING is NP-complete for  $P_r$ -free graphs. This is also unknown for 3-PRECOLOURING EXTENSION and LIST 3-COLOURING. As observed by Golovach et al. [48], an affirmative answer for one of the three problems leads to an affirmative answer for the other two. We also note that there is no graph  $H$  and integer  $k$  known for which the computational complexity of the problems  $k$ -COLOURING,  $k$ -PRECOLOURING EXTENSION and LIST  $k$ -COLOURING differs for  $H$ -free

graphs (whether such a graph  $H$  exists was posed as an open problem by Huang, Johnson and Paulusma [66]).

**Parameterized Complexity Theory** Parameterized complexity theory is a framework that offers a refined analysis of NP-hard algorithmic problems. We measure the complexity of a problem not only in terms of the input length but also in terms of a parameter, which is a numerical value not necessarily dependent on the input length. The instance of a parameterized problem is a pair  $(I, p)$ , where  $I$  is the problem instance and  $p$  is the parameter. The choice of parameter will depend on the structure of the problem (and there might be many possible choices).

The central notion in parameterized complexity theory is the concept of *fixed-parameter tractability*. A problem is called *fixed-parameter tractable* (FPT) if every instance  $(I, p)$  can be solved in time  $f(p)|I|^{O(1)}$  where  $f$  is a computable function that only depends on  $p$ . The complexity class FPT is the class of all fixed-parameter tractable problems. The complexity class XP is the class of all problems that can be solved in time  $|I|^{f(p)}$ .

By definition  $\text{FPT} \subseteq \text{XP}$ , but a collection of intermediate complexity classes has been defined as well. It is known as the **W**-hierarchy:

$$\text{FPT} = \text{W}[0] \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \dots \subseteq \text{W}[P] \subseteq \text{XP}.$$

It is widely believed that  $\text{FPT} \neq \text{W}[1]$ . Hence, if a problem is hard for some class  $\text{W}[i]$ , then it is considered to be fixed-parameter intractable.

A problem is *para-NP-complete* when it is NP-complete for some fixed value of the parameter. Such a problem is not in XP (and so not in FPT) unless  $\text{P} = \text{NP}$ . We refer the reader to the textbook of Niedermeier [93] for further details.

For COLOURING and its variants, the natural parameter is the number of available colours  $k$ . Few parameterized results for COLOURING restricted to  $H$ -free graphs are known. Below we survey some initial results.

**Theorem 9.** *Let  $H$  be a graph. Then the following hold:*

- (i) COLOURING is para-NP-complete for  $H$ -free graphs when parameterized by  $k$  if  $H$  is not a linear forest or if  $H$  contains an induced subgraph isomorphic to  $P_6$ .
- (ii) COLOURING is polynomial-time solvable on  $H$ -free graphs if  $H$  is an induced subgraph of  $P_1 + P_3$  or of  $P_4$ .
- (iii) LIST COLOURING is FPT for  $H$ -free graphs when parameterized by  $k + r$  if  $H = rP_1 + P_2$ .
- (iv) LIST COLOURING is FPT for  $H$ -free graphs when parameterized by  $k$  if  $H$  is an induced subgraph of  $P_1 + P_3$  or of  $P_4$ .

*Proof.* The first part follows from Theorem 7 (i). The second part is a restatement of Theorem 3 (i) (stated again to provide a complete statement on parameterized complexity). The third part is a result of Couturier et al. [23] (they also showed that COLOURING restricted to  $(rP_1 + P_2)$ -free graphs admits a polynomial kernel (see [93] for a definition) for every  $r \geq 2$ , when parameterized by  $k$ ). Couturier et al. [23] also proved the LIST COLOURING result for  $(P_1 + P_3)$ -free graphs; the result for  $P_4$ -free graphs was shown by Jansen and Scheffler [70], who described a linear time algorithm.  $\square$

These results tell us (also see [23]) the smallest open cases:

**Open Problem 2** *Is COLOURING FPT for  $2P_2$ -free graphs or for  $(2P_1 + P_3)$ -free graphs when parameterized by  $k$ ?*

The same question can also be asked for PRECOLOURING EXTENSION. In fact, we can also see from Theorem 9 that the cases  $H = 2P_2$  and  $H = 2P_1 + P_3$  are the two smallest open cases when we consider LIST COLOURING for  $H$ -free graphs parameterized by the number of colours. Another natural parameter for LIST COLOURING is the list size. However, Theorem 3 (iii) shows that in that

case LIST COLOURING is para-NP-complete for  $H$ -free graphs whenever  $H$  is not isomorphic to  $P_3$  (and polynomial-time solvable otherwise).

Hoàng et al. [60] asked whether COLOURING is FPT for  $P_5$ -free graphs when parameterized by  $k$ . In the light of Open Problem 2, we slightly reformulate their open problem.

**Open Problem 3** *Is COLOURING, when parameterized by  $k$ , W[1]-hard for  $P_5$ -free graphs?*

Another interesting problem is to determine whether 3-COLOURING is W[1]-hard for  $P_r$ -free graphs when parameterized by  $r$  (as we have noted though, we currently do not know whether there exists an integer  $r$  such that 3-COLOURING is NP-complete for  $P_r$ -free graphs).

**Certifying Algorithms** Just as with NP-hard problems it is natural to try to refine our understanding by asking about fixed-parameter tractability, for problems in P, we ask for polynomial-time algorithms that not only find solutions but also provide certificates which demonstrate the correctness of solutions and can be “easily” verified. These algorithms are called *certifying* (see, for example, the survey of McConnell, Mehlhorn, Näher and Schweitzer [92]).

For COLOURING, if the input graph  $G = (V, E)$  does have the sought  $k$ -colouring, then a certifying algorithm can give the colouring as a certificate. If  $G$  does not have a  $k$ -colouring, then it must have an induced subgraph that is  $(k + 1)$ -vertex-critical (just delete vertices until one is reached). If for some class of graphs that is closed under vertex deletion, it is possible to construct the set of all the  $(k + 1)$ -vertex-critical graphs (and this set is finite), then a certifying algorithm for  $k$ -COLOURING for that graph class can, when the input graph  $G$  is not  $k$ -colourable, give as a certificate a graph. To verify the certificate, one must check that it is an induced subgraph of  $G$  and that it is one of the  $(k + 1)$ -vertex-critical graphs for the class.

We say that a graph  $G$  is  $(k + 1)$ -critical with respect to a graph class  $\mathcal{G}$  if  $\chi(G) = k + 1$  and every proper subgraph of  $G$  that belongs to  $\mathcal{G}$  is  $k$ -colourable. We will not go through the details, but clearly one can take a similar approach as above using  $(k + 1)$ -critical graphs (rather than  $(k + 1)$ -vertex-critical graphs). We note that Hoàng, Moore, Recoskie, Sawada and Vatshelle [63] observed that if a graph class has a finite number of  $(k + 1)$ -critical graphs, then it has a finite number of  $(k + 1)$ -vertex-critical graphs.

Due to Theorem 7 (iii):4, the case  $H = P_5$  is a natural starting point. Two certifying algorithms exist for 3-COLOURING on  $P_5$ -free graphs. The first one is due to Bruce, Hoàng, and Sawada [17]. They showed that there exist six 4-critical  $P_5$ -free graphs in total and gave an explicit construction of these graphs. The same authors asked whether there exists an algorithm faster than brute force for checking whether a graph contains one of these six 4-critical  $P_5$ -free graphs as a subgraph. The second certifying algorithm is due to Maffray and Morel [85]. They showed that there exist twelve 4-vertex-critical  $P_5$ -free graphs in total and gave an explicit construction of these graphs. The running time of the corresponding certifying algorithm of Maffray and Morel [85] is linear (and as such answered the question posed by Bruce et al. [17]).

For all  $k \geq 5$ , Hoàng, Moore, Recoskie, Sawada and Vatshelle [63] constructed an infinite set of  $k$ -vertex-critical  $P_5$ -free graphs which, as noted, implies that the set of  $k$ -critical  $P_5$ -free graphs is also infinite. For the case  $k = 5$ , they used an exhaustive computer search to construct an infinite set of  $k$ -critical  $P_5$ -free graphs.

Chudnovsky, Goedgebeur, Schaudt and Zhong [19] proved that there exist 24 4-critical  $P_6$ -free graphs and 80 4-vertex-critical  $P_6$ -free graphs. Hence, their result implies the existence of a certifying algorithm that solves 3-COLOURING for  $P_6$ -free graphs (which answers an open problem of a previous version of this survey). The same authors also proved that there are infinitely many 4-critical  $P_7$ -free graphs. Moreover, they observed that there are infinitely many 4-critical  $H$ -free graphs if  $H$  contains a cycle or a claw (which was to be expected given Theorem 7 (i):1-2). Hence they showed that for a connected graph  $H$ , there are finitely many 4-critical  $H$ -free graphs if and only if  $H$  is a subgraph of  $P_6$ .

**Open Problem 4** Determine all linear forests  $H$ , for which there exists a certifying algorithm that solves 3-COLOURING for  $H$ -free graphs.

## 2.2 Choosability

Golovach and Heggernes [43] showed that CHOOSABILITY is NP-hard for  $P_5$ -free graphs. Their work was continued by Golovach, Heggernes, van 't Hof and Paulusma who implicitly showed the following result in the proof of [44, Theorem 2]<sup>4</sup> (adding a dominating vertex to a graph means creating a new vertex and making it adjacent to every existing vertex).

**Theorem 10.** *Let  $\mathcal{G}$  be a graph class that is closed under adding dominating vertices. If COLOURING is NP-hard for  $\mathcal{G}$ , then CHOOSABILITY is NP-hard for  $\mathcal{G}$ .*

Golovach et al. [44] then used Theorem 10 to prove the following result.

**Theorem 11.** *Let  $H$  be a graph. Then the following hold:*

- (i) *If  $H \notin \{K_{1,3}, P_1, 2P_1, 3P_1, P_1 + P_2, P_1 + P_3, P_2, P_3, P_4\}$  then CHOOSABILITY is NP-hard for  $H$ -free graphs.*
- (ii) *If  $H \in \{P_1, 2P_1, 3P_1, P_2, P_3\}$  then CHOOSABILITY is polynomial-time solvable for  $H$ -free graphs.*

Note that there are four missing cases in Theorem 11: when  $H \in \{K_{1,3}, P_1 + P_2, P_1 + P_3, P_4\}$ .

The following result is due to Gutner [54].

**Theorem 12.** *3-CHOOSABILITY and 4-CHOOSABILITY are  $\Pi_2^p$ -complete for planar graphs.*

Gutner and Tarsi [55] showed the following result.

**Theorem 13.** *For all  $k \geq 3$ ,  $k$ -CHOOSABILITY is  $\Pi_2^p$ -complete on bipartite graphs.*

Hence, for some graphs  $H$ , Theorem 11 can be strengthened: statements of NP-hardness can be replaced by stronger statements of  $\Pi_2^p$ -hardness.

**Theorem 14.** *Let  $H$  be a graph. Then CHOOSABILITY is  $\Pi_2^p$ -hard for  $H$ -free graphs if  $H$  is non-planar or contains an odd cycle.*

We describe two open problems for CHOOSABILITY. The first asks for the resolution of the missing cases of Theorem 11.

**Open Problem 5** *Is CHOOSABILITY NP-hard for  $H$ -free graphs if  $H \in \{K_{1,3}, P_1 + P_2, P_1 + P_3, P_4\}$ ?*

We observe that for  $H \in \{P_1 + P_2, P_1 + P_3, P_4\}$ , the class of  $H$ -free graphs contains the class of complete bipartite graphs as a subclass. As noted by Golovach et al. [44], the computational complexity of CHOOSABILITY on complete bipartite graphs is still open as well. We discuss the fourth open case  $H = K_{1,3}$  in more detail later.

The second open problem for CHOOSABILITY asks for an extension of Theorem 14:

**Open Problem 6** *Is CHOOSABILITY  $\Pi_2^p$ -hard for all those classes for which it is NP-hard.*

<sup>4</sup> See the Appendix for an explicit proof of this result.

If  $H \in \{P_1 + P_2, P_1 + P_3, P_4\}$ , then the class of  $H$ -free graphs contains the class of complete bipartite graphs as a subclass. Even the complexity status of CHOOSABILITY for complete bipartite graphs is open. This could be a possible direction for further research. We also make the following remark, which shows that another natural approach does *not* work. In contrast to PRECOLOURING EXTENSION, there exist graphs  $H$  for which LIST COLOURING is NP-complete when restricted to  $H$ -free graphs, while COLOURING becomes polynomial-time solvable. However, it is not possible (unfortunately) to strengthen Theorem 10 by replacing the NP-hardness of COLOURING by NP-hardness of LIST COLOURING as a sufficient condition for NP-hardness of CHOOSABILITY. For instance, let  $\mathcal{G}$  be the class of  $(3P_1, P_1 + P_2)$ -free graphs. It is known that LIST COLOURING is NP-complete for this graph class [48], which is closed under adding of dominating vertices, while CHOOSABILITY is polynomial-time solvable even for  $3P_1$ -free graphs due to Theorem 11.

