New Geometric Representations and Domination Problems on Tolerance and Multitolerance Graphs* †

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Abstract

Tolerance graphs model interval relations in such a way that intervals can tolerate a certain amount of overlap without being in conflict. In one of the most natural generalizations of tolerance graphs with direct applications in the comparison of DNA sequences from different organisms, namely *multitolerance* graphs, two tolerances are allowed for each interval – one from the left and one from the right side. Several efficient algorithms for optimization problems that are NP-hard in general graphs have been designed for tolerance and multitolerance graphs. In spite of this progress, the complexity status of some fundamental algorithmic problems on tolerance and multitolerance graphs, such as the *dominating set* problem, remained unresolved until now, three decades after the introduction of tolerance graphs. In this article we introduce two new geometric representations for tolerance and multitolerance graphs, given by points and line segments in the plane. Apart from being important on their own, these new representations prove to be a powerful tool for deriving both hardness results and polynomial time algorithms. Using them, we surprisingly prove that the dominating set problem can be solved in polynomial time on tolerance graphs and that it is APX-hard on multitolerance graphs, solving thus a longstanding open problem. This problem is the first one that has been discovered with a different complexity status in these two graph classes.

Keywords: Tolerance graph, multitolerance graph, geometric representation, dominating set problem, polynomial time algorithm, APX-hard.

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1 Introduction

A graph $G = (V, E)$ on n vertices is a *tolerance* graph if there exists a collection $I = \{I_v \mid v \in V\}$ of intervals on the real line and a set $t = \{t_v \mid v \in V\}$ of positive numbers (the tolerances), such that for any two vertices $u, v \in V$, $uv \in E$ if and only if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$, where $|I|$ denotes the length of the interval I. The pair $\langle I, t\rangle$ is called a *tolerance representation* of G. If G has a tolerance representation $\langle I, t\rangle$, such that $t_v \leq |I_v|$ for every $v \in V$, then G is called a *bounded tolerance* graph.

If we replace in the above definition "min" by "max", we obtain the class of *max-tolerance* graphs. Both tolerance and max-tolerance graphs have attracted many research efforts [2,4,7,9,10, 12,14–17] as they find numerous applications, especially in bioinformatics, among others [10,12,14]; for a more detailed account see the book on tolerance graphs [11]. One of their major applications is in the comparison of DNA sequences from different organisms or individuals by making use of a

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software tool like BLAST [1]. However, at some parts of the above genomic sequences in BLAST, we may want to be more tolerant than at other parts, since for example some of them may be biologically less significant or we have less confidence in the exact sequence due to sequencing errors in more error prone genomic regions. This concept leads naturally to the notion of *multitolerance* graphs which generalize tolerance graphs [11, 15, 19]. The main idea is to allow two different tolerances for each interval, one to each of its sides. Then, every interval tolerates in its interior part the intersection with other intervals by an amount that is a convex combination of these two border-tolerances.

Formally, let $I = [l, r]$ be an interval on the real line and $l_t, r_t \in I$ be two numbers between l and r, called *tolerant points*. For every $\lambda \in [0, 1]$, we define the interval $I_{l_t,r_t}(\lambda) = [l + (r_t - l)\lambda, l_t + (r - l_t)\lambda]$, which is the convex combination of $[l, l_t]$ and $[r_t, r]$. Furthermore, we define the set $\mathcal{I}(I, l_t, r_t) = \{I_{l_t,r_t}(\lambda) \mid \lambda \in [0, 1]\}$ of intervals. That is, $\mathcal{I}(I, l_t, r_t)$ is the set of all intervals that we obtain when we linearly transform $[l, l_t]$ into $[r_t, r]$. For an interval I, the *set of tolerance-intervals* τ of I is defined either as $\tau = \mathcal{I}(I, l_t, r_t)$ for some values $l_t, r_t \in I$ (the case of a *bounded* vertex), or as $\tau = \{\mathbb{R}\}$ (the case of an *unbounded* vertex). A graph $G = (V, E)$ is a *multitolerance* graph if there exists a collection $I = \{I_v \mid v \in V\}$ of intervals and a family $t = \{\tau_v \mid v \in V\}$ of sets of tolerance-intervals, such that: for any two vertices $u, v \in V$, $uv \in E$ if and only if $Q_u \subseteq I_v$ for some $Q_u \in \tau_u$, or $Q_v \subseteq I_u$ for some $Q_v \in \tau_v$. Then, the pair $\langle I, t\rangle$ is called a *multitolerance representation* of G. If G has a multitolerance representation with only bounded vertices, i.e., with $\tau_v \neq \{\mathbb{R}\}$ for every vertex v, then G is called a *bounded multitolerance* graph.

For several optimization problems that are NP-hard in general graphs, such as the coloring, clique, and independent set problems, efficient algorithms are known for tolerance and multitolerance graphs. However, only few of them have been derived using the (multi)tolerance representation (e.g. [10,19]), while most of these algorithms appeared as a consequence of the containment of tolerance and multitolerance graphs to weakly chordal (and thus also to perfect) graphs [20]. To design efficient algorithms for (multi)tolerance graphs, it seems to be essential to assume that a suitable representation of the graph is given along with the input, as it has been recently proved that the recognition of tolerance graphs is NP-complete [17]. Recently two new geometric intersection models in the 3-dimensional space have been introduced for both tolerance graphs (the *parallelepiped* representation [16]) and multitolerance graphs (the *trapezoepiped* representation [15]), which enabled the design of very efficient algorithms for such problems, in most cases with (optimal) $O(n \log n)$ running time [15,16]. In spite of this, the complexity status of some algorithmic problems on tolerance and multitolerance graphs still remains open, three decades after the introduction of tolerance graphs in [8]. Arguably the two most famous and intriguing examples of such problems are the *minimum dominating set* problem and the *Hamilton cycle* problem (see e.g. [20, page 314]). Both these problems are known to be NP-complete on the greater class of weakly chordal graphs [3, 18] but solvable in polynomial time in the smaller classes of bounded tolerance and bounded multitolerance (i.e., trapezoid) graphs [6,13]. The reason that these problems resisted solution attempts over the years seems to be that the existing representations for (multi)tolerance graphs do not provide enough insight to deal with these problems.

