Corrupt Bookmaking in a Fixed Odds Illegal Betting Market*

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Abstract

A problem of underground betting in a two-team sports contest is studied with player sabotage instigated by a monopolist bookmaker. Whereas punters hold beliefs about the teams' winning chances correlated with Nature's draw, the bookmaker observes this information noise-free. The enforcement authority investigates potential match-fixing with a higher probability, the greater the upset in the contest outcome. In such an environment, if punters do not suspect match-fixing, contests will often be fixed by targeting the favourite, thus creating upsets and intensifying subsequent investigations. The match-fixing result continues to hold even when punters are rational, provided that the bookie's beliefs are noisy: the bookie may resort to fixing by bribing the team he thinks is the favourite, and the bettors still bet on their respective perceived favourites.

***JEL* Classification:** D42, K42. **Key Words:** Illegal (sports) betting, bookie, punters, match-fixing, sabotage, correlated beliefs, rational bettors, big upsets.

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1 Introduction

Deliberate underperformance can occur in any contest for a variety of reasons. In many professional sports, such as football, cricket, tennis, snooker and horse racing, alleged underperformance under the influence of unscrupulous bettors is a common occurrence.[1] Haberfeld and Sheehan (2013) present a rich ensemble of match-fixing studies from Europe, mostly surrounding football. See also Forrest and Simmons (2003) for cricket's centuries-old association with betting, Hill (2009) for some accounts of the fixers' modus operandi, and Preston and Szymansky (2003) for an insightful discussion of match-fixing and other possible cheating instances.

In legal betting markets, corrupt influence may be spotted by bookmakers or the gambling regulatory authority. However, in countries where gambling is illegal but people still gamble, the bookmakers themselves can try to manipulate the outcome of a contest. Some of the spot-fixing controversies in India's high profile cricket league, IPL, suggest that underground bookmakers themselves were involved in fixing (see Hawkins, 2013). News headlines such as "Football's authorities fighting $1 trillion crime wave powered by illegal betting markets in Asia" suggest that illegal gambling-related corruption is too big an issue to be ignored.[2]

For horse races, Shin (1991, 1992) modelled the problem of insider betting in fixed odds markets under monopoly and competitive bookmaking.[3] However, insider betting has more to do with using privileged information rather than exerting influence to alter the outcome of a contest, which match-fixing is all about. Extending Shin's framework, Bag and Saha (2011, 2016) modelled match-fixing under competition and monopoly, respectively. In both papers, the bookmakers are honest, and their pricing strategy recognizes the threat of match-fixing coming from an anonymous punter.[4] The nature of the market equilibrium and the admissibility of match-fixing were the key focus in those two papers.

In this paper, we consider an environment where betting is organised through a secretive network due to legal prohibition on gambling. The secret network allows the bookmaker to not only enjoy sufficient market power, a monopoly setting, but also exert corrupt influence

[1] BBC online news have many such reports – http://news.bbc.co.uk/1/hi/programmes/panorama/2290356.stm (horse races). A recent (10 October 2016) headline, "Daniel Garza: Mexican tennis player banned for match-fixing offence," and similar sporting corruption news abound: http://www.bbc.co.uk/sport/tennis/37698466.

Formal studies of sports corruption include those by Wolfers (2006), and Duggan and Levitt (2002). Strumpf (2003) and Winter and Kukuk (2008) study the betting markets and discuss how betting odds may be affected if illegal activities have occurred.

[2] http://www.telegraph.co.uk/sport/football/international/9848868/Footballs-authorities-fighting-1trillion-crime-wave-powered-by-illegal-betting-markets-in-Asia.html

[3] See also Glosten and Milgrom (1985) for insider trading in financial markets and Ottaviani and Sorensen (2005, 2008) for analysis of parimutuel betting markets.

[4] Konrad (2000) presents a theoretical analysis of sabotage in general contests with no reference to betting.

on a contest if he so wishes. The contest in question is a two-team sports match, with the bookmaker offering fixed-odds bets on either team's win. We ask the following: Would the monopolist stay honest or resort to corrupt bookmaking, and if he turns corrupt, what type of contest is he likely to fix?

To answer the above question we consider two models. In our first model, the bookmaker has the precise knowledge of each team's probability of winning, as drawn by Nature, and the bettors' beliefs are noisy but correlated with the true probability. The correlation is such that the bettors' beliefs are distributed within a band around Nature's draw and that the average bettor's belief is exactly equal to Nature's draw. The bookie has links to corrupt players of either team, of which the bettors are completely unaware. Thus, the bettors in this model are naive. They stubbornly hold on to their initial beliefs. We will refer to this model as the *correlated beliefs* model.

In our second model, both the bettors and the bookie observe Nature's draw with identically distributed noise, which is modelled as a binary discrete signal. The bookie's links with corrupt players are not certain; he can access them only with some probability. More importantly, the bettors in this model are rational in the sense that (i) they are aware of potential match-fixing, (ii) they know that the bookie's signal is noisy, and (iii) they update their belief of a team's winning chance using every available information including the prices. We will refer to this formulation as the *rational bettors* model.

Previously, Shin (1991, 1992) and Bag and Saha (2011, 2016) used mainly naive bettors, whose beliefs were uncorrelated with Nature's draw but the bookie was honest. Bag and Saha (2016) also studied strategic bettors who recognize the possibility that the bookmaker could be indirectly complicit in match-fixing. In this paper, there are two distinct features compared to the studies cited above. First, the bookie is directly involved in corruption. Second, the bettors' beliefs are not uncorrelated with Nature's draw. Bettors still can be naive (as in our first model) or fully rational (as in our second model). These features allow us to study the problem in a more general environment.

■ **Results and intuitions.** In our correlated beliefs formulation, because the bettors' beliefs are distributed within a band around Nature's draw, there are always some optimistic bettors about one or the other team's prospect, and the bookie can trade with them honestly. However, his expected profit is likely to be low because the bettors' beliefs are never too far away from Nature's draw.

Profit can significantly improve if the bookie can bribe a team and fix the match, i.e., reduce its probability of win and in turn widen the gap between his own belief and the optimistic bettors' beliefs. Then, by manipulating the prices, he can induce more bettors to bet on the bribed team and increase his profit far above the honest bookmaking level.

Match-fixing is even more rewarding when the target team is the favourite because then the reduction in the probability of the bribed team's win is proportionately greater, so much so that almost all bettors, including the most pessimistic ones, can be induced to bet on the bribed team.

In the rational bettors' formulation, our main objective is to show that match-fixing can be an equilibrium phenomenon, even if bettors are fully rational and the bookie and the bettors have strategic interactions. We construct an equilibrium in which the bookie's optimal strategy is to induce bets with confounding possibilities in the bettors' minds. The bettors would reason, after seeing a cheap bet, that either (i) the bookie has observed a more pessimistic signal about the concerned team or (ii) the team has been bribed. The bookie bribes his perceived favourite, if he has obtained access to it, but does not fully reveal through his prices whether he indeed obtained the access. In response the bettors bet on their perceived favourite, which can be divergent due to their independent signals. Some bettors would hope that the match has not been fixed, but others would wish the opposite. However, we do not address the question of the optimality of match-fixing; nor do we formally analyse the equilibrium under honest bookmaking. We restrict our task to showing that match-fixing and rational betting can be compatible.

One key assumption of this paper is that investigation of match-fixing is *exogenous*, although with different degrees of sensitivity. In the model with unsuspecting bettors, the probability of investigation is increasing in the degree of upset in the contest outcome. In the rational bettors model, the investigation probability is fixed. This approach is simple, but one can extend the model to endogenous enforcement. In this regard, our analysis may still be relevant for an important enforcement question: Should enforcement be *outcome dependent*, say, investigate only if there is a big upset? A recent work by Chassang and Miquel (2013) on principal-agent with a whistleblower monitor reporting agent-corruption has argued that enforcement should not be too sensitive to whistleblower's report because that would encourage the agent to retaliate against the whistleblower. In our context, in the absence of a whistleblower, that may not be the case; making investigation more likely for a bigger upset should weaken the incentive for corruption, thus supporting conventional wisdom on monitoring.

In Section 2, we present the model. The main analysis appears in Sections 3 and 4. Section 5 explains how our model assumptions can be relaxed before discussing some policy implications. An appendix contains several proofs. A supplementary file reports the formal argument behind Proposition 5, the main result in Section 4.

2 The Model

The most likely scenario of our model is the one where gambling is illegal and betting is organised through a secretive and personalized network, thus giving rise to a captive (i.e., monopoly) market. The sole bookmaker, called the bookie, sets odds on each of two teams winning a contest, a sports match for example. Odds setting is equivalent to setting the prices of two tickets; ticket i with price π_i yields a dollar if team i wins the contest and yields nothing if team i loses. The match being drawn is not a possibility (by assumption). Later on, we discuss a likely modification of our model if draw is permitted.

In the absence of any external influence, the probability that team 1 will win, as drawn by Nature, is p_1. This probability is precisely known to the bookie, the competing players and the enforcement authority.[5]

Both teams have some corrupt players who are willing to underperform for secret monetary gains. The bookmaker has links with the corrupt players and may bribe a team of his choice to reduce its probability of winning from p_i to λp_i, $0 \leq \lambda < 1$, where λ is exogenously given.[6] After fixing the match, the bookie posts the prices, following which the punters bet.

There are a continuum of punters of mass 1, who do not suspect any foul play and go by their private signal or belief q of team 1's winning probability. The private signal is drawn from an interval $[p_1 - \delta, p_1 + \delta]$ where $0 < \delta < 1/2$. The punters' beliefs, although imprecise, are correlated with Nature's draw. Both Individually and collectively, punters have <u>one</u> dollar to bet.

We assume the distribution of q to be uniform; that is, its density function is $\frac{1}{2\delta}$ over the interval $[p_1 - \delta, p_1 + \delta]$, yielding the mean belief precisely p_1, i.e., on average the market is accurate. However, for consistency, this density function applies only to the interval $\delta \leq p_1 \leq 1 - \delta$. For p_1 outside this interval, the density function is modified. For $p_1 < \delta$, the support is $[0, p_1 + \delta)$ and the density is $1/(p_1 + \delta)$, and for $p_1 > 1 - \delta$ they are $(p_1 - \delta, 1]$ and $1/(p_2 + \delta)$, respectively.

Throughout, we prevent 'free money' by imposing the following *Dutch-book restriction*, which rules out splitting the total wager on two bets to ensure a certain win:

ASSUMPTION 1. *The bookie must choose prices* $0 \leq \pi_1, \pi_2 \leq 1$ *such that* $\pi_1 + \pi_2 \geq 1$.

■ **Enforcement and penalties.** An enforcement authority is aware of likely illegal sports betting and match-fixing. It acts on random tip-off that may be received with a fixed

[5] By assumption, the bookie is skilful in predicting the match outcome, as noted by Levitt (2004).

[6] λ can be made endogenous by allowing sabotage to be sensitive to the bribe amount. For a cleaner treatment, we do not take this approach.

probability $0 < \alpha_0 < 1$. The enforcement then imposes an *ex post* penalty s'_B on the bookie (a fine or prison term).[7] We denote $\alpha_0 s'_B$ by s. For match-fixing and bribery, we consider a more responsive enforcement (still exogenous) as follows:

Following the sporting contest, the authority investigates only the losing team i *with probability* $\alpha_i(p_1)$*, where* $\alpha'_1(p_1) \geq 0$ *and* $\alpha'_2(p_1) \leq 0$*. Upon investigation, match-fixing and bribery are uncovered with certainty.*

In fact, we will use the following linear form for the $\alpha_i(p_1)$ function:

Assumption 2. $\alpha_1(p_1) = \underline{\alpha} + \gamma p_1 \leq 1$ *and* $\alpha_2(p_1) = \underline{\alpha} + (1 - p_1)\gamma \leq 1$*, where* $\underline{\alpha} \geq 0$ *and* $\gamma \geq 0$*.*

Further, given the underground nature of betting, it is not very plausible to let $\alpha_i(.)$ depend on betting odds or λ. Later, we comment on making $\alpha_i(.)$ sensitive to betting odds.

