Colouring Diamond-free Graphs*

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Abstract. The Colouring problem is that of deciding, given a graph G
and an integer k, whether G admits a (proper) k-colouring. For all graphs H
up to five vertices, we classify the computational complexity of Colouring
for (diamond, H)-free graphs. Our proof is based on combining known results
together with proving that the clique-width is bounded for (diamond, $P_1 + 2P_2$)-
free graphs. Our technique for handling this case is to reduce the graph under
consideration to a k-partite graph that has a very specific decomposition. As
a by-product of this general technique we are also able to prove boundedness
of clique-width for four other new classes of (H_1, H_2)-free graphs. As such,
our work also continues a recent systematic study into the (un)boundedness
of clique-width of (H_1, H_2)-free graphs, and our five new classes of bounded
clique-width reduce the number of open cases from 13 to 8.

1 Introduction

The Colouring problem is that of testing whether a given graph can be coloured with
at most k colours for some given integer k, such that any two adjacent vertices receive
different colours. The complexity of Colouring is fully understood for general graphs:
it is NP-complete even if $k = 3$ [35]. Therefore it is natural to study its complexity
when the input is restricted. A classic result in this area is due to Grötschel, Lovász,
and Schrijver [26], who proved that Colouring is polynomial-time solvable for perfect
graphs.

As surveyed in [14,20,25,43], Colouring has been well studied for hereditary
graph classes, that is, classes that can be defined by a family $\mathcal{H}$ of forbidden induced
subgraphs. For a family $\mathcal{H}$ consisting of one single forbidden induced subgraph H, the
complexity of Colouring is completely classified: the problem is polynomial-time
solvable if H is an induced subgraph of P_4 or $P_1 + P_3$ and NP-complete otherwise [34].
Hence, many papers (e.g. [13,18,29,34,37,40,41,45]) have considered the complexity
of Colouring for bigenic hereditary graph classes, that is, graph classes defined by
families $\mathcal{H}$ consisting of two forbidden graphs H_1 and H_2; such classes of graphs are
also called (H_1, H_2)-free. This classification is far from complete (see [25] for the state
of art). In fact there are still an infinite number of open cases, including cases where
both H_1 and H_2 are small. For instance, Lozin and Malyshev [37] determined the
computational complexity of Colouring for (H_1, H_2)-free graphs for all graphs H_1
and H_2 up to four vertices except when $(H_1, H_2) \in \{(K_{1,3}, 4P_1), (K_{1,3}, 2P_1 + P_2),$
$(C_4, 4P_1)\}$ (we refer to Section 2 for notation and terminology).

The *diamond* is the graph $\overline{2P_1 + P_2}$, that is, the graph obtained from the complete
graph on four vertices by removing an edge. Diamond-free graphs are well studied in the

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literature. For instance, Tucker [46] gave an $O(kn^2)$ time algorithm for COLOURING for perfect diamond-free graphs. It is also known that that COLOURING is polynomial-time solvable for diamond-free graphs that contain no induced cycle of even length [32] as well as for diamond-free graphs that contain no induced cycle of length at least 5 [8]. Diamond-free graphs also played an important role in proving that the class of P_6-free graphs contains 24 minimal obstructions for 4-COLOURING [15] (that is, the COLOURING problem for $k = 4$).

1.1 Our Main Result

In this paper we focus on COLOURING for (diamond, H)-free graphs where H is a graph on at most five vertices. It is known that COLOURING is NP-complete for (diamond, H)-free graphs when H contains a cycle or a claw [34] and polynomial-time solvable for $H = sP_1 + P_2$ $(s \geq 0)$ [18], $H = 2P_1 + P_3$ [5], $H = P_1 + P_4$ [11], $H = P_2 + P_3$ [19] and $H = P_5$ [1]. Hence, the only graph H on five vertices that remains is $H = P_1 + 2P_2$, for which we prove polynomial-time solvability in this paper. This leads to the following result.

Theorem 1. *Let H be a graph on at most five vertices. Then COLOURING is polynomial-time solvable for (diamond, H)-free graphs if H is a linear forest and NP-complete otherwise.*

To solve the case $H = P_1 + 2P_2$, one could try to reduce to a subclass of diamond-free graphs, for which COLOURING is polynomial-time solvable, such as the aforementioned results of [8,32,46]. This would require us to deal with the presence of small cycles up to C_7, which may not be straightforward. Instead we aim to identify tractability from an underlying property: we show that the class of (diamond, $P_1 + 2P_2$)-free graphs has bounded clique-width. This approach has several advantages and will lead to a number of additional results, as we will discuss in the remainder of Section 1.

Clique-width is a graph decomposition that can be constructed via vertex labels and four specific graph operations, which ensure that vertices labelled alike will always keep the same label and thus behave identically. The clique-width of a graph G is the minimum number of different labels needed to construct G using these four operations (we refer to Section 2 for a precise definition). A graph class $\mathcal{G}$ has *bounded* clique-width if there exists a constant c such that every graph from $\mathcal{G}$ has clique-width at most c.

Clique-width is a well-studied graph parameter (see, for instance, the surveys [27,31]). An important reason for the popularity of clique-width is that a number of classes of NP-complete problems, such as those that are definable in Monadic Second Order Logic using quantifiers on vertices but not on edges, become polynomial-time solvable on any graph class $\mathcal{G}$ of bounded clique-width (this follows from combining results from [16,23,33,44] with a result from [42]). The COLOURING problem is one of the best-known NP-complete problems that is solvable in polynomial time on graph classes of bounded clique-width [33]; another well-known example of such a problem is HAMILTON PATH [23].

1.2 Methodology

The key technique for proving that (diamond, P_1+2P_2)-free graphs have bounded clique-width is the use of a certain graph decomposition of k-partite graphs. We obtain this decomposition by generalizing the so-called canonical decomposition of bipartite graphs, which decomposes a bipartite graph into two smaller bipartite graphs such that edges between these two smaller bipartite graphs behave in a very restricted way. Fouquet,

Giakoumakis and Vanherpe [24] introduced this decomposition and characterized exactly those bipartite graphs that can recursively be canonically decomposed into graphs isomorphic to K_1. Such bipartite graphs are said to be totally decomposable by canonical decomposition. We say that k-partite graphs are totally k-decomposable if they can be, according to our generalized definition, recursively k-decomposed into graphs isomorphic to K_1. We show that totally k-decomposable graphs have clique-width at most $2k$. We prove this result in Section 3, where we also give a formal definition of canonical decomposition, along with our generalization.

Our goal is to transform (diamond, $P_1 + 2P_2$)-free graphs into graphs in some class for which we already know that the clique-width is bounded. Besides the class of totally k-decomposable graphs, we will also reduce to other known graph classes of bounded clique-width, such as the class of (diamond, $P_2 + P_3$)-free graphs [19] and certain classes of H-free bipartite graphs [21]. Of course, our transformations must not change the clique-width by "too much". We ensure this by using certain graph operations (described in Section 2) that are known to preserve (un)boundedness of clique-width [31,38].

## 1.3   Consequences for Clique-Width

There are numerous papers (as listed in, for instance, [22,27,31]) that determine the (un)boundedness of the clique-width or variants of it (see e.g. [4,28]) of special graph classes. Due to the complex nature of clique-width, proofs of these results are often long and technical, and there are still many open cases. In particular, gaps exist in a number of dichotomies on the (un)boundedness of clique-width for graph classes defined by one or more forbidden induced subgraphs. As such our paper also continues a line of research [5,6,19,21,22] in which we focus on these gaps in a *systematic* way. It is known [22] that the class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of P_4. Over the years many partial results [2,7,9,10,11,12,20,39] on the (un)boundedness of clique-width have appeared for classes of (H_1, H_2)-free graphs, but until recently [22] it was not even known whether the number of missing cases was bounded. Combining these older results with recent progress [5,18,19,22] reduced the number of open cases to 13 (up to an equivalence relation) [22].

As a by-product of our general methodology, we are able not only to settle the case $(H_1, H_2) = $ (diamond, $P_1 + 2P_2$), but in fact we solve **five** of the remaining 13 open cases by proving that the class of (H_1, H_2)-free graphs has bounded clique-width if

> **1–4:** $H_1 = K_3$ and $H_2 \in \{P_1 + 2P_2, P_1 + P_2 + P_3, P_1 + P_5, S_{1,2,2}\}$ or
>
> **5:** $H_1 = $ diamond and $H_2 = P_1 + 2P_2$.

The above graphs are displayed in Fig. 1. Note that the $(K_3, P_1 + 2P_2)$-free graph case is properly contained in all four of the other cases. These four other newly solved cases are pairwise incomparable. In Section 4 we use our key technique on totally k-decomposable graphs to find a number of sufficient conditions for a graph class to have bounded clique-width. We use these conditions in Section 5 to prove Results 1–4 and we then prove Result 5 (which relies on Result 1) in Section 6.

Updating the classification (see [22]) with our five new results gives the following theorem. Here, $\mathcal{S}$ is the class of graphs each connected component of which is either a subdivided claw or a path, and we write $H \subseteq_i G$ if H is an induced subgraph of G; see Section 2 for notation that we have not formally defined yet.

K_3 diamond $P_1 + 2P_2$ $P_1 + P_2 + P_3$ $P_1 + P_5$ $S_{1,2,2}$
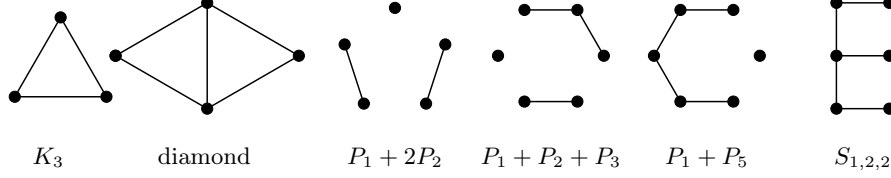
Fig. 1. The forbidden graphs considered in this paper.

Theorem 2. *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then:*

(i) *$\mathcal{G}$ has bounded clique-width if it is equivalent[1] to a class of (H_1, H_2)-free graphs such that one of the following holds:*

     1. H_1 or $H_2 \subseteq_i P_4$;
     2. $H_1 = sP_1$ and $H_2 = K_t$ for some s, t;
     3. $H_1 \subseteq_i P_1 + P_3$ and $\overline{H_2} \subseteq_i K_{1,3} + 3P_1$, $K_{1,3} + P_2$, $P_1 + P_2 + P_3$, $P_1 + P_5$,
        $P_1 + S_{1,1,2}$, P_6, $S_{1,1,3}$ or $S_{1,2,2}$;
     4. $H_1 \subseteq_i 2P_1 + P_2$ and $\overline{H_2} \subseteq_i P_1 + 2P_2$, $2P_1 + P_3$, $3P_1 + P_2$ or $P_2 + P_3$;
     5. $H_1 \subseteq_i P_1 + P_4$ and $\overline{H_2} \subseteq_i P_1 + P_4$ or P_5;
     6. $H_1 \subseteq_i 4P_1$ and $\overline{H_2} \subseteq_i 2P_1 + P_3$;
     7. $H_1, \overline{H_2} \subseteq_i K_{1,3}$.

(ii) *$\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*

     1. $H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$;
     2. $\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$;
     3. $H_1 \supseteq_i K_{1,3}$ or $2P_2$ and $\overline{H_2} \supseteq_i 4P_1$ or $2P_2$;
     4. $H_1 \supseteq_i 2P_1 + P_2$ and $\overline{H_2} \supseteq_i K_{1,3}$, $5P_1$, $P_2 + P_4$ or P_6;
     5. $H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1 + 2P_2$, $2P_1 + P_4$, $4P_1 + P_2$, $3P_2$ or $2P_3$;
     6. $H_1 \supseteq_i 4P_1$ and $\overline{H_2} \supseteq_i P_1 + P_4$ or $3P_1 + P_2$.

1.4   Future Work

Naturally we would like to extend Theorem 1 and solve the following open problem.

Open Problem 1. *What is the computational complexity of the* Colouring *problem for* (*diamond, H*)-*free graphs when H is a graph on at least six vertices?*

Solving Open Problem 1 is highly non-trivial. It is known that 4-Colouring is NP-complete for (C_3, P_{22})-free graphs [30]. Hence, the polynomial-time results in Theorem 1 cannot be extended to all linear forests. The first open case to consider would be $H = P_6$, for which only partial results are known. Indeed, Colouring is polynomial-time solvable for (C_3, P_6)-free graphs [9], but its complexity is unknown for (C_3, P_7)-free graphs (on a side note, a recent result for the latter graph class is that 3-Colouring is polynomial-time solvable [3]).

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[1] Given four graphs H_1, H_2, H_3, H_4, the class of (H_1, H_2)-free graphs and the class of (H_3, H_4)-free graphs are *equivalent* if the unordered pair H_3, H_4 can be obtained from the unordered pair H_1, H_2 by some combination of the operations (i) complementing both graphs in the pair and (ii) if one of the graphs in the pair is K_3, replacing it with $\overline{P_1 + P_3}$ or vice versa. If two classes are equivalent, then one of them has bounded clique-width if and only if the other one does (see [22]).

We observe that boundedness of the clique-width of (diamond, $P_1 + 2P_2$)-free graphs implies boundedness of the clique-width of $(2P_1 + P_2, \overline{P_1 + 2P_2})$-free graphs (recall that the diamond is the complement of the graph $2P_1 + P_2$). Hence our results imply that COLOURING can also be solved in polynomial time for graphs in this class. After incorporating the consequences of our new results and this additional observation, there are 13 classes of (H_1, H_2)-free graphs for which COLOURING could potentially still be solved in polynomial time by showing that their clique-width is bounded (see also [25]):

Open Problem 2. *Is* COLOURING *polynomial-time solvable for (H_1, H_2)-free graphs when:*

1. $\overline{H_1} \in \{3P_1, P_1 + P_3\}$ *and* $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$;
2. $H_1 = 2P_1 + P_2$ *and* $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$;
3. $H_1 = diamond$ *and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$;
4. $H_1 = P_1 + P_4$ *and* $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$;
5. $\overline{H_1} = P_1 + P_4$ *and* $H_2 \in \{P_1 + 2P_2, P_2 + P_3\}$;
6. $H_1 = \overline{H_2} = 2P_1 + P_3$.

As mentioned in Section 1.3, after updating the list of remaining open cases for clique-width from [22], we find that eight non-equivalent open cases remain for clique-width. These are the following cases.