As an aside, there also exist graph classes for which PRECOLOURING EXTENSION is NP-hard but CHOOSABILITY is polynomial-time solvable. Galvin [39] showed that every line graph of a bipartite graph is  $k$ -choosable if and only if it is  $k$ -colourable. Because line graphs of bipartite graphs are perfect [38], and COLOURING can be solved in polynomial time on perfect graphs [53], this means that CHOOSABILITY is polynomial-time solvable on such graphs. However, PRECOLOURING EXTENSION is NP-complete even for line graphs of complete bipartite graphs, as shown by Hujter and Tuza [67].

We now consider  $k$ -CHOOSABILITY. Golovach and Heggernes [43] showed that  $k$ -CHOOSABILITY is linear-time solvable on  $P_5$ -free graphs. Golovach et al. [44] extended this result and proved statement (i) of Theorem 15 below. Statement (ii) of this theorem follows from Theorem 13, whereas statement (iii) follows from Theorem 12. Also recall that 2-CHOOSABILITY is polynomial-time solvable for general graphs by Theorem 1.

**Theorem 15.** *Let  $H$  be graph. Then the following three statements hold for  $H$ -free graphs:*

- (i) *For all  $k \geq 1$ ,  $k$ -CHOOSABILITY is linear-time solvable if  $H$  is a linear forest.*
- (ii) *For all  $k \geq 3$ ,  $k$ -CHOOSABILITY is  $\Pi_2^p$ -hard if  $H$  contains an odd cycle.*
- (iii) *For  $3 \leq k \leq 4$ ,  $k$ -CHOOSABILITY is  $\Pi_2^p$ -hard if  $H$  is non-planar.*

Theorem 15 leads to the following open problem.

**Open Problem 7** *For all  $k \geq 3$ , determine the complexity of  $k$ -CHOOSABILITY on  $H$ -free graphs when  $H$  is a bipartite graph that is not a linear forest.*

Open Problem 7 seems difficult due to its connection to the well-known and long-standing List Colouring Conjecture, for which the aforementioned result of Galvin [39] is a special case. This conjecture states that every line graph is  $k$ -choosable if and only if it is  $k$ -colourable. This conjecture is usually attributed to Vizing (cf. [58]). As observed by Golovach et al. [44],  $k$ -CHOOSABILITY is NP-hard on  $K_{1,3}$ -free graphs for every  $k \geq 3$  if the List Colouring Conjecture is true. This could mean that Theorem 15 (i) is best possible.

### 3 Results and Open Problems for $(H_1, H_2)$ -Free Graphs

When we forbid two induced subgraphs, only partial results are known for COLOURING and its variants. We survey these results below. First we need some other other results starting with the following theorem of Maffray and Preissmann [87].

**Theorem 16.** *3-COLOURING is NP-complete for  $C_3$ -free graphs of maximum degree at most 4.*

For  $1 \leq h \leq i \leq j$ , let  $S_{h,i,j}$  denote the tree that is the union of paths of lengths  $h$ ,  $i$  and  $j$  whose only common vertex is an end-vertex of each. Observe that  $S_{1,1,1} = K_{1,3}$ ,  $S_{1,1,2}$  is the chair and  $S_{1,2,2}$  is the ‘‘E’’-graph (see Figure 2). Let  $A_{h,i,j}$  denote the line graph of  $S_{h,i,j}$ . Schindl [105] showed the following result.

**Theorem 17.** Let  $\{H_1, \dots, H_p\}$  be a finite set of graphs. Then COLOURING is NP-complete for  $(H_1, \dots, H_p)$ -free graphs if the complement of each  $H_i$  has a connected component that is isomorphic neither to any graph  $A_{h,i,j}$ , for  $1 \leq h \leq i \leq j$ , nor to any path  $P_r$  for  $r \geq 1$ .

We also need the following result due to Gravier, Hoàng and Maffray [51] (which is a slight improvement on a similar result of Gyárfás [56]).

**Theorem 18.** Let  $r, t \geq 1$  be two integers. Then every  $(K_r, P_t)$ -free graph can be coloured with at most  $(t - 2)^{r-2}$  colours.

We note that Theorem 18 has been improved by Esperet, Lemoine, Maffray and Morel [34] for the case  $r = 4, t = 5$ ; they showed that every  $(K_4, P_5)$ -free graph is 5-colourable.

It can be seen that COLOURING is polynomial-time solvable on any graph class of bounded clique-width by combining two results: Kobler and Rotics [75] showed that for any constant  $q$ , COLOURING is polynomial-time solvable if a  $q$ -expression is given (they also showed that in that case LIST  $k$ -COLOURING is linear-time solvable for all  $k \geq 1$ ), and Oum [95] showed that a  $(8^p - 1)$ -expression for any  $n$ -vertex graph with clique-width at most  $p$  can be found in  $O(n^3)$  time.

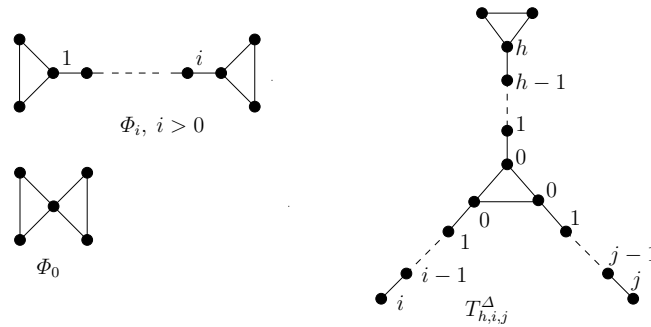
**Theorem 19.** Let  $\mathcal{G}$  be a graph class of bounded clique-width. The following two statements hold:

- (i) COLOURING can be solved in polynomial time on  $\mathcal{G}$ .
- (ii) For all  $k \geq 1$ , LIST  $k$ -COLOURING can be solved in polynomial time on  $\mathcal{G}$ .

As an aside, the statement of Theorem 19 (i) is valid neither for PRECOLOURING EXTENSION nor for LIST COLOURING. For instance, Bonomo, Durán and Marengo [7] proved that PRECOLOURING EXTENSION is NP-complete for distance-hereditary graphs, which have clique-width at most 3 [50], whereas, by Theorem 2, even 3-LIST COLOURING is NP-complete for complete bipartite graphs, which have clique-width at most 2 [21].

The graph  $\overline{P_1 + P_3}$  is called the *paw* (see Figure 2); we also denote it by  $C_3^+$ . By using a result of Olariu [94], which states that a graph is  $C_3^+$ -free if and only if it is  $C_3$ -free or a complete multipartite graph, Král' et al. [76] observed the following.

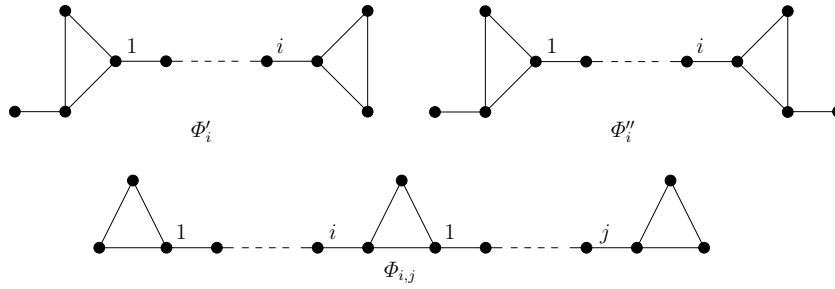
**Theorem 20.** Let  $H$  be a graph. Then COLOURING is polynomial-time solvable on  $(C_3, H)$ -free graphs if and only if it is polynomial-time solvable for  $(C_3^+, H)$ -free graphs.



**Figure 3.** The graphs  $\Phi_i$  and  $T_{h,i,j}^\Delta$ .

Theorem 21 below summarizes results on COLOURING for graph classes defined by two forbidden induced subgraphs. In order to state this theorem, we need to define the following graphs. The graph  $\overline{2P_1 + P_2}$  is also called a *diamond*. The graph  $\overline{P_1 + P_4}$  is also called the *gem*. The graph  $\overline{P_5}$  is also called the *house*. The graph  $\overline{C_4 + P_1}$  is also called the *butterfly*. These graphs are all shown in

Figure 2 as are the *hammer* and the *bull* which we also denote by  $C_3^*$  and  $C_3^{++}$  respectively (recall that  $C_3^+ = \overline{P_1 + P_3}$  denotes the paw). The graph  $\Phi_i$ ,  $i \geq 0$ , is composed of a path  $P$  on  $i$  edges with end-vertices  $u$  and  $v$  and a  $K_3$  that intersects  $P$  in  $u$  and a  $K_3$  that intersects  $P$  in  $v$  (notice that if  $i = 0$ , then  $u = v$  so  $\Phi_0$  is the butterfly). The graph  $T_{h,i,j}^\Delta$ ,  $h, i, j \geq 0$ , is composed of a  $\Phi_h$ , a  $\Phi_i$  and a  $\Phi_j$  which all intersect in a  $K_3$  in such a way that each of its vertices has degree at most 3. Both graphs are illustrated in Figure 3. The graph  $\Phi_{i,j}$ ,  $i, j \geq 0$ , is composed of a path  $P_1$  on  $i$  edges with end-vertices  $u$  and  $v$ , a path  $P_2$  on  $j$  edges with end-vertices  $w$  and  $x$  and a  $K_3$  that intersects  $P_1$  in  $u$ , a  $K_3$  that intersects  $P_1$  in  $v$  and  $P_2$  in  $w$ ,  $v \neq w$ , and a  $K_3$  that intersects  $P_2$  in  $x$  (notice that possibly  $u = v$  or  $w = x$ ). The graph  $\Phi'_i$ ,  $i \geq 0$ , is formed from  $\Phi_i$  by adding a new vertex of degree 1 adjacent to one of the two  $K_3$ s but not to the path between them. The graph  $\Phi''_i$ ,  $i \geq 0$ , is formed from  $\Phi_i$  by adding two new vertices of degree 1; each one is adjacent to a different one of the two  $K_3$ s but not to the path between them. These graphs are illustrated in Figure 4.



**Figure 4.** The graphs  $\Phi'_i$ ,  $\Phi''_i$  and  $\Phi_{i,j}$ .

A (partial) proof of Theorem 21 can be found in the papers of Golovach and Paulusma [45] and Dabrowski, Golovach and Paulusma [25]. Note that, by symmetry, the graphs  $H_1$  and  $H_2$  may be swapped in each of the subcases of Theorem 21.

**Theorem 21.** *Let  $H_1$  and  $H_2$  be two graphs. Then the following hold:*

(i) COLOURING is NP-complete for  $(H_1, H_2)$ -free graphs if

1.  $H_1 \supseteq_i C_r$  for  $r \geq 3$ , and  $H_2 \supseteq_i C_s$  for  $s \geq 3$
2.  $H_1 \supseteq_i K_{1,3}$ , and  $H_2 \supseteq_i K_{1,3}$  or  $H_2 \supseteq_i \overline{2P_1 + P_2}$  or  $H_2 \supseteq_i C_r$  for  $r \geq 4$  or  $H_2 \supseteq_i K_4$  or  $H_2 \supseteq_i \Phi_{i,j}$  for  $i, j \geq 0$ , both even or  $H_2 \supseteq_i \Phi'_i$  for  $i \geq 1$  odd or  $H_2 \supseteq_i \Phi''_i$  for  $i \geq 0$  even
3.  $H_1 \supseteq_i \Phi_i$  for  $i \geq 1$ , and  $H_2$  contains a spanning subgraph of  $2P_2$  as an induced subgraph
4.  $H_1$  and  $H_2$  contain a spanning subgraph of  $2P_2$  as an induced subgraph
5.  $H_1 \supseteq_i C_3^{++}$ , and  $H_2 \supseteq_i K_{1,4}$  or  $H_2 \supseteq_i \overline{C_4 + P_1}$
6.  $H_1 \supseteq_i C_3$ , and  $H_2 \supseteq_i K_{1,r}$  for  $r \geq 5$
7.  $H_1 \supseteq_i C_3$ , and  $H_2 \supseteq_i P_{22}$
8.  $H_1 \supseteq_i C_r$  for  $r \geq 5$ , and  $H_2$  contains a spanning subgraph of  $2P_2$  as an induced subgraph
9.  $H_1 \supseteq_i C_r + P_1$  for  $3 \leq r \leq 4$  or  $H_1 \supseteq_i \overline{C_r}$  for  $r \geq 6$ , and  $H_2$  contains a spanning subgraph of  $2P_2$  as an induced subgraph
10.  $H_1 \supseteq_i K_5$ , and  $H_2 \supseteq_i P_7$
11.  $H_1 \supseteq_i K_6$ , and  $H_2 \supseteq_i P_6$ .