After match-fixing is uncovered, both the bookie and the participating players are fined. The players' fine is s_P. The bookie's fine includes a specific fine for match-fixing f and also s'_B, if the illegal act of organising betting has not been uncovered separately. Thus, the expected fine of the bookie from the match-fixing investigation is $s_B = f + (1 - \alpha_0)s'_B$. This leads to an expected overall penalty of organising betting and match-fixing (by team i) as

$$(1 - \lambda p_i)\alpha_i(p_1)[f + (1 - \alpha_0)s'_B] + \alpha_0 s'_B \equiv (1 - \lambda p_i)\alpha_i(p_1)s_B + s.$$

■ **Bribery.** There are several issues regarding bribery. First, how should the match-fixing agreement be enforced? Second, should bribe be paid beforehand or afterwards? Third, how much bribe is to be paid? To address these issues, we take the approach of paying the bribe *ex post*, only if the contacted team loses. The amount to be paid is agreed *ex ante*, and we assume that the bookie can commit to honour his promise. The amount of bribe is decided by the bookie as a first-and-final offer. Later, we show that to a large extent, our qualitative results will survive, even if we allow bargaining or if the bribe is paid beforehand.

The arrangement of paying the bribe *ex post* makes the underperformance incentive compatible for a corrupt player. Without loss of generality, consider the case of bribing team 1. By underperforming and losing, the corrupt player receives a bribe B but forgoes a prize money w, and with probability $\alpha_1(p_1)$, he will have to pay s_P as a fine. In contrast, by not underperforming (i.e., by cheating on the bookie), he will receive w with probability p_1 and B with probability $(1 - p_1)$ with some risk of detection. Underperformance is incentive

[7] Alternative modes of discovery are also possible – a raid or surveillance by the enforcement authority. We do not consider them. In any case, we do not model the authority's enforcement decision.

compatible if

$$\begin{aligned}(1-\lambda p_1)[B-\alpha_1(p_1)s_P]+\lambda p_1 w &\geq (1-p_1)[B-\alpha_1(p_1)s_P]+p_1 w\\ \text{or,} \qquad B &\geq w+\alpha_1(p_1)s_P.\end{aligned}$$

We set $B_1 = w + \alpha_1(p_1)s_P$ as bribe for team 1. Likewise, $B_2 = w + \alpha_2(p_1)s_P$ for team 2.

■ **Bribery game Γ.** We now describe the bribery game.

Stage 1. Nature draws p_1 and reveals it to the bookie, the players and the enforcement authority; and the punters draw their respective private signals q, which is correlated to p_1.

Stage 2. The bookie decides whether to engage in match-fixing.

Stage 3. Prices (π_1, π_2) are set for the tickets on respective team's win; $0 \leq \pi_1, \pi_2 \leq 1$.

Stage 4. Punters place bets according to their beliefs. The match is played out according to the winning probabilities $(p_1, 1-p_1)$ or $(\lambda p_1, 1-\lambda p_1)$ (where team 1 is bribed), or $(1-\lambda p_2, \lambda p_2)$ (where team 2 is bribed) and the match outcome is realised.

Stage 5. Finally, the enforcement authority follows its investigation policy. ||

For comparison with a different formulation in Section 4, we also draw the following time line of the game Γ:

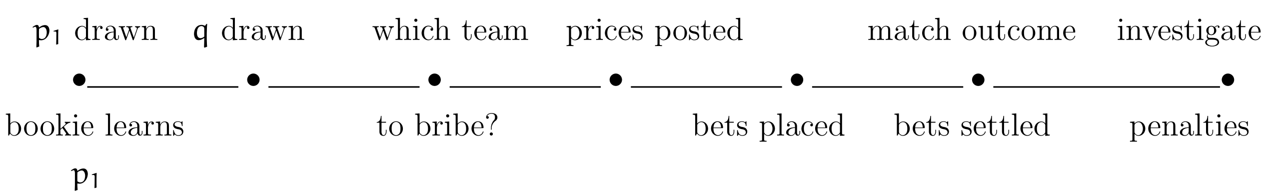

Figure 1: Time line

3 To Bribe or not to Bribe?

We now analyse the bookie's decision problem regarding which team to bribe, which contests to fix and what prices to set. In setting the prices, the bookie must consider the punters' betting behaviour.

Punters' betting decision. The risk-neutral punters maximise their expected return by the following rule: Bet on team 1 if and only if $\frac{q}{\pi_1} \geq \max\{\frac{1-q}{\pi_2}, 1\}$; bet on team 2 if and only if $\frac{1-q}{\pi_2} \geq \max\{\frac{q}{\pi_1}, 1\}$.

To elaborate, when prices (π_1, π_2) are such that $\pi_1 \leq p_1 + \delta$ and $1 - \pi_2 \geq p_1 - \delta$, punters with $q \in [\pi_1, p_1 + \delta]$ bet on team 1, punters with $q \in [p_1 - \delta, 1 - \pi_2]$ bet on team 2, and the remaining punters whose beliefs fall between $1 - \pi_2$ and π_1 do not bet (this interval is non-empty due to Assumption 1).

Combining the punters' betting rule and the distribution of their beliefs, we write the market shares or the mass of bettors betting on team 1 and team 2, respectively, as

$$n_1 = \begin{cases} \frac{p_1+\delta-\pi_1}{p_1+\delta}, & \forall p_1 \in [0, \delta) \\ \frac{p_1+\delta-\pi_1}{2\delta}, & \forall p_1 \in [\delta, 1-\delta] \\ \frac{1-\pi_1}{1-p_1+\delta}, & \forall p_1 \in (1-\delta, 1], \end{cases} \quad n_2 = \begin{cases} \frac{1-\pi_2}{p_1+\delta}, & \forall p_1 \in [0, \delta) \\ \frac{1-\pi_2-p_1+\delta}{2\delta}, & \forall p_1 \in [\delta, 1-\delta] \\ \frac{1-\pi_2-p_1+\delta}{1-p_1+\delta}, & \forall p_1 \in (1-\delta, 1]. \end{cases} \tag{1}$$

■ **Honest bookmaking.** We first consider honest bookmaking. The bookie solves the following profit maximisation problem:

$$\max_{\pi_1, \pi_2} E\Pi \equiv E\Pi_0 - \alpha_0 s_B' = \left[n_1(\pi_1)(1 - \frac{p_1}{\pi_1}) + n_2(\pi_2)(1 - \frac{p_2}{\pi_2}) \right] - s \tag{2}$$

subject to

$$\begin{aligned} \max\{0, p_1 - \delta\} &\leq \pi_1 \leq \min\{p_1 + \delta, 1\}, \\ \max\{0, p_1 - \delta\} &\leq 1 - \pi_2 \leq \min\{p_1 + \delta, 1\}, \\ \pi_1 + \pi_2 &\geq 1. \end{aligned} \tag{3}$$

The first constraint in (3) acknowledges that π_1 should not be set below $p_1 - \delta$ because that would be wasteful. At the same time, π_1 must not be set above $p_1 + \delta$ for the market share of the bets on team 1 to be positive. The second constraint applies the same logic to π_2 vis-à-vis the market share of the bets on team 2. The third one is the Dutch-book restriction.

While we assume $0 < \delta < 1/2$, it will be helpful to restrict δ below a critical level $\hat{\delta}$ so the bookie's profit curve (as derived in the proof of Proposition 1) will be well-behaved at all p_1, including $p_1 \leq \delta$ and $p_1 \geq 1 - \delta$.

ASSUMPTION 3. $0 < \delta < \hat{\delta} = 0.376$, *where* $\hat{\delta}$ *solves* $2(\sqrt{2} - 1) - (2\sqrt{2} - 1)\delta - \delta^2 = 0$.

The solution to the bookie's problem is given in Proposition 1. It suggests that the optimal prices will generally be unconstrainted at all p_1, except at very high and very low values. Furthermore, of the two bounds on the two prices (i.e., the first two constraints of (3)), only the upper bound may bind on π_1, and only the lower bound may bind on $1 - \pi_2$.

PROPOSITION 1. *The optimal prices under honest bookmaking are*

$$\pi_1^0 = \begin{cases} \sqrt{p_1(p_1+\delta)}, & \forall p_1 \in [0, 1-\delta) \\ \sqrt{p_1}, & \forall p_1 \in [1-\delta, 1], \end{cases} \quad \pi_2^0 = \begin{cases} \sqrt{p_2}, & \forall p_1 \in [0, \delta) \\ \sqrt{p_2(p_2+\delta)} & \forall p_1 \in [\delta, 1], \end{cases} \tag{4}$$

resulting in the following profit:

$$E\Pi_0 = \begin{cases} \frac{1}{p_1+\delta}\left[2+p_1+\delta-2\sqrt{p_1(p_1+\delta)}-2\sqrt{p_2}\right]-s & \textit{for } p_1 < \delta \\ \frac{1}{\delta}\left[1+\delta-\sqrt{p_1(p_1+\delta)}-\sqrt{p_2(p_2+\delta)}\right]-s & \textit{for } \delta \le p_1 \le 1-\delta \\ \frac{1}{p_2+\delta}\left[2+p_2+\delta-2\sqrt{p_1}-2\sqrt{p_2(p_2+\delta)}\right]-s & \textit{for } 1-\delta < p_1. \end{cases} \tag{5}$$

$E\Pi_0$ *is symmetric and U-shaped with its minimum occurring at* $p_1 = 1/2$.

The explanation for the U-shaped profit curve is that when the contests are very uneven, the bookie offers the bettors a low return on the favourite and a high return on the longshot. However, because the probability of paying out on the longshot is also low, the overall expected profit is dictated by the low return on the favourite. This explains the increasing segment of $E\Pi_b$ for $p_1 > 1/2$. In contrast, when the contests are near-even, almost equal bets are placed on both sides, with comparable chances of paying out on either of them. Hence, profit falls to a minimum.

While retaining the same shape as that of Shin (1991), our profit curve will be flatter in the middle, specifically at $\delta \le p_1 \le 1-\delta$, but steeper elsewhere.

■ **Match-fixing.** Without loss of generality, consider the case of bribing team 1. Upon bribery, p_1 is secretly reduced to λp_1. The bettors do not suspect foul play, and their betting rule remains unchanged. The bookie determines the optimal prices by maximizing the following objective function subject to the constraints in (3):

$$E\Pi_b = y\left[n_1(\pi_1)(1-\frac{\lambda p_1}{\pi_1}) + n_2(\pi_2)(1-\frac{1-\lambda p_1}{\pi_2})\right] - (1-\lambda p_1)c(p_1) - s, \tag{6}$$

where $c(p_1) = B_1 + \alpha_1 s_B = w + \alpha_1(p_1)(s_P + s_B)$.

PROPOSITION 2. *The optimal match-fixing prices are*

$$\pi_1^b = \begin{cases} \max\{p_1-\delta,\ \sqrt{\lambda p_1(p_1+\delta)}\}, & \forall p_1 \in [0, 1-\delta) \\ \max\{p_1-\delta,\ \sqrt{\lambda p_1}\}, & \forall p_1 \in [1-\delta, 1], \end{cases} \tag{7}$$

$$\pi_2^b = \begin{cases} \sqrt{1-\lambda p_1}, & \forall p_1 \in [0, \delta) \\ \min\{p_2+\delta,\ \sqrt{(1-\lambda p_1)(p_2+\delta)}\}, & \forall p_1 \in [\delta, 1]. \end{cases} \tag{8}$$

The constraints on the prices may now bind over a significant range of p_1, depending on the value of λ, which means that the market shares of both tickets may not always be positive. Below, we discuss this and other implications to establish the attractiveness of match-fixing from the bookie's perspective.

- **When the impact of sabotage is maximum.** First, a special case to take note of is $\lambda = 0$. Here, team 1 will lose with certainty, regardless of p_1, and the bookie will always set $\pi_1 = p_1 - \delta$, which means that all bettors will bet on team 1, although both tickets will be on offer.[8] Profit is $E\Pi_b = 1 - c(p_1) - s$. Because $c'(p_1) = \alpha_1'(p_1)[s_P + s_B] \geq 0$, $E\Pi_b(\lambda = 0)$ is non-increasing in p_1; specifically, it is a flat line at all p_1 if $\alpha_1'(p_1) = 0$. However, more interesting cases arise when $\lambda > 0$.