Open Problem 3. *Does the class of (H_1, H_2)-free graphs have bounded or unbounded clique-width when:*

1. $H_1 = 3P_1$ *and* $\overline{H_2} \in \{P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,3}\}$;
2. $H_1 = 2P_1 + P_2$ *and* $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$;
3. $H_1 = P_1 + P_4$ *and* $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ *or*
4. $H_1 = \overline{H_2} = 2P_1 + P_3$.

Bonomo, Grippo, Milanič and Safe [4] determined all pairs of connected graphs H_1, H_2 for which the class of (H_1, H_2)-free graphs has power-bounded clique-width. In order to compare their result with our results for clique-width, we would only need to solve the single open case $(H_1, H_2) = (K_3, S_{1,2,3})$, which is equivalent to the (open) case $(H_1, H_2) = (3P_1, \overline{S_{1,2,3}})$ mentioned in Open Problem 3. This follows because our new result for the case $(H_1, H_2) = (K_3, S_{1,2,2})$ has reduced the number of open cases (H_1, H_2) with H_1, H_2 both connected from two to one.

2 Preliminaries

Throughout our paper we only consider finite, undirected graphs without multiple edges or self-loops. Below we define further graph terminology.

The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG. The *complement* of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. We use $G \setminus S$ to denote the graph obtained from G by deleting every vertex in S, i.e. $G \setminus S = G[V(G) \setminus S]$. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G.

The graphs $C_r, K_r, K_{1,r-1}$ and P_r denote the cycle, complete graph, star and path on r vertices, respectively. The graph $K_{1,3}$ is also called the *claw*. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. The graph $S_{1,2,2}$ is also known as the E, since it can be drawn like a capital letter E (see Fig. 1). Recall that the graph $\overline{2P_1 + P_2}$ is known as the *diamond*. The graphs K_3 and $\overline{P_1 + 2P_2}$ are also known as the *triangle* and the 5-vertex *wheel*, respectively. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph G is $(H_1, \ldots, H_p)$-*free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write H_1-free instead of (H_1)-free.

Let X be a set of vertices in a graph $G = (V, E)$. A vertex $y \in V \setminus X$ is *complete* to X if it is adjacent to every vertex of X and *anti-complete* to X if it is non-adjacent to every vertex of X. Similarly, a set of vertices $Y \subseteq V \setminus X$ is *complete* (*anti-complete*) to X if every vertex in Y is complete (anti-complete) to X. A vertex y or a set Y is *trivial* to X if it is either complete or anti-complete to X. Note that if Y contains both vertices complete to X and vertices not complete to X, we may have a situation in which every vertex in Y is trivial to X, but Y itself is not trivial to X.

For a graph $G = (V, E)$, the set $N(u) = \{v \in V \mid uv \in E\}$ denotes the neighbourhood of $u \in V$. Let X and Y be disjoint sets of vertices in a graph $G = (V, E)$. If every vertex of X has at most one neighbour in Y and vice versa then we say that the edges between X and Y form a *matching*. If every vertex of X has exactly one neighbour in Y and vice versa then we say that the edges between X and Y form a *perfect matching*.

A graph is k-*partite* if its vertex set can be partitioned into k independent sets (some of which may be empty). A graph is *bipartite* if it is 2-partite. A graph is *complete bipartite* if its vertex set can be partitioned into two independent sets that are complete to each other. For integers $r, s \geq 0$, the *biclique* $K_{r,s}$ is the complete bipartite graph with sets in the partition of size r and s respectively. The *bipartite complement* of a bipartite graph G with bipartition (X, Y) is the graph obtained from G by replacing every edge from a vertex in X to a vertex in Y by a non-edge and vice versa.

Clique-Width. The *clique-width* of a graph G, denoted $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

1. creating a new graph consisting of a single vertex v with label i;
2. taking the disjoint union of two labelled graphs G_1 and G_2;
3. joining each vertex with label i to each vertex with label j ($i \neq j$);
4. renaming label i to j.

An algebraic term that represents such a construction of G and uses at most k labels is said to be a k-*expression* of G (i.e. the clique-width of G is the minimum k for which G has a k-expression). Recall that a class of graphs $\mathcal{G}$ has bounded clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. We define the following operations. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets S and T in G, the *bipartite complementation* operation with respect to S and T acts on G by replacing every edge with one end-vertex in S and the other one in T by a non-edge and vice versa.

We now state some useful facts about how the above operations (and some other ones) influence the clique-width of a graph. We will use these facts throughout the paper. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k, γ)-*obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and

(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [38].

Fact 2. Subgraph complementation preserves boundedness of clique-width [31].

Fact 3. Bipartite complementation preserves boundedness of clique-width [31].

The following lemma is easy to show.

Lemma 1. *The clique-width of a graph of maximum degree at most* 2 *is at most* 4.

Two vertices are *false twins* if they have the same neighbourhood (note that such vertices must be non-adjacent). The following lemma follows immediately from the definition of clique-width.

Lemma 2. *If a vertex x in a graph G has a false twin then* $\mathrm{cw}(G) = \mathrm{cw}(G \setminus \{x\})$.

We will also make use of the following two results.

Lemma 3 ([19]). *The class of* (*diamond*, $P_2 + P_3$)*-free graphs has bounded clique-width.*

Lemma 4 ([21]). *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if*

- $H = sP_1$ *for some $s \geq 1$;*
- $H \subseteq_i K_{1,3} + 3P_1$;
- $H \subseteq_i K_{1,3} + P_2$;
- $H \subseteq_i P_1 + S_{1,1,3}$ *or*
- $H \subseteq_i S_{1,2,3}$.

In some of our proofs we will use the fact that $S_{1,2,3}$-free bipartite graphs have bounded clique-width, which follows from Lemma 4. Alternatively we could have used the result of Lozin [36], who showed that $S_{1,2,3}$-free bipartite graphs have clique-width at most 5.

3   Totally k-Decomposable Graphs

In this section we describe our key technique, which is based on a decomposition of bipartite graphs introduced by Fouquet, Giakoumakis and Vanherpe [24], which is defined as follows.
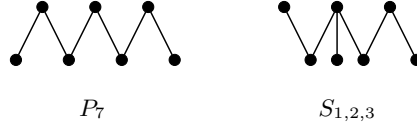
Let G be a bipartite graph with a vertex bipartition (V_1, V_2). A 2-*decomposition of G* with respect to (V_1, V_2) consists of two *non-empty* graphs $G[V_1' \cup V_2']$ and $G[V_1'' \cup V_2'']$ such that:

(i) for $i \in \{1, 2\}$, $V_i' \cup V_i'' = V_i$ and $V_i' \cap V_i'' = \emptyset$;

(ii) V_1' is either complete or anti-complete to V_2'' in G;

(iii) V_2' is either complete or anti-complete to V_1'' in G.

Note that $V_1' \cup V_1''$ and $V_2' \cup V_2''$ are independent sets in G and that the last two conditions imply that each of $G[V_1' \cup V_2'']$ and $G[V_1'' \cup V_2']$ is either an independent set or a biclique. Observe that we do not impose restrictions on the bipartite graphs $G' = G[V_1' \cup V_2']$ and $G'' = G[V_1'' \cup V_2'']$. If G has a 2-decomposition G', G'' with respect to some bipartition, we say that G can be 2-*decomposed* into G' and G''. A graph G is *totally decomposable by canonical decomposition* if it can be recursively 2-decomposed into graphs isomorphic to K_1. Note that if G has a 2-decomposition G', G'' with respect to some bipartition (V_1, V_2), this does not force us to decompose G' and G'' with respect to a sub-partition of (V_1, V_2). As we will see, this distinction does not make a difference for bipartite graphs, but it will become an issue when we extend the notion to k-partite graphs when $k \geq 3$.

Fouquet, Giakoumakis and Vanherpe proved the following characterization, which we will need for our proofs (see Fig. 2 for pictures of P_7 and $S_{1,2,3}$).

Lemma 5 ([24]). *A bipartite graph is totally decomposable by canonical decomposition if and only if it is $(P_7, S_{1,2,3})$-free.*

P_7 $S_{1,2,3}$

Fig. 2. The forbidden graphs from Lemma 5.

For our purposes we need to generalize the notion of totally decomposable bipartite graphs to k-partite graphs for $k \geq 2$, and we will also need to partially classify graphs with this modified notion, in effect generalizing Lemma 5.

Let G be a k-partite graph with a *fixed* vertex k-partition $(V_1, \ldots, V_k)$. A k-*decomposition of G* with respect to the partition $(V_1, \ldots, V_k)$ consists of two non-empty graphs, each with their own partition: $G' = G[V_1' \cup \cdots \cup V_k']$ with partition $(V_1', V_2', \ldots, V_k')$ and $G'' = G[V_1'' \cup \cdots \cup V_k'']$ with partition $(V_1'', V_2'', \ldots, V_k'')$, such that:

(i) for $i \in \{1, \ldots, k\}$, $V_i' \cup V_i'' = V_i$ and $V_i' \cap V_i'' = \emptyset$;
(ii) for all $i, j \in \{1, \ldots, k\}$, V_i' is either complete or anti-complete to V_j'' in G.

Note that the last condition holds for $i = j$ by definition, since $V_i = V_i' \cup V_i''$ is an independent set in G. Also note that in the above definition, $(V_1', V_2', \ldots, V_k')$ and $(V_1'', V_2'', \ldots, V_k'')$ are sub-partitions of $(V_1, V_2, \ldots, V_k)$, in the sense that $V_i' = V_i \cap V(G')$ and $V_i'' = V_i \cap V(G)$ for $i \in \{1, \ldots, k\}$, so the original partition on G uniquely specifies the partitions on G' and G''.

If a graph G with a fixed k-partition has a k-decomposition with respect to this partition into two graphs G' and G'' (with their associated sub-partitions), we say that G can be k-*decomposed* into G' and G'' (with each of these subgraphs getting the appropriate sub-partition). We say that G is *totally k-decomposable* with respect to some fixed partition if G can be recursively k-decomposed *with respect to this fixed partition* into graphs isomorphic to K_1. Note that by definition, if a graph H appears in a total k-decomposition of G with respect to some fixed partition $(V_1, \ldots, V_k)$, then the k-partition $(V_1^H, V_2^H, \ldots, V_k^H)$ of H used to partition H satisfies $V_i^H = V_i \cap V(H)$ for $i = 1, \ldots, k$. This property will be necessary for us to be able to use inductive arguments "safely."

To compare graphs that are totally decomposable by canonical decomposition and graphs that are totally 2-decomposable, we observe that every connected bipartite graph G has a unique bipartition (up to isomorphism and swapping the two independent sets in the bipartition). Also, if G is totally decomposable by canonical decomposition, then this decomposition can recursively be done component-wise. Hence, in each step of the recursion, we may decompose with respect to an arbitrary bipartition of the graph under consideration. This means that the definitions of total canonical decomposability and total 2-decomposability are equivalent. However, for $k > 2$, a connected graph can have multiple k-partitions, even up to isomorphism and permuting the independent sets of the partition. Therefore, unlike for $k = 2$, we need to fix the partition of the subgraphs G' and G'' in the definition of total k-decomposability.

As mentioned, for our proofs we need to generalize Lemma 5. It seems difficult to give a full characterization of totally k-decomposable graphs for $k \geq 3$. However, the following lemma is sufficient for our purposes.

Lemma 6. *A 3-partite graph G is totally 3-decomposable with respect to a 3-partition (V_1, V_2, V_3) if the following two conditions are both satisfied:*

- *$G[V_1 \cup V_2], G[V_1 \cup V_3]$ and $G[V_2 \cup V_3]$ are all $(P_7, S_{1,2,3})$-free, and*
- *for every $v_1 \in V_1$, every $v_2 \in V_2$ and every $v_3 \in V_3$, the graph $G[v_1, v_2, v_3]$ is isomorphic neither to K_3 nor to $3P_1$.*

Proof. Let G be a 3-partite graph with a 3-partition (V_1, V_2, V_3) such that both conditions are satisfied. Note that any induced subgraph H of G (with partition $(V(H) \cap V_1, V(H) \cap V_2, V(H) \cap V_3)$) also satisfies the hypotheses of the lemma. This enables us to apply induction. It is therefore sufficient to show that G has a 3-decomposition with respect to the given 3-partition.

If V_1 is empty then G is a $(P_7, S_{1,2,3})$-free bipartite graph and is therefore totally 2-decomposable with respect to the partition (V_2, V_3) by Lemma 5 (and is thus totally 3-decomposable with respect to the partition (V_1, V_2, V_3)). By symmetry, we may therefore assume that every set V_i is non-empty.

Now $G[V_1, V_2]$ is a bipartite $(P_7, S_{1,2,3})$-free graph, so by Lemma 5, $G[V_1 \cup V_2]$ is totally 2-decomposable. Since V_1 and V_2 are both non-empty, it follows that V_1 can be partitioned into two sets V_1' and V_1'' and V_2 can be partitioned into two sets V_2' and V_2'', such that V_1' is either complete or anti-complete to V_2'', and V_2' is either complete or anti-complete to V_1''. Since the graphs $G[V_1' \cup V_2']$ and $G[V_1'' \cup V_2'']$ in this decomposition must be non-empty, it follows that $V_1' \cup V_2'$ and $V_1'' \cup V_2''$ must be non-empty. Since for $i \in \{1, 2\}$ we know that $V_i = V_i' \cup V_i''$ is non-empty, at least one of V_i' and V_i'' is non-empty. Hence, combining these two observations, we may assume without loss of generality that V_1' and V_2'' are non-empty. Assume that these sets are maximal, that is, no vertex of V_1'' (respectively V_2') can be moved to V_1' (respectively V_2''). Note that V_1'' or V_2' may be empty.

We will prove that we can partition V_3 into sets V_3' and V_3'', such that for all $i, j \in \{1, 2, 3\}$, V_i' is complete or anti-complete to V_j''. Note that we already know that V_1' (respectively V_2') is complete or anti-complete to V_2'' (respectively V_1''). Also note that for $i \in \{1, 2, 3\}$, V_i' is automatically anti-complete to V_i'', since V_i is an independent set.

First suppose that V_1' is complete to V_2''. If a vertex of V_3 has a neighbour in both V_1' and V_2'' then these three vertices would form a forbidden K_3, so every vertex in V_3 is anti-complete to V_1' or V_2''. Let V_3' be the set of vertices in V_3 that are anti-complete to V_2'' and let $V_3'' = V_3 \setminus V_3'$. Note that V_3'' must be anti-complete to V_1'. Suppose, for contradiction, that $z \in V_3'$ has a non-neighbour $v \in V_1''$. Since V_1' is maximal, v must have a non-neighbour $w \in V_2''$. This means that $G[v, w, z]$ is a $3P_1$.