(ii) COLOURING is polynomial-time solvable for  $(H_1, H_2)$ -free graphs if

1.  $H_1$  or  $H_2$  is an induced subgraph of  $P_1 + P_3$  or of  $P_4$
2.  $H_1 \subseteq_i K_{1,3}$ , and  $H_2 \subseteq_i C_3^{++}$  or  $H_2 \subseteq_i C_3^*$  or  $H_2 \subseteq_i P_5$
3.  $H_1 \neq K_{1,5}$  is a forest on at most six vertices or  $H_1 = K_{1,3} + 3P_1$ , and  $H_2 \subseteq_i C_3^+$
4.  $H_1 \subseteq_i sP_2$  or  $H_1 \subseteq_i sP_1 + P_5$  for  $s \geq 1$ , and  $H_2 = K_t$  for  $t \geq 4$



5.  $H_1 \subseteq_i sP_2$  or  $H_1 \subseteq_i sP_1 + P_5$  for  $s \geq 1$ , and  $H_2 \subseteq_i C_3^+$
6.  $H_1 \subseteq_i P_1 + P_4$  or  $H_1 \subseteq_i P_5$ , and  $H_2 \subseteq_i \overline{P_1 + P_4}$
7.  $H_1 \subseteq_i P_1 + P_4$  or  $H_1 \subseteq_i P_5$ , and  $H_2 \subseteq_i \overline{P_5}$
8.  $H_1 \subseteq_i \overline{2P_1 + P_2}$ , and  $H_2 \subseteq_i \overline{P_1 + 2P_2}$  or  $H_2 \subseteq_i \overline{2P_1 + P_3}$  or  $H_2 \subseteq_i \overline{P_2 + P_3}$
9.  $H_1 \subseteq_i \overline{2P_1 + P_2}$ , and  $H_2 \subseteq_i P_1 + 2P_2$  or  $H_2 \subseteq_i 2P_1 + P_3$  or  $H_2 \subseteq_i P_2 + P_3$
10.  $H_1 \subseteq_i sP_1 + P_2$  for  $s \geq 0$  or  $H_1 = P_5$ , and  $H_2 \subseteq_i \overline{tP_1 + P_2}$  for  $t \geq 0$
11.  $H_1 \subseteq_i 4P_1$  and  $H_2 \subseteq_i \overline{2P_1 + P_3}$
12.  $H_1 \subseteq_i P_5$ , and  $H_2 \subseteq_i C_4$  or  $H_2 \subseteq_i \overline{2P_1 + P_3}$ .

*Proof.* In each case we either refer back to an earlier result, or give a reference. The results quoted can clearly be seen to imply the statements of the theorem.

(i) We first consider the NP-completeness results.

1. By Theorem 4, for  $k \geq 3$ ,  $k$ -COLOURING is NP-complete for  $(C_r, C_s)$ -free graphs for all  $r \geq 3$  and  $s \geq 3$ .
2. These cases were proved for 3-COLOURING by Lozin and Purcell [83]. (This is sufficient but we note also that by Theorem 5, for  $k \geq 3$ ,  $k$ -COLOURING is NP-complete for claw-free graphs and that Král' et al. [76] showed that 3-COLOURING is NP-complete for  $(C_r, K_{1,3})$ -free graphs whenever  $r \geq 4$  and for  $(K_4, K_{1,3}, \overline{2P_1 + P_2})$ -free graphs.)
3. This follows from a result of Lozin and Malyshev [82] who proved that COLOURING is NP-complete for  $(C_3 + P_1, 2P_2, 2P_1 + P_2, 4P_1, C_5, \overline{C_6}, \dots, \overline{C_p}, \overline{\Phi_0}, \dots, \overline{\Phi_p})$ -free graphs for every integer  $p \geq 6$ .
4. This is a result of Král' et al. [76].
5. Malyshev [89] proved that 3-COLOURING is NP-complete for  $(C_3^{++}, \overline{C_4 + P_1}, K_{1,4})$ -free graphs after previously proving that 3-COLOURING is NP-complete for  $(C_3^{++}, K_{1,4})$ -free graphs [88]. Note that Theorem 21 (i).3 already implies that 3-COLOURING is NP-complete for  $(\overline{C_4 + P_1}, K_{1,4})$ -free graphs.
6. By Theorem 16, 3-COLOURING is NP-complete for  $(C_3, K_{1,r})$ -free graphs for all  $r \geq 5$ .
7. Huang, Johnson and Paulusma [66] proved that 4-COLOURING is NP-complete for  $(C_3, P_{22})$ -free graphs, thereby improving a result of Golovach et al. [47] who showed that 4-COLOURING is NP-complete for  $(C_3, P_{164})$ -free graphs.
8. This is a result of Král' et al. [76].
9. This follows from Theorem 17.
10. This follows from Theorem 7 (i) and the fact that  $K_6$  is not 5-colourable.
11. This follows from Theorem 7 (i) and the fact that  $K_5$  is not 4-colourable.

(ii) We now consider the tractable cases.

1. This follows from Theorem 3 (i).
2. This was proved by Malyshev [88] for  $(K_{1,3}, C_3^*)$ -free graphs and  $(K_{1,3}, P_5)$ -free graphs and by Malyshev [90] for  $(K_{1,3}, C_3^{++})$ -free graphs.
3. First we consider the case when  $H_1$  is a forest on at most six vertices not isomorphic to  $K_{1,5}$  and  $H_2 \subseteq_i C_3$ . Dabrowski, Lozin, Raman and Ries [27] proved that COLOURING is polynomial-time solvable for  $(H_1, C_3)$ -free graphs by combining a number of new results with known results for  $H_1 = K_{1,4}$  [76],  $H_1 = S_{1,2,2}$  [98],  $H_1 = P_2 + P_4$  [15],  $H_1 = 2P_3$  [16],  $H_1 = P_6$  [9],  $H_1$  is the cross [99] and  $H_1$  is the ‘‘H’’-graph [98] (see Figure 2 for pictures of the cross and the ‘‘H’’-graph). Then they applied Theorem 20. Dabrowski and Paulusma [28] proved that the class of  $(K_{1,3} + 3P_1, C_3^+)$ -free graphs has bounded clique-width, so Theorem 19 (i) can be applied.
4. Theorem 18 implies that for all  $r \geq 1$ , COLOURING is polynomial-time solvable on  $(K_r, F)$ -free graphs for some linear forest  $F$  if  $k$ -COLOURING is polynomial-time solvable on  $F$ -free graphs for all  $k \geq 1$ . The latter is true for  $F = sP_1 + P_5$  and  $F = sP_2$ , for all  $s \geq 1$ , by Theorem 7 (iii).
5. This is obtained by combining the arguments of the previous case with Theorem 20.

6. The classes of  $(P_1 + P_4, \overline{P_1 + P_4})$ -free graphs [11] and  $(P_5, \overline{P_1 + P_4})$ -free graphs [10] have bounded clique-width. Hence, COLOURING is polynomial-time solvable for these two graph classes by Theorem 19 (i).
7. For the class of  $(P_1 + P_4, \overline{P_5})$ -free graphs, we again note they have bounded clique-width [10]. Hoàng and Lazzarato [61] showed that COLOURING is polynomial-time solvable on  $(P_5, \overline{P_5})$ -free graphs (in fact they show that the weighted variant of COLOURING is polynomial-time solvable). Previously, this result was known only for  $(2P_2, \overline{P_5})$ -free graphs, as Hoàng, Maffray and Mechebbek [62] showed that these graphs are b-perfect, and in the same paper they proved that COLOURING is polynomial-time solvable for b-perfect graphs.
8. Dabrowski, Dross and Paulusma [24] proved that the class of  $(2P_1 + P_2, \overline{P_1 + 2P_2})$ -free graphs has bounded clique-width. Dabrowski, Huang and Paulusma [26] showed that the class of  $(2P_1 + P_2, \overline{2P_1 + P_3})$ -free graphs and the class of  $(2P_1 + P_2, \overline{P_2 + P_3})$ -free graphs have bounded clique-width.
9. This is due to Dabrowski et al. [24, 26] as well.
10. Dabrowski, Golovach and Paulusma [25] proved that for every two integers  $s \geq 0$  and  $t \geq 0$ , COLOURING is polynomial-time solvable for  $(sP_1 + P_2, \overline{tP_1 + P_2})$ -free graphs. Malyshev and Lobanova [91] proved that, for all  $t \geq 0$ , COLOURING is polynomial-time solvable for  $(P_5, \overline{tP_1 + P_2})$ -free graphs, which generalizes an earlier result of Dabrowski, Golovach and Paulusma [25] for the class of  $(2P_2, \overline{tP_1 + P_2})$ -free graphs.
11. The class of  $(4P_1, \overline{2P_1 + P_3})$ -free graphs has bounded clique-width [8], hence we apply Theorem 19 (i).
12. This was proved by Malyshev [88] for  $(P_5, C_4)$  and by Malyshev [90] for  $(P_5, \overline{2P_1 + P_3})$ . Previously this was known for  $(P_5, \overline{2P_1 + P_2})$ -free graphs [3].  $\square$

We pose the following problem.

**Open Problem 8** *Complete the classification of the complexity of COLOURING for  $(H_1, H_2)$ -free graphs.*

**Some Important Subproblems** A classification of the complexity of COLOURING for  $(H_1, H_2)$ -free graphs is already problematic when  $H_1$  and  $H_2$  are small. Lozin and Malyshev [82] determined the computational complexity of COLOURING restricted to  $\mathcal{H}$ -free graphs for every finite set  $\mathcal{H}$  that consists only of 4-vertex graphs except in the following four cases, which are still open:

- $\mathcal{H} = \{K_{1,3}, 4P_1\}$
- $\mathcal{H} = \{K_{1,3}, 2P_1 + P_2, 4P_1\}$
- $\mathcal{H} = \{K_{1,3}, 2P_1 + P_2\}$
- $\mathcal{H} = \{C_4, 4P_1\}$ .

The same authors showed that the cases  $\mathcal{H} = \{K_{1,3}, 2P_1 + P_2\}$  and  $\mathcal{H} = \{K_{1,3}, 2P_1 + P_2, 4P_1\}$  are polynomially equivalent (hence three cases remain). Fraser, Hamel and Hoàng [37] continued this study by showing that COLOURING is polynomial-time solvable for a subclass of  $(K_{1,3}, 4P_1)$ -free graphs, namely for (a superclass of)  $4P_1$ -free line graphs.

The above open cases are part of a larger set of open subproblems. As we will see (Theorem 24), for all  $k, r, s, t \geq 1$ ,  $k$ -COLOURING can be solved in linear time for the class of  $(K_{r,s}, P_t)$ -free graphs. However, for COLOURING restricted to  $(K_{r,s}, P_t)$ -free graphs much less is known. Besides the aforementioned open cases, also the case of  $(C_4, P_6)$ -free graphs is a natural open case to consider, as COLOURING is polynomial-time solvable for  $(C_4, P_5)$ -free graphs due to Theorem 21 (ii).12. In addition, the case of  $(C_3, P_7)$ -free graphs still needs to be solved (we discuss this case in more detail later).

Finally, Dabrowski et al. [24] listed all 13 classes of  $(H_1, H_2)$ -free graphs, for which COLOURING could still be solved in polynomial time by showing that their clique-width is bounded. These classes are

1.  $\overline{H_1} \in \{3P_1, P_1 + P_3\}$  and  $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$ ;
2.  $H_1 = 2P_1 + P_2$  and  $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$ ;
3.  $H_1 = 2\overline{P_1} + \overline{P_2}$  and  $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$ ;
4.  $H_1 = P_1 + P_4$  and  $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ ;
5.  $\overline{H_1} = P_1 + P_4$  and  $H_2 \in \{P_1 + 2P_2, P_2 + P_3\}$ ;
6.  $H_1 = \overline{H_2} = 2P_1 + P_3$ .

We now discuss what is known for PRECOLOURING EXTENSION restricted to  $(H_1, H_2)$ -free graphs. By Theorem 3 (ii), PRECOLOURING EXTENSION can be solved in polynomial time on  $(H_1, H_2)$ -free graphs whenever, for some  $i \in \{1, 2\}$ ,  $H_i \subseteq_i P_4$  or  $H_i \subseteq_i P_1 + P_3$ , and, of course, the NP-completeness results from Theorem 21 also hold for PRECOLOURING EXTENSION. This is all that seems to be known of PRECOLOURING EXTENSION on  $(H_1, H_2)$ -free graphs. Let us give an example of a class of  $(H_1, H_2)$ -free graphs for which the complexities of COLOURING and PRECOLOURING EXTENSION are different (unless  $P=NP$ ): Case (ii):3 of Theorem 21, which shows that COLOURING is polynomial-time solvable for  $(C_3, P_6)$ -free graphs, can be compared with the following result.