- **Profit per bettor.** From the bookie's perspective, the expected profit per bettor is now greater on ticket 1 and smaller on ticket 2 (due to $\lambda < 1$) compared to the honest bookmaking case. The expected profit per bettor from ticket 1 is $(1 - \sqrt{\frac{\lambda p_1}{p_1+\delta}})$ or $(1 - \sqrt{\lambda p_1})$ or $(1 - \frac{\lambda p_1}{p_1 - \delta})$, and that from ticket 2 is $(1 - \sqrt{\frac{1-\lambda p_1}{p_2+\delta}})$ or $(1 - \sqrt{1 - \lambda p_1})$.

- **Market shares.** The market share of ticket 2 is not always positive. If λ is not too large and p_1 exceeds a critical level, the bookie will set $\pi_2 = p_2 + \delta$ so no bettor will bet on team 2. Suppose $\lambda \leq 1 - \delta$; then, by setting $\pi_2 = \sqrt{(1 - \lambda p_1)(p_2 + \delta)} \geq (p_2 + \delta)$, we see that at all $p_1 \geq \frac{\delta}{1-\lambda}$ betting on team 2 would be reduced to zero. In Fig. 2, we show such combinations of λ and p_1 in regions A and B.

  In contrast, the market share of ticket 1 will not only always be positive but will increase with p_1, eventually reaching the maximum (refer to region A in Fig. 2). This is a complete reversal of the honest bookmaking case. However, if p_1 is small (such as $p_1 < \delta$) or if the impact of sabotage is small, for instance if $\lambda > 1 - \delta$, betting on both teams will be active. This refers to region C in Fig. 2. The betting pattern in this region does not drastically differ from the honest bookmaking case. A full description of the market shares is provided in the Appendix.

  In sum, a combination of the following two factors enables the bookie to reverse the betting pattern. First, as long as $0 < \lambda < 1$, the extent of reduction in the winning chance of team 1 increases with p_1, and as do the gains from bribery. Second, the distribution of the punters' beliefs tracks Nature's draw. Thus, when team 1 is the favourite, most bettors will be optimistic about team 1; therefore, a secret reduction in p_1 via match-fixing can make offering bets even to the least optimistic bettor profitable.

[8] This assumes $p_1 > \delta$. For $p_1 \leq \delta$, optimal π_1 would be arbitrarily close to zero.

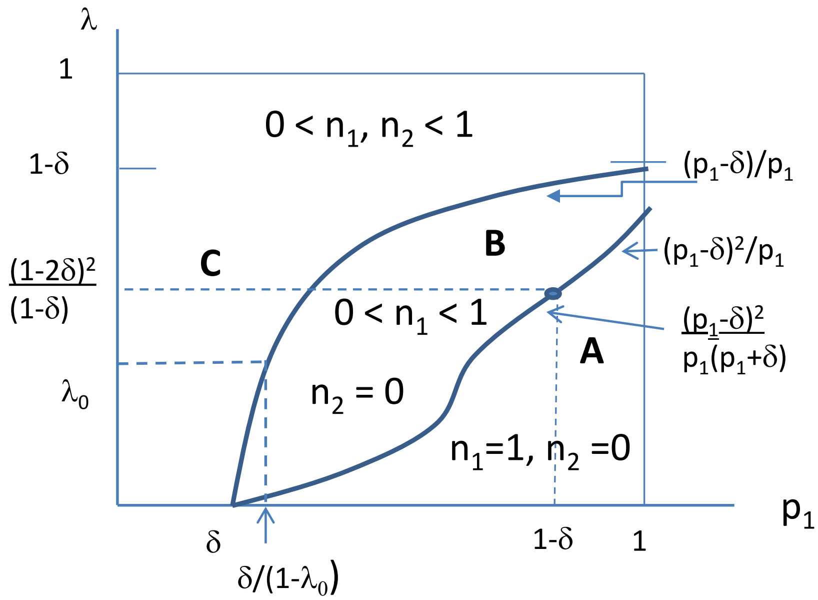

Figure 2: Betting patterns under match-fixing

All this requires is a sufficient reduction in the price of ticket 1. However, when team 1 is a longshot, most bettors will be pessimistic about team 1. Hence, it will be too difficult to induce them to bet on team 1; in anticipation of that difficulty, bribing the longshot will not be worthwhile.

In light of the above discussion, it is clear that our analysis will be too lengthy if we consider the entire range of p_1. To present our formal analysis in a clear manner we will restrict our attention to $p_1 \in [\delta, 1-\delta]$. Later on, we provide some simulation results for the whole range of p_1. We also focus on $\lambda \in [0, 1-\delta]$ wherever necessary because match-fixing is not worthwhile at very high values of λ.

For $p_1 \in [\delta, 1-\delta]$ and $\lambda \in [0, 1)$, the expected gross profit under match-fixing is[9]

$$E\Pi_b^G(p_1;\lambda) = \begin{cases} 1 - \frac{\lambda p_1}{p_1-\delta} & \forall\, \lambda \leq \frac{(p_1-\delta)^2}{p_1(p_1+\delta)} \\ \frac{1}{2\delta}\left[p_1(1+\lambda) + \delta - 2\sqrt{\lambda p_1(p_1+\delta)}\right] & \forall\, \frac{(p_1-\delta)^2}{p_1(p_1+\delta)} < \lambda < \frac{p_1-\delta}{p_1} \\ \frac{1}{\delta}\left[1 + \delta - \sqrt{\lambda p_1(p_1+\delta)} - \sqrt{(1-\lambda p_1)(p_2+\delta)}\right] & \forall\, \frac{p_1-\delta}{p_1} \leq \lambda. \end{cases} \tag{9}$$

After subtracting the bribe and the possible penalties, we derive the net profit as

$$E\Pi_b(p_1;\lambda) = E\Pi_b^G(p_1,\lambda) - (1-\lambda p_1)c(p_1) - s.$$

[9]The detailed profit function for all possible (p_1, λ) can be derived with the help of (n_1, R_1) and (n_2, R_2), as in the Appendix.

Using Assumption 2 and denoting $s_B + s_P = F$, write:

$$E\Pi_b(p_1; \lambda) = E\Pi_b^G(p_1, \lambda) - (1 - \lambda p_1)[w + (\underline{\alpha} + \gamma p_1)F] - s. \tag{10}$$

There are some key properties of $E\Pi_b$ that we would like to highlight in several lemmas. They concern the behaviours of $E\Pi_b$ with respect to p_1, λ, and some of the fixed components of the bribe costs, such as w or F, and the minimum point of the $E\Pi_b$ curve itself. Identifying these properties is essential to compare profits under bribery and honest pricing.

Generally speaking, subject to some (mild) conditions, the expected profit curve implied by (10) will shift downward if the impact of sabotage falls (i.e., λ increases) or the player's prize money w increases, which calls for a bigger bribe at all p_1. This is established in the following lemma.

LEMMA 1. $E\Pi_b(p_1; \lambda, w)$ *is decreasing in* w *at all* $\lambda \in [0, 1)$. *It is also decreasing in* $\lambda \in [0,\ 1 - \delta]$, *if* $w + F < \frac{\sqrt{1-\delta(1-\delta)} - (1-\delta)}{2\delta(1-\delta)\sqrt{1-\delta(1-\delta)}}$ $(\equiv \bar{c})$.

However, the behaviour of the profit curve is less clear-cut with respect to p_1. There are two sources of uncertainty. The first source is the expected cost of bribery, $Ec = (1 - \lambda p_1)[w + (\underline{\alpha} + \gamma p_1)F]$, which may increase or decrease with p_1, depending on λ:

$$\begin{aligned} \frac{\partial Ec}{\partial p_1} &= -\lambda[w + (\underline{\alpha} + 2\gamma p_1)F] + \gamma F, \\ \frac{\partial Ec}{\partial p_1} &\leq (>)\ 0 \quad \text{if} \quad \lambda \geq (<) \frac{\gamma F}{w + (\underline{\alpha} + 2\gamma p_1)F}. \end{aligned}$$

Low levels of λ imply a greater ex ante risk of detection and, in turn, a greater expected cost.

The second source is the way in which the *slope* of the profit function (i.e., $\partial E\Pi_b / \partial p_1$) behaves. In particular, if the profit curve has an interior minimum, we want to know where the minimum occurs, e.g., at p_1 less or greater than $1/2$. This, in part, depends on whether $c'(p_1) = 0$ or $c'(p_1) > 0$. When $c'(p_1) = 0$, i.e., if the probability of investigation is fixed, the $E\Pi_b$ curve can have an interior minimum only in the region C, as shown in Fig. 2. This means the bookie will do strictly better in regions B and A of Fig. 2, sharply highlighting the attractiveness of bribing the favourite. This is shown in the following lemma.

LEMMA 2. *Suppose* $\gamma = 0$ *so that* $c(p_1) = \underline{\alpha}$, *and* $E\Pi_b(.) > 0$ *at all* $p_1 \in [\delta, 1 - \delta]$. *Then if* $E\Pi_b$ *has an interior minimum, betting on team 2 (the winning team) must be active. If betting on team 2 is not active,* $E\Pi_b$ *must be increasing in* $p_1 \in [\delta, 1 - \delta]$.

When $c'(p_1) > 0$, we cannot say that the expected profit under bribery will always be increasing in regions B and A of Fig. 2. In that sense, the attractiveness of bribing the favourite is less prominent. A larger exogenous bribe (such as w) forces the profit function to fall sharply and reach its minimum rather quickly (i.e., at some p_1 that is well below $1/2$). However, a greater λ only slows the fall in profit to arrive at the minimum of $E\Pi_b$ at some $p_1 > 1/2$. Thus, we try to strike a balance between the two opposite effects by setting a lower bound on the expected cost and ensure at least that the interior minimum of $E\Pi_b$ occurs at some $p_1 < 1/2$. Lemma 3 does precisely that. This suffices to guarantee that bribing the favourite will be more rewarding than bribing the longshot.

LEMMA 3. *Suppose* $E\Pi_b(p_1;\lambda) > 0$*, and it has an interior minimum at* p_1^**. Then* p_1^* *is inversely related to* w*. Further, if* $\gamma F < \delta/(1-\delta)^3$*, and* w *is above a critical level* $\underline{w} = \max\{\underline{w}_1, \underline{w}_2\}$ *(where* $\underline{w}_1$, $\underline{w}_2$ *are given by Eqs.* (17) *and* (21)*, respectively),* p_1^* *is strictly less than* $1/2$ *at all* $0 < \lambda < 1$.

The restrictions on the costs are motivated as follows. First, when the impact of sabotage is very strong (very low λ), the entire market is captured by ticket 1; in this case, the only cost the bookie needs to worry about is the (increasing) risk of investigation. As long as the marginal fine cost γF is not too large, the profit function will still be rising. Second, when the market is not fully captured by ticket 1, profit may initially fall and then rise, depending on p_1 and λ. If w, a key component of the bribe, is above some threshold, the profit will fall quite rapidly before turning around to rise. The threshold level of w varies depending on whether both bets are active or only one bet is active (with incomplete market coverage). These two thresholds are denoted as $\underline{w}_1$ and $\underline{w}_2$.

The key point of Lemma 3 is that compared to the honest pricing case, the shape of the expected profit curve under bribery will become twisted in a certain way, as shown in the two panels of Fig. 3. If the bribery cost is fairly large and the impact of sabotage is moderate, the match-fixing profit quickly drops to zero or even becomes negative; it then becomes positive again only after p_1 exceeds some level. This is shown in panel a of Fig. 3, thus supporting a common perception that longshots are unattractive for fixing.

The second possibility, as shown in panel b of Fig. 3, is that the bribery cost may be moderate (but not below the threshold level), so that the bribery profit remains positive throughout, a scenario discussed in Lemma 3. In both panels, the minimum of the match-fixing profit curves never occurs at $p_1 > 1/2$. This is a result of persistent steering of betting toward team 1 in the complete reversal of the betting pattern under honest bookmaking.

■ **Match-fixing or honest bookmaking?** We now compare bookie's profit from match-fixing with that from honest bookmaking. First, note two extreme cases: $\lambda = 0$ and $\lambda \to 1$.