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This contradiction means that V_1'' is complete to V_3'. Similarly, V_2' is complete to V_3''. Therefore $G[V_1' \cup V_2' \cup V_3']$ and $G[V_1'' \cup V_2'' \cup V_3'']$ form the required 3-decomposition of G.

Now suppose that V_1' is anti-complete to V_2''. If a vertex of V_3 has a non-neighbour in both V_1' and V_2'' then these three vertices would induce a forbidden $3P_1$, so every vertex in V_3 is complete to V_1' or V_2''. Let V_3' be the set of vertices in V_3 that are complete to V_2'' and let $V_3'' = V_3 \setminus V_3'$. Note that V_3'' must be complete to V_1'. By using similar arguments to those in the previous case, we find that V_1'' is anti-complete to V_3' and V_2' is anti-complete to V_3''. Hence, $G[V_1' \cup V_2' \cup V_3']$ and $G[V_1'' \cup V_2'' \cup V_3'']$ form the required 3-decomposition of G. This completes the proof. $\qquad\square$

We also need the following lemma.

Lemma 7. *Let G be a k-partite graph with vertex partition $(V_1, \ldots, V_k)$. If G is totally k-decomposable with respect to this partition, then the clique-width of G is at most $2k$. Moreover, there is a $2k$-expression for G that assigns, for $i \in \{1, \ldots, k\}$, label i to every vertex of V_i.*

Proof. We prove the lemma by induction on the number of vertices. If G contains only one vertex then the lemma holds trivially. Suppose that the lemma is true for all k-partite graphs H on at most n vertices and for all k-partitions $(V_1^H, \ldots, V_2^H)$ with respect to which H is totally k-decomposable. Let G be a graph on $n+1$ vertices that is totally k-decomposable with respect to a vertex partition $(V_1, \ldots, V_k)$. Then, we can partition every set V_i into two sets V_i' and V_i'' in such a way that each set V_i' is either complete or anti-complete to each set V_j'' for all $i, j \in \{1, \ldots, k\}$ and $G' = G[V_1' \cup \ldots \cup V_k']$ and $G'' = G[V_1'' \cup \ldots \cup V_k'']$ are totally k-decomposable with respect to the partitions $(V_1', \ldots, V_k')$ and $(V_1'', \ldots, V_k'')$, respectively.

As both G' and G'' are smaller graphs that G, we can apply the induction hypothesis. Hence, we can find a $2k$-expression that constructs G' such that the vertices in each set V_i' have label i for $i \in \{1, \ldots, k\}$. Similarly, we can find a $2k$-expression that constructs G'' such that the vertices in each set V_j'' have label $k+j$ for $j \in \{1, \ldots, k\}$. We take the disjoint union of these two constructions. Next, for $i, j \in \{1, \ldots, k\}$, we join the vertices with label i to the vertices with label $k+j$ if and only if V_i' is complete to V_j'' in G. Finally, for $i \in \{1, \ldots, k\}$, we relabel the vertices with label $k+i$ to have label i. This completes the proof of the lemma. $\qquad\square$

4 Sufficient Conditions for $(K_3, S_{1,2,3})$-free Graphs

We observe that the classes of $(K_3, P_1 + 2P_2)$-free, $(K_3, P_1 + P_2 + P_3)$-free, $(K_3, P_1 + P_5)$-free and $(K_3, S_{1,2,2})$-free graphs are all subclasses of the class of $(K_3, S_{1,2,3})$-free graphs. In order to prove that each of the four subclasses has bounded clique-width, we investigate, in this section, sufficient conditions for a subclass of $(K_3, S_{1,2,3})$-free graphs to be of bounded clique-width. We present these conditions in Corollary 1 and Lemma 9. Corollary 1 follows from a structural result (Lemma 8), which we prove first. The proof of Lemma 9 uses the results from the previous section. We will not use Corollary 1 and Lemma 9 directly when proving that the class of (diamond, $P_1 + 2P_2$)-free graphs has bounded clique-width. However, our proof of that result does rely on these two results indirectly, as it depends on the $(K_3, P_1 + 2P_2)$-free case.

Lemma 8. *Let G be a connected $(K_3, C_5, S_{1,2,3})$-free graph that does not contain a pair of false twins. Then G is either bipartite or an induced cycle.*

Proof. Let G be a connected $(K_3, C_5, S_{1,2,3})$-free graph that does not contain a pair of false twins. We may assume that G is not bipartite, otherwise we are done. We know that G is (C_3, C_5)-free (since $C_3 = K_3$). We may therefore assume that G contains an induced odd cycle C on k vertices, say $v_1 - v_2 - \cdots - v_k - v_1$, where $k \geq 7$. Assume that C is an odd cycle of minimum length in G.

Suppose that not every vertex of G is in C. Since G is connected, we may assume that there is a vertex v not in C that has a neighbour in C. Suppose v is adjacent to precisely one vertex of C. If v is adjacent to v_3, but has no other neighbours on C then $G[v_3, v, v_2, v_1, v_4, v_5, v_6]$ is an $S_{1,2,3}$, a contradiction. By symmetry, it follows that v must be adjacent to at least two vertices of C. Note that since G is K_3-free, no vertex outside of C can be adjacent to two consecutive vertices of C.

Suppose that v is adjacent to v_1 and v_i and non-adjacent to $v_2, \ldots, v_{i-1}$ for some even i with $i \leq k - 2$. Then $G[v, v_1, v_2, \ldots, v_i]$ would be an odd cycle on less than k vertices, contradicting the minimality of k. By a parity argument, since C is an odd cycle, it follows that v must be adjacent to precisely two vertices of C, which must be at distance 2 away from each other on the cycle.

Let V_i be the set of vertices outside of C that are adjacent to v_{i-1} and v_{i+1} (subscripts interpreted modulo k) and let U be the set of vertices that have no neighbour in C. Suppose, for contradiction, that U is non-empty. Since G is connected, without loss of generality there is a vertex $u \in U$ that has a neighbour $v \in V_1$. Then $G[v_2, v_1, v, u, v_3, v_4, v_5]$ is an $S_{1,2,3}$, a contradiction. We conclude that U must be empty.

Now since G is K_3-free, for every i the set V_i is anti-complete to the set V_{i+2}. Moreover, if i and j are such that the vertices v_i and v_j are at distance more than 2 on the cycle, then V_i and V_j must be anti-complete, as otherwise there would be a smaller odd cycle than C in G, which would contradict the minimality of k.

Note that every set V_i is independent in G, since G is K_3-free. If a vertex $x_1 \in V_1$ is non-adjacent to a vertex $x_2 \in V_2$ then $G[v_3, x_2, v_2, x_1, v_4, v_5, v_6]$ is an $S_{1,2,3}$, a contradiction. Therefore a vertex $x_i \in V_i$ is adjacent to a vertex $x_j \in V_j$ if and only if v_i and v_j are consecutive vertices of C. In other words, for every i, every vertex in V_i is a false twin of v_i. Therefore every set V_i must be empty, so G is an induced odd cycle. This completes the proof. $\qquad\square$

We immediately get the following corollary, which implies that the four triangle-free cases in our new results hold when the graph class under consideration is in addition C_5-free.

Corollary 1. *The class of $(K_3, C_5, S_{1,2,3})$-free graphs has bounded clique-width.*

Proof. Let G be a $(K_3, C_5, S_{1,2,3})$-free graph. If G contains a pair of false twins then by Lemma 2 we may delete one of them. By Lemma 8, every component of the resulting graph is either a bipartite graph or an induced cycle. In the first case, such a component is an $S_{1,2,3}$-free bipartite graph, so it has bounded clique-width by Lemma 4. In the second case, the component has clique-width at most 4 by Lemma 1. The corollary follows. $\qquad\square$

In our second lemma we state a number of sufficient conditions for a subclass of $(K_3, S_{1,2,3})$-free graphs to be of bounded clique-width when C_5 is no longer a forbidden induced subgraph. To prove it we will need Lemmas 6 and 7.

Lemma 9. *Let $\mathcal{G}$ be the subclass of $(K_3, S_{1,2,3})$-free graphs for which the vertices in each graph $G \in \mathcal{G}$ can be partitioned into ten independent sets $V_1, \ldots, V_5, W_1, \ldots, W_5$, such that the following seven conditions hold (we interpret subscripts modulo 5):*

(i) for all i, V_i is anti-complete to $V_{i-2} \cup V_{i+2} \cup W_{i-1} \cup W_{i+1}$;

(ii) for all i, W_i is complete to $W_{i-1} \cup W_{i+1}$;

(iii) for all i, every vertex of V_i is trivial to at least one of the sets V_{i+1} and V_{i-1};

(iv) for all i, every vertex in V_i is trivial to W_i;

(v) for all i, W_i is trivial to W_{i-2} and to W_{i+2};

(vi) for all i, j, the graphs induced by $V_i \cup V_j$ and $V_i \cup W_j$ are P_7-free;

(vii) for all i, there are no three vertices $v \in V_i$, $w \in V_{i+1}$ and $x \in W_{i+3}$ such that v, w and x are pairwise non-adjacent.

Then $\mathcal{G}$ has bounded clique-width.

Proof. Let G be a $(K_3, S_{1,2,3})$-free graph with such a partition that satisfies Conditions (i)–(vii) of the lemma. Note that for all i, every vertex $v \in V_i$ is trivial to $V_{i-2}, V_{i+2}, W_{i-1}, W_{i+1}, W_i$ and either trivial to V_{i-1} or trivial to V_{i+1}. Therefore a vertex $v \in V_i$ can only be non-trivial to W_{i-2}, W_{i+2} and at most one of V_{i-1} and V_{i+1}. Likewise, every vertex $w \in W_i$ is trivial to $W_{i-1}, W_{i+1}, W_{i-2}, W_{i+2}, V_{i-1}$ and V_{i+1}. Therefore, a vertex $w \in W_i$ can only be non-trivial to V_i, V_{i-2} and V_{i+2} (and every vertex in V_i is trivial to W_i).

For $i \in \{1, \dots, 5\}$, let W_i' be the set of vertices in W_i that are non-trivial to both V_{i-2} and V_{i+2}, let V_i' be the set of vertices in V_i that are non-trivial to both V_{i+1} and W_{i-2} and let V_i'' be the set of vertices in V_i that are non-trivial to both V_{i-1} and W_{i+2}. Note that $V_i' \cap V_i'' = \emptyset$ by Condition (iii).

We say that an edge is *irrelevant* if one of its end-vertices is in a set V_i, V_i', V_i'', W_i or W_i', and its other end-vertex is complete to this set, otherwise we say that the edge is *relevant*. We will now show that for $i \in \{1, \dots, 5\}$, the graph $G[V_i' \cup V_{i+1}'' \cup W_{i-2}']$ can be separated from the rest of G by using a bounded number of bipartite complementations. To do this, we first prove the following claim.

Claim 1. *If $u \in V_i' \cup V_{i+1}'' \cup W_{i-2}'$ and $v \notin V_i' \cup V_{i+1}'' \cup W_{i-2}'$ are adjacent then uv is an irrelevant edge.*

We split the proof of Claim 1 into the following cases.

Case 1: $u \in V_i'$.
Since u is in V_i, v must be in $V_{i-1} \cup V_{i+1} \cup W_{i-2} \cup W_{i+2}$, otherwise uv would be irrelevant by Condition (i) or (iv). We consider the possible cases for v.

Case 1a: $v \in V_{i-1}$.
Since u is in V_i', it is non-trivial to V_{i+1}, so by Condition (iii), u is trivial to V_{i-1}. Therefore uv is irrelevant.

Case 1b: $v \in V_{i+1}$.
Suppose, for contradiction, that v is complete to W_{i-2}. Let $w \in W_{i-2}$ be a neighbour of u (such a vertex w exists, since u is non-trivial to W_{i-2}). Then $G[u, v, w]$ is a K_3, a contradiction, so v cannot be complete to W_{i-2}. Now suppose, for contradiction that v is anti-complete to W_{i-2}. We may assume that v has a non-neighbour $u' \in V_i'$, otherwise v would be trivial to V_i', in which case uv would be irrelevant. Since $u' \in V_i'$, u' is non-trivial to W_{i-2}, so it must have a non-neighbour $w \in W_{i-2}$. Then, since v is anti-complete to W_{i-2}, it follows that $G[u', v, w]$ is a $3P_1$, contradicting Condition (vii). We may therefore assume that v is non-trivial to W_{i-2}. We know that $v \notin V_{i+1}''$. Therefore v must be trivial to V_i, so uv is irrelevant.

Case 1c: $v \in W_{i-2}$.
Reasoning as in the previous case, we find that v cannot be complete or anti-complete to V_{i+1}. Hence, as $v \notin W_{i-2}'$, v must be trivial to V_i, so uv is irrelevant.

Case 1d: $v \in W_{i+2}$.
Since u is non-trivial to W_{i-2} (by definition of V_i'), there is a vertex $w \in W_{i-2}$ that is

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adjacent to u. By Condition (ii), w is adjacent to v. Therefore $G[u, v, w]$ is a K_3. This contradiction implies that $v \notin W_{i+2}$. This completes Case 1.

Now assume that $u \notin V_i'$. Then, by symmetry, $u \notin V_{i+1}''$. This means that the following case holds.

Case 2: $u \in W_{i-2}'$.
We argue similarly to Case 1b. We may assume that v is non-trivial to W_{i-2}', otherwise uv would be irrelevant. By Conditions (i), (ii), (iv) and (v), it follows that $v \in V_i \cup V_{i+1}$. Without loss of generality assume that $v \in V_i$. Since $v \notin V_i'$ and v is non-trivial to W_{i-2}, it follows that v is trivial to V_{i+1}. If v is complete to V_{i+1} then since u is non-trivial to V_{i+1}, there must be a vertex $w \in V_{i+1}$ adjacent to u, in which case $G[u, v, w]$ is a K_3, a contradiction. Therefore v must be anti-complete to V_{i+1}. Since v is non-trivial to W_{i-2}', there must be a vertex $u' \in W_{i-2}'$ that is non-adjacent to v. Since $u' \in W_{i-2}'$, u' must have a non-neighbour $w \in V_{i+1}$. Then $G[u', v, w]$ is a $3P_1$, contradicting Condition (vii). This completes Case 2.

We conclude that, if $u \in V_i' \cup V_{i+1}'' \cup W_{i-2}'$ and $v \notin V_i' \cup V_{i+1}'' \cup W_{i-2}'$ are adjacent, then uv is an irrelevant edge. Hence we have proven Claim 1.