**Theorem 22.** PRECOLOURING EXTENSION is NP-complete for  $(C_3, P_6)$ -free graphs.

*Proof.* We reduce from the restriction of LIST COLOURING to complete bipartite graphs which is NP-complete by Theorem 2. Let  $G = (V, E)$  be a complete bipartite graph with list assignment  $L$ . Let  $X = \bigcup_{u \in V} L(u)$ . For each  $u \in V$ , add  $|X| - |L(u)|$  new vertices, add an edge from each to  $u$ , and assign each a different colour from  $X \setminus L(u)$ . Let  $G'$  be the resulting graph, let  $W$  be the set of vertices in  $G' - V$  and let  $k = |X|$ , and notice that in the previous sentence we have defined a  $k$ -precolouring of  $G'$  in which a vertex has a colour if and only if it is in  $W$ . It is readily seen that  $G'$  is  $(C_3, P_6)$ -free, and that  $G$  has a colouring that respects  $L$  if and only if the  $k$ -precolouring of  $G'$  can be extended to a  $k$ -colouring.  $\square$

Hujter and Tuza asked for which graph classes PRECOLOURING EXTENSION is NP-complete (Problem 1.1 in [68]). We pose the following problem.

**Open Problem 9** Complete the classification of the complexity of PRECOLOURING EXTENSION for  $(H_1, H_2)$ -free graphs.

By combining a number of known hardness results on LIST COLOURING for complete bipartite graphs [70], complete split graphs [43] and  $(3P_1, P_1 + P_2)$ -free graphs [48] with a number of new hardness results, Golovach and Paulusma [45] completely classified the complexity of LIST COLOURING and  $\ell$ -LIST COLOURING,  $\ell \geq 3$ , for  $(H_1, H_2)$ -free graphs. Note that, by symmetry, the graphs  $H_1$  and  $H_2$  may be swapped in each of the three subcases of Theorem 23.

**Theorem 23.** Let  $H_1$  and  $H_2$  be two graphs. Then LIST COLOURING is polynomial-time solvable for  $(H_1, H_2)$ -free graphs in the following cases:

1.  $H_1 \subseteq_i P_3$  or  $H_2 \subseteq_i P_3$
2.  $H_1 \subseteq_i C_3$  and  $H_2 \subseteq_i K_{1,3}$
3.  $H_1 = K_r$  for some  $r \geq 3$  and  $H_2 = sP_1$  for some  $s \geq 3$ .

In all other cases, even 3-LIST COLOURING is NP-complete for  $(H_1, H_2)$ -free graphs.

The computational complexity classification of  $k$ -COLOURING,  $k$ -PRECOLOURING EXTENSION and LIST  $k$ -COLOURING restricted to  $(H_1, H_2)$ -free graphs is not complete either. Tractability for many cases is obtained from Theorem 7 (iii)–(iv). Moreover, as mentioned in the proof of Theorem 21, Cases (i):1, 2, 4–6, 10 of Theorem 21 hold for 3-COLOURING and Case (i):7 holds for 4-COLOURING. In particular, the case in which  $H_1$  is a cycle and  $H_2$  is a path has been studied for all three variants [47, 59, 66].

We survey the known results for these two cases below. In order to do this we need three additional results. The first additional result was proven by Golovach et al. [47], which extends a corresponding result of Lozin and Rautenbach [84] from  $r = 1$  to arbitrary  $r \geq 1$ .

**Theorem 24.** *For all  $k, r, s, t \geq 1$ , LIST  $k$ -COLOURING can be solved in linear time for  $(K_{r,s}, P_t)$ -free graphs.*

Theorem 24 implies that for all  $g \geq 5, k \geq 1$  and  $t \geq 1$ , LIST  $k$ -COLOURING can be solved in linear time for  $P_t$ -free graphs of girth at least  $g$ , or equivalently  $(C_3, \dots, C_{g-1}, P_t)$ -free graphs (contrast with Theorem 4 on  $k$ -COLOURING). Huang et al. [66] showed that when  $C_4 = K_{2,2}$  is no longer forbidden the computational complexity changes again by proving that for all  $k \geq 4$  and  $g \geq 6$ , there exists a constant  $t_k^g$  such that  $k$ -COLOURING is NP-complete for  $(C_3, C_5, \dots, C_{g-1}, P_{t_k^g})$ -free graphs.

We also need another result of Huang et al. [66].

**Theorem 25.** *LIST 4-COLOURING is NP-complete for  $(C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4}, P_6)$ -free graphs.*

The third additional result was also proven by Huang et al. [66]. It strengthens a result of Kratochvíl [77] who showed that 5-PRECOLOURING EXTENSION is NP-complete for  $P_{14}$ -free bipartite graphs.

**Theorem 26.** *For all  $k \geq 4$ ,  $k$ -PRECOLOURING EXTENSION is NP-complete for  $P_{10}$ -free bipartite graphs.*

We are now ready to state Theorem 27. A proof of this theorem was given by Huang et al. [66]; as it is obtained by combining a number of results from different papers we include it here as well.

**Theorem 27.** *Let  $k, s, t$  be three integers. The following statements hold for  $(C_s, P_t)$ -free graphs.*

(i) LIST  $k$ -COLOURING is NP-complete if

1.  $k \geq 4, s = 3$  and  $t \geq 8$
2.  $k \geq 4, s \geq 5$  and  $t \geq 6$ .

LIST  $k$ -COLOURING is polynomial-time solvable if

3.  $k \leq 2, s \geq 3$  and  $t \geq 1$
4.  $k = 3, s = 3$  and  $t \leq 7$
5.  $k = 3, s = 4$  and  $t \geq 1$
6.  $k = 3, s \geq 5$  and  $t \leq 6$
7.  $k \geq 4, s = 3$  and  $t \leq 6$
8.  $k \geq 4, s = 4$  and  $t \geq 1$
9.  $k \geq 4, s \geq 5$  and  $t \leq 5$ .

(ii)  $k$ -PRECOLOURING EXTENSION is NP-complete if

1.  $k = 4, s = 3$  and  $t \geq 10$
2.  $k = 4, s = 5$  and  $t \geq 7$
3.  $k = 4, s = 6$  and  $t \geq 7$
4.  $k = 4, s = 7$  and  $t \geq 8$
5.  $k = 4, s \geq 8$  and  $t \geq 7$
6.  $k \geq 5, s = 3$  and  $t \geq 10$
7.  $k \geq 5, s \geq 5$  and  $t \geq 6$ .

$k$ -PRECOLOURING EXTENSION is polynomial-time solvable if

8.  $k \leq 2, s \geq 3$  and  $t \geq 1$
9.  $k = 3, s = 3$  and  $t \leq 7$

10.  $k = 3, s = 4$  and  $t \geq 1$
11.  $k = 3, s \geq 5$  and  $t \leq 6$
12.  $k \geq 4, s = 3$  and  $t \leq 6$
13.  $k \geq 4, s = 4$  and  $t \geq 1$
14.  $k \geq 4, s \geq 5$  and  $t \leq 5$ .

(iii)  $k$ -COLOURING is NP-complete if

1.  $k = 4, s = 3$  and  $t \geq 22$
2.  $k = 4, s = 5$  and  $t \geq 7$
3.  $k = 4, s = 6$  and  $t \geq 7$
4.  $k = 4, s = 7$  and  $t \geq 9$
5.  $k = 4, s \geq 8$  and  $t \geq 7$
6.  $k \geq 5, s = 3$  and  $t \geq t_k$  where  $t_k$  is a constant that only depends on  $k$
7.  $k \geq 5, s = 5$  and  $t \geq 7$
8.  $k \geq 5, s \geq 6$  and  $t \geq 6$ .

$k$ -COLOURING is polynomial-time solvable if

9.  $k \leq 2, s \geq 3$  and  $t \geq 1$
10.  $k = 3, s = 3$  and  $t \leq 7$
11.  $k = 3, s = 4$  and  $t \geq 1$
12.  $k = 3, s \geq 5$  and  $t \leq 7$
13.  $k = 4, s = 3$  and  $t \leq 6$
14.  $k = 4, s = 4$  and  $t \geq 1$
15.  $k = 4, s = 5$  and  $t \leq 6$
16.  $k = 4, s \geq 6$  and  $t \leq 5$
17.  $k \geq 5, s = 3$  and  $t \leq k + 2$
18.  $k \geq 5, s = 4$  and  $t \geq 1$
19.  $k \geq 5, s \geq 5$  and  $t \leq 5$ .

*Proof.* Again we either refer back to an earlier result, or give a reference and the results quoted can clearly be seen to imply the statements of the theorem.

We first consider the intractable cases of LIST  $k$ -COLOURING. For (i).1, we note that Huang et al. [66] showed that LIST 4-COLOURING is NP-complete for  $(C_3, P_8)$ -free graphs. Theorem 25 implies that LIST 4-COLOURING is NP-complete for the class of  $(C_5, C_6, P_6)$ -free graphs which proves (i).2.

We now consider the tractable cases of LIST  $k$ -COLOURING. Theorem 1 (i) implies (i).3, whereas (i).4 follows from Theorem 7 (iii). Theorem 24 implies (i).5 and (i).8, whereas (i).6 and (i).9 follow from Theorem 7 (iii). Recall that the class of  $(C_3, P_6)$ -free graphs has bounded clique-width, as shown by Brandstädt, Klemmt and Mahfud [9]. Combining this with Theorem 19 (ii) we find that LIST  $k$ -COLOURING is polynomial-time solvable on  $(C_3, P_6)$ -free graphs for all  $k \geq 1$ . This proves (i).7

We now consider  $k$ -PRECOLOURING EXTENSION. As the tractable cases all follow from Theorem 27 (i), we are left to consider the NP-complete cases. Theorem 26 implies (ii).1 and (ii).6. Huang et al. [66] proved that 4-PRECOLOURING EXTENSION is NP-complete for  $(C_7, P_8)$ -free graphs, which implies (ii).4, and they also proved (ii).7. We observe that (ii).2, (ii).3 and (ii).5 follow immediately from corresponding results for  $k$ -COLOURING as shown by Hell and Huang [59].

Finally, we consider  $k$ -COLOURING; first the NP-complete cases. Recall that (iii).1 has been shown by Huang et al. [66], who also proved (iii).6; they showed that  $t_k \leq k + (k + 1)(3 \cdot 2^{k-1} - 1)$  for all  $k \geq 5$ . Golovach et al. [47] proved that for all  $s \geq 5$ , there exists a constant  $t(s)$  such that 4-COLOURING is NP-complete for  $(C_5, \dots, C_s, P_{t(s)})$ -free graphs. In particular, they showed that 4-COLOURING is NP-complete for  $(C_5, P_{23})$ -free graphs, and this result has been strengthened by Hell and Huang [59] who proved all the other NP-completeness subcases.

We now consider the tractable cases of  $k$ -COLOURING. Theorem 7 (iii) implies (iii).9, (iii).10, (iii).12, (iii).16 and (iii).19. Theorem 24 implies (iii).11, (iii).14 and (iii).18. Case (ii):3 of Theorem 21 implies (iii).13. Chudnovsky, Maceli, Stacho and Zhong [20] proved (iii).15. Theorem 18 implies (iii).17.  $\square$

Theorem 27 leaves a number of cases open (also see Huang et al. [66]).

**Open Problem 10** *Determine the complexity of the missing cases from Theorem 27 for  $(C_s, P_t)$ -free graphs, which are:*

(i) for LIST  $k$ -COLOURING when

- $k = 3, s = 3$  and  $t \geq 7$
- $k = 3, s \geq 5$  and  $t \geq 7$
- $k \geq 4, s = 3$  and  $t = 7$ .

(ii) for  $k$ -PRECOLOURING EXTENSION when

- $k = 3, s = 3$  and  $t \geq 7$
- $k = 3, s \geq 5$  and  $t \geq 7$
- $k = 4, s = 3$  and  $7 \leq t \leq 9$
- $k = 4, s \geq 5$  and  $t = 6$
- $k = 4, s = 7$  and  $t = 7$
- $k \geq 5, s = 3$  and  $7 \leq t \leq 9$

(iii) for  $k$ -COLOURING when

- $k = 3, s = 3$  and  $t \geq 8$
- $k = 3, s \geq 5$  and  $t \geq 8$
- $k = 4, s = 3$  and  $7 \leq t \leq 21$
- $k = 4, s \geq 6$  and  $t = 6$
- $k = 4, s = 7$  and  $7 \leq t \leq 8$
- $k \geq 5, s = 3$  and  $k + 3 \leq t \leq t_k - 1$
- $k \geq 5, s = 5$  and  $t = 6$ .

**Some Important Subproblems** Note that as a consequence of Theorem 25, LIST 4-COLOURING is NP-complete for  $(C_5, P_6)$ -free graphs. As 4-COLOURING is polynomial-time solvable for  $(C_5, P_6)$ -free graphs by Theorem 27, there exists an integer  $k$  and two graphs  $H_1$  and  $H_2$  (namely  $k = 4, H_1 = C_5$  and  $H_2 = P_6$ ) for which the complexity of  $k$ -COLOURING and LIST  $k$ -COLOURING is not the same when restricted to  $(H_1, H_2)$ -free graphs. Recall that such a situation is not known when we forbid only one induced graph  $H$ . Moreover, when forbidding two graphs  $H_1$  and  $H_2$  no such complexity jump is known, for any integer  $k$ , between  $k$ -COLOURING and  $k$ -PRECOLOURING EXTENSION restricted to  $(H_1, H_2)$ -free graphs. Hence, it would also be interesting to determine the complexity of 4-PRECOLOURING EXTENSION for  $(C_5, P_6)$ -free graphs, which is one of the cases in Open Problem 10.