Our contribution. In this article we introduce a new geometric representation for multitolerance graphs, which we call the *shadow representation*, given by a set of line segments and points in the plane. In the case of tolerance graphs, this representation takes a very special form, in which all line segments are horizontal, and therefore we call it the *horizontal shadow representation*. Note that both the shadow and the horizontal shadow representations are *not* intersection models for multitolerance graphs and for tolerance graphs, respectively, in the sense that two line segments may not intersect in the representation although the corresponding vertices are adjacent. However, the main advantage of these two new representations is that they provide substantially new insight for tolerance and multitolerance graphs and they can be used to interpret optimization problems (such as the dominating set problem and its variants) using computational geometry terms.

Apart from being important on their own, these new representations enable us to establish the complexity of the *minimum dominating set* problem on both tolerance and multitolerance graphs, thus solving a longstanding open problem. Given a horizontal shadow representation of a tolerance graph G, we present an algorithm that computes a minimum dominating set in polynomial time. On the other hand, using the shadow representation, we prove that the minimum dominating set problem is APX-hard on multitolerance graphs by providing a reduction from a special case of the set cover problem. That is, there exists no Polynomial Time Approximation Scheme (PTAS) for this problem unless P=NP. This is the first problem that has been discovered with a different complexity status in these two graph classes. Therefore, given the (seemingly) small difference between the definition of tolerance and multitolerance graphs, this dichotomy result appears to be surprising.

Organization of the paper. In Section 2 we briefly revise the 3-dimensional intersection models for tolerance graphs [16] and multitolerance graphs [15], which are needed in order to present our new geometric representations. In Section 3 we introduce our new geometric representation for multitolerance graphs (the *shadow representation*) and its special case for tolerance graphs (the *horizontal shadow representation*). In Section 4 we prove that DOMINATING SET on multitolerance graphs is APX-hard. Then, in Sections 5-7 we present our polynomial algorithm for the dominating set problem on tolerance graphs, using the horizontal shadow representation (cf. Algorithms 1, 2, and 3). In particular, we first present Algorithm 1 in Section 5, which solves a variation of the dominating set problem on tolerance graphs, called BOUNDED DOMINATING SET. Then we present Algorithm 2 in Section 6, which uses Algorithm 1 as a subroutine in order to solve a slightly modified version of BOUNDED DOMINATING SET on tolerance graphs, namely RESTRICTED BOUNDED DOMINATING SET. In Section 7 we present our main algorithm (Algorithm 3) which solves DOMINATING SET on tolerance graphs in polynomial time, using Algorithms 1 and 2 as subroutines. Finally, in Section 8 we discuss the presented results and some interesting further research questions.

Notation. In this article we consider simple undirected graphs with no loops or multiple edges. In an undirected graph G the edge between two vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. We denote by $N(u) = \{v \in V : uv \in E\}$ the set of neighbors of a vertex u in G, and $N[u] = N(u) \cup \{u\}$. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the vertices in S. A subset $S \subseteq V$ is a *dominating set* of G if every vertex $v \in V \setminus S$ has at least one neighbor in S. Finally, given a set $X \subseteq \mathbb{R}^2$ of points in the plane, we denote by $H_{\text{convex}}(X)$ the *convex hull* defined by the points of X, and by $\overline{X} = \mathbb{R}^2 \setminus X$ the complement of X in $\mathbb{R}^2$. For simplicity of the presentation we make the following notational convention throughout the paper: whenever we need to compute a set S with the smallest cardinality among a family $\mathcal{S}$ of sets, we write $S = \min\{\mathcal{S}\}$.

2 Tolerance and multitolerance graphs

In this section we briefly revise the 3-dimensional intersection model for tolerance graphs [16] and its generalization to multitolerance graphs [15], together with some useful properties of these models that are needed for the remainder of the paper. Since the intersection model of [16] for tolerance graphs is a special case of the intersection model of [15] for multitolerance graphs, we mainly focus below on the more general model for multitolerance graphs.

Consider a multitolerance graph $G = (V, E)$ that is given along with a multitolerance representation R. Let V_B and V_U denote the set of bounded and unbounded vertices of G in this representation, respectively. Consider now two parallel lines L_1 and L_2 in the plane. For every vertex $v \in V = V_B \cup V_U$, we appropriately construct a trapezoid $\overline{T}_v$ with its parallel lines on L_1

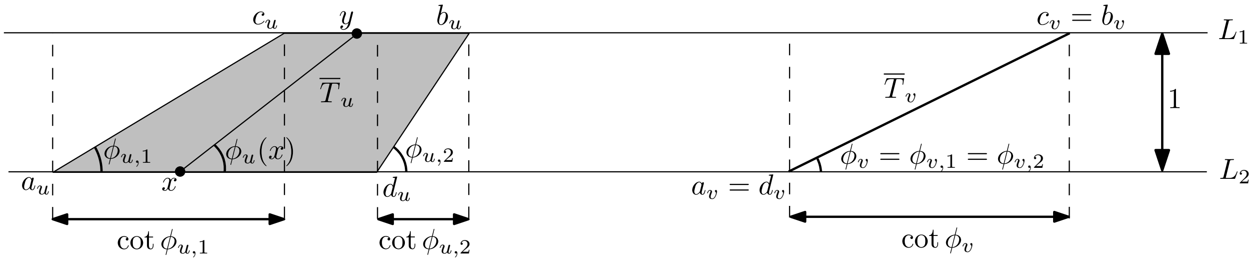

Figure 1: The trapezoid $\overline{T}_u$ corresponds to the bounded vertex $u \in V_B$, while the line segment $\overline{T}_v$ corresponds to the unbounded vertex $v \in V_U$.