By Claim 1 we find that if $u \in V_i' \cup V_{i+1}'' \cup W_{i-2}'$ and $v \notin V_i' \cup V_{i+1}'' \cup W_{i-2}'$ are adjacent then u or v is complete to some set V_j, V_j', V_j'', W_j or W_j' that contains v or u, respectively. By applying a bounded number of bipartite complements (which we may do by Fact 3), we can separate $G[V_i' \cup V_{i+1}'' \cup W_{i-2}']$ from the rest of G. By Conditions (vi) and (vii) and the fact that G is $(K_3, S_{1,2,3})$-free, Lemmas 6 and 7 imply that $G[V_i' \cup V_{i+1}'' \cup W_{i-2}']$ has clique-width at most 6. Repeating this argument for each i, we may assume that $V_i' \cup V_{i+1}'' \cup W_{i-2}' = \emptyset$ for every i.

For $i \in \{1, \ldots, 5\}$ let V_i^* be the set of vertices in V_i that are either non-trivial to V_{i+1} or non-trivial to W_{i+2} and let V_i^{**} be the set of the remaining vertices in V_i. For $i \in \{1, \ldots, 5\}$, let W_i^* be the set of vertices that are non-trivial to V_{i+2} and let W_i^{**} be the set of the remaining vertices in W_i.

We claim that every vertex in V_i that is non-trivial to V_{i-1} or that is non-trivial to W_{i-2} is in V_i^{**}. Indeed, if $v \in V_i$ is non-trivial to V_{i-1} then by Condition (iii), v is trivial to V_{i+1} and since V_i'' is empty, v must be trivial to W_{i+2}. If $v \in V_i$ is non-trivial to W_{i-2} then v must be trivial to V_{i+1} since V_i' is empty. Moreover, in this case v must also be trivial to W_{i+2}, otherwise, by Condition (ii) the vertex v, together with a neighbour of v in each of W_{i+2} and W_{i-2}, would induce a K_3 in G. It follows that every vertex in V_i that is non-trivial to V_{i-1} or that is non-trivial to W_{i-2} is indeed in V_i^{**}. Similarly, for all i, since W_i' is empty, every vertex in W_i that is non-trivial to V_{i-2} is in W_i^{**}.

We say that an edge uv is *insignificant* if u or v is in some set V_i^*, V_i^{**}, W_i^* or W_i^{**} and the other vertex is trivial to this set; all other edges are said to be *significant*. We prove the following claim.

Claim 2. *If $u \in W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$ and $v \notin W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$ are adjacent then the edge uv is insignificant.*
To prove this claim suppose, for contradiction, that uv is a significant edge. We split the proof into two cases.

Case 1: $u \in W_i$.
We will show that $v \in V_{i+2}^{**}$ or $v \in V_{i-2}^*$ if $u \in W_i^*$ or $u \in W_i^{**}$, respectively. By Conditions (i), (ii), (iv) and (v) we know that u is trivial to $V_{i-1}, V_{i+1}, W_{i-1}, W_{i+1}, W_{i-2}$ and W_{i+2}, and that every vertex of V_i is trivial to W_i. Furthermore, u is trivial to $W_i^{**} \setminus \{u\}$ since W_i is independent. Therefore $v \in V_{i-2} \cup V_{i+2}$. Note that v is non-trivial to W_i (by choice of v). If $u \in W_i^*$ then u must be trivial to V_{i-2}, since W_i' is

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empty. Therefore $v \in V_{i+2}$. Now if $v \in V_{i+2}^*$ then v is non-trivial to V_{i-2} or non-trivial to W_{i-1}. In the first case v is non-trivial to both V_{i-2} and W_i, contradicting the fact that V_{i+2}' is empty. In the second case v has a neighbour $w \in W_{i-1}$. By Condition (ii), w is adjacent to u, so $G[u, v, w]$ is a K_3. This contradiction implies that if $u \in W_i^*$ then $v \in V_{i+2}^{**}$, contradicting the choice of v. Now suppose $u \in W_i^{**}$. Then u is trivial to V_{i+2}, so $v \in V_{i-2}$. If $v \in V_{i-2}^{**}$ then v is trivial W_i (by definition of V_{i-2}^{**}). Therefore if $u \in W_i^{**}$ then $v \in V_{i-2}^*$, contradicting the choice of v.

We conclude that for every $i \in \{1, \ldots, 5\}$ the vertex u is not in W_i. Similarly, we may assume $v \notin W_i$. This means that the following case holds.

Case 2: $u \in V_i$, $v \in V_j$ *for some* i, j.
Then $i \neq j$, since V_i is an independent set. By Condition (i), $j \notin \{i-2, i+2\}$. Without loss of generality, we may therefore assume that $j = i+1$. If $u \in V_i^{**}$ then u is trivial to V_{i+1}, so we may assume that $u \in V_i^*$. If $v \in V_{i+1}^*$ then v is non-trivial to V_{i+2}, so by Condition (iii) v is trivial to V_i, contradicting the fact that uv is significant. Therefore $v \in V_{i+1}^{**}$, contradicting the choice of v.

We conclude that if for some i, $u \in W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$ and $v \notin W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$ are adjacent then the edge uv is insignificant. Hence we have proven Claim 2.

Note that $W_i^*, V_{i+2}^{**}, V_{i+1}^*$ and W_{i-2}^{**} are independent sets. By Condition (i), W_i^* is anti-complete to V_{i+1}^* and V_{i+2}^{**} is anti-complete to W_{i-2}^{**}. Therefore $W_i^* \cup V_{i+1}^*$ and $V_{i+2}^{**} \cup W_{i-2}^{**}$ are independent sets. Thus $G[W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}]$ is an $S_{1,2,3}$-free bipartite graph, which has bounded clique-width by Lemma 4. Applying a bounded number of bipartite complementations (which we may do by Fact 3), we can separate $G[W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}]$ from the rest of the graph. We may thus assume that $W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**} = \emptyset$. Repeating this process for each i we obtain the empty graph. This completes the proof. $\square$

5 The Four Triangle-free Cases

We can now give the following result, which also implies the $(K_3, P_1 + 2P_2)$-free case.

Theorem 3. *For $H \in \{P_1 + P_5, S_{1,2,2}, P_1 + P_2 + P_3\}$, the class of (K_3, H)-free graphs has bounded clique-width.*

The proofs for all three cases are broadly similar. We will prove the $H = P_1 + P_2 + P_3$ case separately, as it is a little more involved than the other two cases.

5.1 Proof of the $H = P_1 + P_5$ and $H = S_{1,2,2}$ Cases.

Proof. Let $H \in \{P_1 + P_5, S_{1,2,2}\}$ and consider a (K_3, H)-free graph G. We may assume that G is connected.

By Corollary 1, we may assume that G contains an induced cycle on five vertices, say $C = v_1 - v_2 - \cdots - v_5 - v_1$. Again, we will interpret subscripts on vertices and vertex sets modulo 5.

Since G is K_3-free, no vertex v is adjacent to two consecutive vertices of the cycle. Therefore every vertex of G has either zero, one or two neighbours on the cycle and if it has two neighbours then they must be non-consecutive vertices of the cycle.

We partition the vertices of G that are not on C as follows:

- U: the set of vertices adjacent to no vertices of C,
- W_i: the set of vertices whose unique neighbour in C is v_i and

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– V_i: the set of vertices adjacent to v_{i-1} and v_{i+1}.

In the remainder of the proof we will show how to modify the graph using operations that preserve boundedness of clique-width, such that in the resulting graph the set U is empty and the partition $V_1, \ldots, V_5, W_1, \ldots, W_5$ satisfies Conditions (i)–(vii) of Lemma 9. In order to do this we prove a number of claims.

The first two claims follow immediately from the fact that G is K_3-free.

Claim 1. *For all i, V_i and W_i are independent sets.*
Claim 2. *For all i, V_i is anti-complete to $V_{i-2} \cup V_{i+2} \cup W_{i-1} \cup W_{i+1}$.*

Claim 3. *We may assume that U is empty.*
We prove Claim 3 as follows. First consider the case where $H = S_{1,2,2}$ and suppose, for contradiction, that U is not empty. Since G is connected there must be a vertex $u \in U$ that is adjacent to a vertex $v \notin U$ that has a neighbour on the cycle C. Without loss of generality, we may assume that $v \in V_1 \cup W_2$, in which case v is adjacent to v_2 and non-adjacent to v_1, v_3 and v_4. Now $G[v_2, v_1, v_3, v_4, v, u]$ is an $S_{1,2,2}$. This contradiction means that $U = \emptyset$ if $H = S_{1,2,2}$.

Now consider the case where $H = P_1 + P_5$ and suppose that U is non-empty. Suppose, for contradiction, that there are two vertices $u, u' \in U$ that do not have the same neighbourhood in some set V_i or W_i. Without loss of generality, assume $v \in V_1 \cup W_2$ is adjacent to u, but not u'. Note that v is adjacent to v_2, but non-adjacent to v_1, v_3 and v_4. Then $G[v_4, u', u, v, v_2, v_1]$ is a $P_1 + P_5$ if u and u' are adjacent and $G[u', u, v, v_2, v_3, v_4]$ is a $P_1 + P_5$ if they are not. This contradiction means that every vertex in U has the same neighbourhood in every set V_i and every set W_i. Since G is connected there must be a vertex v in some V_i or W_i that is adjacent to every vertex of U. Since G is K_3-free, U must therefore be an independent set. Applying a bipartite complementation (which we may do by Fact 3) between U and the vertices adjacent to the vertices of U disconnects U from the rest of the graph. Since $G[U]$ is independent, it has clique-width at most 1. We may therefore assume that U is empty.

Claim 4. *For all i, W_i is complete to $W_{i-1} \cup W_{i+1}$.*
Suppose, for contradiction, that $v \in W_1$ has a non-neighbour $w \in W_2$. Then $G[w, v, v_1, v_5, v_4, v_3]$ is a $P_1 + P_5$ and $G[v_1, v, v_2, w, v_5, v_4]$ is an $S_{1,2,2}$. This contradiction proves the claim.

See Fig. 3 for an illustration of the graph G.

Claim 5. *For all i, every vertex of V_i is trivial to at least one of the sets V_{i+1} and V_{i-1}.*
Suppose, for contradiction that the claim is false. Without loss of generality, there is a vertex $v \in V_2$ with non-neighbours $u \in V_1$ and $w \in V_3$. By Claim 2, u and w must be non-adjacent. Then $G[v_5, u, v_1, v, v_4, w]$ is an $S_{1,2,2}$ and $G[u, v_1, v, v_3, v_4, w]$ is a $P_1 + P_5$. This contradiction completes the proof of the claim.

**Claim 6.** *For all i, every vertex in V_i is trivial to W_i.*
Suppose, for contradiction, that the claim is false. Without loss of generality, we may assume there are vertices $v \in V_1$ and $w, w' \in W_1$ such that v is adjacent to w, but not to w'. Then $G[v_2, v, v_1, w', v_3, v_4]$ is an $S_{1,2,2}$ and $G[w', w, v, v_2, v_3, v_4]$ is a $P_1 + P_5$. This contradiction completes the proof of the claim.

Claim 7. *For all i, W_i is trivial to W_{i-2} and to W_{i+2}.*
Suppose, for contradiction, that this does not hold. Without loss of generality, assume $v \in W_1$ is adjacent to $w \in W_3$ and non-adjacent to $w' \in W_3$. Then $G[v_1, v_2, v_5, v_4, v, w]$ is an $S_{1,2,2}$ and $G[w', w, v, v_1, v_5, v_4]$ is a $P_1 + P_5$. This contradiction proves the claim.

Claim 8. *For all i, j, the graphs induced by $V_i \cup V_j$ and $V_i \cup W_j$ are P_7-free.*
Note that $P_1 + P_5$ is an induced subgraph of P_7. Therefore if $H = P_1 + P_5$ then the

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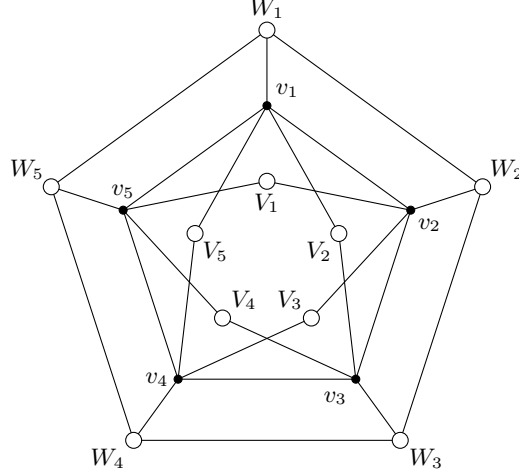
Fig. 3. The graph G. The black points are the vertices of the cycle C. The circles are (possibly empty) independent sets of vertices and the lines are complete bipartite graphs. Note that G may contain additional edges that are not represented in this figure.

claim follows immediately. Now suppose $H = S_{1,2,2}$. Without loss of generality, we may assume $i = 1$. Suppose that $G[V_1 \cup V_j]$ or $G[V_1 \cup W_j]$ contains an induced P_7, for some i, j. By Claims 1, 2 and 6 and symmetry, we may assume that $G[V_1 \cup V_2]$ or $G[V_1 \cup W_3]$ contains this P_7. This P_7 contains an induced subgraph isomorphic to $2P_2$, say on vertices v, v', w, w'. Then $G[v_5, v_4, v, v', w, w']$ is an $S_{1,2,2}$. This contradiction completes the proof of the claim.

Claim 9. *For all i, there are no three vertices $v \in V_i$, $w \in V_{i+1}$ and $x \in W_{i+3}$ such that v, w and x are pairwise non-adjacent.*
Suppose, for contradiction that such pairwise non-adjacent vertices exist, say with $v \in V_1, w \in V_2$ and $x \in W_4$. Then $G[v_4, x, v_3, w, v_5, v]$ is an $S_{1,2,2}$ and $G[x, v_3, w, v_1, v_5, v]$ is a $P_1 + P_5$. This contradiction completes the proof of the claim.

We now consider the graph obtained G' from G by removing the five vertices of C. Claims 1 and 3 show that we may assume $V_1, \ldots, V_5, W_1, \ldots, W_5$ are independent sets that form a partition of the vertex set of G'. Claims 2 and 4–9 correspond to the seven conditions of Lemma 9. Therefore G' has bounded clique-width. By Fact 1, G also has bounded clique-width. This completes the proof. $\qquad\square$

5.2 Proof of the $H = P_1 + P_2 + P_3$ Case.

Proof. Consider a $(K_3, P_1 + P_2 + P_3)$-free graph G. We may assume that G is connected.