In addition to the above, we point out another case in Open Problem 10: that of determining the complexity of 4-COLOURING restricted to  $(C_3, P_7)$ -free graphs. By Theorem 18, every  $(C_3, P_7)$ -free graph is 5-colourable. By Theorem 27 (iii),  $k$ -COLOURING is polynomial-time solvable for  $(C_3, P_7)$ -free graphs if  $k \leq 3$ . Hence, the problems 4-COLOURING and COLOURING are polynomially equivalent for  $(C_3, P_7)$ -free graphs.

We now discuss a number of results for  $k$ -COLOURING restricted to  $(H_1, H_2)$ -free graphs when  $(H_1, H_2)$  is not a cycle and a path.

First we consider pairs of graphs  $(H_1, H_2)$  with the property that every  $(H_1, H_2)$ -free graph is 3-colourable. Because 2-COLOURING is polynomial-time solvable, such results imply polynomial-time solvability of 3-COLOURING for  $(H_1, H_2)$ -free graphs.

We note that only when  $H \in \{P_1, P_2\}$  is every  $H$ -free graph 3-colourable. Thus for all graphs  $H_2$ , every  $(P_1, H_2)$ -free graph and every  $(P_2, H_2)$ -free graph is 3-colourable. Also Wagon [110]

showed that every  $(K_r, 2P_2)$ -free graph is  $\frac{1}{2}r(r-1)$ -colourable, which implies that every  $(C_3, 2P_2)$ -free graph is 3-colourable.

We focus now on the case where  $H_1$  and  $H_2$  are connected and show that this is *almost* completely understood. A pair of graphs  $(H_1, H_2)$  is called *good* if every  $(H_1, H_2)$ -free graph is 3-colourable, and, moreover, the class of  $(H_1, H_2)$ -free graphs is properly contained in the classes of  $H_1$ -free graphs and  $H_2$ -free graphs.

A good pair  $(H_1, H_2)$  is *saturated* if there is no good pair  $(H'_1, H'_2)$  with  $H_1 \subsetneq H'_1$  and  $H_2 \subsetneq H'_2$ . We note in passing that Sumner [106] showed that every  $(C_3, P_5)$ -free graph is 3-colourable. However, the pair  $(C_3, P_5)$  is not saturated. This follows from this result of Randerath [98] (see Figure 2 for the names of small graphs):

- If  $(K_3, \text{fork})$  is a good pair, then  $(K_3, \text{fork})$ ,  $(K_3, \text{“H”-graph})$  and  $(K_4, P_4)$  are the only saturated pairs of connected graphs.
- If  $(K_3, \text{fork})$  is not a good pair, then  $(K_3, \text{cross})$ ,  $(K_3, \text{“E”-graph})$ ,  $(K_3, \text{“H”-graph})$  and  $(K_4, P_4)$  are the only saturated pairs of connected graphs.

Note that the cross and “E”-graph are the two maximal connected proper induced subgraphs of the fork. Hence the following open problem remains (which is Conjecture 6 in [98] and Conjecture 44 in [101]).

**Open Problem 11** *Is every  $(K_3, \text{fork})$ -free graph 3-colourable?*

Recently, Fan, Xu, Ye and Yu [35] made progress in answering this question by proving that every  $(C_5, K_3, \text{fork})$ -free graph is 3-colourable.

The natural next question is, of course, to ask when  $(H_1, H_2)$ -free graphs are  $k$ -colourable for  $k \geq 4$ . A little is known. For  $r \geq 2$  and  $s \geq 1$ , the *broom*  $B(r, s)$  (also called a *mop*) is the graph obtained from a star  $K_{1, s+1}$  after subdividing one of its edges  $r-2$  times, so  $B(2, s) = K_{1, s+1}$ ,  $B(r, 1) = P_{r+1}$  and  $B(r, 2) = S_{1, 1, r-1}$ . Gyarfas, Szemeredi and Tuza [57] proved that every  $(C_3, B(r, s))$ -free graph is  $(r+s-1)$ -colourable. Hence, every  $(C_3, P_r)$ -free graph is  $(r-1)$ -colourable. Randerath and Schiermeyer [101] improved this implication by showing that for all  $r \geq 4$ , every  $(C_3, P_r)$ -free graph is  $(r-2)$ -colourable. This result has been extended by Wang and Wu [111], who proved that every connected  $(C_3, B(r, s))$ -free graph that is not an odd cycle is  $(r+s-2)$ -colorable. For the cases  $r=2, s \geq 8$  and  $r=3, s \in \{2, 3\}$  they were able to reduce this bound to  $r+s-3$ .

The above results imply that every  $(C_3, P_6)$ -free graph is 4-colourable (this also follows from Theorem 18). Brandt [12] showed that every  $(C_3, sP_2)$ -free graph is  $(2s-2)$ -colourable for any  $s \geq 3$ . This means that every  $(C_3, 3P_2)$ -free graphs is 4-colourable. Pyatkin [97] showed that every  $(C_3, 2P_3)$ -free graph is 4-colourable, whereas Broersma et al. [16] showed that every  $(C_3, P_2 + P_4)$ -free graph is 4-colourable.

**Open Problem 12** *Determine all pairs  $(H_1, H_2)$  that have the property that every  $(H_1, H_2)$ -free graph is 4-colourable.*

One problem that has had considerable attention is the classification of the computational complexity of 3-COLOURING for  $(K_{1,3}, H)$ -free graphs. As noted in Theorem 21, Lozin and Purcell [83] showed that 3-COLOURING on  $(K_{1,3}, H)$ -free graphs is NP-complete whenever  $H \supseteq_i K_{1,3}$  or  $H \supseteq_i 2P_1 + P_2$  or  $H \supseteq_i C_r$  for  $r \geq 4$  or  $H \supseteq_i K_4$  or  $H \supseteq_i \Phi_{i,j}$  for  $i, j \geq 0$ , both even or  $H \supseteq_i \Phi'_i$  for  $i \geq 1$  odd or  $H \supseteq_i \Phi''_i$  for  $i \geq 0$  even. They also observed that 3-COLOURING is polynomial-time solvable on  $(K_{1,3}, H)$ -free graphs if every connected component of  $H$  contains at most one triangle. So what about the remaining cases where  $H$  has a connected component containing two triangles? Randerath, Schiermeyer and Tewes [102] proved that 3-COLOURING is polynomial-time solvable on  $(K_{1,3}, \Phi_0)$ -free graphs, and later Kaminski and Lozin [73] gave a linear-time algorithm. The latter authors also showed that 3-COLOURING is polynomial-time solvable on  $(K_{1,3}, T_{0,0,j}^\Delta)$ -free graphs for all  $j \geq 0$ , and Lozin and Purcell [83] showed that 3-COLOURING is polynomial-time solvable on  $(K_{1,3}, \Phi_1)$ -free graphs and  $(K_{1,3}, \Phi_3)$ -free graphs.

**Open Problem 13** Complete the classification of the complexity of 3-COLOURING for  $(K_{1,3}, H)$ -free graphs.

Malyshev [89] characterized exactly those pairs of graphs  $H_1$  and  $H_2$  each with at most five vertices for which 3-COLOURING is polynomial-time solvable when restricted to  $(H_1, H_2)$ -free graphs. His main result, obtained by combining known results with new results, can be summarized as follows. Recall that  $C_3^{++}$  is the bull. If at least one of  $H_1, H_2$  is a forest and at least one of  $H_1, H_2$  is the line graph of a forest with maximum degree at most 3 and  $(H_1, H_2) \neq \{(K_{1,4}, \overline{C_4 + P_1}), (K_{1,4}, C_3^{++})\}$ , then 3-COLOURING is polynomial-time solvable for  $(H_1, H_2)$ -free graphs; otherwise it is NP-complete.

We recall that the complexity status of 4-COLOURING is still open for the class of  $P_6$ -free graphs. Randerath et al. [102] showed that every  $(C_3, P_6)$ -free graph is 4-colourable and gave a polynomial-time algorithm for finding a 4-colouring of a  $(C_3, P_6)$ -free graph. The latter result also follows from Theorem 27 (i):4 and can be extended to  $(\overline{P_1 + P_3}, P_6)$ -free graphs due to Theorem 21 (ii):3. Besides the aforementioned results that 4-COLOURING is polynomial-time solvable for  $(C_5, P_6)$ -free graphs (due to Theorem 27 (iii):15) and  $(K_{r,s}, P_6)$ -free graphs for any two positive integers  $r, s$  (due to Theorem 24), the following results are known as well. Let  $C_4^+$  denote the banner, which is the graph obtained from  $C_4$  after adding a pendant vertex. Huang [65] proved that 4-COLOURING is polynomial-time solvable for  $(C_4^+, P_6)$ -free graphs. Brause, Schiermeyer, Holub, Ryjáček, Vrána and Krivoš-Belluš [13] proved that 4-COLOURING is polynomial-time solvable for  $(P_6, S_{1,1,2})$ -free graphs. The same authors also considered subclasses of  $(C_3^{++}, P_6)$ -free graphs and showed that 4-COLOURING is polynomial-time solvable for  $(C_3^+, C_3^{++}, P_6)$ -free graphs and for  $(C_3^{++}, P_6, \overline{S_{1,1,2}})$ -free graphs. Maffray and Pastor [86] generalized these two results by proving that 4-COLOURING is polynomial-time solvable for  $(C_3^{++}, P_6)$ -free graphs.

**Certifying Algorithms** Recall from the previous section that Hoàng, Moore, Recoskie, Sawada and Vatschelle [63] showed that the number of 5-critical  $P_5$ -free graphs and the number of 5-vertex-critical  $P_5$ -free graphs is infinite. They also showed that there exist exactly eight 5-critical  $(C_5, P_5)$ -free graphs. Dhaliwal et al. [29] proved that, for all  $k \geq 1$ , the number of  $k$ -vertex-critical  $(P_5, \overline{P_5})$ -free graphs is finite. They showed that their result implies a certifying algorithm for  $k$ -COLOURING on  $(P_5, \overline{P_5})$ -free graphs for all  $k \geq 1$ . Randerath, Schiermeyer and Tewes [102] proved that the Grötzsch graph is the only 4-critical  $(C_3, P_6)$ -free graph. Hell and Huang [59] showed that, for all  $k \geq 1$ , the number of  $k$ -vertex-critical  $(C_4, P_6)$ -free graphs is finite. Moreover, they gave an explicit construction of all four 4-vertex-critical  $(C_4, P_6)$ -free graphs and of all thirteen 5-vertex-critical  $(C_4, P_6)$ -free graphs. Hence, they obtained certifying algorithms for 3-COLOURING and 4-COLOURING on  $(C_4, P_6)$ -free graphs. For all  $k \geq 6$ , explicit constructions of all  $k$ -vertex-critical graphs are unknown (for  $k \geq 5$ , no certifying algorithm is known for  $k$ -COLOURING on  $(C_4, P_6)$ -free graphs). Goedgebeur and Schaudt [42] determined all 4-critical  $(C_4, P_7)$ -free graphs, all 4-critical  $(C_5, P_7)$ -free graphs and all 4-critical  $(C_4, P_8)$ -free graphs leading to certifying algorithms for 3-COLOURING restricted to these three graph classes.

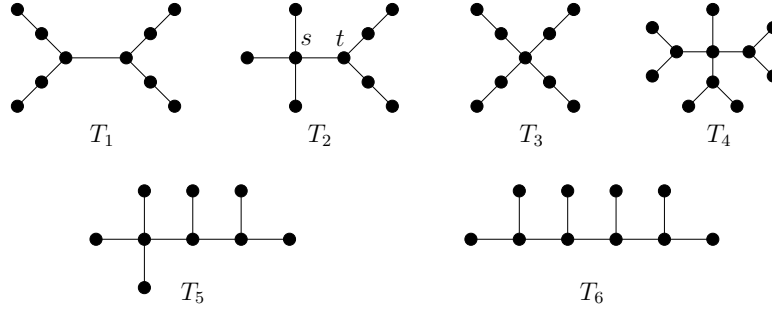
We conclude this section by noting that, as far as we are aware, there are no additional results for CHOOSABILITY and  $k$ -CHOOSABILITY known for  $(H_1, H_2)$ -free graphs other than those that follow directly from previously mentioned theorems and two results of Esperet, Gyárfás and Maffray [33] who proved that every  $(K_{1,3}, K_4)$ -free graph is 4-choosable and that every  $(K_{1,3}, K_5)$ -free graph is 7-choosable.

## 4 Graph Classes Defined by Other Forbidden Patterns

In this section we consider a number of other graph classes. We first consider strongly  $H$ -free graphs. Recall that, given a graph  $H$ , the class of strongly  $H$ -free graphs contains those graphs that do not contain  $H$  as a subgraph.