and L_2, respectively (for details of this construction of the trapezoids we refer to [15]). According to this construction, for every unbounded vertex $v \in V_U$ the trapezoid $\overline{T}_v$ is trivial, i.e., a line [15]. For every vertex $v \in V = V_B \cup V_U$ we denote by a_v, b_v, c_v, d_v the lower left, upper right, upper left, and lower right endpoints of the trapezoid $\overline{T}_v$, respectively. Note that for every unbounded vertex $v \in V_U$ we have $a_v = d_v$ and $c_v = b_v$, since $\overline{T}_v$ is just a line segment. An example is depicted in Figure 1, where $\overline{T}_u$ corresponds to a bounded vertex u and $\overline{T}_v$ corresponds to an unbounded vertex v.

We now define the left and right angles of these trapezoids. For every angle ϕ, the values $\tan \phi$ and $\cot \phi = \frac{1}{\tan \phi}$ denote the tangent and the cotangent of ϕ, respectively. Furthermore, $\phi = arc\cot x$ is the angle ϕ, for which $\cot \phi = x$.

Definition 1 ([15]) *For every vertex $v \in V = V_B \cup V_U$, the values $\phi_{v,1} = arc\cot(c_v - a_v)$ and $\phi_{v,2} = arc\cot(b_v - d_v)$ are the* left angle *and the* right angle *of $\overline{T}_v$, respectively. Moreover, for every unbounded vertex $v \in V_U$, $\phi_v = \phi_{v,1} = \phi_{v,2}$ is the* angle *of $\overline{T}_v$.*

Note here that, if G is given along with a *tolerance* representation R (i.e., if G is a tolerance graph), then for every bounded vertex u we have that $\phi_{u,1} = \phi_{u,2}$, and thus the corresponding trapezoid $\overline{T}_u$ always becomes a *parallelogram* [15] (see also [16]).

Without loss of generality we can assume that all endpoints and angles of the trapezoids are distinct, i.e., $\{a_u, b_u, c_u, d_u\} \cap \{a_v, b_v, c_v, d_v\} = \emptyset$ and $\{\phi_{u,1}, \phi_{u,2}\} \cap \{\phi_{v,1}, \phi_{v,2}\} = \emptyset$ for every $u, v \in V$ with $u \neq v$, as well as that $0 < \phi_{v,1}, \phi_{v,2} < \frac{\pi}{2}$ for all angles $\phi_{v,1}, \phi_{v,2}$ [15]. It is important to note here that this set of trapezoids $\{\overline{T}_v : v \in V = V_B \cup V_U\}$ is *not* an intersection model for the graph G, as two trapezoids $\overline{T}_v, \overline{T}_w$ may have a non-empty intersection although $vw \notin E$. However the subset of trapezoids $\{\overline{T}_v : v \in V_B\}$ that corresponds to the *bounded* vertices (i.e., to the vertices of V_B) is an intersection model of the induced subgraph $G[V_B]$.

In order to construct an intersection model for the whole graph G (i.e., including also the set V_U of the unbounded vertices), we exploit the third dimension as follows. Let $\Delta = \max\{b_v : v \in V\} - \min\{a_u : u \in V\}$ (where we consider the endpoints b_v and a_u as real numbers on the lines L_1 and L_2, respectively). First, for every unbounded vertex $v \in V_U$ we construct the line segment $T_v = \{(x, y, z) : (x, y) \in \overline{T}_v, z = \Delta - \cot \phi_v\}$. For every bounded vertex $v \in V_B$, denote by $\overline{T}_{v,1}$ and $\overline{T}_{v,2}$ the left and the right line segment of the trapezoid $\overline{T}_v$, respectively. We construct two line segments $T_{v,1} = \{(x, y, z) : (x, y) \in \overline{T}_{v,1}, z = \Delta - \cot \phi_{v,1}\}$ and $T_{v,2} = \{(x, y, z) : (x, y) \in \overline{T}_{v,2}, z = \Delta - \cot \phi_{v,2}\}$. Then, for every $v \in V_B$, we construct the 3-dimensional object T_v as the convex hull $H_{\text{convex}}(\overline{T}_v, T_{v,1}, T_{v,2})$; this 3-dimensional object T_v is called the *trapezoepiped* of vertex $v \in V_B$. The resulting set $\{T_v : v \in V = V_B \cup V_U\}$ of objects in the 3-dimensional space is called the *trapezoepiped representation* of the multitolerance graph G [15]. This is an *intersection model* of G, i.e., two vertices v, w are adjacent if and only if $T_v \cap T_w \neq \emptyset$. For a proof of this fact and for more details about the trapezoepiped representation of multitolerance graphs we refer to [15].