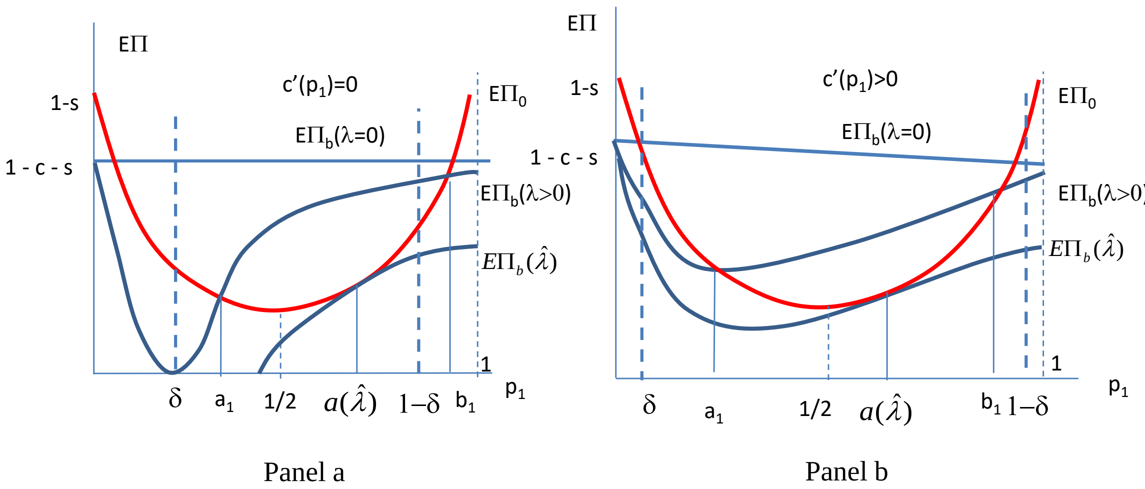

Figure 3: Regions of bribery

As already noted, when $\lambda = 0$, the bookie earns $E\Pi_b = 1 - c(p_1) - s$. If $c'(p_1) = 0$, his profit is given by a flat line as shown in panel a of Fig. 3, and if $c'(p_1) > 0$ then it will be a declining line as in panel b of Fig. 3. Compare this bribery profit curve in both panels of Fig. 3 with the profit curve from honest bookmaking, $E\Pi_0$. We may find all contests in the $[\delta, 1-\delta]$ range preferred for fixing. At the same time, it is noteworthy that some extremely uneven contests, such as contests close to $p_1 = 0$ and $p_1 = 1$, will never be fixed.

At the other extreme, when $\lambda \to 1$, the profit from match-fixing must be strictly less than the profit from honest bookmaking, due to the positive match-fixing cost. Let us assume that at all $\lambda \in [1-\delta, 1)$, $E\Pi_b < E\Pi_0$. By Lemma 1 the $E\Pi_b$ curve will shift upwards if we successively reduce λ below $1-\delta$. At some point we will have tangency between the $E\Pi_b$ and $E\Pi_0$ curves. The specific λ at which the tangency occurs is denoted $\hat{\lambda}$, as shown in Fig. 3.

It is obvious now that the region of bribery $[a_1(\lambda), b_1(\lambda)]$ is largest at $\lambda = 0$ and smallest (i.e., a single point with $a_1 = b_1$) at $\hat{\lambda}$. That is to say, the region of bribery consistently shrinks with the lessening of sabotage. Let us call $a(\hat{\lambda})$ the limiting contest that is optimal over the maximal range of λ. This limiting contest must be greater than $1/2$, because the minimum of $E\Pi_b$ occurs at $p_1 < 1/2$, compared to the minimum of $E\Pi_0$, which occurs at $p_1 = 1/2$. That $a(\hat{\lambda}) > 1/2$ indicates the sustainability of a favourite for bribing.

There is another point to note. If $c'(p_1) = 0$, i.e., the probability of investigation is fixed, the set of contests optimal for fixing will have a favourite bias. That is, we will have $a_1 > 1 - b_1$, and more than half of the fixed contests will feature team 1 as the favourite. In contrast, if the investigation risk is increasing in p_1, then the bias in the set of fixed contests will depend on the value of λ. At low values of λ the majority of these contests will feature team 1 as the underdog (i.e., $a_1 < 1 - b_1$) and after a critical value of λ team 1 will be the favourite in most of the fixed contests. This occurs because the marginal effect of the investigation risk is overturned only after λ exceeds a critical value.

PROPOSITION 3 (**Match-fixing**). *Suppose that some contests are optimal to fix at* $\lambda = 0$ *and no contest is optimal to fix at* $\lambda = 1 - \delta$. *Then, given Assumptions 1–3 and conditions in Lemma 1-3, there exists a critical* λ, *say* $\hat{\lambda} < 1 - \delta$, *such that match-fixing occurs over a range of contests* $p_1 \in [a_1, b_1]$ *at all* $\lambda < \hat{\lambda}$. *With an increase in* λ *the match-fixing set of contests will shrink (i.e.,* $a_1'(\lambda) > 0$, $b_1'(\lambda) < 0$*), eventually at* $\hat{\lambda}$ *collapsing to the limiting contest* $a(\hat{\lambda})$ *which must be greater than* $1/2$.

Furthermore, if $c'(p_1) = 0$, *the match-fixing set will have a favorite bias* ($a_1 > 1 - b_1$). *However, if* $c'(p_1) > 0$ *then the match-fixing set will exhibit longshot (favourite) bias at* λ *below (above) a critical value.*

■ **Which team to bribe?** Now we ask: when both teams are accessible for fixing, which team is the bookie going to pick? Based on the argument made for bribing team 1, and symmetrically defining the bribery region of team 2 as $[a_2(\lambda), b_2(\lambda)]$, we can see that $a_2 = 1 - b_1$ and $a_1 = 1 - b_2$. Drawing two bribery profit curves on the same graph and comparing them with the profit curve under honest bookmaking we can see that there are two possible configurations – either $a_2 \leq a_1 < 1/2 < b_2 \leq b_1$, or $a_2 < b_2 \leq 1/2 \leq a_1 < b_1$.

In the first case, bribing team 2 is more profitable over the interval $[a_2, \frac{1}{2})$, and bribing team 1 is more profitable over the interval $[\frac{1}{2}, b_1]$. In the second case, the preferred intervals for bribery change to (a_2, b_2) for team 2 and (a_1, b_1) for team 1.

PROPOSITION 4 (**Bribe the favourite**). *Suppose either team is corruptible, and the bookie can select the team to bribe at will. Then, there exists a range of* λ *and a range of contests such that bribery is preferred to honest bookmaking and the bribed team will be the favourite.*

■ **Illustrative simulation.** We illustrate the key points of bribery with some numerical examples in Table 1, assuming that only team 1 is bribed. In the numerical examples we cover the entire range of p_1 and λ.

We present four sets of simulations based on three specifications of $\alpha_1(p_1)$. In the first section of the table, we present the case of $\alpha_1 = \underline{\alpha}$, which is the fixed probability of investigation case. The next two sections present the case of $\alpha_1 = p_1^2/2$ for two different values of w. The last section of the table considers $\alpha_1 = p_1$, the linear investigation probability case assumed in our theoretical model.

Throughout, bettors' collective wealth is 1 and $\delta = 0.2$. In the top section we assume $w + \alpha_1 F = 0.2$. The maximum λ up to which bribery is optimal is $\hat{\lambda} = 0.626$. The

Table 1: Bribe inducement range of p_1

	$\alpha_1 = \underline{\alpha}$	$\delta = 0.20$	$w + \underline{\alpha}F = 0.20$	$\hat{\lambda} = 0.626$	$\hat{p}_1 = 0.795$	
λ	0	0.20	0.40	0.50	0.6	0.626
a_1	0.002	0.009	0.044	0.33	0.60	0.795
b_1	0.998	0.989	0.965	0.935	0.81	0.795
	$\alpha_1(p_1) = p_1^2/2$	$\delta = 0.2$	$w = 0.2$	$F = 0.2$	$\hat{\lambda} = 0.581$	$\hat{p}_1 = 0.77$
λ	0	0.20	0.40	0.50	0.581	
a_1	0.003	0.01	0.039	0.37	0.77	
b_1	0.994	0.982	0.951	0.91	0.77	
	$\alpha_1(p_1) = p_1^2/2$	$\delta = 0.2$	$w = 0.25$	$F = 0.2$	$\hat{\lambda} = 0.542$	$\hat{p}_1 = 0.76$
λ	0	0.20	0.40	0.45	0.50	0.542
a_1	0.004	0.017	0.293	0.395	0.534	0.76
b_1	0.992	0.978	0.943	0.92	0.89	0.76
	$\alpha_1(p_1) = p_1$	$\delta = 0.2$	$w = 0.20$	$F = 0.15$	$\hat{\lambda} = 0.538$	$\hat{p}_1 = 0.795$
λ	0	0.20	0.40	0.415	0.50	0.538
a_1	0.002	0.01	0.045	0.065	0.537	0.795
b_1	0.993	0.978	0.942	0.935	0.882	0.795

bribery region starts with $[0.006, 0.994]$ at $\lambda = 0$ and eventually contracts to a singular point $p_1 = 0.795$. Note that the match-fixing set $[a_1, b_1]$ is biased toward the favourite at all $\lambda > 0$, as we claim in Proposition 3.

Next, in the convex and increasing probability of investigation case of $\alpha_1(p_1) = p_1^2/2$, we first assume $w = 0.20$ and $F = 0.20$. In this case, as we argue in Proposition 3 (although for a slightly different $\alpha_1(.)$ function) that the match-fixing set will be initially (at low values of λ) biased towards the underdog and then flip towards the favourite, the numbers bear out the pattern. At some λ between 0.40 and 0.50, the flip occurs, with the match-fixing set being predominantly concentrated on the favourite after the flip. The limiting contest is 0.77, which occurs at $\lambda = 0.581$. A second example of the same specification of $\alpha_1(.)$ is provided with $w = 0.25$. With a higher bribe cost, the flip of the match-fixing set occurs between 0.20 and 0.30, and the limiting contest falls to 0.76 at a slightly smaller λ. Both examples show, as expected, that with the increasing risk of investigation $\hat{\lambda}$ becomes smaller compared to the case of the fixed probability of investigation.

Finally, for the linear investigation probability $\alpha_1 = p_1$, we set $w = 0.20$ and $F = 0.15$ and find that the limiting contest increases again to 0.795, largely because the expected cost rises at a constant rate, instead of an increasing rate. However, the pattern is similar. Here too, the match-fixing set $[a_1, b_1]$ is initially biased towards the underdog, and then after $\lambda = 0.415$ it is heavily tilted towards the favourite.

4 Rational Bettors

Our model has, so far, relied on two key assumptions: the bettors are not rational, and the bookie holds superior information. In this section, we show that even if bettors are rational, our central result on bribery of the favourite may still hold, as long as the bookie does not have informational superiority. We suggest a variant model with binary (uncorrupted) winning odds and binary signals with the following assumptions: (i) bettors are rational in the sense that they will update their beliefs about the teams' winning chances in cognisance of match-fixing, (ii) the bookie observes a noisy signal about Nature's draw just like any other bettor, and (iii) the bookie is not guaranteed to have an access to a team for bribery.

Assumption (ii) is critical to create a scope for gainful trade between the bookie and the bettors, and it allows them to have divergent beliefs within the same information hierarchy. Otherwise we will run into the well-known 'no-trade' result of Milgrom and Stokey (1982).

Below, we outline the model, report a result and discuss its intuition. The formal analysis, given its detailed nature, is contained in a separate supplementary file.

■ **Flat (prior) information hierarchy model.** As above, consider a two-team contest with a win-or-loss outcome. Let p_1, the probability of team 1's win, be drawn from $\{p_\ell, p_h\}$ such that $0 < p_\ell < 1/2 < p_h < 1$. Assume that $\wp = \Pr(p_1 = p_\ell) < 1/2$, so the prior favours team 1.

There is a continuum of rational bettors of mass and collective wealth 1, and they draw conditionally independent signals $\sigma \in \{\sigma_\ell, \sigma_h\}$ according to the following distribution:

$$\beta = \Pr(\sigma = \sigma_\ell | p_1 = p_\ell) = \Pr(\sigma = \sigma_h | p_1 = p_h), \quad 1-\beta = \Pr(\sigma = \sigma_h | p_1 = p_\ell) = \Pr(\sigma = \sigma_\ell | p_1 = p_h),$$

where $1/2 < \beta < 1$. For match-fixing, consider for simplicity $\lambda = 0$ such that fixing ensures losing. Also assume $\gamma = 0$ such that the probability of investigation is fixed. Furthermore, $c = w + \underline{\alpha} \cdot (s_p + s_B)$ and $s = \alpha_0 s'_B$ are "small" to ensure a scope for match-fixing.