Contrast with  $H$ -free graphs where the graph  $H$  is forbidden as an *induced* subgraph: forbidding a graph  $H$  as an induced subgraph is equivalent to forbidding  $H$  as a subgraph if and only if  $H$  is a complete graph. So Theorem 3 tells us that COLOURING is NP-complete for strongly  $H$ -free graphs if  $H$  is a complete graph. Golovach, Paulusma and Ries [46] extended this result. Let  $T_1, \dots, T_6$  be the trees displayed in Figure 5. For an integer  $p \geq 0$ , let  $T_2^p$  be the tree obtained from  $T_2$  after subdividing the edge  $st$   $p$  times; note that  $T_2^0 = T_2$ .



**Figure 5.** The trees  $T_1, \dots, T_6$ .

**Theorem 28.** *Let  $H$  be a graph. Then the following two statements hold:*

- (i) COLOURING is polynomial-time solvable for strongly  $H$ -free graphs if
  1.  $H$  is a forest with  $\Delta(H) \leq 3$  in which each connected component has at most one vertex of degree 3, or
  2.  $H$  is a forest with  $\Delta(H) \leq 4$  and  $|V(H)| \leq 7$ .
- (ii) Even 3-COLOURING is NP-complete for strongly  $H$ -free graphs if
  1.  $H$  contains a cycle, or
  2.  $\Delta(H) \geq 5$ , or
  3.  $H$  has a connected component with at least two vertices of degree 4, or
  4.  $H$  contains a subdivision of the tree  $T_1$  as a subgraph, or
  5.  $H$  contains the tree  $T_2^p$  as a subgraph for some  $0 \leq p \leq 9$ , or
  6.  $H$  contains one of the trees  $T_3, T_4, T_5, T_6$  as a subgraph.

Theorems 3 and 28 show that COLOURING behaves differently on  $H$ -free graphs and strongly  $H$ -free graphs. Theorem 28 implies the following classification for graphs  $H$  of at most seven vertices (also see [46]).

**Theorem 29.** *Let  $H$  be a graph. If  $|V(H)| \leq 7$ , then COLOURING is polynomial-time solvable on strongly  $H$ -free graphs if  $H$  is a forest of maximum degree at most 4, and NP-complete otherwise.*

The classification of PRECOLOURING EXTENSION for  $H$ -free graphs is still open. For LIST COLOURING, Golovach and Paulusma [45] gave a complete complexity classification even for graph classes defined by more than two forbidden subgraphs.

**Theorem 30.** *Let  $\{H_1, \dots, H_p\}$  be a finite set of graphs. Then LIST COLOURING is polynomial-time solvable for strongly  $(H_1, \dots, H_p)$ -free graphs if at least one of the  $H_i$ ,  $1 \leq i \leq p$ , is a forest of maximum degree at most 3, every connected component of which has at most one vertex of degree 3. In all other cases, even LIST 3-COLOURING is NP-complete for  $(H_1, \dots, H_p)$ -free graphs.*

Thus for strongly  $H$ -free graphs, we have the following:

**Open Problem 14** Complete the classification of the complexity of the problems COLOURING and PRECOLOURING EXTENSION for strongly  $H$ -free graphs.

We also note that the classifications of the complexity of the problems  $k$ -COLOURING and  $k$ -PRECOLOURING EXTENSION restricted to strongly  $H$ -free graphs have yet to be finished. In particular, it would be interesting to find out whether there exists a graph  $H$  such that for strongly  $H$ -free graphs 3-COLOURING is polynomial-time solvable but COLOURING is NP-complete.

We now consider graphs that are  $H$ -minor-free, that is, they do not contain some graph  $H$  as a minor. Robertson and Seymour showed that every class of  $H$ -minor-free graphs can be recognized in cubic time [104]. We present some results that will allow us to determine the complexity of colouring problems on  $H$ -minor-free graphs. The first is also by Robertson and Seymour [103].

**Theorem 31.** *Let  $H$  be any planar graph. Then the class of  $H$ -minor free graphs has bounded treewidth.*

The second result was proved by Jansen and Scheffler [70].

**Theorem 32.** *Let  $\mathcal{G}$  be a graph class of treewidth at most  $t$ . Then LIST COLOURING can be solved in time  $O(nk^{t+1})$  on a graph of  $\mathcal{G}$  with  $n$  vertices and a  $k$ -list assignment.*

The third and final result we need is from Garey, Johnson, and Stockmeyer [41].

**Theorem 33.** *3-COLOURING is NP-complete for planar graphs.*

In the next theorem, we present a dichotomy for  $H$ -minor-free graphs. The first statement follows from Theorems 31 and 32, and the second from Theorem 33 (after observing that the class of planar graphs is closed under taking minors).

**Theorem 34.** *Let  $H$  be a fixed graph. Then LIST COLOURING is polynomial-time solvable for  $H$ -minor-free graphs if  $H$  is planar. Even 3-COLOURING is NP-complete for  $H$ -minor-free graphs if  $H$  is non-planar.*

Let  $H$  be a graph. Then a graph is  $H$ -topological-minor-free if it does not contain  $H$  as a topological minor. Grohe, Kawarabayashi, Marx and Wollan showed that every class of  $H$ -topological-minor-free graphs can be recognized in cubic time [52].

By Theorem 33, and the fact that the class of planar graphs is also closed under taking topological minors, we see that 3-COLOURING is NP-complete for  $H$ -topological-minor-free graphs whenever  $H$  is a non-planar graph. For every graph  $H$ , the class of  $H$ -topological-minor-free graphs is a subclass of the class of strongly  $H$ -free graphs. Hence the analogue of Theorem 28:(i) for  $H$ -topological-minor-free graphs is true. However, assuming  $\mathbf{P} \neq \mathbf{NP}$ , we cannot have a dichotomy equivalent to that of Theorem 34; that is, the complexity of COLOURING for  $H$ -minor-free graphs and  $H$ -topological-minor-free graphs may be different. By Theorem 34, COLOURING is polynomial-time solvable for  $K_{1,5}$ -minor-free graphs. However, every graph of maximum degree at most 4 does not contain  $K_{1,5}$  as a topological minor, and even 3-COLOURING is NP-complete for graphs of maximum degree at most 4 according to Garey, Johnson, and Stockmeyer [41]. Similarly, the complexity of COLOURING for strongly  $H$ -free graphs and  $H$ -topological-minor-free graphs may be different as Theorem 28 (ii):1 and the following example show.

**Theorem 35.** *For all  $r \geq 3$ , COLOURING is polynomial-time solvable on  $C_r$ -topological-minor-free graphs.*

*Proof.* Let  $r \geq 3$ , and let  $G$  be a  $C_r$ -topological-minor-free graph. We may assume, without loss of generality, that  $G$  is 2-connected. Suppose that  $G$  contains a path  $P$  on  $r$  vertices. Because  $G$  is 2-connected, there exists another path  $P'$  between the end-vertices of  $P$  that is internally vertex-disjoint from  $P$  by Menger's Theorem. Then the subgraph of  $G$  induced by  $V(P) \cup V(P')$  contains a cycle on at least  $r$  vertices. Consequently,  $G$  contains  $C_r$  as a topological minor, which is not possible. Thus  $G$  is strongly  $P_r$ -free. We apply Theorem 28 (i):1.  $\square$

**Open Problem 15** Complete the classification of the complexity of COLOURING, PRECOLOURING EXTENSION and LIST COLOURING for  $H$ -topological-minor-free graphs.

It remains to consider CHOOSABILITY restricted to the graph classes considered in this section. Because strongly  $H$ -free graph classes are not closed under adding dominating vertices, we cannot just combine Theorem 28 (ii) with Theorem 10 but instead need some additional results. The first follows from a result of Bienstock, Robertson, Seymour and Thomas [5].

**Theorem 36.** Let  $H$  be a forest with  $\Delta(H) \leq 3$ , in which each connected component has at most one vertex of degree 3. Then every  $H$ -minor-free graph has pathwidth at most  $|V(H)| - 2$ .

The next result is from Fellows et al. [36].

**Theorem 37.** CHOOSABILITY can be solved in linear time for any graph class of bounded treewidth.

Theorems 36 and 37 imply the first statement of the following theorem after observing that a forest  $H$  in which each connected component is either a path or a subdivided claw is a subgraph of a graph  $G$  if and only if it is a minor of  $G$ . The second statement follows from Theorems 12 and 13.

**Theorem 38.** Let  $H$  be a graph. Then the following two statements hold:

- (i) CHOOSABILITY is linear-time solvable for strongly  $H$ -free graphs if  $H$  is a forest with  $\Delta(H) \leq 3$ , in which each connected component has at most one vertex of degree 3.
- (ii) Even 3-CHOOSABILITY is  $\Pi_2^p$ -hard for strongly  $H$ -free graphs if  $H$  is non-planar or contains an odd cycle.

We pose the following open problem.

**Open Problem 16** Complete the classification of the complexity of CHOOSABILITY for strongly  $H$ -free graphs.

When we consider  $H$ -minor-free graphs we obtain a full dichotomy result by using Theorem 31, Theorem 37 and Theorem 12 and recalling that the class of planar graphs is closed under taking minors.

**Theorem 39.** Let  $H$  be a fixed graph. Then CHOOSABILITY is linear-time solvable for  $H$ -minor-free graphs if  $H$  is planar, whereas even 3-CHOOSABILITY is  $\Pi_2^p$ -hard for  $H$ -minor-free graphs if  $H$  is non-planar.

By Theorem 12 again, and the fact that the class of planar graphs is also closed under taking topological minors, we have that 3-CHOOSABILITY is  $\Pi_2^p$ -hard for  $H$ -topological-minor-free graphs whenever  $H$  is non-planar. And as, for every graph  $H$ , the class of  $H$ -topological-minor-free graphs is a subclass of the class of strongly  $H$ -free graphs, the analogue of Theorem 38:(i) for  $H$ -topological-minor-free graphs holds.

**Open Problem 17** Complete the classification of the complexity of CHOOSABILITY for  $H$ -topological-minor-free graphs.

From Theorems 38 and 39, we see that the complexity of CHOOSABILITY for strongly  $H$ -free graphs and  $H$ -minor-free graphs may be different: for instance when  $H$  is an odd cycle. It would be interesting to determine whether there exists a graph  $H$  for which the complexity of CHOOSABILITY is different for strongly  $H$ -free graphs and  $H$ -topological-minor-free graphs, and whether there exists a graph  $H^*$  for which the complexity of CHOOSABILITY is different for  $H^*$ -minor-free graphs and  $H^*$ -topological-minor-free graphs.

*Acknowledgments.* We thank Konrad Dabrowski, François Dross, Oliver Schaudt and two anonymous reviewers for helpful comments.