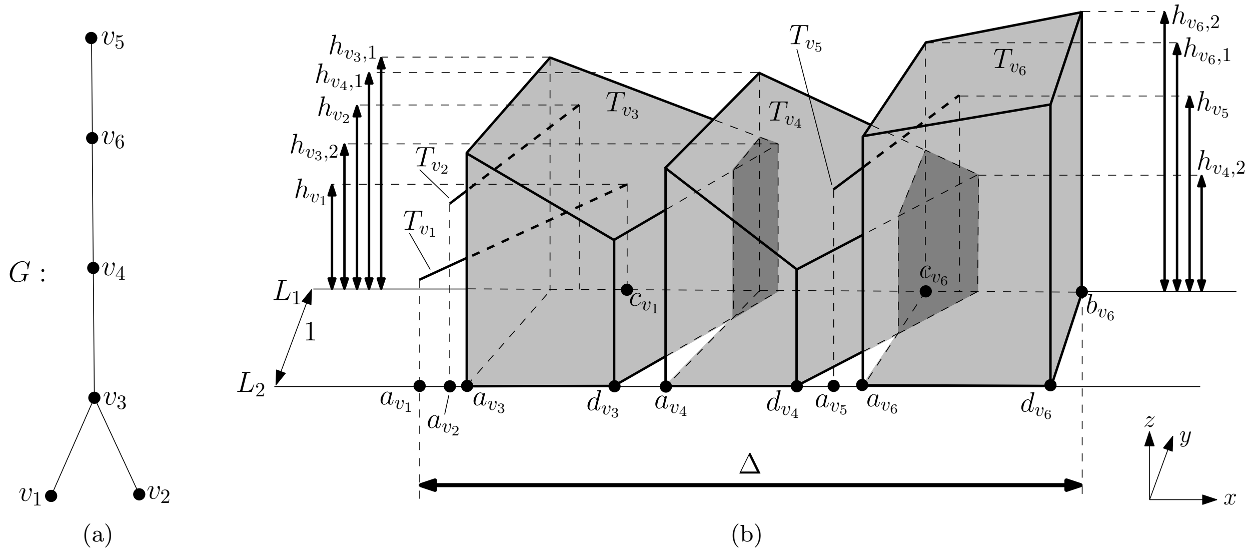

Figure 2: (a) A multitolerance graph G and (b) a trapezoepiped representation R of G. Here, $h_{v_i,j} = \Delta - \cot \phi_{v_i,j}$ for every bounded vertex $v_i \in V_B$ and $j \in \{1, 2\}$, while $h_{v_i} = \Delta - \cot \phi_{v_i}$ for every unbounded vertex $v_i \in V_U$.

Recall that, if G is a tolerance graph, given along with a tolerance representation R, then $\phi_{u,1} = \phi_{u,2}$ for every bounded vertex u. Therefore, in the above construction, for every bounded vertex u the trapezoepiped T_u becomes a *parallelepiped*, and in this case the resulting trapezoepiped representation is called a *parallelepiped representation* [15, 16].

An example of the construction of a trapezoepiped representation is given in Figure 2. A multitolerance graph G with six vertices $\{v_1, v_2, \ldots, v_6\}$ is depicted in Figure 2(a), while the trapezoepiped representation of G is illustrated in Figure 2(b). The set of bounded and unbounded vertices in this representation are $V_B = \{v_3, v_4, v_6\}$ and $V_U = \{v_1, v_2, v_5\}$, respectively.

Definition 2 ([15]) *An unbounded vertex $v \in V_U$ is* inevitable *if replacing T_v by $H_{convex}(T_v, \overline{T}_v)$ creates a new edge uv in G; then u is a* hovering vertex *of v and the set $H(v)$ of all hovering vertices of v is the* hovering set *of v. A trapezoepiped representation of a multitolerance graph G is called* canonical *if every unbounded vertex is inevitable.*

In the example of Figure 2, v_2 and v_5 are inevitable unbounded vertices, v_1 and v_4 are hovering vertices of v_2 and v_5, respectively, while v_1 is not an inevitable unbounded vertex. Therefore, this representation is not canonical for the graph G. However, if we replace T_{v_1} by $H_{\text{convex}}(T_{v_1}, a_{v_1}, c_{v_1})$, we get a canonical representation for G in which vertex v_1 is bounded.

Lemma 1 ([15]) *Let $v \in V_U$ be an inevitable unbounded vertex of a multitolerance graph G. Then $N(v) \subseteq N(u)$ for every hovering vertex $u \in H(v)$ of v.*

Lemma 2 ([15]) *Let R be a canonical representation of a multitolerance graph G and $v \in V_U$ be an (inevitable) unbounded vertex of G. Then there exists a hovering vertex u of v, which is bounded.*

Recall that $\{\overline{T}_v : v \in V_B\}$ is an intersection model of the induced subgraph $G[V_B]$ on the bounded vertices of G, i.e., $uv \in E$ if and only if $\overline{T}_u \cap \overline{T}_v \neq \emptyset$ where $u, v \in V_B$. Furthermore, although $\{\overline{T}_v : v \in V = V_B \cup V_U\}$ is *not* an intersection model of G, it still provides the whole information about the adjacencies of the vertices of G, cf. Lemma 3. For Lemma 3 we need the next definition of the angles $\phi_u(x)$, where $u \in V_B$ and $a_u \leq x \leq d_u$, cf. Figure 1 for an illustration.