The bookie also observes, independently, a noisy binary signal $\sigma_b \in \{\sigma_\ell, \sigma_h\}$ of Nature's draw according to the same conditional distribution $(\beta, 1-\beta)$. Given that the bookie is *no better in predicting* the teams' winning chances, the two parties – the bookie and the bettors – can possibly hope to trade profitably. Specifically, with the bookie's signal being noisy, bettors might not be able to tell from a 'cheap bet' whether the bookie has fixed the match or just received a signal indicating a low chance of the corresponding team's win.

The bookie can access either only team 1 or only team 2 each with an identical probability θ, $0 < \theta < 1/2$, and fails to have access to either team with probability $1 - 2\theta$. This last possibility will add further uncertainty about match-fixing.

The game proceeds as follows:

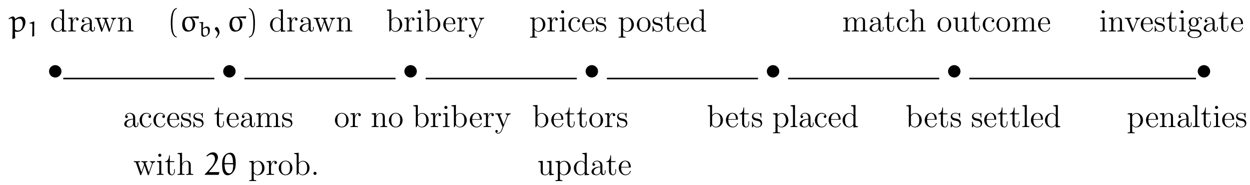

Figure 4: Time line

We do not intend to present a full-blown analysis of this model. Our sole aim is to construct *an* equilibrium that admits match-fixing.

■ **Strategies.** To be able to give a sense of the model's workings leading to a match-fixing equilibrium, we now describe the qualitative nature of the bookie's strategy. The strategy is the decision of bribery/no bribery of a team conditional on access, and a pair of prices. Our primary interest will be in a *partial pooling (perfect Bayesian) equilibrium.* With this in mind, we partition the bookie's information set into two sets of nodes as follows:

$$\mathcal{I}_1 \equiv \begin{cases} I_1 = \{\text{access team } 1, \sigma_b = \sigma_h\} \\ I_2 = \{\text{access team } 1, \sigma_b = \sigma_\ell\} \\ I_3 = \{\text{no access}, \sigma_b = \sigma_\ell\} \end{cases} \tag{11}$$

$$\mathcal{I}_2 \equiv \begin{cases} I_4 = \{\text{access team } 2, \sigma_b = \sigma_\ell\} \\ I_5 = \{\text{access team } 2, \sigma_b = \sigma_h\} \\ I_6 = \{\text{no access}, \sigma_b = \sigma_h\}. \end{cases} \tag{12}$$

Our equilibrium will involve two distinct price pairs – one each for $\mathcal{I}_1$ and $\mathcal{I}_2$ – seeing which the bettors can identify the information set at play but not the individual nodes. The bettors' strategy is a mapping from price pairs and privately observed signal to beliefs (a probability distribution over individual nodes) and a corresponding optimal betting.

In the supplementary material, we derive conditions (including the bettors' equilibrium and out-of-equilibrium beliefs) for the following equilibrium:

PROPOSITION 5 (**Rational betting under suspicion of match-fixing**). *Under suitable parameters* $(p_\ell, p_h, \wp, \beta, \theta, c, s)$*, the following will be true:*

(i) *There exist two distinct price pairs,* (π_1^*, π_2^*) *for information set* $\mathcal{I}_1$ *and* (π_1^{**}, π_2^{**}) *for information set* $\mathcal{I}_2$*, such that at* $\mathcal{I}_1$*, the bookie bribes only team 1 (on access), the favourite team according to his private signal, and at* $\mathcal{I}_2$*, he bribes only team 2 (on access), his perceived favourite team.*

(ii) *Bettors will place bets as follows:*

1) If the announced prices are (π_1^*, π_2^*)*, then those who have observed* $\sigma = \sigma_h$ *will bet on team 1 and those who have observed* $\sigma = \sigma_\ell$ *will bet on team 2.*

2) If the announced prices are (π_1^{**}, π_2^{**})*, then those who have observed* $\sigma = \sigma_\ell$ *will bet on team 2, and those who have observed* $\sigma = \sigma_h$ *will bet on team 1.*

REMARKS. (i) In the proposed equilibrium, the bookie creates a doubt about whether he has bribed but leaves no doubt as to which team he might have bribed, if he bribed at all. In addition, whenever he bribes a team, it is also the team that he reckons to be the favourite. (ii) Bettors also bet on their perceived favourites, hoping that the bookie has received an opposite signal. (iii) One group of bettors will hope that the match has not been fixed, whereas the other group would wish the opposite. For instance, at $\mathcal{I}_1$ the bettors with signal σ_h hope that the match has not been fixed, and the bookie has observed σ_ℓ. However, the bettors with signal σ_ℓ would hope that the bookie has observed σ_h and fixed the match. (iv) Finally, the constructed equilibrium is not necessarily the best equilibrium from the bookie's point of view, because we do not address the equilibrium selection issue.

EXAMPLES. We are able to report the following two sets of parameter values confirming Proposition 5. A supplementary file provides the corresponding numerical values of the bookie and bettor payoffs, based on the derived analytical equilibrium conditions.[10]

Table 2: Parameters

$\lambda = 0$	$p_h = 0.653286$	$p_\ell = 0.363761$	$\wp = 0.336077$
$\beta = 0.625798$	$\theta = 0.160131$	$c = 0.147717$	$s = 0.014554$
$\pi_1^* = 0.506417$	$\pi_2^* = 0.569213$	$\pi_1^{**} = 0.655001$	$\pi_2^{**} = 0.373474$

Table 3: Parameters

$\lambda = 0$	$p_h = 0.745445$	$p_\ell = 0.329499$	$\wp = 0.094006$
$\beta = 0.730153$	$\theta = 0.145988$	$c = 0.118418$	$s = 0.008619$
$\pi_1^* = 0.6477$	$\pi_2^* = 0.494873$	$\pi_1^{**} = 0.745207$	$\pi_2^{**} = 0.257546$

5 Further Discussions

We discuss how our correlated beliefs model will change if we relax or modify some of the assumptions that underpinned the analysis of Section 2.

[10] Simulation has been done using **R** programming.

1. **A general distribution function of bettors' beliefs.** Suppose that q, the bettors' belief about team 1's winning chance, is distributed over $[p_1 - \delta,\ p_1 + \delta]$ according to a probability distribution function $G(q)$ with density $g(q)$. Assume that $G(q)$ has a single peak and that $g(q) > 0$. Bag and Saha (2016) modelled this case in a slightly different context, where an honest bookie addresses an anonymous fixer-cum-bettor.

   Based on Bag and Saha (2016), we can say that if certain conditions are met by $G(q)$, our qualitative results are likely to be maintained.[11] Intuitively, majority of the bettors' beliefs will be concentrated around p_1, so the profit will be far less from honest bookmaking, making match-fixing an attractive option.

   However, we may obtain different results if the distribution function is bi-modal and the two modes appear on either side of p_1. In this case, one group of bettors is optimistic about team 1 and the other group is pessimistic. Then, inducing the pessimists to bet on team 1 is not easy unless the reduction in p_1 via bribery is really substantial. Thus, match-fixing is less likely to be optimal.

2. **Possibility of draw.** Suppose Nature picks (p_1, p_2) probabilities of team 1 and team 2 winning respectively, with $p_3 = 1-(p_1+p_2)$ being the probability of a draw (or tie). The bettors draw a pair of private signals (q_1, q_2) from the set $[p_1-\delta, p_1+\delta]\times[p_2-\delta, p_2+\delta]$ according to some distribution. Their belief of a tie is $1 - q_1 - q_2$. They will, however, bet on only one outcome.

   Suppose team 1 is bribed to lower its probability of winning to λp_1. But team 2's win probability will not automatically rise by $(1-\lambda)p_1$. Rather, part of $(1-\lambda)p_1$ will be transferred to the probability of a tie as well. This is likely to be so, even if $\lambda = 0$. The bookie must consider (for bribing) how the probabilities of the two other events are altered, relative to the distribution of bettors' beliefs. We believe the intuition for bribing the favourite remains true because the average bettor's belief will coincide with Nature's draw, and therefore, turning the most likely event less likely will continue to be profitable.

   In addition, match-fixing may invite less attention from the enforcement authority, which is more likely to react if the favourite loses than when there is a tie.[12] Furthermore, if λ can be freely chosen, then a moderate value of λ may possibly make a tie

[11] We need that $G(q)$ is single-peaked, exhibits decreasing hazard rate, and in addition, $G(q)/g(q)$ should be increasing in q. These conditions will ensure some regularity properties of the optimal prices.

[12] Of course in tournaments where the outcome of an individual contest between two teams determines all teams' relative standings possibly affecting promotion and relegation of teams, even a tie could signify foul play and intense investigation.

more likely than a loss. In that sense, the integrity of sports will be compromised more often, though possibly with less severity.

3. **Scoreline/margin betting.** Betting on the margin (of victory) can be modelled by extending our win-loss framework as follows. Suppose that it is common knowledge that team 1 is the favourite, i.e., $p_1 > p_2$, but that the bettors might differ in their perception of how strong team 1 is. This can be represented by letting $p_1 \in (1/2, 1)$ be drawn according to some distribution. Then, the margin of victory can be thought of as some increasing function $m = f(\frac{p_1}{p_2})$ (e.g., $m = \ln(\frac{p_1}{p_2})$). The bookie and the players observe p_1 precisely while bettors independently draw a signal q that is positively correlated with p_1 with $E[q|p_1] = p_1$.

   Now, given any draw of p_1, the bookie will be able to estimate the true (non-corrupt) expected margin of victory which will also agree with the average bettor's estimate. In such a situation, without corruption the bookie will not be able to make much profit because bets on two sides of the expected margin will mostly cancel out. Therefore, through bribery the bookie might be able to distance the margin of victory from the market's expectation, thus opening up profit opportunities.

   Betting on the margin is very profitable where betting on the winning or losing is not permitted as in many American sports, as well as where one team is particularly strong. The bookie can then divide the bettors between the high-belief and the low-belief categories by offering betting odds on margins. When only margin betting is considered our model is more appropriate in the sense that $\lambda > 0$ implies a range of 'point shaving', which has been empirically investigated in Wolfers (2006) for the United States.

4. **Bribe paid beforehand.** Suppose, different from our model, the bribe is given unconditionally before the outcome. Assuming that the bribe-taker will not renege due to reputation cost, taking the bribe (and honouring it by underperforming) is optimal for the player if

$$B_1 + (1 - \lambda p_1)\alpha_1(p_1)s_P + \lambda p_1 w \geq p_1 w \quad \text{or,} \quad B_1 \geq p_1(1 - \lambda)w + (1 - \lambda p_1)\alpha_1(p_1)s_P.$$

   If $B_1 = p_1(1 - \lambda)w + (1 - \lambda p_1)\alpha_1(p_1)s_P$, then we see that the bribe amount here is much smaller than that in Section 2. This alone suggests that match-fixing will be more profitable. The expected cost of the bookie, bribe plus his own fine, will be $Ec = p_1(1-\lambda)w + (1-\lambda p_1)\alpha_1(p_1)F$, and it will have similar relationships with p_1 and λ. Therefore the qualitative results will not be different.