## References

1. N. Alon, Restricted colorings of graphs, *Surveys in combinatorics*, London Mathematical Society Lecture Note Series 187 (1993) 1–33.
2. K. Appel and W. Haken, Every planar map is four colorable, *Contemporary Mathematics* 89, AMS Bookstore, 1989.
3. C. Arbib and R. Mosca, On  $(P_5, \text{diamond})$ -free graphs, *Discrete Mathematics* 250 (2002) 1–22.
4. E. Balas and C. S. Yu, On graphs with polynomially solvable maximum-weight clique problem, *Networks* 19 (1989) 247–253.
5. D. Bienstock, N. Robertson, P. D. Seymour, and R. Thomas, Quickly excluding a forest, *Journal of Combinatorial Theory, Series B* 52 (1991) 274–283.
6. F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein, and M. Zhong, Three-coloring and list three-coloring of graphs without induced paths on seven vertices, *Manuscript*.
7. F. Bonomo, G. Durán and J. Marenco, Exploring the complexity boundary between coloring and list-coloring, *Annals of Operations Research* 169 (2009) 3–16.
8. A. Brandstädt, K.K. Dabrowski, S. Huang and D. Paulusma. Bounding the clique-width of  $H$ -free chordal graphs, *Proc. MFCS 2015*, LNCS 9235, 139–150.
9. A. Brandstädt, T. Klemmt and S. Mahfud,  $P_6$ - and triangle-free graphs revisited: structure and bounded clique-width, *Discrete Mathematics & Theoretical Computer Science* 8 (2006) 173–188.
10. A. Brandstädt and D. Kratsch, On the structure of  $(P_5, \text{gem})$ -free graphs, *Discrete Applied Mathematics* 145 (2005) 155–166.
11. A. Brandstädt, H.-O. Le and R. Mosca, Gem- and co-gem-free graphs have bounded clique-width, *International Journal of Foundations of Computer Science* 15 (2004), 163–185.
12. S. Brandt, Triangle-free graphs and forbidden subgraphs, *Discrete Applied Mathematics* 120 (2002) 25–33.
13. C. Brause, I. Schiermeyer, P. Holub, Z. Ryjáček, P. Vrána and R. Krivoš-Belluš, 4-Colorability of  $P_6$ -free graphs, *Proc. EuroComb 2015*, ENDM 49 (2015) 37–42.
14. H.J. Broersma, F.V. Fomin, P.A. Golovach and D. Paulusma, Three complexity results on coloring  $P_k$ -free graphs, *European Journal of Combinatorics* 34 (2013) 609–619.
15. H.J. Broersma, P.A. Golovach, D. Paulusma and J. Song, Determining the chromatic number of triangle-free  $2P_3$ -free graphs in polynomial time, *Theoretical Computer Science* 423 (2012) 1–10.
16. H.J. Broersma, P.A. Golovach, D. Paulusma and J. Song, Updating the complexity status of coloring graphs without a fixed induced linear forest, *Theoretical Computer Science* 414 (2012) 9–19.
17. D. Bruce, C.T. Hoàng, and J. Sawada, A certifying algorithm for 3-colorability of  $P_5$ -free graphs, *Proc. ISAAC 2009*, LNCS 5878, 595–604.
18. M. Chudnovsky, Coloring graphs with forbidden induced subgraphs, *Proc. ICM 2014 vol IV*, 291–302.
19. M. Chudnovsky, J. Goedgebeur, O. Schaudt and M. Zhong, Obstructions for three-coloring graphs without induced paths on six vertices, *Proc. SODA 2016*, 1774–1783.
20. M. Chudnovsky, P. Maceli, J. Stacho and M. Zhong, 4-coloring  $P_6$ -free graphs with no induced 5-cycles, *Journal of Graph Theory*, to appear.
21. B. Courcelle and S. Olariu, Upper bounds to the clique width of graphs, *Discrete Applied Mathematics* 101 (2000) 77–144.
22. J.F. Couturier, P.A. Golovach, D. Kratsch and D. Paulusma, List coloring in the absence of a linear forest, *Algorithmica* 71 (2015) 21–35.
23. J.F. Couturier, P.A. Golovach, D. Kratsch and D. Paulusma, On the parameterized complexity of coloring graphs in the absence of linear forest, *J. Discrete Algorithms* 15 (2012) 56–62.
24. K.K. Dabrowski, F. Dross and D. Paulusma, Narrowing the gap in the clique-width dichotomy for  $(H_1, H_2)$ -free graphs, *Manuscript*, arXiv:1512.07849.
25. K.K. Dabrowski, P.A. Golovach and D. Paulusma, Colouring of graphs with Ramsey-type forbidden subgraphs, *Theoretical Computer Science* 522 (2014) 34–43.
26. K.K. Dabrowski, S. Huang and D. Paulusma, Bounding clique-width via perfect graphs, *Proc. LATA 2015*, LNCS 8977 (2015) 676–688.
27. K.K. Dabrowski, V. Lozin, R. Raman and B. Ries, Colouring vertices of triangle-free graphs without forests, *Discrete Mathematics* 312 (2012) 1372–1385.
28. K.K. Dabrowski and D. Paulusma, Clique-width of graph classes defined by two forbidden induced subgraphs, *The Computer Journal*, to appear.
29. H.S. Dhaliwal, A.M. Hamel, C.T. Hoàng, F. Maffray, T.J.D. McConnell and S.A. Panait, On color-critical  $(P_3, \overline{P_5})$ -free graphs, *Manuscript*, arXiv:1403.8027.

30. R. Diestel, Graph Theory. Springer-Verlag, Electronic Edition, 2005.
31. T. Emden-Weinert, S. Hougardy and B. Kreuter, Uniquely colourable graphs and the hardness of colouring graphs of large girth, *Combinatorics, Probability & Computing* 7 (1998) 375–386.
32. P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing* (Humboldt State Univ., Arcata, Calif., 1979), *Congress. Numer.*, XXVI, Winnipeg, Man., 1980, *Utilitas Math.*, pp. 125–157.
33. L. Esperet, A. Gyárfás and F. Maffray, List-coloring claw-free graphs with small clique number, *Graphs and Combinatorics* (2014)30 365-375.
34. L. Esperet, L. Lemoine, F. Maffray and G. Morel, The chromatic number of  $\{P_5, K_4\}$ -free graphs, *Discrete Mathematics* (2013) 313, 743–754.
35. G. Fan, B. Xu, T. Ye and X. Yu, Forbidden subgraphs and 3-colorings, *SIAM Journal on Discrete Mathematics* 28 (2014) 1226–1256.
36. M.R. Fellows, F.V. Fomin, D. Lokshtanov, F. Rosamond, S. Saurabh, S. Szeider and C. Thomassen, On the complexity of some colorful problems parameterized by treewidth, *Information and Computation* 209 (2011) 143–153.
37. D.J. Fraser, A.M. Hamel and C.T. Hoàng, A coloring algorithm for  $4K_1$ -free line graphs, *Manuscript*, arXiv:1506.05719.
38. T. Gallai, Maximum-minimum Sätze über Graphen, *Acta Math. Acad. Sci. Hungar.* 9 (1958) 395–434.
39. F. Galvin, The list chromatic index of a bipartite multigraph, *Journal of Combinatorial Theory, Series B* 63 (1995)153–158.
40. M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, San Fransisco (1979).
41. M.R. Garey, D.S. Johnson, and L.J. Stockmeyer, Some simplified NP-complete graph problems, *Proc. STOC 1974*, 47–63.
42. J. Goedgebeur and O. Schaudt, Exhaustive generation of  $k$ -critical  $H$ -free graphs, *Manuscript*, arXiv:1506.03647.
43. P.A. Golovach and P. Heggernes, Choosability of  $P_5$ -free graphs, *Proc. MFCS 2009, LNCS 5734* (2009) 82–391.
44. P.A. Golovach, P. Heggernes, P. van 't Hof and D. Paulusma, Choosability on  $H$ -free graphs, *Information Processing Letters* 113 (2013) 107–110.
45. P.A. Golovach and D. Paulusma, List coloring in the absence of two subgraphs, *Discrete Applied Mathematics* 166 (2014) 123–130.
46. P.A. Golovach, D. Paulusma and B. Ries, Coloring graphs characterized by a forbidden subgraph, *Discrete Applied Mathematics* 180 (2015) 101–110.
47. P.A. Golovach, D. Paulusma and J. Song, Coloring graphs without short cycles and long induced paths, *Discrete Applied Mathematics* 167 (2014) 107–120.
48. P.A. Golovach, D. Paulusma and J. Song, Closing complexity gaps for coloring problems on  $H$ -free graphs, *Information and Computation* 237 (2014) 204–21.
49. P.A. Golovach, D. Paulusma and J. Song, 4-Coloring  $H$ -free graphs when  $H$  is small, *Discrete Applied Mathematics* 161 (2013) 140–150.
50. M.C. Golumbic and U. Rotics, On the clique-width of some perfect graph classes, *International Journal of Foundations of Computer Science* 11 (2000) 423–443.
51. S. Gravier, C.T. Hoàng and F. Maffray, Coloring the hypergraph of maximal cliques of a graph with no long path, *Discrete Mathematics* 272 (2003) 285–290.
52. M. Grohe, K. Kawarabayashi, D. Marx, and P. Wollan, Finding topological subgraphs is fixed-parameter tractable, *Proc. STOC 2011*, 479–488.
53. M. Grötschel, L. Lovász, and A. Schrijver, Polynomial algorithms for perfect graphs. *Annals of Discrete Mathematics* 21 (1984) 325–356.
54. S. Gutner, The complexity of planar graph choosability, *Discrete Mathematics* 159 (1996) 119–130.
55. S. Gutner and M. Tarsi, Some results on  $(a:b)$ -choosability, *Discrete Mathematics* 309 (2009), 2260–2270.
56. A. Gyárfás, Problems from the world surrounding perfect graphs, *Zastosowania Matematyki Applicationes Mathematicae XIX*, 3–4 (1987) 413–441.
57. A. Gyárfás, E. Szemerédi and Zs. Tuza, Induced subtrees in graphs of large chromatic number, *Discrete Mathematics* 30 (1980) 235–244.
58. R. Häggkvist and A. Chetwynd, Some upper bounds on the total and list chromatic numbers of multigraphs, *Journal of Graph Theory* 16 (1992) 503–516.
59. P. Hell and S. Huang, Complexity of coloring graphs without paths and cycles, *Discrete Applied Mathematics*, to appear.

60. C.T. Hoàng, M. Kamiński, V. Lozin, J. Sawada, and X. Shu, Deciding  $k$ -colorability of  $P_5$ -free graphs in polynomial time, *Algorithmica* 57 (2010) 74–81.
61. C.T. Hoàng and D. A. Lazzarato, Polynomial-time algorithms for minimum weighted colorings of  $(P_5, \overline{P_5})$ -free graphs and similar graph classes, *Discrete Applied Mathematics* 186 (2015) 106–111.
62. C.T. Hoàng, F. Maffray and M. Mechebbek, A characterization of  $b$ -perfect graphs, *Journal of Graph Theory* 71 (2012) 95–122.
63. C.T. Hoàng, B. Moore, D. Recoskie, J. Sawada and M. Vatselle, Constructions of  $k$ -critical  $P_5$ -free graphs, *Discrete Applied Mathematics* 182 (2015) 91–98.
64. I. Holyer, The NP-completeness of edge-coloring, *SIAM J. Comput.* 10 (1981) 718–720.
65. S. Huang, Improved complexity results on  $k$ -coloring  $P_t$ -free graphs, *European Journal of Combinatorics* 51 (2016) 336–346.
66. S. Huang, M. Johnson and D. Paulusma, Narrowing the complexity gap for colouring  $(C_s, P_t)$ -Free Graphs, *The Computer Journal* 58 (2015) 3074–308.
67. M. Hujter and Zs. Tuza, Precoloring extension. II. Graph classes related to bipartite graphs, *Acta Math. Univ. Comenianae* Vol. LXII (1993) 1–11.
68. M. Hujter and Zs. Tuza, Precoloring extension. III. Classes of perfect graphs. *Combinatorics, Probability and Computing* 5 (1996) 35–56.
69. K. Jansen, Complexity Results for the Optimum Cost Chromatic Partition Problem, Universität Trier, *Mathematik/Informatik, Forschungsbericht* 96–41, 1996.
70. K. Jansen and P. Scheffler, Generalized coloring for tree-like graphs, *Discrete Applied Mathematics* 75 (1997) 135–155.
71. T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley Interscience, 1995.
72. M. Kamiński and V.V. Lozin, Coloring edges and vertices of graphs without short or long cycles, *Contributions to Discrete Mathematics* 2 (2007) 61–66.
73. M. Kamiński and V.V. Lozin, Vertex 3-colorability of Claw-free Graphs, *Algorithmic Operations Research* 2 (2007) 15–21.
74. R.M. Karp, Reducibility among combinatorial problems, In: *Complexity of Computer Computations* (1972) 85–103.
75. D. Kobler and U. Rotics, Edge dominating set and colorings on graphs with fixed clique-width, *Discrete Applied Mathematics* 126 (2003) 197–221.
76. D. Král', J. Kratochvíl, Zs. Tuza, and G.J. Woeginger, Complexity of coloring graphs without forbidden induced subgraphs, *Proc. WG 2001, LNCS 2204* (2001) 254–262.
77. J. Kratochvíl, Precoloring extension with fixed color bound. *Acta Mathematica Universitatis Comenianae* 62 (1993) 139–153.
78. J. Kratochvíl, Zs. Tuza and M. Voigt, New trends in the theory of graph colorings: choosability and list coloring, *Proc. DIMATIA-DIMACS Conference, DIMACS Series in Discrete Mathematics and Theoretical Computer Science* 49 (1999) 183–197.
79. V.B. Le, B. Randerath and I. Schiermeyer, On the complexity of 4-coloring graphs without long induced paths, *Theoretical Computer Science* 389 (2007) 330–335.
80. D. Leven and Z. Galil, NP completeness of finding the chromatic index of regular graphs, *Journal of Algorithms* 4 (1983) 35–44.
81. L. Lovász, Coverings and coloring of hypergraphs, *Proc. 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Utilitas Math.* (1973) 3–12.
82. V.V. Lozin and D.S. Malyshev, Vertex coloring of graphs with few obstructions, *Discrete Applied Mathematics*, to appear.
83. V.V. Lozin and C. Purcell, Coloring vertices of claw-free graphs in three colors, *Journal of Combinatorial Optimization* 28 (2014) 462–479.
84. V. V. Lozin and D. Rautenbach, Some results on graphs without long induced paths, *Information Processing Letters* 88 (2003) 167–171.
85. F. Maffray and G. Morel, On 3-Colorable  $P_5$ -free graphs, *SIAM Journal on Discrete Mathematics* 26 (2012) 1682–1708.
86. F. Maffray and L. Pastor, 4-Coloring  $(P_6, \text{bull})$ -free graphs, *Manuscript*, arXiv:1511.08911.
87. F. Maffray and M. Preissmann, On the NP-completeness of the  $k$ -colorability problem for triangle-free graphs, *Discrete Mathematics* 162 (1996) 313–317.
88. D.S. Malyshev, The coloring problem for classes with two small obstructions, *Optimization Letters* 8 (2014) 2261–2270.
89. D.S. Malyshev, The complexity of the 3-colorability problem in the absence of a pair of small forbidden induced subgraphs, *Discrete Mathematics* 338 (2015) 1860–1865.