By Corollary 1, we may assume that G contains an induced cycle on five vertices, say $C = v_1 - v_2 - \cdots - v_5 - v_1$. Again, we will interpret subscripts on vertices and vertex sets modulo 5.

Since G is K_3-free, no vertex v is adjacent to two consecutive vertices of C. Therefore every vertex of G has either zero, one or two neighbours on C and if it has two neighbours then they must be non-consecutive vertices of C.

We partition the vertices of G that are not on C as follows:

- U: the set of vertices adjacent to no vertices of C,
- W_i: the set of vertices whose unique neighbour in C is v_i and

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- V_i: the set of vertices adjacent to v_{i-1} and v_{i+1}.

In the remainder of the proof we will show how to modify the graph using operations that preserve boundedness of clique-width, such that in the resulting graph the set U is empty and the partition $V_1, \ldots, V_5, W_1, \ldots, W_5$ satisfies Conditions (i)–(vii) of Lemma 9.

The first two claims follow immediately from the fact that G is K_3-free.

Claim 1. *For all i, V_i and W_i are independent sets.*
Claim 2. *For all i, V_i is anti-complete to $V_{i-2} \cup V_{i+2} \cup W_{i-1} \cup W_{i+1}$.*

Claim 3. *We may assume that U is empty.*
In order to proof Claim 3, we first suppose that there are two adjacent vertices $u, u' \in U$. Since G is connected, we may assume without loss of generality that u is adjacent to some vertex $v \in V_1 \cup W_2$. Then u' must be non-adjacent to v, otherwise $G[u, u', v]$ would be a K_3. Note that v is adjacent to v_2, but not to v_1, v_3 or v_4. Now $G[v_1, v_3, v_4, u', u, v]$ is a $P_1 + P_2 + P_3$. This contradiction implies that U must be an independent set.

Now suppose, for contradiction, that a vertex $u \in U$ has two neighbours in some set $V_i \cup W_{i+1}$. Without loss of generality assume that u is adjacent to $v, v' \in V_1 \cup W_2$. Note that v and v' are adjacent to v_2, but not adjacent to v_1, v_3 and v_4. Now $G[v_1, v_3, v_4, v, u, v']$ is a $P_1 + P_2 + P_3$. This contradiction implies that every vertex of U has at most one neighbour in $V_i \cup W_{i+1}$ for each i. In particular, this means that every vertex of U has degree at most 5. Therefore, if $u \in U$ then we delete $\{u\} \cup N(u)$ (a set of at most 6 vertices). This gives us a $(K_3, P_2 + P_3)$-free graph, which has bounded clique-width by Lemma 3. By Fact 1, we may therefore assume that U is empty, that is, we have proven Claim 3.

We say that a set V_i or W_i is *large* if it contains at least two vertices and *small* if it contains exactly one vertex. If any set V_i is not large then by Fact 1 we may assume that it is empty. (Later in the proof, we may delete vertices from some sets V_i or W_i. In doing so, some sets that were previously large may become small. If this happens, we will simply repeat the argument. We will only do this a bounded number of times, so boundedness of clique-width will be preserved.)

Claim 4. *For all i, W_i is complete to $W_{i-1} \cup W_{i+1}$.*
Suppose, for contradiction, that $v \in W_1$ has a non-neighbour $w \in W_2$. Since W_2 is non-empty, it must be large, so it must contain a vertex w' distinct from w. Then $G[w, v_3, v_4, v_1, v, w']$ is a $P_1 + P_2 + P_3$ if v and w' are adjacent and $G[v, v_4, v_5, w, v_2, w']$ is a $P_1 + P_2 + P_3$ if they are not. This contradiction completes the proof of Claim 4.

Claim 5. *For all i, every vertex of V_i is trivial to at least one of the sets V_{i+1} and V_{i-1}.*
Suppose, for contradiction that the claim is false. Without loss of generality, there is a vertex $v \in V_2$ with non-neighbours $u \in V_1$ and $w \in V_3$ and neighbour $u' \in V_1$. By Claim 2, w and must be non-adjacent to both u and u'. Then $G[u, v_4, w, v_1, v, u']$ is a $P_1 + P_2 + P_3$. This contradiction completes the proof of Claim 5.

Claim 6. *For all i, every vertex in V_i is trivial to W_i.*
In fact we will prove a stronger statement, namely that for all i, V_i is trivial to W_i. Suppose, for contradiction, that this is not the case. Without loss of generality, assume that V_1 is not trivial to W_1. First suppose that there are vertices $w \in W_1$ and $v, v' \in V_1$ such that w is adjacent to v, but not to v'. Then $G[v', v_3, v_4, v_1, w, v]$ is a $P_1 + P_2 + P_3$. Therefore every vertex in W_1 must be trivial to V_1. Since we assumed that V_1 is not trivial to W_1, there must therefore be vertices $v \in V_1$ and $w, w' \in W_1$ such that v is adjacent to w, but not to w'. Since V_1 is non-empty, it must be large, so there must be another vertex $v' \in V_1$. Since every vertex of W_1 is trivial to V_1, v' must be

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adjacent to w and non-adjacent to w'. Then $G[w', v_3, v_4, v, w, v']$ is a $P_1 + P_2 + P_3$. This contradiction completes the proof of Claim 6.

Claim 7. *We may assume that for all i, W_i is anti-complete to W_{i-2} and to W_{i+2}.*
We start by showing that the edges between W_i and W_{i+2} form a matching. Indeed, suppose for contradiction that there is a vertex $v \in W_1$ with two neighbours $w, w' \in W_3$. Then $G[v_2, v_4, v_5, w, v, w']$ is a $P_1 + P_2 + P_3$, a contradiction. By symmetry, no vertex of W_3 has two neighbours in W_1. We conclude that the edges between W_i and W_{i+2} form a matching.

Let W_1' be the set of vertices in W_1 that have a neighbour in W_3. Similarly, let W_3'' be the set of vertices in W_3 that have a neighbour in W_1. Note that $|W_1'| = |W_3''|$ since the edges between W_1' and W_3'' form a perfect matching. We will show that every vertex of $G \setminus (W_1' \cup W_3'')$ is trivial to W_1' and W_3''. This follows immediately if $|W_1'| = |W_3''| = 1$.

Assume $|W_1'| = |W_3''| \geq 2$. Suppose there is a vertex $w \in V(G) \setminus (W_1' \cup W_3'')$ that is non-trivial to W_1'. Then we may choose $u, u' \in W_1'$ and $v, v' \in W_3''$ such that u is adjacent to v and w, but non-adjacent to v' while u' is adjacent to v', but non-adjacent to v and w. Since w is non-trivial to W_1, it cannot be in W_1 (by Claim 1), $V_2 \cup V_5$ (by Claim 2), $W_2 \cup W_5$ (by Claim 4), V_1 (by Claim 6) or W_3 (since we assumed $w \notin W_3''$). Furthermore, $w \notin C$ by definition of W_1. Therefore $w \in V_4 \cup W_4 \cup V_3$. By Claims 2, 4 and 6 respectively, we conclude that w is trivial to W_3. Since u is adjacent to v and w, it follows that w must be non-adjacent to v, otherwise $G[u, v, w]$ would be a K_3, a contradiction. Therefore w must be anti-complete to W_3. If $w \in V_3 \cup W_4$, let $z = v_5$ and otherwise (if $w \in V_4$) let $z = v_4$. Then z is non-adjacent to u, u', v, v' and w. Now $G[z, u', v', v, u, w]$ is a $P_1 + P_2 + P_3$, a contradiction. Therefore every vertex in $V(G) \setminus (W_1' \cup W_3'')$ is trivial to W_1'. By symmetry, every vertex in $V(G) \setminus (W_1' \cup W_3'')$ is trivial to W_3''.

Therefore, by applying a bipartite complementation (which we may do by Fact 3) between W_1' and the vertices in $V(G) \setminus W_3''$ that are complete to W_1' and another bipartite complementation between W_3'' and the vertices in $V(G) \setminus W_1'$ that are complete to W_3'', we separate $G[W_1' \cup W_3'']$ from the rest of the graph. Since $G[W_1' \cup W_3'']$ is a perfect matching, it has clique-width at most 2. We may therefore assume that $W_1' \cup W_3''$ is empty i.e. that W_1 is anti-complete to W_3. Repeating this argument for each $i \in \{1, \ldots, 5\}$, we show that we may assume that W_i is anti-complete to W_{i-2} for every i. This completes the proof of Claim 7.

Note that when applying Claim 7 we may delete vertices in some sets W_i, which may cause some large sets to become small. In this case, as stated earlier, we may simply delete the small sets as before. Thus we may assume that every set W_i is either large or empty.

Claim 8. *For all i, j, the graphs induced by $V_i \cup V_j$ and $V_i \cup W_j$ are P_7-free.*
Suppose, for contradiction, that the claim is false. Then there is an i and a j such that $G[V_i \cup V_j]$ or $G[V_i \cup W_j]$ contains an induced P_7, say on vertices $u_1, \ldots, u_7$. There must be a vertex $v_k \in C$ that is non-adjacent to every vertex of $V_i \cup V_j$ or $V_i \cup W_j$, respectively (since every vertex not in C has at most two neighbours in C). Then $G[v_k, u_1, u_2, u_4, u_5, u_6]$ is a $P_1 + P_2 + P_3$, a contradiction. This completes the proof of Claim 8.

Claim 9. *For all i, if there are vertices $v \in V_i$, $w \in V_{i+1}$ and $x \in W_{i+3}$ such that v, w and x are pairwise non-adjacent then G has bounded clique-width.*
Suppose that such pairwise non-adjacent vertices exist, say with $v \in V_1, w \in V_2$ and $x \in W_4$. We start by showing that $V_3 \cup V_4 \cup V_5 \cup W_1 \cup W_2 \cup W_3 \cup W_5$ is empty.

First suppose there is a vertex $y \in V_3$. Then y is non-adjacent to v and x by Claim 2. Then $G[x, v, v_5, v_3, w, y]$ or $G[v, v_1, w, x, v_4, y]$ is a $P_1 + P_2 + P_3$ if y is adjacent

or non-adjacent to w, respectively. This contradiction implies that V_3 is empty. By symmetry V_5 is also empty.

Next, suppose there is a vertex $y \in V_4$. Then y is non-adjacent to v and w by Claim 2. Then $G[v, v_1, w, v_4, x, y]$ or $G[y, v_4, x, v_2, v_1, w]$ is a $P_1 + P_2 + P_3$ if y is adjacent or non-adjacent to x, respectively. This contradiction implies that V_4 is empty.

Next, suppose there is a vertex $y \in W_1$. Then y is non-adjacent to w and x by Claims 2 and 7, respectively. Then $G[w, x, v_4, v_2, v, y]$ or $G[v, x, v_4, w, v_1, y]$ is a $P_1 + P_2 + P_3$ if y is adjacent or non-adjacent to v, respectively. This contradiction implies that W_1 is empty. By symmetry W_2 is also empty.

Finally, suppose that W_3 is not empty. Then W_3 must be large, so it contains two vertices, say y and y'. Then y and y' are each non-adjacent to w and adjacent to x by Claims 2 and 4, respectively. If y is non-adjacent to v then $G[v, v_1, w, v_4, x, y]$ would be a $P_1 + P_2 + P_3$, a contradiction. Therefore y is adjacent to v, and similarly y' is adjacent to v. Now $G[v_4, v_1, w, y, v, y']$ is a $P_1 + P_2 + P_3$. This contradiction implies that W_3 is empty. By symmetry, we may assume that W_5 is also empty.

The above means that $V_3 \cup V_4 \cup V_5 \cup W_1 \cup W_2 \cup W_3 \cup W_5$ is indeed empty, so $V(G) = V_1 \cup V_2 \cup W_4 \cup V(C)$.

Let V_1' and V_1'' be the set of vertices in V_1 that are anti-complete or complete to $\{w, x\}$, respectively. Let V_2' and V_2'' be the set of vertices in V_2 that are anti-complete or complete to $\{v, x\}$, respectively. Let W_4' and W_4'' be the set of vertices in W_4 that are anti-complete or complete to $\{v, w\}$, respectively. Observe that $v \in V_1', w \in V_2'$ and $x \in W_4'$. We will show that $V_1', V_1'', V_2', V_2'', W_4'$ and W_4'' form a partition of $V(G) \setminus V(C)$.

Suppose, for contradiction, that there is a vertex $v' \in V_1$ with exactly one neighbour in $\{w, x\}$. Then $G[v, v_4, x, v', w, v_1]$ or $G[v, v_1, w, v_4, x, v']$ is a $P_1 + P_2 + P_3$ if this neighbour is w or x, respectively. Therefore every vertex of V_1 is in $V_1' \cup V_1''$. Similarly, every vertex of V_2 is in $V_2' \cup V_2''$.

Suppose, for contradiction, that there is a vertex $x' \in W_4$ with exactly one neighbour in $\{v, w\}$. Without loss of generality, suppose that x' is adjacent to v, but not to w. Then $G[x, w, v_3, v_5, v, x']$ is a $P_1 + P_2 + P_3$. Therefore every vertex of W_4 is in $W_4' \cup W_4''$. Thus every vertex of $V(G) \setminus V(C)$ is in $V_1' \cup V_1'' \cup V_2' \cup V_2'' \cup V_4' \cup V_4''$.

Observe that the remarks made above for v, w and x also hold if one of these is replaced by a vertex of V_1', V_2' or W_4', respectively. Indeed, suppose $v' \in V_1' \setminus \{v\}$, then every vertex of W_4 must be either complete or anti-complete to $\{v', w\}$. Since the vertices of W_4' are non-adjacent to w, but the vertices of W_4'' are adjacent to w, it follows that W_4' is anti-complete to $\{v', w\}$ and that W_4'' is complete to $\{v', w\}$. Therefore W_4' is anti-complete to V_1', and W_4'' is complete to V_1'. Since G is K_3-free and every vertex of $V_1'' \cup V_2''$ is adjacent to x, it follows that V_1'' is anti-complete to V_2''. Similarly, we conclude that V_1', V_2' and W_4' are pairwise anti-complete, V_1'', V_2'' and W_4'' are pairwise anti-complete and for every pair of sets $S \in \{V_1', V_2', W_4'\}$ and $T \in \{V_1'', V_2'', W_4''\}$ such that $(S, T) \notin \{(V_1', V_1''), (V_2', V_2''), (W_4', W_4'')\}$, S and T are complete to each-other.