5. **Bribe sensitive to prices.** In our model, the bribe enters the bookie's expected profit as a fixed cost, sensitive to p_1 but not to (π_1, π_2). Suppose the bribe includes the minimum amount necessary for underperformance, plus a share (τ) of the bookie's *ex post* profit. Assuming that team 1 is bribed and paid *ex post* only in the event of the team's loss, the bribe will be given by

$$B_1 = w + \alpha_1(p_1)s_P + \tau n_1(\pi_1).$$

The expected profit of the bookie from bribery is then

$$E\Pi_b = n_1(\pi_1)\left(1 - \frac{\lambda p_1}{\pi_1}\right) + n_2(\pi_2)\left(1 - \frac{1-\lambda p_1}{\pi_2}\right) - (1-\lambda p_1)[w + \alpha_1(p_1)F + \tau n_1(\pi_1)] - s.$$

Optimal π_1 would be given by

$$\left(1 - \frac{\lambda p_1}{\pi_1}\right) n_1'(\pi_1) + n_1(\pi_1)\frac{\lambda p_1}{\pi_1^2} = (1-\lambda p_1)\tau n_1'(\pi_1).$$

Because the right-hand side is negative (as $n_1'(\pi_1) < 0$), optimal π_1 will be smaller (or no greater) than what is found under the fixed bribe. Optimal π_2 will not change. This means the market share of bets on team 1 will be even greater, which will only reinforce the qualitative results of our fixed-bribe model.

6. **Legal betting environment.** What if the bookie is corrupt, but betting is legal? In this environment, one interesting possibility is that the enforcement authority can condition its decision to investigate on both p_1 and posted prices. The bookie's decision to engage in corruption and the enforcement authority's decision to investigate should then be modelled as strategic interactions.

   Strategic interactions between these two agents can also be permitted in our current model by letting $\alpha_i(p_1)$ to be endogenous by modelling the enforcement authority's objectives and constraints. In anticipation of the endogenous enforcement, the bookie may randomize over his decision to fix the match or stay honest. Match-fixing may then emerge as a mixed strategy equilibrium.

7. **Policy implications.** One policy question concerns legalization of betting. Although the tax collected on betting could be a strong and obvious motive for legalization, our study suggests that there are some more benefits to follow. The bookmakers will have a long-term interest in the industry and will thus have less incentive to engage in corruption. They can also be regulated. In addition, because prices will be public,

enforcement can be 'smart' in the sense described above. However, the authority also needs data on teams' initial chances, the size of the betting market and the spread of the market (online or spatial), as laid out in our model. Thus, close coordination is needed between police, sports regulation bodies and betting companies, often over multiple jurisdictions. The coordination issue has been discussed at length in Haberfeld and Sheehan (2013), both in general terms as well as with some specific experience of Brazilian soccer. Our theoretical exercise suggests a similar strategy.

Appendix: Omitted proofs

Proof of Proposition 1. Substitute $n_1(\pi_1)$ and $n_2(\pi_2)$ from (1) into the objective function (2) for any given interval of p_1 and then derive the unconstrained solutions as in (4).

Now, verify that they are valid within the relevant interval of p_1 ensuring positive bets on both teams. Consider first $p_1 \in [\delta, 1-\delta]$. That $\pi_1^0 = \sqrt{p_1(p_1+\delta)} < p_1 + \delta$ and $\sqrt{p_1(p_1+\delta)} > p_1 - \delta$ when $p_1 > \delta/3$ are clear, because $\delta/3 < \delta$. Similarly, $1 - \pi_2^0 > p_1 - \delta$ implies $\pi_2^0 < p_2 + \delta$. That $\sqrt{p_2(p_2+\delta)} < p_2 + \delta$ is obvious, and $1 - \pi_2^0 < p_1 + \delta$ (or $\pi_2^0 > p_2 - \delta$) implies $\sqrt{p_2(p_2+\delta)} > p_2 - \delta$ which is valid for all $\delta, p_1 > 0$. Elsewhere, clearly $\sqrt{p_1} < 1$ and $\sqrt{p_2} > 0$. Therefore, $0 < n_1 < 1$ and $0 < n_2 < 1$ at all $p_1 \in (0,1)$.

In this context we also note the following to be true at (π_1^0, π_2^0):

$$\begin{array}{llll} \text{for } 0 \le p_1 < \delta, & n_1 = 1 - \sqrt{\frac{p_1}{p_1+\delta}} & \text{and} & n_2 = \frac{1-\sqrt{p_2}}{p_1+\delta} \\ \text{for } \delta \le p_1 \le 1-\delta, & n_1 = \frac{p_1+\delta}{2\delta}\left(1 - \sqrt{\frac{p_1}{p_1+\delta}}\right) & \text{and} & n_2 = \frac{p_2+\delta}{2\delta}\left(1 - \sqrt{\frac{p_2}{p_2+\delta}}\right) \\ \text{for } 1-\delta < p_1 \le 1, & n_1 = \frac{1-\sqrt{p_1}}{p_2+\delta} & \text{and} & n_2 = 1 - \sqrt{\frac{p_2}{p_2+\delta}}. \end{array} \tag{13}$$

Differentiating n_1 and n_2 with respect to p_1, we obtain

$$\begin{array}{llll} \text{for } 0 \le p_1 < \delta, & n_1'(p_1) = -\frac{\delta}{2\pi_1(p_1+\delta)} < 0 & \text{and} & n_2'(p_1) = \frac{(1-\sqrt{p_2})^2+\delta}{2(p_1+\delta)^2\sqrt{p_2}} > 0 \\ \text{for } \delta \le p_1 \le 1-\delta, & n_1'(p_1) = \frac{2\pi_1-(2p_1+\delta)}{4\delta\pi_1} < 0 & \text{and} & n_2'(p_1) = \frac{2\pi_2-(2p_2+\delta)}{4\delta\pi_2} \times p_2'(p_1) > 0 \\ \text{for } 1-\delta < p_1 \le 1, & n_1'(p_1) = -\frac{(1-\sqrt{p_1})^2+\delta}{2(p_2+\delta)^2\sqrt{p_1}} < 0 & \text{and} & n_2'(p_1) = \frac{\delta}{2\pi_2(p_2+\delta)} > 0. \end{array} \tag{14}$$

In the range $\delta \le p_1 \le 1-\delta$, $n_1'(p_1) < 0$ because $2p_1 + \delta > 2\pi_1 = 2\sqrt{p_1(p_1+\delta)}$. This can be verified by checking that $(2p_1+\delta)^2 > 4p_1(p_1+\delta)$. Then, by symmetry $n_2'(p_1) > 0$.

Next, substituting (π_1^0, π_2^0) in $E\Pi_0$, we derive Eq. (5). Differentiating $E\Pi_0$ w.r.t. p_1, we obtain

$$E\Pi_0'(p_1) = \begin{cases} \left[\frac{\pi_1\{(1+\delta)+\pi_2(\pi_1+\pi_2)\}-\pi_2\{(p_1+\delta)^2+2\pi_1\}}{(p_1+\delta)^2\pi_1\pi_2}\right] < 0 & \text{for } \; p_1 < \delta \\ \frac{1}{2\delta}\left[-\frac{2p_1+\delta}{\pi_1} + \frac{2p_2+\delta}{\pi_2}\right] \geq (<) 0 \quad \text{if } \; p_1 \geq (<) \, p_2 & \text{for } \; \delta \leq p_1 \leq 1-\delta \\ \left[\frac{\pi_2\{(1+\delta)+\pi_1(\pi_1+\pi_2)\}-\pi_1\{(p_2+\delta)^2+2\pi_2\}}{(p_2+\delta)^2\pi_1\pi_2}\right] > 0 & \text{for } \; 1-\delta < p_1. \end{cases} \tag{15}$$

That $E\Pi_0'(p_1) < 0$ at all $0 \leq p_1 < \delta$ and $E\Pi_0'(p_1) > 0$ at all $1-\delta < p_1 \leq 1$ is ensured by Assumption 3. To ascertain the sign of $E\Pi_0'(p_1)$ at $p_1 \in [\delta, 1-\delta]$ we show that $\frac{2p_1+\delta}{\pi_1} > (<)\frac{2p_2+\delta}{\pi_2}$ if $p_1 < (>)p_2$. Consider the inequality $\frac{2p_1+\delta}{\pi_1} > \frac{2p_2+\delta}{\pi_2}$, and write $\pi_1 = \sqrt{p_1(p_1+\delta)}$ and $\pi_2 = \sqrt{p_2(p_2+\delta)}$ to obtain

$$\sqrt{\frac{p_1}{p_1+\delta}} + \sqrt{\frac{p_1+\delta}{p_1}} > \sqrt{\frac{p_2}{p_2+\delta}} + \sqrt{\frac{p_2+\delta}{p_2}}$$

$$\text{or,} \quad \left(\sqrt{\frac{p_1}{p_1+\delta}} + \sqrt{\frac{p_1+\delta}{p_1}}\right)^2 > \left(\sqrt{\frac{p_2}{p_2+\delta}} + \sqrt{\frac{p_2+\delta}{p_2}}\right)^2$$

$$\text{or,} \quad \frac{p_1}{p_1+\delta} + \frac{p_1+\delta}{p_1} > \frac{p_2}{p_2+\delta} + \frac{p_2+\delta}{p_2} \qquad \text{or,} \quad \frac{p_1+\delta}{p_1} - \frac{p_2+\delta}{p_2} > \frac{p_2}{p_2+\delta} - \frac{p_1}{p_1+\delta}$$

$$\text{or,} \quad \frac{(p_2-p_1)\delta}{p_1p_2} > \frac{(p_2-p_1)\delta}{(p_1+\delta)(p_2+\delta)} \qquad \text{or,} \quad \frac{p_2+\delta}{p_2} > \frac{p_1}{p_1+\delta} \qquad \text{(because } p_2 > p_1).$$

It can also be verified that $E\Pi_0''(p_1) > 0$ at all p_1. For the sake of economy, we do not report the detailed expression of $E\Pi_0''(p_1)$. That $E\Pi_0(p_1)$ is a convex curve proves that $E\Pi_0$ is at a minimum at $p_1 = 1/2$. **Q.E.D.**

■ **Market shares and per-bettor profit under match-fixing.** Using the market share rule given in Eq. (1) and optimal prices given in Eqs. (7)–(8), in what follows, we derive the market shares and per-bettor profits.

Market shares.

- If $\lambda = 0$, $n_1 = 1$ and $n_2 = 0$ for all $p_1 \in [0, 1]$.
- If $\lambda > 0$ and $p_1 < \delta$ then the market shares (for any $\lambda > 0$) are:

$$n_1 = \left(1 - \sqrt{\frac{\lambda p_1}{p_1+\delta}}\right) \qquad \text{and} \qquad n_2 = \frac{1-\sqrt{1-\lambda p_1}}{p_1+\delta}.$$

- If $p_1 \in [\delta, 1-\delta]$, then the following function describes the market share of ticket 1:

$$n_1 = \begin{cases} \left(1 - \sqrt{\frac{\lambda p_1}{p_1+\delta}}\right) \cdot \frac{p_1+\delta}{2\delta} & \text{for } \lambda \geq \frac{(p_1-\delta)^2}{p_1(p_1+\delta)} \\ 1 & \text{for } \lambda < \frac{(p_1-\delta)^2}{p_1(p_1+\delta)}. \end{cases}$$

- If $p_1 \in (1-\delta, 1]$, then the market share of ticket 1 is

$$n_1 = \begin{cases} \frac{1-\sqrt{\lambda p_1}}{p_2+\delta} & \text{for } \lambda \geq \frac{(p_1-\delta)^2}{p_1} \\ 1 & \text{for } \lambda < \frac{(p_1-\delta)^2}{p_1}. \end{cases}$$

- For the market share of ticket 2, the following rule applies when $p_1 > \delta$ and $\lambda > 0$:

$$n_2 = \begin{cases} \left(1 - \sqrt{\frac{1-\lambda p_1}{p_2+\delta}}\right) \cdot \frac{p_2+\delta}{2\delta} & \text{for } \lambda \geq \frac{(p_1-\delta)}{p_1} \text{ and } p_1 \leq 1-\delta \\ \left(1 - \sqrt{\frac{1-\lambda p_1}{p_2+\delta}}\right) & \text{for } \lambda \geq \frac{(p_1-\delta)}{p_1} \text{ and } p_1 > 1-\delta \\ 0 & \text{for } \lambda < \frac{(p_1-\delta)}{p_1}. \end{cases}$$

Per-bettor profit. Let us denote per-bettor profit for ticket 1 as R_1 and the same for ticket 2 as R_2.