Definition 3 ([15]) *Let* $u \in V_B$ *be a bounded vertex and* a_u, b_u, c_u, d_u *be the endpoints of the trapezoid* $\overline{T}_u$*. Let* $x \in [a_u, d_u]$ *and* $y \in [c_u, b_u]$ *be two points on the lines* L_2 *and* L_1*, respectively, such that* $x = \lambda a_u + (1-\lambda)d_u$ *and* $y = \lambda c_u + (1-\lambda)b_u$ *for the same value* $\lambda \in [0,1]$*. Then* $\phi_u(x)$ *is the angle of the line segment with endpoints* x *and* y *on the lines* L_2 *and* L_1*, respectively.*

Lemma 3 ([15]) *Let* $u \in V_B$ *and* $v \in V_U$ *in a trapezoepiped representation of a multitolerance graph* $G = (V, E)$*. Let* a_u*,* d_u*, and* $a_v = d_v$ *be the endpoints of* $\overline{T}_u$ *and* $\overline{T}_v$*, respectively, on* L_2*. Then:*

- *if* $a_v < a_u$*, then* $uv \in E$ *if and only if* $\overline{T}_u \cap \overline{T}_v \neq \emptyset$*,*
- *if* $a_u < a_v < d_u$*, then* $uv \in E$ *if and only if* $\phi_v \leq \phi_u(a_v)$*,*
- *if* $d_u < a_v$*, then* $uv \notin E$*.*

3 The new geometric representations

In this section we introduce new geometric representations on the plane for both tolerance and multitolerance graphs. The new representation of tolerance graphs is called the *horizontal shadow representation*, which is given by a set of points and horizontal line segments in the plane. The horizontal shadow representation can be naturally extended to general multitolerance graphs, in which case the line segments are not necessarily horizontal; we call this representation of multitolerance graphs the *shadow representation*. In the remainder of this section, we present the shadow representation of general multitolerance graphs, since the horizontal shadow representation of tolerance graphs is just the special case, in which every line segment is horizontal.

Definition 4 (shadow representation) *Let* $G = (V, E)$ *be a multitolerance graph,* R *be a trapezoepiped representation of* G*, and* V_B, V_U *be the sets of bounded and unbounded vertices of* G *in* R*, respectively. We associate the vertices of* G *with points and line segments in the plane as follows:*

- *for every* $v \in V_B$*, the points* $p_{v,1} = (a_v, \Delta - \cot \phi_{v,1})$ *and* $p_{v,2} = (d_v, \Delta - \cot \phi_{v,2})$ *and the line segment* $L_v = (p_{v,1}, p_{v,2})$*,*
- *for every* $v \in V_U$*, the point* $p_v = (a_v, \Delta - \cot \phi_v)$*.*

The tuple $(\mathcal{P}, \mathcal{L})$*, where* $\mathcal{L} = \{L_v : v \in V_B\}$ *and* $\mathcal{P} = \{p_v : v \in V_U\}$*, is the* shadow representation *of* G*. If* $\phi_{v,1} = \phi_{v,2}$ *for every* $v \in V_B$*, then* $(\mathcal{P}, \mathcal{L})$ *is the* horizontal shadow representation *of the tolerance graph* G*. Furthermore, the representation* $(\mathcal{P}, \mathcal{L})$ *is* canonical *if the initial trapezoepiped representation* R *is also canonical.*

Note by Definition 4 that, given a trapezoepiped (resp. parallelepiped) representation of a multitolerance (resp. tolerance) graph G with n vertices, we can compute a shadow (resp. horizontal shadow) representation of G in $O(n)$ time. As an example for Definition 4, we illustrate in Figure 3 the shadow representation $(\mathcal{P}, \mathcal{L})$ of the multitolerance graph G of Figure 2.

Observation 1 *In Definition 4,* $L_v = \{(x, \Delta - \cot \phi_v(x)) : a_v \leq x \leq d_v\}$ *for every bounded vertex* $v \in V_B$ *of the multitolerance graph* G*.*

Now we introduce the notions of the *shadow* and the *reverse shadow* of points and of line segments in the plane; an example is illustrated in Figure 4.

Definition 5 (shadow) *For an arbitrary point* $t = (t_x, t_y) \in \mathbb{R}^2$ *the* shadow *of* t *is the region* $S_t = \{(x, y) \in \mathbb{R}^2 : x \leq t_x,\ y - x \leq t_y - t_x\}$*. Furthermore, for every line segment* L_u*, where* $u \in V_B$*, the* shadow *of* L_u *is the region* $S_u = \bigcup_{t \in L_u} S_t$*.*

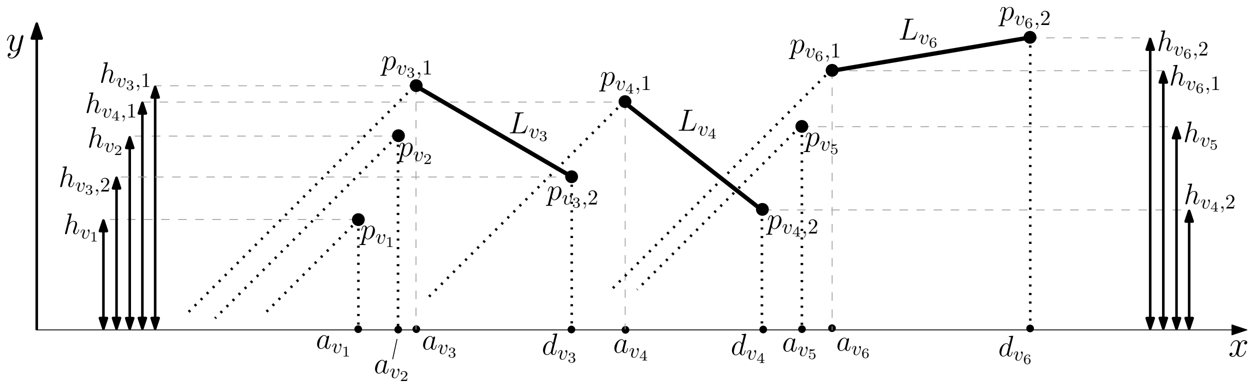

Figure 3: The shadow representation $(\mathcal{P}, \mathcal{L})$ of the multitolerance graph G of Figure 2. The unbounded vertices $V_U = \{v_1, v_2, v_5\}$ and the bounded vertices $V_B = \{v_3, v_4, v_6\}$ are associated with the points $\mathcal{P} = \{p_{v_1}, p_{v_2}, p_{v_5}\}$ and with the line segments $\mathcal{L} = \{L_{v_1}, L_{v_2}, L_{v_5}\}$, respectively.