Now if we delete the vertices of C (which we may do by Fact 1) and apply bipartite complementations between $V_1' \& V_2''$, $V_1' \& W_4''$, $V_2' \& V_1''$, $V_2' \& W_4''$, $W_4' \& V_1''$ and $W_4' \& V_2''$, we obtain an edgeless graph, which therefore has clique-width at most 1. By Fact 3, it follows that G has bounded clique-width. This completes the proof of Claim 9.

We now consider the graph G' obtained from G by removing the five vertices of C. Claims 1 and 3 show that we may assume $V_1, \ldots, V_5, W_1, \ldots, W_5$ are independent sets that form a partition of the vertex set of G'. Claims 2 and 4–9 correspond to the seven conditions of Lemma 9. Therefore G' has bounded clique-width. By Fact 1, G also has bounded clique-width. This completes the proof. □

6   The Diamond-free Case

In this section, we prove that (diamond, $P_1 + 2P_2$)-free graphs have bounded clique-width. In order to do this, we first need to prove the following two lemmas.

Lemma 10. *The class of disconnected (diamond, $P_1 + 2P_2$)-free graphs has bounded clique-width.*

Proof. If G is a disconnected (diamond, $P_1 + 2P_2$)-free graph then it contains at least two components. Therefore every component of G must be (diamond, $2P_2$)-free and thus has bounded clique-width by Lemma 3. We conclude that G has bounded clique-width.

Lemma 11. *The class of (diamond, $P_1 + 2P_2$)-free graphs that contain a K_4 has bounded clique-width.*

Proof. Let G be a (diamond, $P_1 + 2P_2$)-free graph containing an induced K_4. By Lemma 10, we may assume that G is connected. Let K be a maximum clique of G and note that $|K| \geq 4$. We may assume that G contains vertices outside K, otherwise G is a clique on at least four vertices, in which case it has clique-width 2.

Suppose there is a vertex v in G that is not in K, but has at least two neighbours $x, y \in K$. By maximality of K, there must be a vertex $z \in K$ that is not adjacent to v. However this means that $G[x, y, v, z]$ is a diamond, a contradiction. Therefore every vertex not in K has at most one neighbour in K.

Choose $v_1, v_2, v_3, v_4 \in K$ arbitrarily. For $i \in \{1, 2, 3, 4\}$, let V_i be the set of vertices not in K whose unique neighbour in K is v_i. Let U be the set of vertices not in K that do not have a neighbour in $\{v_1, v_2, v_3, v_4\}$. Note that vertices of U may have neighbours in $K \setminus \{v_1, v_2, v_3, v_4\}$.

Claim 1. *For $i, j \in \{1, 2, 3, 4\}$, $G[U \cup V_i \cup V_j]$ must be $(P_1 + P_2)$-free.*
Indeed, if $G[U \cup V_1 \cup V_2]$ contains an induced $P_1 + P_2$ on vertices y_1, y_2, y_3, say, then $G[y_1, y_2, y_3, v_3, v_4]$ is a $P_1 + 2P_2$, a contradiction. The claim follows by symmetry.

Claim 2. *For $i \in \{1, 2, 3, 4\}$, we may assume $G[V_i]$ is either a clique on at most two vertices or an independent set.*
If $G[V_1]$ contains an induced P_3 on vertices y_1, y_2, y_3, say, then $G[v_1, y_2, y_1, y_3]$ is a diamond, a contradiction. Therefore $G[V_1]$ is a disjoint union of cliques. Claim 1 implies that $G[V_1]$ is either a clique, or else every clique in $G[V_1]$ contains at most one vertex i.e. V_1 is an independent set.

Suppose, for contradiction, that V_1 is a clique on at least three vertices. We will show that the clique-width of G is bounded in this case. First suppose, for contradiction, that there is a vertex $u \in U \cup V_2 \cup V_3 \cup V_4$. Since $G[\{u\} \cup V_1]$ is $(P_1 + P_2)$-free by Claim 1, u must be adjacent to all but at most one vertex of V_1. Let $x, y \in V_1$ be neighbours of u. Then $G[x, y, u, v_1]$ is a diamond, a contradiction. We conclude that $U \cup V_2 \cup V_3 \cup V_4 = \emptyset$, so $V(G) = K \cup V_1$. Deleting v_1 we obtain a disconnected (diamond, $P_1 + 2P_2$)-free graph, which has bounded clique-width by Lemma 10. Therefore G has bounded clique-width by Fact 1. Therefore if V_1 is a clique then it contains at most two vertices. The claim follows by symmetry.

Claim 3. *For distinct $i, j \in \{1, 2, 3, 4\}$, if V_i is an independent set then every vertex of V_j is either complete or anti-complete to V_i.*
Indeed, this follows directly from Claim 1, which states that $G[V_i \cup V_j]$ is $(P_1 + P_2)$-free. (Note that if V_j is a clique then it may contain a vertex that is complete to V_i and another that is anti-complete to V_i.)

Claim 4. *We may assume U contains at least three vertices.*
Suppose that U has at most two vertices. By Fact 1 and Claim 2, we may remove every

vertex of U and every vertex of V_i for those V_i that are cliques. After this, by Claim 2, every set V_i will either be empty or an independent set. Furthermore, for distinct $i, j \in \{1, 2, 3, 4\}$, by Claim 3, every vertex of V_i is trivial to V_j and vice versa, so V_i is complete or anti-complete to V_j. By Fact 3, we may apply a bipartite complementation between V_i and V_j if they are complete. By Fact 1, we may delete v_1, v_2, v_3, v_4. We obtain a graph that is the disjoint union of a clique and at most four independent sets and therefore has clique-width at most 2. It follows that the graph G must also have had bounded clique-width. We may therefore assume that U contains at least three vertices. This completes the proof of the claim.

We now consider a number of cases:

Case 1: *Every vertex of K has at most one neighbour outside of K.*
By Fact 2, we may remove all the edges connecting pairs of vertices in K. Let G' be the resulting graph and note that in G', every vertex of K has at most one neighbour. Then $\mathrm{cw}(G') \leq \mathrm{cw}(G' \setminus K) + 1$. (Given a k-expression for $G' \setminus K$, whenever we create a vertex v that has a neighbour w in K, we immediately create w with a special new label $*$, take the disjoint union and join v to w by an edge. For any vertices in K with no neighbours outside of K, we simply add them with label $*$ at the end of the process. This will give a $(k+1)$-expression for G'.) Now $G' \setminus K = G \setminus K$. Since V_1 contains at most one vertex, by Fact 1, it is sufficient to show that $G \setminus (V_1 \cup K)$ has bounded clique-width. However, $G \setminus (V_1 \cup K)$ is (diamond, $2P_2$)-free, since if it contained an induced $2P_2$ then this, together with v_1 would induce a $P_1 + 2P_2$ in G. Therefore $G \setminus (V_1 \cup K)$ has bounded clique-width by Lemma 3 and therefore G also has bounded clique-width. This completes the proof of this case.

We may now assume that at least one vertex of K has at least two neighbours outside of K.

Case 2: *Exactly one vertex of K has neighbours outside K.*
Suppose that v_1 is the only vertex of K that has neighbours outside of K (at least one vertex of K has a neighbour outside of K since G is connected and not a clique). Now $G \setminus \{v_1\}$ is a disconnected (diamond, $P_1 + 2P_2$)-free graph, so it has bounded clique-width by Lemma 10. By Fact 1, G also has bounded clique-width. This completes the proof of this case.

We may now assume that at least two vertices of K have neighbours outside of K. Without loss of generality, we may therefore assume that the following case holds.

Case 3: V_1 *contains at least two vertices and* V_2 *contains at least one vertex.*
Fix $x, y, z \in V_1 \cup V_2$, with two of these vertices in V_1 and one in V_2. If these vertices are pairwise adjacent then $G[x, y, v_1, z]$ would be a diamond, a contradiction. We may therefore assume that x and y are non-adjacent. Now every vertex of $v \in U$ is either complete or anti-complete to $\{x, y\}$, otherwise $G[v, x, y]$ would be a $P_1 + P_2$ in $G[U \cup V_1 \cup V_2]$, which would contradict Claim 1.

Suppose $u, v \in U$. If u and v are adjacent then they cannot both be complete to $\{x, y\}$, otherwise $G[u, v, x, y]$ would be a diamond and they cannot both be anti-complete to $\{x, y\}$, otherwise $G[x, u, v]$ would be a $P_1 + P_2$ in $G[U \cup V_1 \cup V_2]$, which would contradict Claim 1. Therefore if u and v are adjacent then one of them is complete to $\{x, y\}$ and the other is anti-complete to $\{x, y\}$. If u and v are non-adjacent then they must either both be complete to $\{x, y\}$ or both be anti-complete to $\{x, y\}$. Indeed, suppose for contradiction that u is complete to $\{x, y\}$ and v is anti-complete to $\{x, y\}$. Then $G[v, u, x]$ would be an induced $P_1 + P_2$ in $G[U \cup V_1 \cup V_2]$, which would contradict Claim 1. The above holds for every pair of vertices $u, v \in U$. This implies that $G[U]$ is a complete bipartite graph with one of the sets in the bipartition consisting

of the vertices complete to $\{x, y\}$ and the other consisting of the vertices anti-complete to $\{x, y\}$. (Note that one of the parts of the complete bipartite graph $G[U]$ may be empty, as we allow the case where U is an independent set.)

Note that the arguments in the above paragraph only used the facts that $G[U \cup V_1 \cup V_2]$ is $(P_1 + P_2, \text{diamond})$-free and that $V_1 \cup V_2$ contains two non-adjacent vertices. Let U_1 and U_2 be the independent sets that form the bipartition of U. Note that since U contains at least three vertices (by Claim 4), we may assume without loss of generality that U_1 contains at least two vertices. If U_2 contains exactly one vertex, by Fact 1, we may delete it. (Note that this may cause U to contain only two vertices, rather than at least three, however this does not affect our later arguments.) We may therefore assume that U_2 is either empty or contains at least two vertices. Repeating the argument in the previous paragraph with the roles of U and $V_1 \cup V_2$ reversed, we find that $G[V_1 \cup V_2]$ is a complete bipartite graph, with one side of the bipartition complete to U_1 and the other anti-complete to U_1 and if U_2 is non-empty then one side of the bipartition is complete to U_2 and the other is anti-complete to U_2. Similarly, for each pair of distinct $i, j \in \{1, 2, 3, 4\}$, the same argument shows that $G[V_i \cup V_j]$ is also a complete bipartite graph with a similar bipartition.

We now proceed as follows: if V_i is a clique for some i then it contains at most two vertices, in which case we delete them and make V_i empty. For every pair of distinct $i, j \in \{1, 2, 3, 4\}$ (V_i or V_j may be empty) $G[V_i \cup V_j]$ must then be an independent set, in which case we do nothing, or a complete bipartite graph with bipartition (V_i, V_j), in which case we apply a bipartite complementation between V_i and V_j. Now every set V_i is either complete or anti-complete to U_1 and complete or anti-complete to U_2. Applying at most $4 \times 2 = 8$ bipartite complementations, we can remove all edges between $V_1 \cup \cdots \cup V_4$ and U. Next, we apply a bipartite complementation between U_1 and U_2. Finally, we apply a complementation to the clique K. Let G' be the resulting graph and note that $G'[V_1 \cup \cdots \cup V_4 \cup U]$ and $G'[K]$ are independent sets and that in G' every vertex in $V_1 \cup \cdots \cup V_4 \cup U$ has at most one neighbour in K. Therefore G' is a disjoint union of stars, and so has clique-width at most 2. By Facts 1, 2 and 3, it follows that G also has bounded clique-width. This completes proof for this case and therefore completes the proof of the lemma. □

To prove the main result of this section, we will need an additional notion. Let G be a graph. For each set T that induces a triangle in G, let U^T be the set of vertices in G that have no neighbour in T. Let $\mathcal{U} = \{u \in U^T \mid T \text{ induces a triangle in } G\}$. We say that the graph G is *basic* if we can partition the vertices of $G \setminus \mathcal{U}$ into three sets V_1, V_2, V_3 and also into sets $T^1, W_1, T^2, W_2, \ldots, T^p, W_p$ for some p such that the following properties hold:

  (i) No triangle in G contains a vertex of $\mathcal{U}$.
 (ii) For every triangle T, the set U^T is independent and there is a vertex $x \in V(T)$ such that $N(x) = N(u) \cup (V(T) \setminus \{x\})$ for all $u \in U^T$.
(iii) V_1, V_2 and V_3 are independent.
(iv) $\{G[T^1], \ldots, G[T^p]\}$ is the set of all induced triangles in G and each of them has exactly one vertex in each of V_1, V_2 and V_3.
 (v) $G[W_i]$ is $(P_1 + 2P_2)$-free and does not contain an induced $3P_1$ with one vertex in each of V_1, V_2 and V_3.
(vi) If $i < j$ and $k + 1 \not\equiv \ell \pmod 3$ then:
    1. $T^i \cap V_k$ is anti-complete to $T^j \cap V_\ell$,
    2. $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$,
    3. $W_i \cap V_k$ is anti-complete to $T^j \cap V_\ell$ and
    4. $W_i \cap V_k$ is anti-complete to $W_j \cap V_\ell$,

(vii) If $i < j$ and $k + 1 \equiv \ell \pmod 3$ then:
   1. $T^i \cap V_k$ is complete to $T^j \cap V_\ell$,
   2. $T^i \cap V_k$ is complete to $W_j \cap V_\ell$ and
   3. $W_i \cap V_k$ is complete to $T^j \cap V_\ell$.

(viii) If $i + 1 < j$ and $k + 1 \equiv \ell \pmod 3$ then:
   1. $W_i \cap V_k$ is complete to $W_j \cap V_\ell$.

(ix) If $i + 1 = j$ and $k + 1 \equiv \ell \pmod 3$ then:
   1. $W_i \cap V_k$ is either complete or anti-complete to $W_j \cap V_\ell$.

(x) If $i = j$ and $k + 1 \equiv \ell \pmod 3$ then:
   1. $T^i \cap V_k$ is complete to $W_j \cap V_\ell$.

(xi) If $i = j$ and $k + 1 \not\equiv \ell \pmod 3$ then:
   1. $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.

Next, we show that basic graphs have bounded clique-width.