- If $\lambda = 0$, $R_1 = 1$ and $R_2 = 0$ for all $p_1 \in [0, 1]$.
- If $\lambda > 0$ and $p_1 < \delta$, then the per-bettor profit from ticket 1 and ticket 2, respectively, (for any $\lambda > 0$) are:

$$R_1 = \left(1 - \sqrt{\frac{\lambda p_1}{p_1 + \delta}}\right) \quad \text{and} \quad R_2 = 1 - \sqrt{1 - \lambda p_1}.$$

- If $p_1 \in [\delta, 1]$, then the following is the per-bettor profit from ticket 1:

$$R_1 = \begin{cases} 1 - \sqrt{\frac{\lambda p_1}{p_1+\delta}} & \text{for } \lambda \geq \frac{(p_1-\delta)^2}{p_1(p_1+\delta)} \text{ and } p_1 \leq 1-\delta \\ 1 - \sqrt{\lambda p_1} & \text{for } \lambda \geq \frac{(p_1-\delta)^2}{p_1} \text{ and } p_1 > 1-\delta \\ 1 - \frac{\lambda p_1}{p_1-\delta} & \text{for } \lambda < \frac{(p_1-\delta)^2}{p_1(p_1+\delta)}. \end{cases}$$

- If $p_1 \in [\delta, 1]$, the per-bettor profit from ticket 2 is given as follows:

$$R_2 = \begin{cases} 1 - \sqrt{\frac{1-\lambda p_1}{p_2+\delta}} & \text{for } \lambda > \frac{(p_1-\delta)}{p_1} \\ 0 & \text{for } \lambda \leq \frac{(p_1-\delta)}{p_1}. \end{cases}$$

Proof of Lemma 1. It is obvious that $E\Pi_b$ is decreasing in w. For the effect of λ, there are three cases to consider.

Case 1. $\lambda \leq \frac{(p_1-\delta)^2}{p_1(p_1+\delta)}$.

From Eqs. (9) and (10), we see that

$$\frac{\partial E\Pi_b}{\partial \lambda} = \left[-\frac{1}{p_1-\delta} + c(p_1)\right] p_1 < 0, \text{ because } \frac{1}{p_1-\delta} > 1 > c(p_1).$$

Case 2. $\frac{(p_1-\delta)^2}{p_1(p_1+\delta)} < \lambda < \frac{p_1-\delta}{p_1}$.

From Eqs. (9) and (10), we derive

$$\frac{\partial E\Pi_b}{\partial \lambda} = \frac{p_1}{2\delta}\left[1 + 2\delta c(p_1) - \frac{\sqrt{p_1+\delta}}{\sqrt{\lambda p_1}}\right].$$

We want $\frac{\partial E\Pi_b}{\partial \lambda} < 0$ at all $p_1 \in [\delta, 1-\delta]$ and all $\lambda \in (\frac{(p_1-\delta)^2}{p_1(p_1+\delta)}, \frac{p_1-\delta}{p_1})$. Because $\frac{\sqrt{p_1+\delta}}{\sqrt{\lambda p_1}}$ is decreasing in both λ and p_1 and $c(p_1)$ is increasing in p_1, we want to ensure that $\frac{\sqrt{p_1+\delta}}{\sqrt{\lambda p_1}} > 1 + 2\delta c(p_1)$ holds at $\lambda = \frac{p_1-\delta}{p_1}$ and $p_1 = 1-\delta$. Substituting these two largest values of λ and p_1, we write

$$\frac{\partial E\Pi_b}{\partial \lambda} = \frac{1-\delta}{2\delta}\left[1 + 2\delta c(1-\delta) - \frac{1}{\sqrt{1-2\delta}}\right] < 0 \quad \text{if and only if} \quad c(1-\delta) < \frac{1-\sqrt{1-2\delta}}{2\delta\sqrt{1-2\delta}}.$$

Because $c(1-\delta) = w + [\underline{\alpha} + \gamma(1-\delta)]F < w + F$, our sufficient condition $w + F < \frac{\sqrt{1-\delta(1-\delta)}-(1-\delta)}{2\delta(1-\delta)\sqrt{1-\delta(1-\delta)}}$ ensures that $c(1-\delta) < \frac{1-\sqrt{1-2\delta}}{2\delta\sqrt{1-2\delta}}$ because $\frac{1-\sqrt{1-2\delta}}{2\delta\sqrt{1-2\delta}} > \frac{\sqrt{1-\delta(1-\delta)}-(1-\delta)}{2\delta(1-\delta)\sqrt{1-\delta(1-\delta)}}$.

Case 3. $\frac{p_1-\delta}{p_1} < \lambda < 1-\delta$.

From Eqs. (9) and (10), we now obtain

$$\frac{\partial E\Pi_b}{\partial \lambda} = \frac{p_1}{2\delta}\left[-\frac{\sqrt{p_1+\delta}}{\sqrt{\lambda p_1}} + \frac{\sqrt{p_2+\delta}}{\sqrt{1-\lambda p_1}} + 2\delta c(p_1)\right] < 0,$$

if and only if

$$\frac{\sqrt{p_1+\delta}}{\sqrt{\lambda p_1}} > \frac{\sqrt{p_2+\delta}}{\sqrt{1-\lambda p_1}} + 2\delta c(p_1).$$

Note that vis-à-vis p_1 the LHS of the above inequality is smallest at the largest value of p_1, i.e., $p_1 = 1-\delta$. On the RHS, the first term is largest at $p_1 = \delta$ (because $\frac{p_2+\delta}{1-\lambda p_1}$ is inversely related to p_1), and the second term is largest at $p_1 = 1-\delta$. However, vis-à-vis λ, the LHS is smallest at the highest value of λ, i.e., $\lambda = 1-\delta$, and the RHS is largest also at $\lambda = 1-\delta$. Therefore, by substituting the appropriate values of p_1 and λ on both sides, we set the LHS as smallest and the RHS as largest and arrive at the following inequality:

$$\frac{1}{1-\delta} > \frac{1}{\sqrt{1-\delta(1-\delta)}} + 2\delta c(1-\delta).$$

This requires $c(1-\delta) < \frac{\sqrt{1-\delta(1-\delta)}-(1-\delta)}{2\delta(1-\delta)\sqrt{1-\delta(1-\delta)}}$, which would be easily met by our condition on $w+F$, because $c(1-\delta) < w+F$. **Q.E.D.**

Proof of Lemma 2. Suppose $\gamma = 0$ and $\underline{\alpha} > 0$; thus, we denote $c(p_1)$ simply as c. Furthermore, assume $E\Pi_b(.) > 0$.

Case 1 (Region C**).** Consider the scenario where betting is active on both sides. This refers to the interval $\lambda \in (\frac{p_1-\delta}{p_1}, 1)$, where $0 < n_1 < 1$ and $0 < n_2 < 1$. To minimise $E\Pi_b$, set:

$$\frac{\partial E\Pi_b}{\partial p_1} = \frac{1}{2\delta}\left[-\frac{\lambda(2p_1+\delta)}{\pi_1} + \frac{\lambda(2p_2+\delta)+(1-\lambda)}{\pi_2}\right] + \lambda c \;=\; 0, \tag{16}$$

where $\pi_1 = \sqrt{\lambda p_1(p_1+\delta)}$ and $\pi_2 = \sqrt{(1-\lambda p_1)(p_2+\delta)}$.

Suppose that p_1^* solves Eq. (16). To check the second-order condition, rewrite $\lambda(2p_2+\delta)+(1-\lambda) = \lambda(p_2+\delta)+(1-\lambda p_1)$ in the expression for $\frac{\partial E\Pi_b}{\partial p_1}$ above, and then derive

$$\begin{aligned}
\frac{\partial^2 E\Pi_b}{\partial p_1^2} &= \frac{1}{2\delta}\left[-\frac{\lambda}{\pi_1^2}\left\{2\pi_1 - \frac{\lambda(2p_1+\delta)^2}{2\pi_1}\right\} + \frac{1}{\pi_2^2}\left\{-2\pi_2\lambda + \frac{(\lambda(p_2+\delta)+(1-\lambda p_1))^2}{2\pi_2}\right\}\right] \\
&= \frac{1}{4\delta}\left[\frac{\lambda^2\delta^2}{\pi_1^3} + \frac{((1-\lambda p_1)-\lambda(p_2+\delta))^2}{\pi_2^3}\right] = \frac{1}{4\delta}\left[\frac{\lambda^2\delta^2}{\pi_1^3} + \frac{(1-\lambda(1+\delta))^2}{\pi_2^3}\right] > 0.
\end{aligned}$$

Therefore, p_1^* must give a minimum, and it is unique because multiple minima cannot occur without altering the sign of the second-order derivative.

Case 2 (Region B**).** Consider $\frac{(p_1-\delta)^2}{p_1(p_1+\delta)} < \lambda < \frac{p_1-\delta}{p_1}$, where $n_2 = 0$ and $n_1 < 1$. Here, $E\Pi_b$ must be increasing in p_1. To establish this, we first show that $E\Pi_b$ is a strictly convex function at all $p_1 \in [\delta, 1-\delta]$ for any λ in the specified interval.

From Eqs. (9) and (10), we derive

$$\frac{\partial E\Pi_b}{\partial p_1} = \frac{1}{2\delta}\left[1 + \lambda - \frac{\lambda(2p_1+\delta)}{\pi_1}\right] + \lambda c,$$

where $\pi_1 = \sqrt{\lambda p_1(p_1+\delta)}$. Differentiating further, we obtain

$$\frac{\partial^2 E\Pi_b}{\partial p_1^2} = -\frac{\lambda}{2\delta\pi_1^2}\left[2\pi_1 - \frac{\lambda(2p_1+\delta)^2}{2\pi_1}\right] = -\frac{\lambda}{4\delta\pi_1^3}\left[4\pi_1^2 - \lambda(2p_1+\delta)^2\right] = \frac{\lambda^2\delta}{4\pi_1^3} > 0.$$

Thus, the slope of $E\Pi_b$ must be rising with p_1. If we consider the lowest value of p_1 in this range and ascertain the slope of $E\Pi_b$ to be positive, then we know that at all higher values of p_1 the slope must also remain positive. The lowest value of p_1 to consider is $p_1 = \delta/(1-\lambda)$ (which follows from the upper bound on λ).

Now evaluate $\frac{\partial E\Pi_b}{\partial p_1}$ at $p_1 = \delta/(1-\lambda)$ as follows:

$$\frac{\partial E\Pi_b}{\partial p_1} = \frac{1}{2\delta\sqrt{2-\lambda}}\left[\sqrt{(2-\lambda)}(1+\lambda) - \sqrt{\lambda}(3-\lambda)\right] + \lambda c.$$

For the term inside the bracket, it can be easily verified that the inequality $\sqrt{(2-\lambda)}(1+\lambda) > \sqrt{\lambda}(3-\lambda)$ reduces to $(1-\lambda)^2 > \lambda(1-\lambda)^2$ as $\lambda < 1$. Hence, $E\Pi_b$ is increasing.

Case 3 (Region A**).** Consider $\lambda \leq \frac{(p_1-\delta)^2}{p_1(p_1+\delta)}$. In this range $n_2 = 0$ and $n_1 = 1$. From Eqs. (9) and (10), we see that

$$\frac{\partial E\Pi_b}{\partial p_1} = \frac{\lambda\delta}{(p_1-\delta)^2} + \lambda c > 0 \qquad (\text{for } \lambda > 0)\,.$$ **Q.E.D.**

Proof of Lemma 3. Suppose that $E\Pi_b(p_1, \lambda)$ has an interior minimum between δ and $1-\delta$. Denote it as p_1^*, which can be obtained by solving

$$\frac{\partial E\Pi_b(.)}{\partial p_1} = \frac{\partial E\Pi_b^G(.)}{\partial p_1} - \frac{\partial Ec(.)}{\partial p_1} = 0,$$

with $\frac{\partial^2 E\Pi_b(.)}{\partial p_1^2} > 0$ holding at p_1^*. Because

$$\frac{\partial^2 E\Pi_b(.)}{\partial p_1^2} = \frac{\partial^2 E\Pi_b^G(.)}{\partial p_1^2} - \frac{\partial^2 Ec(.)}{\partial p_1^2} = \frac{\partial^2 E\Pi_b^G(.)}{\partial p_1^2} + 2\lambda\gamma F,$$

the second-order condition for the minimum boils down to satisfying $\frac{\partial^2 E\Pi_b^G(.)}{\partial p_1^2} > 0$. Assuming that is the case, it is easy to see that p_1^* must decline in w:

$$\frac{\partial p_1^*}{\partial w} = -\frac{\partial^2 E\Pi_b/\partial p_1 \partial w}{\partial^2 E\Pi_b/\partial p_1^2} = -\frac{\lambda}{\partial^2 E\Pi_b/\partial p_1^2} < 0.$$

We wish to exploit this property and set a lower bound on w such that p_1^* is always less than $1/2$. However, p_1^* is also sensitive to λ and we cannot ascertain the sign of $\partial p_1^*/\partial\lambda$; as we see,

$$\frac{\partial p_1^*}{\partial \lambda} = -\frac{\partial^2 E\Pi_b/\partial p_1 \partial \lambda}{\partial^2 E\Pi_b/\partial p_1^2} \;>\; (\leq)\; 0 \quad \text{if} \quad \frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda} \;<\; (\geq)\; 0,$$

and in general the sign of $\frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda}$ is ambiguous.