90. D.S. Malyshev, Two cases of polynomial-time solvability for the coloring problem, *Journal of Combinatorial Optimization* 31 (2016) 833–845.
91. D.S. Malyshev and O.O. Lobanova, The coloring problem for  $\{P_5, \overline{P_5}\}$ -free graphs and  $\{P_5, K_p - e\}$ -free graphs is polynomial, Manuscript, arXiv:1503.02550.
92. R. M. McConnell, K. Mehlhorn, S. Näher and P. Schweitzer, Certifying algorithms, *Computer Science Review* 5 (2011) 119–161.
93. R. Niedermeier, Invitation to fixed-parameter algorithms, vol. 31 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2006.
94. S. Olariu, Paw-free graphs, *Information Processing Letters* 28 (1988) 53–54.
95. S.-I. Oum, Approximating rank-width and clique-width quickly, *ACM Transactions on Algorithms* 5 (2008).
96. D. Paulusma, Open problems on graph coloring for special graph classes, Proc. WG 2015, LNCS, to appear.
97. A.V. Pyatkin, Triangle-free  $2P_3$ -free graphs are 4-colorable, *Discrete Mathematics* 313 (2013) 715–720.
98. B. Randerath, 3-colorability and forbidden subgraphs. I., Characterizing pairs, *Discrete Mathematics* 276 (2004) 313–325.
99. B. Randerath and I. Schiermeyer, A note on Brook’s theorem for triangle-free graphs, *Australas. J. Combin.* 26 (2002) 3–9.
100. B. Randerath and I. Schiermeyer, 3-Colorability  $\in P$  for  $P_6$ -free graphs, *Discrete Applied Mathematics* 136 (2004) 299–313.
101. B. Randerath and I. Schiermeyer, Vertex colouring and forbidden subgraphs - a survey, *Graphs Combin.* 20 (2004) 1–40.
102. B. Randerath, I. Schiermeyer and M.Tewes, Three-colorability and forbidden subgraphs. II: polynomial algorithms, *Discret Mathematics* 251 (2002) 137–153.
103. N. Robertson and P.D. Seymour, Graph minors V. Excluding a planar graph, *Journal of Combinatorial Theory, Series B* 41 (1986) 92–114.
104. N. Robertson and P.D. Seymour, Graph minors. XIII. The disjoint paths problem. *Journal of Combinatorial Theory, Series B* 63 (1995) 65–110.
105. D. Schindl, Some new hereditary classes where graph coloring remains NP-hard, *Discrete Mathematics* 295 (2005) 197–202.
106. D.P. Sumner, Subtrees of a graph and the chromatic number, *Proceedings of the 4th Int. Conf. on Theory and Applications of Graphs* (1980) 557–576.
107. S. Tsukiyama, M. Ide, H. Ariyoshi and I. Shirakawa, A new algorithm for generating all the maximal independent sets, *SIAM Journal on Computing* 6 (1977) 505–517.
108. Zs. Tuza, Graph colorings with local restrictions - a survey, *Discuss. Math. Graph Theory* 17 (1997) 161–228.
109. V.G. Vizing, Coloring the vertices of a graph in prescribed colors, in *Diskret. Analiz.*, no. 29, *Metody Diskret. Anal. v. Teorii Kodov i Shem* 101 (1976) 3–10.
110. S. Wagon, A bound on the chromatic number of graphs without certain induced subgraphs, *Journal of Combinatorial Theory, Series B* 29 (1980) 345–346.
111. X. Wang and B. Wu, Upper bounds on the chromatic number of triangle-free graphs with a forbidden subtree, *Journal of Combinatorial Optimization*, to appear.
112. G.J. Woeginger and J. Sgall, The complexity of coloring graphs without long induced paths, *Acta Cybernetica* 15 (2001) 107–117.

## Appendix

Three of the known results mentioned in our survey are not made explicit in the literature (as at the time the focus was more on  $k$ -COLOURING and  $k$ -PRECOLOURING EXTENSION for  $H$ -free graphs than on LIST  $k$ -COLOURING). For completeness, we give the proofs of these three results here.

The first two theorems translate statements from [14, 16] for 3-PRECOLOURING EXTENSION into statements for LIST 3-COLOURING. As we show these results are implicit in [14, 16] or follow immediately from the proof methods used therein.

**Theorem 40.** *Let  $H$  be a graph. If LIST 3-COLOURING is polynomial-time solvable for  $H$ -free graphs, then it is also polynomial-time solvable for  $(P_1 + H)$ -free graphs.*

*Proof.* This result can be proven by using the same arguments as the ones that Broersma et al. [16] used for proving that 3-PRECOLOURING EXTENSION is polynomial-time solvable. Let  $G$  be an  $(H + P_1)$ -free graph with a 3-list assignment  $L$ . If  $G$  is  $H$ -free, we are done. Suppose  $G$  contains an induced subgraph  $H'$  that is isomorphic to  $H$ . Because  $G$  is  $(H + P_1)$ -free, every vertex in  $V(G) \setminus V(H')$  must be adjacent to a vertex in  $H'$ . We guess a colouring of  $V(H')$  that respects the lists. Afterwards we apply Theorem 1 (ii). Since  $H'$  has a fixed size, the number of guesses is polynomially bounded.  $\square$

**Theorem 41.** *For every integer  $s \geq 1$ , LIST 3-COLOURING is polynomial-time solvable on  $sP_3$ -free graphs.*

*Proof.* Theorem 6 of Broersma et al. [16] states that 3-PRECOLOURING EXTENSION can be solved in polynomial time on  $sP_3$ -graphs for any fixed  $s \geq 1$ . In the proof of this theorem a polynomial-time algorithm is presented that takes as input a graph  $G = (V, E)$  and a set of precoloured vertices  $W \subseteq V$ . We can copy the proof when the input is a graph and a 3-list assignment after defining  $W$  to be the set of all vertices with a list of at most 2 admissible colours.  $\square$

The last result we prove in this appendix is Theorem 10, which has been shown implicitly by Golovach et al. [44]. The proof below is only a slight adjustment of their original proof.

**Theorem 10.** *Let  $\mathcal{G}$  be a graph class that is closed under adding dominating vertices. If COLOURING is NP-hard for  $\mathcal{G}$ , then CHOOSABILITY is NP-hard for  $\mathcal{G}$ .*

*Proof.* Let  $\mathcal{G}$  be a graph class that is closed under adding dominating vertices, for which COLOURING is NP-complete. Consider an instance  $(G, k)$  of COLOURING where  $G$  belongs to  $\mathcal{G}$  and  $k \geq 1$  is an integer. We may assume without loss of generality that  $\deg_G(u) \geq k$  for all  $u \in V(G)$ , as otherwise we add dominating vertices to  $G$  and increase  $k$  accordingly, in order to obtain a pair  $(G', k')$  such that  $G'$  is  $k'$ -colourable if and only if  $G$  is  $k$ -colourable, and by the definition of  $\mathcal{G}$ ,  $G'$  would belong to  $\mathcal{G}$  as well.

We now define  $k^* = k + \sum_{u \in V(G)} (\deg_G(u) - k + 1)$  and construct a graph  $G^*$  from  $G$  by adding a set of  $k^* - k$  vertices  $T = \{t_1, \dots, t_{k^* - k}\}$  that are adjacent to each other and to every vertex of  $G$ . By the definition of  $\mathcal{G}$ , we derive that  $G^*$  belongs to  $\mathcal{G}$ . We prove that  $G$  is  $k$ -colourable if and only if  $G^*$  is  $k^*$ -choosable.

First suppose that  $G^*$  is  $k^*$ -choosable. Then  $G^*$  has a colouring  $c$  that respects the list assignment  $\mathcal{L}^* = \{L^*(u) \mid u \in V(G^*)\}$  with  $L^*(u) = \{1, \dots, k^*\}$  for all  $u \in V(G^*)$ . Because the  $k^* - k$  vertices in  $T$  are mutually adjacent, they are all coloured differently by  $c$ . Moreover, because every vertex of  $T$  is adjacent to every vertex of  $G$ , no vertex in  $G$  has the same colour as a vertex in  $T$ . Hence, by taking the restriction of  $c$  to  $V(G)$ , we find that  $G$  is  $k$ -colourable.

Now suppose that  $G$  is  $k$ -colourable. We prove that  $G^*$  is  $k^*$ -choosable. In order to do this, let  $\mathcal{L}^* = \{L^*(u) \mid u \in V(G^*)\}$  be an arbitrary  $k^*$ -list assignment of  $G^*$ . We will construct a colouring of  $G^*$  that respects  $\mathcal{L}^*$ . We start by colouring the vertices of  $T$  and, if possible, reducing  $G^*$  by applying the following procedure:

1. As long as there is an uncoloured vertex  $t_j \in T$  such that  $L^*(t_j)$  contains an unused colour  $x$  and there is a vertex  $u \in V(G)$  with  $x \notin L^*(u)$ , do as follows: give  $t_j$  colour  $x$  and delete all vertices  $u \in V(G)$  for which at least  $\deg_G(u) - k + 1$  used colours are not in  $L^*(u)$ .
2. Afterwards, consider the vertices of the remaining set  $T' \subseteq T$  one by one and give them any unused colour from their list.

It is possible to colour all vertices of  $T$  by this procedure, because  $|L^*(t_j)| = k^*$  for  $j = 1, \dots, k^* - k$  and  $|T| = k^* - k \leq k^*$ . We must show that the procedure is correct. Let  $u \in V(G)$ . After colouring all vertices of  $T$  we can partition  $T$  into two sets  $A_u$  and  $B_u$ , where  $A_u$  consists of those vertices of  $T$  that received a colour not in  $L^*(u)$  and  $B_u = T \setminus A_u$  consists of those vertices of  $T$  that received a colour from  $L^*(u)$ . Then the number of available colours for  $u$  is



$k^* - |B_u| = k^* - (|T| - |A_u|) = k^* - (k^* - k - |A_u|) = k + |A_u|$ , whereas  $u$  still has  $\deg_G(u)$  uncoloured neighbours in  $G^*$ . If  $k + |A_u| \geq \deg_G(u) + 1$ , or equivalently, if  $|A_u| \geq \deg_G(u) - k + 1$ , then we may delete  $u$ ; after colouring all vertices of  $V(G^*) \setminus \{u\}$ , we are guaranteed that there exists at least one colour in  $L^*(u)$  that is not used on the neighbourhood of  $u$  in  $G^*$ , and we can give  $u$  this colour.

After colouring the vertices in  $T$  as described above, we let  $U$  denote the subset of vertices of  $V(G)$  that were not deleted while colouring  $T$ . Recall the set  $T'$  defined in the procedure. We distinguish two cases.

First suppose  $T' = \emptyset$ . Then every  $t \in T$  received a colour that does not appear in the list  $L^*(u)$  for at least one vertex  $u \in V(G)$  that was not yet deleted from the graph at the moment  $t$  was coloured. Consequently, the size of some set  $A_u$  increases by 1 whenever a vertex of  $T$  receives a colour. Recall that a vertex  $u \in U$  is deleted from the graph as soon as the size of  $A_u$  reaches  $\deg_G(u) - k + 1$ . Since  $|T| = k^* - k = \sum_{u \in V(G)} (\deg_G(u) - k + 1)$ , every vertex of  $V(G)$  is deleted from the graph at some point during the procedure. Hence  $U = \emptyset$ , implying that  $G^*$  is  $k^*$ -choosable due to the correctness of our procedure.

Now suppose  $T' \neq \emptyset$  and let  $t' \in T'$ . Because  $|L^*(t')| = k^*$  and  $|T| = k^* - k$ , the list  $L^*(t')$  contains a set  $D$  of  $k$  colours that are not used as a colour for any vertex in  $T$  (including  $t'$  itself). We will show that  $D \subseteq L^*(u)$  for every  $u \in U$ . For contradiction, suppose there exists a colour  $y \in D$  and a vertex  $w \in U$  such that  $y \notin L^*(w)$ . By the definition of  $T'$ , vertex  $t'$  received a colour  $z$  that appears in the list  $L^*(u)$  for every  $u \in U$ . But according to our procedure, we would not have coloured  $t'$  with colour  $z$  if colour  $y$  was also available; note that  $y$  is not used to colour any vertex in  $T \setminus \{t'\}$  by the definition of  $D$ . This yields the desired contradiction, implying that  $D \subseteq L^*(u)$  for every  $u \in U$ . By symmetry of the colours, we may assume that  $D = \{1, \dots, k\}$ . We assumed that  $G$  is  $k$ -colourable, so  $G$  has a colouring  $c : V(G) \rightarrow \{1, \dots, k\}$ , and we can safely assign colour  $c(u)$  to each  $u \in U$ . Due to this and the correctness of our procedure, we conclude that  $G^*$  is also  $k$ -choosable when  $T' \neq \emptyset$ .  $\square$