Lemma 12. *If G is a basic graph then it has clique-width at most 9.*

Proof. Let G be a graph with vertices partitioned into sets as above. This means that we have sets of vertices $T^1, W_1, T^2, W_2, \ldots, T^p, W_p$ in order, such that if X and Y are sets in this order with X coming before Y then $X \cap V_k$ is complete to $Y \cap V_\ell$ if $k + 1 \equiv \ell \pmod 3$ and anti-complete otherwise in all cases except where $X = W_i, Y = W_{i+1}$ for some i, in which case $X \cap V_k$ may either be complete or anti-complete to $Y \cap V_{k+1}$. Also recall that U^{T^i} is an independent set for every i and there is a vertex $x \in T^i$ such that every vertex of U^{T^i} has the same neighbourhood in $G \setminus T^i$ as x.

Note that $W_i \subseteq V_1 \cup V_2 \cup V_3$. Then $G[W_i]$ is a 3-partite graph with 3-partition $(W_i \cap V_1, W_i \cap V_2, W_i \cap V_3)$. Furthermore, $G[W_i]$ is K_3-free, and contains no induced $3P_1$ with exactly one vertex in each V_j. Since $G[W_i]$ is $(P_1 + 2P_2)$-free it must therefore be $(P_7, S_{1,2,3})$-free. Therefore, by Lemma 6, the graph $G[W_i]$ is totally 3-decomposable with respect to this partition. By Lemma 7, we can construct $G[W_i]$ using at most six labels such that the resulting labelled graph has all vertices in W_i labelled with label i for $i \in \{1, 2, 3\}$.

We are now ready to describe how to construct G. We do this by constructing $G[T^i \cup U^{T^i}]$ then $G[W_i]$ for each $i \in \{1, \ldots, p\}$ in turn and adding it to the graph. More formally, we start with the empty graph, then for $i = 1, \ldots, p$ in turn, we do the following:

1. Let $\{x_1^i, x_2^i, x_3^i\} = T^i$, where $x_j^i \in V_j$ for $j \in \{1, 2, 3\}$. Add vertices x_1^i, x_2^i and x_3^i with labels $4, 5$ and 6, respectively, then add edges between vertices labelled $4\&5, 5\&6$ and $4\&6$.
2. If U^{T^i} is non-empty then the vertices in this set have the same neighbourhood in $G \setminus T^i$ as x_1^i, x_2^i or x_3^i. Add the vertices of U^{T^i} with label $4, 5$ or 6, respectively.
3. Add edges between vertices labelled $1\&5, 2\&6$ and $3\&4$.
4. Relabel vertices labelled $4, 5$ or 6 to have labels $1, 2$ or 3, respectively.
5. Construct $G[W_i]$ with vertices labelled $4, 5$ or 6, if they are in V_1, V_2 or V_3, respectively.
6. Add edges between vertices labelled $1\&5, 2\&6$ and $3\&4$.
7. If $i > 1$ then add edges between vertices labelled $4\&9, 5\&7$ and $6\&8$ if $V_k \cap W_i$ is complete to $V_{k-1} \cap W_{i-1}$ for $k = 1, 2, 3$, respectively.
8. Relabel vertices labelled $7, 8$ or 9 to have labels $1, 2$ or 3, respectively.
9. Relabel vertices labelled $4, 5$ or 6 to have labels $7, 8$ or 9, respectively.

Note that at the end of any iteration of the above procedure, the vertices of W_i will have labels in $\{7, 8, 9\}$ and all other constructed vertices will have labels in $\{1, 2, 3\}$.

This construction builds a copy of G using at most nine labels. Thus G has clique-width at most 9. This concludes the proof of the lemma. □

We are now ready to prove our main theorem of this section. To do so, we show that if a graph G is (diamond, $P_1 + 2P_2$)-free then either we can show that G has bounded clique-width directly (possibly by applying some graph operations that do not change the clique-width the graph by "too much") or else the (unmodified) graph G is itself basic (in which case it has clique-width at most 9).

Theorem 4. *The class of* (*diamond*, $P_1 + 2P_2$)*-free graphs has bounded clique-width.*

Proof. Let G be a (diamond, $P_1 + 2P_2$)-free graph. By Lemma 10, we may assume that G is connected. By Theorem 3, we may assume that G contains an induced K_3. By Lemma 11, we may assume that G is K_4-free.

Let T be an arbitrary induced triangle (i.e. K_3) in G with vertices v_1^T, v_2^T and v_3^T. Since G is (diamond, K_4)-free, every vertex not in T has at most one neighbour in T. For $i \in \{1, 2, 3\}$ let V_i^T be the set of vertices not in T whose unique neighbour in T is v_i^T and let U^T be the set of vertices that have no neighbour in T. We will now prove a series of claims. More formally, we will show that if the conditions of any of these claims are not satisfied, then either we obtain a contradiction or we can directly prove that G has bounded clique-width, in which case we are done.

Claim 1. *For every triangle T, the sets V_1^T, V_2^T and V_3^T each contain at least three vertices.*
If for some i the set V_i^T contains at most two vertices then v_i^T has at most four neighbours in G. If we delete every vertex in $N(v_i^T)$, then v_i^T has no neighbours in the resulting graph. Therefore either G has at most five vertices (in which case it has clique-width at most 5), or $G \setminus N(v_i^T)$ is a disconnected (diamond, $P_1 + 2P_2$)-free graph, so it has bounded clique-width by Lemma 10. By Fact 1, it follows that G has bounded clique-width. This completes the proof of the claim.

Claim 2. *For every triangle T, the sets V_1^T, V_2^T and V_3^T are independent.*
Suppose, for contradiction, that V_1^T is not an independent set. Since G is K_4-free and every vertex of V_1^T is adjacent to v_1^T, it follows that $G[V_1^T]$ is K_3-free. Since V_1^T contains at least three vertices by Claim 1, there must be vertices $x, y, z \in V_1^T$ such that x is adjacent to y, but not to z. Then $G[v_1^T, y, x, z]$ is a diamond if y and z are adjacent and $G[z, x, y, v_2^T, v_3^T]$ is a $P_1 + 2P_2$ if they are not. This contradiction implies that V_1^T is an independent set. The claim follows by symmetry.

Claim 3. *Every pair of triangles in G is vertex-disjoint.*
Consider a triangle T with vertex v_1^T. The neighbourhood of v_1^T is $V_1^T \cup \{v_2^T, v_3^T\}$. Now V_1^T is independent by Claim 2 and anti-complete to $\{v_2^T, v_3^T\}$ by definition. Therefore, if a triangle in G contains v_1^T then it must also contain v_2^T and v_3^T. In other words, v_1^T is contained in only one triangle in G, namely T. The claim follows by symmetry.

Claim 4. *For every triangle T, the set U^T is independent.*
By Claim 1, we can choose $x, y \in V_1^T$ and by Claim 2, x must be non-adjacent to y. If a vertex $u \in U^T$ is adjacent to x, but not to y then $G[y, u, x, v_2^T, v_3^T]$ is a $P_1 + 2P_2$, a contradiction. Therefore every vertex of U^T is either complete or anti-complete to $\{x, y\}$. Suppose $u, v \in U^T$. First suppose u and v are non-adjacent. If x is adjacent to u but v is not, then $G[v, u, x, v_2^T, v_3^T]$ is a $P_1 + 2P_2$, a contradiction. Therefore if $u, v \in U^T$ are non-adjacent, then $\{u, v\}$ is either complete or anti-complete to

$\{x, y\}$. Now suppose u and v are adjacent. Then $G[u, v, x, y]$ is a diamond if $\{u, v\}$ is complete to $\{x, y\}$ and $G[x, u, v, v_2^T, v_3^T]$ is a $P_1 + 2P_2$ if $\{u, v\}$ is anti-complete to $\{x, y\}$. Therefore if $u, v \in U^T$ are adjacent then exactly one of them is complete to $\{x, y\}$ and the other is anti-complete to $\{x, y\}$. This means that $G[U^T]$ is a complete bipartite graph, with partition classes U_1^T and U_2^T, say, and furthermore, one of U_1^T and U_2^T is complete to V_1^T and the other is anti-complete to V_1^T. Similarly, this holds with the same partition (U_1^T, U_2^T) if we replace V_1^T by V_2^T or V_3^T. Thus every vertex of U_1^T (respectively U_2^T) has the same neighbourhood in $V_1^T \cup V_2^T \cup V_3^T$.

Suppose that V_i^T and V_j^T are both complete to U_k^T for some $i, j \in \{1, 2, 3\}$ with $i \neq j$ and some $k \in \{1, 2\}$ and that U_k^T contains at least two vertices, say u and v. If $x \in V_i^T$ and $y \in V_j^T$ are adjacent, then $G[x, y, u, v]$ is a diamond, a contradiction. Therefore V_i^T is anti-complete to V_j^T.

Suppose that U_1^T and U_2^T each contain at least one vertex, say u and v, respectively. We will show that in this case the clique-width of G is bounded. Suppose, for contradiction, that $G \setminus (T \cup \{u, v\})$ contains an induced K_3, say with vertex set T'. Since $G[U^T]$ is a complete bipartite graph with bipartition (U_1^T, U_2^T) and no vertex of a set V_i^T can have neighbours in both U_1^T and U_2^T, at most one vertex of T' can be in U^T. Suppose that U_1^T contains at least two vertices (so $U_1^T \setminus \{u\}$ is non-empty) and that U_1^T is complete to V_i^T and V_j^T for some $i \neq j$ (in which case U_2^T is anti-complete to V_i^T and V_j^T). Then V_i^T and V_j^T must be anti-complete. We conclude that in this case no vertex of U_1^T can belong to T'. No vertex of U_2^T can belong to T' either, since vertices in U_2^T can only have neighbours in U_1^T and in V_k^T where $k \notin \{i, j\}$ (if U_1^T is anti-complete to V_k^T). Furthermore, since V_i^T is anti-complete to V_j^T, and V_1^T, V_2^T, V_3^T are independent (by Claim 2), there is no induced K_3 in $G[V_1^T \cup V_2^T \cup V_3^T]$. Thus T' cannot exist, a contradiction.

The above means that if such a triangle T' does exist and a set U_i^T contains at least two vertices, then U_i^T must be anti-complete to at least two distinct sets V_j^T and V_k^T (in which case U_i^T cannot contain a vertex of T'). Since T' consists of vertices of $G \setminus (T \cup \{u, v\})$, this means that no vertex of U^T is in T' (if U_i^T contains a single vertex for some i then by definition T' does not include it). By Claim 2, it follows that T' must consist of vertices $x \in V_1^T$, $y \in V_2^T$ and $z \in V_3^T$. Since each set V_i^T is anti-complete to exactly one of U_1^T and U_2^T, we may assume without loss of generality that U_1^T (and therefore u) is complete to both V_1^T and V_2^T. Now $G[x, y, z, u]$ is a K_4 or diamond if u and z are adjacent or non-adjacent, respectively. This contradiction means that $G \setminus (T \cup \{u, v\})$ must in fact be K_3-free. Since $G \setminus (T \cup \{u, v\})$ is a $(K_3, P_1 + 2P_2)$-free graph, it has bounded clique-width by Theorem 3. By Fact 1, we conclude that G also has bounded clique-width. We may therefore assume that either U_1^T or U_2^T is empty. It follows that U^T is an independent set. This completes the proof of the claim.

Claim 5. *For every triangle T, there is a vertex $x \in V(T)$ such that $N(x) = N(u) \cup (V(T) \setminus \{x\})$ for all $u \in U^T$.*

By the previous claim, we may assume that U^T is independent. Note that by the same arguments as for the previous claim, for all $i \in \{1, 2, 3\}$, U^T is trivial to V_i^T. Suppose $u \in U^T$. By the same arguments as for the previous claim, U^T must be anti-complete to at least two distinct sets V_i^T and V_j^T, otherwise $G \setminus (T \cup \{u\})$ would be K_3-free and the clique-width of G would be bounded as before. Since G is connected, it follows that U^T must be complete to at least one set V_i^T. Therefore U^T must be complete to exactly one set V_i^T. It follows that $N(v_i^T) = V_i^T \cup (V(T) \setminus \{v_i^T\}) = N(u) \cup (V(T) \setminus \{v_i^T\})$ for all $u \in U^T$. This completes the proof of the claim.

Claim 6. *No triangle in G contains a vertex of $\mathcal{U}$.*

If $u \in \mathcal{U}$ then $u \in U_T$ for some triangle T. By the previous claim, the neighbourhood

of every vertex of U^T is V_i^T, for some i. Since V_i^T is an independent set, the claim follows immediately.

Claim 7. *If T and T' are distinct triangles in G then the edges between them form an induced matching.*

Suppose T and T' are distinct triangles in G. By Claim 3, T and T' must be vertex-disjoint. By Claim 6, it follows that every vertex of T' is in $V_1^T \cup V_2^T \cup V_3^T$, so every vertex of T' has exactly one neighbour in T. By Claim 2, for $i \in \{1, 2, 3\}$, the set V_i^T is an independent set, so it can contain at most one vertex of T'. Therefore T' has exactly one vertex in each of V_1^T, V_2^T and V_3^T. By definition of V_i^T, this means that every vertex of T' has a different neighbour in T. The claim follows.

Claim 8. *For every triangle T and for every pair of distinct $i, j \in \{1, 2, 3\}$, $G[V_i^T \cup V_j^T]$ is $2P_2$-free.*

Suppose, for contradiction, that $G[V_1^T \cup V_2^T]$ contains an induced $2P_2$. Then this $2P_2$, together with the vertex v_3^T would induce a $P_1 + 2P_2$ in G. The claim follows by symmetry.

Claim 9. *For every triangle T, there is no induced $3P_1$ in G with one vertex in each of V_1^T, V_2^T and V_3^T.*

Suppose that there are three vertices $x \in V_1^T, y \in V_2^T$ and $z \in V_3^T$ that are pairwise non-adjacent. We will show that in this case G has bounded clique-width. Suppose $u \in U^T$. By Claim 5, u has exactly one neighbour in $\{x, y, z\}$. Without loss of generality, assume that u is adjacent to x. Then $G[z, u, x, y, v_2^T]$ is a $P_1 + 2P_2$, a contradiction. We may therefore assume that U^T is empty. If there is a vertex $x' \in V_1^T \setminus \{x\}$ that is adjacent to y, but not to z then $G[x, x', y, v_3^T, z]$ is a $P_1 + 2P_2$ in G. This contradiction means that every vertex of V_1^T is either complete or anti-complete to $\{y, z\}$. Similarly, every vertex of V_2^T is either complete or anti-complete to $\{x, z\}$ and every vertex of V_3^T is either complete or anti-complete to $\{x, y\}$. Note that the above holds for any three pairwise non-adjacent vertices in V_1^T, V_2^T and V_3^T, respectively.