Now, we need to consider three ranges of λ separately to analyse the issue further.

Case 1 (Region A**).** In this range of λ, if p_1^* exists, it will be given by

$$\frac{\partial E\Pi_b}{\partial p_1} = \frac{\lambda\delta}{(p_1-\delta)^2} + \lambda[w + (\underline{\alpha} + 2\gamma p_1)F] - \gamma F = 0.$$

We can check that the second-order condition for a minimum is satisfied only if $\delta/(1-2\delta)^3 < \gamma F$, which violates our assumption. Hence, we rule out an interior minimum in this region.

Case 2 (Region B**).** In this case, if the following is satisfied, the profit function has a minimum:

$$\frac{\partial E\Pi_b}{\partial p_1} = \frac{1}{2\delta}\left[1 + \lambda - \frac{\lambda(2p_1+\delta)}{\sqrt{\lambda p_1(p_1+\delta)}}\right] + \lambda[w + (\underline{\alpha} + 2\gamma p_1)F] - \gamma F = 0,$$

because we can recall from the proof of Lemma 2 that the second-order condition will be met:

$$\frac{\partial^2 E\Pi_b^G(.)}{\partial p_1^2} = \frac{\lambda^2\delta}{4\pi_1^3} > 0 \quad \text{where} \quad \pi_1 = \sqrt{\lambda p_1(p_1+\delta)}.$$

Suppose p_1^* exists and derive

$$\frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda} = \frac{1}{2\delta}\left[1 - \frac{(2p_1^*+\delta)}{2\sqrt{\lambda p_1^*(p_1^*+\delta)}}\right] + w + (\underline{\alpha} + 2\gamma p_1^*)F \quad \leq \quad 0 \quad \text{or} \quad > 0,$$

depending on whether λ is below or above a critical value. That is to say, p_1^* initially rises with λ and then declines after reaching a maximum. Let us denote the maximum value of p_1^* as p_1^{*M} and the corresponding λ as λ^*.

We will now identify the level of w such that $p_1^{*M} = 1/2$. Note that by definition, p_1^* satisfies simultaneously two equations: $\frac{\partial E\Pi_b}{\partial p_1} = 0$ and $\frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda} = 0$. Let us set $p_1^* = 1/2$ in these two equations and write them as

$$\frac{\partial E\Pi_b}{\partial p_1} = 0 \;\Rightarrow\; \frac{1}{2\delta}\left[1 - \frac{(1+\delta)\sqrt{\lambda}}{\sqrt{1+2\delta}}\right] - \gamma F + \lambda\left[\frac{1}{2\delta}\left\{1 - \frac{1+\delta}{\sqrt{\lambda(1+2\delta)}}\right\} + w + (\underline{\alpha}+\gamma)F\right] = 0;$$

$$\frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda} = 0 \;\Rightarrow\; \frac{1}{2\delta}\left\{1 - \frac{1+\delta}{\sqrt{\lambda(1+2\delta)}}\right\} + w + (\underline{\alpha}+\gamma)F = 0.$$

We solve for

$$\lambda^* = \frac{(1+2\delta)}{(1+\delta)^2}(1-2\delta\gamma F)^2, \qquad \underline{w}_1 = \frac{1}{2\delta}\left\{\frac{(1+\delta)^2}{(1+2\delta)(1-2\delta\gamma F)} - 1\right\} - (\underline{\alpha}+\gamma)F. \quad (17)$$

If w is set above $\underline{w}$, then p_1^* will be less than $1/2$ in region B for any $\lambda > 0$.

Case 3 (Region C). In this scenario, $\pi_1 = \sqrt{\lambda p_1(p_1+\delta)}$ and $\pi_2 = \sqrt{(1-\lambda p_1)(p_2+\delta)}$, and p_1^* would satisfy the following equation:

$$\frac{\partial E\Pi_b}{\partial p_1} = \frac{1}{2\delta}\left[-\frac{\lambda(2p_1+\delta)}{\pi_1} + \frac{\lambda(2p_2+\delta)+(1-\lambda)}{\pi_2}\right] + \lambda[w + (\underline{\alpha} + 2\gamma p_1)F] - \gamma F = 0. \quad (18)$$

This would be a minimum because $\partial^2 E\Pi_b^G/\partial p_1^2 > 0$ as already seen in the proof of Lemma 2.

We also find (with some manipulation of terms):

$$\frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda} = \frac{1}{4\delta}\left[-\frac{2p_1^*+\delta}{\pi_1} + \frac{2p_2^*+\delta-1}{\pi_2} + \frac{p_2^*+\delta}{(1-\lambda p_1^*)\pi_2}\right] + w + (\underline{\alpha} + 2\gamma p_1)F. \quad (19)$$

Because π_1 varies inversely and π_2 positively with λ, it is apparent in Eq. (19) that when λ is sufficiently small, the negative term will dominate and that when λ is sufficiently large the positive term will dominate. Thus, we have $\partial p_1^*/\partial\lambda > 0$ up to a critical value of λ, which can be obtained by setting $\frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda} = 0$, and then $\partial p_1^*/\partial\lambda < 0$.

As in Case 2 above, here too we consider two equations – Eq. (18) and $\frac{\partial^2 E\Pi_b}{\partial p_1 \partial \lambda} = 0$ (using Eq. (19)) – and solve for p_1^{*M}, which corresponds to the highest p_1^* at any given w. Then, set $p_1^{*M} = 1/2$ and obtain the following equation to solve for λ^*:

$$(1+2\delta)(2-\lambda)\sqrt{\lambda(2-\lambda)} + \lambda(3+\lambda\delta) - 4 = 0. \quad (20)$$

There exists a λ between 0 and 1 which satisfies Eq. (20), and its uniqueness can be verified.

Then, we determine $\underline{w}_2$ after substituting λ^* in Eq. (18) as follows:

$$\underline{w}_2 = \frac{1}{\delta\sqrt{(1+2\delta)}}\left[\frac{1+\delta}{\sqrt{\lambda^*}} - \frac{1+\delta\lambda^*}{\lambda^*\sqrt{2-\lambda^*}}\right] - (\underline{\alpha}+\gamma)F. \quad (21)$$

If we set $w > \underline{w}_2$, then p_1^* will be strictly less than $1/2$ at all $\lambda > 0$ in region C.

Now combining Case 2 and Case 3, we can say that if $w > \max\{\underline{w}_1, \underline{w}_2\}$, then $p_1^* < 1/2$ is guaranteed at all $\lambda > 0$ and $p_1 \in [\delta, 1-\delta]$. **Q.E.D.**

Proof of Proposition 3. Define

$$g(p_1;\lambda) = E\Pi_b(p_1;\lambda) - E\Pi_0(p_1). \quad (22)$$

By assumption, $g(p_1;\lambda = 1-\delta) < 0$ at all p_1, and by the very nature of the two profit functions $g(p_1;\lambda = 0) > 0$ over a range of p_1, say $(\underline{a}, \underline{b})$ where $\underline{a}$ and $\underline{b}$ are the two roots of $g(p_1;0)=0$. Because $E\Pi_0$ is symmetric and $E\Pi_b(\lambda=0)$ is constant when $c'(p_1)=0$, in that special case, we also have $\underline{a}=1-\underline{b}$.

Now, for $\lambda>0$, define $a_1(\lambda)$ and $b_1(\lambda)$ to be the two solutions of the equation $g(p_1;\lambda)=0$. At all $p_1<a_1(\lambda)$ and all $p_1>b_1(\lambda)$, $g(p_1;\lambda)<0$, and at all $p_1\in(a_1(\lambda),b_1(\lambda))$, $g(p_1;\lambda)>0$. Consider $g(a_1(\lambda);\lambda)=0$, and write

$$\frac{\partial g(.)}{\partial a_1}a_1'(\lambda)+\frac{\partial g(.)}{\partial \lambda}=0.$$

Note that $\frac{\partial g(.)}{\partial a_1}$ is nothing but $\frac{\partial g(.)}{\partial p_1}$ evaluated at $p_1=a_1$. By definition, at $p_1=a_1$, $\frac{\partial g(.)}{\partial p_1}>0$ as $E\Pi_b>E\Pi_0$ locally to the right of a_1. In addition, we know $\frac{\partial g(.)}{\partial \lambda}=\frac{\partial E\Pi_b}{\partial \lambda}<0$ by Lemma 1. Hence from the above, we derive

$$a_1'(\lambda)=-\frac{\partial E\Pi_b/\partial\lambda}{\partial g(.)/\partial p_1}>0.$$

Similarly, using the fact that $\frac{\partial g(.)}{\partial p_1}<0$ at $p_1=b_1$, as $E\Pi_b<E\Pi_0$ locally to the right of b_1, from $g(b_1(\lambda);\lambda)=0$, we derive

$$b_1'(\lambda)=-\frac{\partial E\Pi_b/\partial\lambda}{\partial g(.)/\partial p_1}<0.$$

Clearly, $\hat{\lambda}$ must exist by the continuity argument, and $a_1(\hat{\lambda})=b_1(\hat{\lambda})$ $(=a(\hat{\lambda})$, say). If $a(\hat{\lambda})$ were not greater than $1/2$, there would be two possibilities: either $a(\hat{\lambda})=1/2$ or $a(\hat{\lambda})<1/2$. The case of $a(\hat{\lambda})=1/2$ is ruled out by the fact that the two profit curves cannot be tangent at any λ (including $\lambda=1$) at $p_1=1/2$ as long as $c>0$. For $a(\hat{\lambda})<1/2$, i.e. two curves to be tangent at $p_1<1/2$, we must have the minimum of $E\Pi_b$ occurring at $p_1>1/2$. However, that is not possible by Lemma 3. Hence, we must have $a(\hat{\lambda})>1/2$.

We argue that if $c'(p_1)=0$, it must be that $a_1(\lambda)>1-b_1(\lambda)$. This follows from the fact that $E\Pi_b$ must have a minimum, and the minimum occurs at some $p_1<1/2$ by Lemma 3. If the two intersection points both correspond to the $E\Pi_b$ curve's rising segment, then our claim obviously holds. When the two intersection points fall on either segment of the $E\Pi_b$ curve, the minimum $E\Pi_b$ must be greater than the minimum of $E\Pi_0$.

Because the minimum of $E\Pi_b$ occurs at $p_1^*<1/2$, at $p_1=1/2$, not only do we have $E\Pi_b>E\Pi_0$, but there will also be a bigger gap between $E\Pi_b$ and $E\Pi_0$ than at p_1^*. That is to say, $g(p_1;\lambda)$ must be positive and rising between p_1^* and $1/2$. Hence, for $E\Pi_0$ to catch up with $E\Pi_b$ it must rise sufficiently. This would imply b_1 must be further away from $1/2$ on the right-hand side than a_1 is on the left-hand side.

When $c'(p_1) > 0$, the $E\Pi_b$ curve will not have an interior minimum at sufficiently small values of λ; it will be a declining curve, which implies that $a_1 < 1 - b_1$. At higher values of λ the $E\Pi_b$ curve will have an interior minimum, leading to $a_1 > 1 - b_1$; the same argument as in the case of $c'(p_1) = 0$ will apply. **Q.E.D.**

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