Definition 6 (reverse shadow) *For an arbitrary point $t = (t_x, t_y) \in \mathbb{R}^2$ the* reverse shadow *of t is the region $F_t = \{(x, y) \in \mathbb{R}^2 : x \geq t_x,\ y - x \geq t_y - t_x\}$. Furthermore, for every line segment L_i, where $u \in V_B$, the* reverse shadow *of L_i is the region $F_i = \bigcup_{t \in L_i} F_t$.*

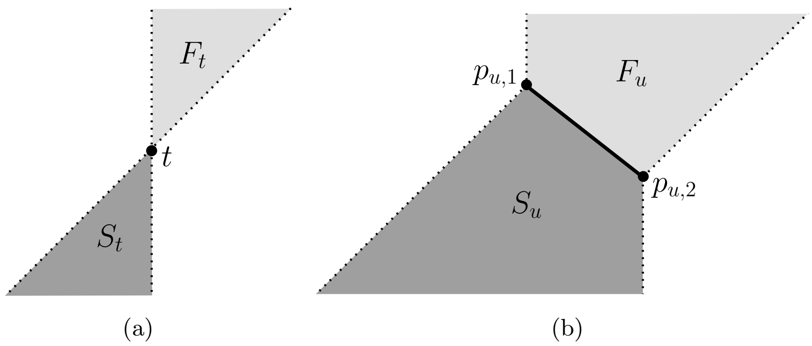

Figure 4: The shadow and the reverse shadow of (a) a point $t \in \mathbb{R}^2$ and (b) a line segment L_u.

Lemma 4 *Let G be a multitolerance graph and $(\mathcal{P}, \mathcal{L})$ be a shadow representation of G. Let $u \in V_B$ be a bounded vertex of G such that the corresponding line segment L_u is not trivial, i.e., L_u is not a single point. Then the angle of the line segment L_u with a horizontal line (i.e., parallel to the x-axis) is at most $\frac{\pi}{4}$ and at least $-\frac{\pi}{2}$.*

Proof. The two endpoints of L_u are the points $(a_u, \Delta - \cot \phi_{u,1})$ and $(d_u, \Delta - \cot \phi_{u,2})$. For the purposes of the proof, denote by ψ the angle of the line segment L_u with a horizontal line (i.e., parallel to the x-axis). To prove that $\psi \geq -\frac{\pi}{2}$ it suffices to observe that $a_u \leq d_u$ (cf. Figure 1). To prove that $\psi \leq \frac{\pi}{4}$ it suffices to show that $(\Delta - \cot \phi_{u,2}) - (\Delta - \cot \phi_{u,1}) \leq d_u - a_u$, or equivalently to show that $(\Delta - (b_u - d_u)) - (\Delta - (c_u - a_u)) \leq d_u - a_u$. The latter inequality is equivalent to $b_u \geq c_u$, which is always true (cf. Figure 1). ■

Recall now that two unbounded vertices $u, v \in V_U$ are never adjacent. The connection between a multitolerance graph G and a shadow representation of it is the following. Two bounded vertices $u, v \in V_B$ are adjacent if and only if $L_u \cap S_v \neq \emptyset$ or $L_v \cap S_u \neq \emptyset$, cf. Lemma 5. A bounded vertex $u \in V_B$ and an unbounded vertex $v \in V_U$ are adjacent if and only if $p_v \in S_u$, cf. Lemma 6.

Lemma 5 *Let $(\mathcal{P}, \mathcal{L})$ be a shadow representation of a multitolerance graph G. Let $u, v \in V_B$ be two bounded vertices of G. Then $uv \in E$ if and only if $L_v \cap S_u \neq \emptyset$ or $L_u \cap S_v \neq \emptyset$.*

Proof. Let R be the trapezoepiped representation of G, from which the shadow representation $(\mathcal{P}, \mathcal{L})$ is constructed, cf. Definition 4.

($\Rightarrow$) Let $uv \in E$. Assume first that the intervals $[a_u, d_u]$ and $[a_v, d_v]$ of the x-axis share at least one common point, say t_x. If $\phi_v(t_x) \leq \phi_u(t_x)$, then the point $(t_x, \Delta - \cot \phi_v(t_x))$ of the line segment L_v belongs to the shadow S_u of the line segment L_u, i.e., $L_v \cap S_u \neq \emptyset$. Otherwise, symmetrically, if $\phi_v(t) > \phi_u(t)$ then $L_u \cap S_v \neq \emptyset$.

Assume now that $[a_u, d_u]$ and $[a_v, d_v]$ are disjoint, i.e., either $d_u < a_v$ or $d_v < a_u$. Without loss of generality we may assume that $d_u < a_v$, as the other case is symmetric. Then, as $uv \in E$ by assumption, it follows that $\overline{T}_u \cap \overline{T}_v \neq \emptyset$ in the trapezoepiped representation R of G. Thus $b_u \geq c_v$, since we assumed that $d_u < a_v$. Therefore $\cot \phi_u = b_u - d_u \geq c_v - d_u = \cot \phi_{v,1} + (a_v - d_u)$. That is, $(\Delta - \cot \phi_{u,2}) - d_u \leq (\Delta - \cot \phi_{v,1}) - a_v$, and thus the point $(d_u, \Delta - \cot \phi_{u,2})$ of the line segment L_u belongs to the shadow S_t of the point $t = (a_v, \Delta - \cot \phi_{v,1})$ of the line segment L_v. Therefore $L_u \cap S_v \neq \emptyset$.