Let $V_1'^T$ and $V_1''^T$ be the sets of vertices in V_1^T that are anti-complete or complete to $\{y, z\}$, respectively. Let $V_2'^T$ and $V_2''^T$ be the sets of vertices in V_2^T that are anti-complete or complete to $\{x, z\}$, respectively. Let $V_3'^T$ and $V_3''^T$ be the sets of vertices in V_3^T that are anti-complete or complete to $\{x, y\}$, respectively. Note that $x \in V_1'^T, y \in V_2'^T$ and $z \in V_3'^T$.

Suppose $x' \in V_1'^T$ and $y' \in V_2'^T$. Since x' is non-adjacent to y and to z, it follows that $G[x', y, z]$ is a $3P_1$. Since y' is non-adjacent to z, it must therefore be anti-complete to $\{x', z\}$. In particular, this means that if $i, j \in \{1, 2, 3\}$ are distinct then $V_i'^T$ is anti-complete to $V_j'^T$.

Suppose $x' \in V_1'^T$ and $y' \in V_2''^T$. Since x' is non-adjacent to y and to z, it follows that $G[x', y, z]$ is a $3P_1$. Since y' is adjacent to z, it must therefore be complete to $\{x', z\}$. In particular, this means that if $i, j \in \{1, 2, 3\}$ are distinct then $V_i'^T$ is complete to $V_j''^T$.

Note that for all $i \in \{1, 2, 3\}$, $V_i'^T$ is anti-complete to $V_i''^T$, since V_i^T is an independent set.

Suppose $x' \in V_1''^T$ and $y' \in V_2''^T$. If x' and y' are non-adjacent then $G[x', y, x, y']$ is a $2P_2$ in $G[V_1^T \cup V_2^T]$, which would contradict Claim 8. This means that if $i, j \in \{1, 2, 3\}$ are distinct then $V_i''^T$ is complete to $V_j''^T$.

We now proceed as follows: from G, we delete the three vertices of T. We then apply a bipartite complementation between every pair of sets $V_i'^T$ and $V_j''^T$ and every pair of distinct sets $V_i''^T$ and $V_j''^T$ (a total of nine bipartite complementations). After doing this, we obtain an edge-less graph, which therefore has clique-width at most 1. By

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Facts 1 and 3, it follows that G must also have bounded clique-width. This completes the proof of the claim.

Claim 10. *G contains at least three vertex-disjoint triangles.*
Suppose, for contradiction, that the claim is false. Then G contains at most two vertex-disjoint triangles, in which case, we can delete at most six vertices to obtain a $(K_3, P_1 + 2P_2)$-free graph, which has bounded clique-width by Theorem 3. By Fact 1, G also has bounded clique-width. This completes the proof of the claim.

We will now assume that the above claims are satisfied and show that this implies that G is basic. We arbitrarily fix a triangle T^1 with vertices $v_1^{T^1}$, $v_2^{T^1}$ and $v_3^{T^1}$. To simplify notation, set $v_i = v_i^{T^1}$ for $i \in \{1, 2, 3\}$. Recall that by Claim 6, no K_3 in G has a vertex in $\mathcal{U}$. By Claim 2, it follows that every K_3 in G apart from T^1 has exactly one vertex in each of $V_1^{T^1} \setminus \mathcal{U}$, $V_2^{T^1} \setminus \mathcal{U}$ and $V_3^{T^1} \setminus \mathcal{U}$. We now set $V_1 = (V_1^{T^1} \cup \{v_2\}) \setminus \mathcal{U}$, $V_2 = (V_2^{T^1} \cup \{v_3\}) \setminus \mathcal{U}$ and $V_3 = (V_3^{T^1} \cup \{v_1\}) \setminus \mathcal{U}$.

Claim 11. *V_1, V_2 and V_3 are independent.*
The vertices in $V_i^{T^1}$ are exactly the vertices outside T^1 whose unique neighbour in T^1 is v_i. The claim follows by Claim 2.

By Claim 3 any two triangles in G must be vertex-disjoint. By Claim 7, the edges between any two triangles in G form a perfect matching. Let $T^x = \{x_1, x_2, x_3\}$ and $T^y = \{y_1, y_2, y_3\}$ be two distinct triangles in G with $x_i, y_i \in V_i$ for $i \in \{1, 2, 3\}$. By Claim 11, x_i is non-adjacent to y_i for $i \in \{1, 2, 3\}$. This means that the set of edges between T^x and T^y is either $\{x_1y_2, x_2y_3, x_3y_1\}$ or $\{x_1y_3, x_2y_1, x_3y_2\}$. We say that $T^x < T^y$ holds in the first case and $T^y < T^x$ holds in the second. Note that exactly one of these statements holds for any two distinct triangles in G. Furthermore, note that if T^x is a triangle other than T^1 then the definition of the sets V_i implies that $T^1 < T^x$.

We show that the relation $<$ is transitive. Suppose, for contradiction, that this is not the case. Then there must be three pairwise distinct triangles in G, say $T^x = \{x_1, x_2, x_3\}, T^y = \{y_1, y_2, y_3\}$ and $T^z = \{z_1, z_2, z_3\}$, where $x_i, y_i, z_i \in V_i^T$ for $i \in \{1, 2, 3\}$, with $T^x < T^y$, $T^y < T^z$ and $T^z < T^x$. Then x_1 is adjacent to y_2, y_2 is adjacent to z_3 and z_3 is adjacent to x_1. Therefore $G[x_1, y_2, z_3]$ is a K_3 which shares exactly one vertex with T^x, which would contradict Claim 3. Therefore $<$ is a transitive, anti-symmetric relation on the triangles in G. We may now order the triangles in G, say $T^1 < T^2 < \cdots < T^p$ for some p. By Claim 10, it follows that $p \geq 3$. We now conclude the following:

Claim 12. *$\{G[T^1], \ldots, G[T^p]\}$ is the set of all induced triangles in G and each of them has exactly one vertex in each of V_1, V_2 and V_3.*
Claim 13. *If $i < j$ and $k + 1 \not\equiv \ell \pmod 3$ then $T^i \cap V_k$ is anti-complete to $T^j \cap V_\ell$.*
Claim 14. *If $i < j$ and $k + 1 \equiv \ell \pmod 3$ then $T^i \cap V_k$ is complete to $T^j \cap V_\ell$.*

Consider a vertex x that is not in any induced triangle in G. If $x \notin \mathcal{U}$ then $x \in V_1 \cup V_2 \cup V_3$ and x must have exactly one neighbour in every triangle in G. Let W be the set of vertices that are not in any triangle in G and have exactly one neighbour in every induced triangle in G.

We extend the relation $<$ as follows: suppose $T = \{x_1, x_2, x_3\}$ is an induced triangle in G with $x_1 \in V_1, x_2 \in V_2$ and $x_3 \in V_3$ and suppose $w \in W$. Then w is a vertex in V_i for some $i \in \{1, 2, 3\}$. By Claim 11, w is not adjacent to x_i. Since $w \in W$, w must be adjacent to exactly one vertex of T. We say that $x < T$ holds if x is adjacent to x_{i+1} and $T < x$ if x is adjacent to x_{i-1} (we interpret indices modulo 3).

Let $w \in W$ and let T and T' be triangles in G such that $w < T$ and $T < T'$. We will show that $w < T'$. Say $T = \{x_1, x_2, x_3\}$ and $T' = \{y_1, y_2, y_3\}$, where $x_i, y_i \in V_i$ for

$i \in \{1, 2, 3\}$. Without loss of generality, assume $w \in V_1$. Since $w < T$, w is adjacent to x_2. Since $T < T'$, x_2 is adjacent to y_3. Since $w \in V_1$, w is non-adjacent to y_1. Now w cannot be adjacent to y_3, otherwise $G[w, x_2, y_3]$ would be a triangle that is not vertex-disjoint from T, which would contradict Claim 3. Since $w \in W$, it must have a neighbour in T', so w must therefore be adjacent to y_2. It follows that $w < T'$. Similarly, if $T < T'$ and $T' < w$ then $T < w$ and if $T < w$ and $w < T'$ then $T < T'$.

This means that we can now partition W into sets $W_1, \ldots, W_p$ where W_i contains the vertices $x \in W$ such that $T^j < x$ for $j \leq i$ and $x < T^j$ for $j > i$. (Note that $T^1 < w$ for all $w \in W$, by construction.) We immediately conclude the following:

Claim 15. *If $i < j$ and $k + 1 \not\equiv \ell \pmod 3$ then $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.*
Claim 16. *If $i = j$ and $k + 1 \not\equiv \ell \pmod 3$ then $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.*
Claim 17. *If $i < j$ and $k + 1 \not\equiv \ell \pmod 3$ then $W_i \cap V_k$ is anti-complete to $T^j \cap V_\ell$.*
Claim 18. *If $i < j$ and $k + 1 \equiv \ell \pmod 3$ then $T^i \cap V_k$ is complete to $W_j \cap V_\ell$.*
Claim 19. *If $i = j$ and $k + 1 \equiv \ell \pmod 3$ then $T^i \cap V_k$ is complete to $W_j \cap V_\ell$.*
Claim 20. *If $i < j$ and $k + 1 \equiv \ell \pmod 3$ then $W_i \cap V_k$ is complete to $T^j \cap V_\ell$.*

We also prove the following claim:

Claim 21. *$G[W_i]$ is $(P_1 + 2P_2)$-free and does not contain an induced $3P_1$ with one vertex in each of V_1, V_2 and V_3.*
Since G is $(P_1 + 2P_2)$-free, it follows that $G[W_i]$ is also $(P_1 + 2P_2)$-free. Since the vertices of W_i do not belong to any triangle of G and do not belong to $\mathcal{U}$, it follows that $W_i \subseteq V_1^{T^1} \cup V_2^{T^1} \cup V_3^{T^1}$. The claim then follows by Claim 9.

It remains to analyse the edges between the sets $W_1, \ldots, W_p$.

Claim 22. *If $i < j$ and $k + 1 \not\equiv \ell \pmod 3$ then $W_i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.*
Let $i, j \in \{1, \ldots, p\}$ be such that $i < j$. Let $T^j = \{x_1, x_2, x_3\}$ with $x_k \in V_k$ for $k \in \{1, 2, 3\}$. Note that if $x \in W_i$ and $y \in W_j$ then $x < T^j$ and $T^j < y$. Now $W_i \cap V_k$ is anti-complete to $W_j \cap V_k$ for $k \in \{1, 2, 3\}$, since V_k is an independent set by Claim 11. Suppose $x \in W_i \cap V_1$ and $y \in W_j \cap V_3$. Then x and y are both adjacent to x_2. Therefore x and y cannot be adjacent, otherwise $G[x_2, x, y]$ would be a triangle which is not vertex-disjoint from T^j, which would contradict Claim 3. By symmetry we conclude that $W_i \cap V_k$ is anti-complete to $W_j \cap V_{k+2}$ for $k \in \{1, 2, 3\}$ (interpreting subscripts modulo 3). This completes the proof of the claim.

The edges between $W_i \cap V_k$ and $W_j \cap V_{k+1}$ for $k \in \{1, 2, 3\}$ are more complicated, as shown in the following two claims:

Claim 23. *If $i + 1 < j$ and $k + 1 \equiv \ell \pmod 3$ then $W_i \cap V_k$ is complete to $W_j \cap V_\ell$.*
Let $i, j \in \{1, \ldots, p\}$ be such that $i + 1 < j$. Suppose, for contradiction, that $x \in W_i \cap V_1$ and $y \in W_j \cap V_2$ are non-adjacent. Since $i + 2 \leq j$ we find that $x < T^{j-1}, x < T^j, T^{j-1} < y$ and $T^j < y$. Let $T^j = \{x_1, x_2, x_3\}$ with $x_k \in V_k$ for $k \in \{1, 2, 3\}$. Let $T^{j-1} = \{y_1, y_2, y_3\}$, where $y_k \in V_k$ for $k \in \{1, 2, 3\}$. Then x is adjacent to y_2, but non-adjacent to x_1, while y is adjacent to x_1, but non-adjacent to y_2. Since $T^{j-1} < T^j$ it follows that y_2 is non-adjacent to x_1. Since $T^1 < x, y$, the vertex v_3 must be non-adjacent to x and y (recall that $v_3 = v_3^{T^1}$ and that this vertex has no neighbours in V_1 or V_2 apart from v_1 and v_2). Now $G[v_3, x, y_2, x_1, y]$ is a $P_1 + 2P_2$, a contradiction. By symmetry this completes the proof of the claim.

Claim 24. *If $i + 1 = j$ and $k + 1 \equiv \ell \pmod 3$ then $W_i \cap V_k$ is either complete or anti-complete to $W_j \cap V_\ell$.*
Let $i, j \in \{1, \ldots, p\}$ with $i + 1 = j$. Let $T^j = \{x_1, x_2, x_3\}$ with $x_k \in V_k$ for $k \in \{1, 2, 3\}$. Assume, for contradiction, that the vertex sets $W_i \cap V_k$ and $W_j \cap V_{k+1}$ are not trivial to each-other for some $k \in \{1, 2, 3\}$. Without loss of generality, we may assume that there

is a vertex x with a neighbour y and a non-neighbour y' such that either $x \in W_i \cap V_1$ and $y, y' \in W_j \cap V_2$ or $y, y' \in W_i \cap V_1$ and $x \in W_j \cap V_2$. Note that x_3 is non-adjacent to x, y and y'. Since $T^1 < x, y, y'$, the vertex v_3 must be non-adjacent to x and y (recall that $v_3 = v_3^T$ and that this vertex has no neighbours in V_1 or V_2 apart from v_1 and v_2). Now $G[y', x, y, v_3, x_3]$ is a $P_1 + 2P_2$, a contradiction. By symmetry this completes the proof of the claim.

The claims proved above imply all the necessary properties for G to be basic. Indeed, Claim 6 implies Property (i) and Claims 4 and 5 imply Property (ii). Claims 11, 12, 21, 13, 15, 17, 22, 14, 18, 20, 23, 24, 19 and 16 imply Properties (iii), (iv), (v), (vi).1, (vi).2, (vi).3, (vi).4, (vii).1, (vii).2, (vii).3, (viii).1, (ix).1, (x).1, (xi).1 and respectively. Therefore G is basic, so it has bounded clique-width by Lemma 12. This completes the proof. $\square$

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