($\Leftarrow$) Let $L_v \cap S_u \neq \emptyset$ or $L_u \cap S_v \neq \emptyset$. Assume first that the intervals $[a_u, d_u]$ and $[a_v, d_v]$ of the x-axis share at least one common point, say t_x. Then $t_x \in [a_u, d_u] \cap [a_v, d_v]$, and thus the trapezoids $\overline{T}_u$ and $\overline{T}_v$ in the trapezoepiped representation R have a common point on the line L_2, i.e., $\overline{T}_u \cap \overline{T}_v \neq \emptyset$. Therefore, since both u and v are bounded vertices, it follows that $uv \in E$.

Assume now that $[a_u, d_u]$ and $[a_v, d_v]$ are disjoint, i.e., either $d_v < a_u$ or $d_u < a_v$. Without loss of generality we may assume that $d_v < a_u$, as the other case is symmetric. Then $L_u \cap S_v = \emptyset$, and thus $L_v \cap S_u \neq \emptyset$. Therefore, by Lemma 4, it follows that the point $t = (d_v, \Delta - \cot \phi_{v,2})$ of L_v must belong to S_u. In particular, this point t of L_v must belong to the shadow $S_{t'}$ of the point $t' = (a_u, \Delta - \cot \phi_{u,1})$ of L_u. That is, $(\Delta - \cot \phi_{v,2}) - d_v \leq (\Delta - \cot \phi_{u,1}) - a_u$. It follows that $(b_v - d_v) = \cot \phi_{v,2} \geq \cot \phi_{u,1} + (a_u - d_v) = (c_u - a_u) + (a_u - d_v)$, and thus $b_v \geq c_u$. Therefore, since $d_v < a_u$, it follows that $\overline{T}_u \cap \overline{T}_v \neq \emptyset$, and thus $uv \in E$. ■

Lemma 6 *Let $(\mathcal{P}, \mathcal{L})$ be a shadow representation of a multitolerance graph G. Let $v \in V_U$ and $u \in V_B$ be two vertices of G. Then $uv \in E$ if and only if $p_v \in S_u$.*

Proof. Let R be the trapezoepiped representation of G, from which the shadow representation $(\mathcal{P}, \mathcal{L})$ is constructed, cf. Definition 4. Furthermore recall that $p_v = (a_v, \Delta - \cot \phi_v)$ by Definition 4.

($\Rightarrow$) Let $uv \in E$. If $d_u < a_v$, then $uv \notin E$ by Lemma 3, which is a contradiction. Therefore $a_v < d_u$. Assume first that $a_u < a_v < d_u$. Then Lemma 3 implies that $\phi_v \leq \phi_u(a_v)$. Thus it follows by Observation 1 that $p_v \in S_u$. Assume now that $a_v < a_u$. Then Lemma 3 implies that $\overline{T}_u \cap \overline{T}_v \neq \emptyset$. Thus $b_v \geq c_u$, since $a_v < a_u$. Therefore $\cot \phi_v = (b_v - a_v) \geq (a_u - a_v) + (c_u - a_u) = (a_u - a_v) + \cot \phi_{u,1}$. That is, $(\Delta - \cot \phi_v) - a_v \leq (\Delta - \cot \phi_{u,1}) - a_u$, and thus the point $p_v = (a_v, \Delta - \cot \phi_v)$ belongs to the shadow S_t, where $t = (a_u, \Delta - \cot \phi_{u,1}) \in L_u$, i.e., $p_v \in S_u$.

($\Leftarrow$) Let $p_v \in S_u$. Then clearly $a_v \leq d_u$. Assume first that $a_u \leq a_v \leq d_u$. Then, since $p_v \in S_u$, it follows by Observation 1 that $\Delta - \cot \phi_v \leq \Delta - \cot \phi_u(a_v)$, and thus $\phi_v \leq \phi_u(a_v)$. Therefore Lemma 3 implies that $uv \in E$.

Assume now that $a_v < a_u$. Then, since $p_v \in S_u$, it follows that $p_v \in S_t$, where $t = (a_u, \Delta - \cot \phi_{u,1}) \in L_u$. Thus $(\Delta - \cot \phi_v) - a_v \leq (\Delta - \cot \phi_{u,1}) - a_u$. That is, $(b_v - a_v) = \cot \phi_v \geq (a_u - a_v) + \cot \phi_{u,1} = (a_u - a_v) + (c_u - a_u)$, and thus $b_v \geq c_u$. Therefore, since $a_v < a_u$, it follows that $\overline{T}_u \cap \overline{T}_v \neq \emptyset$, and thus $uv \in E$ by Lemma 3. ■

Lemmas 5 and 6 show how adjacencies between vertices can be seen in a shadow representation $(\mathcal{P}, \mathcal{L})$ of a multitolerance graph G. The next lemma describes how the hovering vertices of an unbounded vertex $v \in V_U$ (cf. Definition 2) can be seen in a shadow representation $(\mathcal{P}, \mathcal{L})$.

Lemma 7 *Let $(\mathcal{P}, \mathcal{L})$ be a shadow representation of a multitolerance graph G. Let $v \in V_U$ be an unbounded vertex of G and $u \in V \setminus \{v\}$ be another arbitrary vertex. If $u \in V_B$ (resp. $u \in V_U$), then u is a hovering vertex of v if and only if $L_u \cap S_v \neq \emptyset$ (resp. $p_u \in S_v$).*

Proof. Let $G = (V, E)$ and R be the trapezoepiped representation of G, from which the shadow representation $(\mathcal{P}, \mathcal{L})$ is constructed, cf. Definition 4.

($\Leftarrow$) Let u be a hovering vertex of v. That is, if we replace in the trapezoepiped representation R the line segment T_v by $H_{\text{convex}}(T_v, \overline{T}_v)$ (i.e., if we make v a bounded vertex) then the vertices u and v become adjacent in the resulting trapezoepiped representation R'. Denote the new graph by $G' = (V, E \cup \{uv\})$, i.e., R' is a trapezoepiped representation of G'. Note here that, since both T_v and $\overline{T}_v$ are line segments, $H_{\text{convex}}(T_v, \overline{T}_v)$ is a degenerate trapezoepiped which is 2-dimensional.

Consider the shadow representation $(\mathcal{P}', \mathcal{L}')$ of G' that is obtained by this new trapezoepiped representation R'. Note that $\mathcal{P}' = \mathcal{P} \setminus \{p_v\}$ and $\mathcal{L}' = \mathcal{L} \cup \{L_v\}$, where L_v is a trivial line segment that consists of only one point p_v. Assume first that $u \in V_U$. Then, since v is bounded and v is adjacent to u in G', Lemma 6 implies that $p_u \in S_v$. Assume now that $u \in V_B$. Then, since v is bounded and v is adjacent to u in G', Lemma 5 implies that $L_v \cap S_u \neq \emptyset$ or $L_u \cap S_v \neq \emptyset$. That is, $p_v \in S_u$ or $L_u \cap S_v \neq \emptyset$, since $L_v = \{p_v\}$. If $p_v \in S_u$ then u and v are adjacent in G, by Lemma 6, which is a contradiction. Therefore $L_u \cap S_v \neq \emptyset$.

($\Rightarrow$) Consider the shadow representation $(\mathcal{P}', \mathcal{L}')$ that is obtained by the shadow representation $(\mathcal{P}, \mathcal{L})$ of G, such that $\mathcal{P}' = \mathcal{P} \setminus \{p_v\}$ and $\mathcal{L}' = \mathcal{L} \cup \{L_v\}$, where L_v is a trivial line segment that consists of only one point p_v. Then $(\mathcal{P}', \mathcal{L}')$ is a shadow representation of some multitolerance graph G', where the bounded vertices V'_B of G' correspond to the line segments of $\mathcal{L}'$ and the unbounded vertices V'_U of G' correspond to the points of $\mathcal{P}'$. Furthermore note that $V'_B = V_B \cup \{v\}$ and $V'_U = V_U \setminus \{v\}$.

Assume first that $u \in V'_B$ and $L_u \cap S_v \neq \emptyset$. Then, since both $u, v \in V'_B$, Lemma 5 implies that u and v are adjacent in G'. Thus, since u is not adjacent to v in G, it follows that u is a hovering vertex of v. Assume now that $u \in V'_U$ and $p_u \in S_v$. Then, since both $v \in V'_B$, Lemma 6 implies that u and v are adjacent in G'. Thus, similarly, u is a hovering vertex of v. ∎

In the example of Figure 3 the shadows of the points in $\mathcal{P}$ and of the line segments in $\mathcal{L}$ are shown with dotted lines. For instance, $p_{v_2} \in S_{v_3}$ and $p_{v_2} \notin S_{v_4}$, and thus the unbounded vertex v_2 is adjacent to the bounded vertex v_3 but not to the bounded vertex v_4. Furthermore $L_{v_3} \cap S_{v_4} \neq \emptyset$, and thus v_3 and v_4 are adjacent. On the other hand, $L_{v_3} \cap S_{v_6} = L_{v_6} \cap S_{v_3} = \emptyset$, and thus v_3 and v_4 are not adjacent. Finally $p_{v_1} \in S_{v_2}$ and $L_{v_4} \cap S_{v_5} \neq \emptyset$, and thus v_1 is a hovering vertex of v_2 and v_4 is a hovering vertex of v_5. These facts can be also checked in the trapezoepiped representation of the same multitolerance graph G in Figure 2(b).

4 Dominating set is APX-hard on multitolerance graphs

In this section we prove that the dominating set problem on multitolerance graphs is APX-hard. Let us first recall that an optimization problem P_1 is *L-reducible* to an optimization problem P_2 [21] if there exist two functions f and g, which are computable in polynomial time, and two constants $\alpha, \beta > 0$ such that:

- for any instance $\mathcal{I}$ of P_1, $f(\mathcal{I})$ is an instance of P_2 and $\text{OPT}(f(\mathcal{I})) \leq \alpha \cdot \text{OPT}(\mathcal{I})$, and
- for any feasible solution D of $f(\mathcal{I})$, $g(D)$ is a feasible solution of $\mathcal{I}$, and it holds that $|c(g(D)) - \text{OPT}(\mathcal{I})| \leq \beta \cdot |c(D) - \text{OPT}(f(\mathcal{I}))|$, where $c(D)$ and $c(g(D))$ denote the costs of the solutions D and $g(D)$, respectively.

Let us now define a special case of the unweighted set cover problem, namely the Special 3-Set Cover (S3SC) problem [5].

Theorem 1 ([5]) Special 3-Set Cover *is APX-hard.*

Special 3-Set Cover

Input: A pair $(\mathcal{U}, \mathcal{S})$ consisting of a universe $\mathcal{U} = A \cup W \cup X \cup Y \cup Z$, and a family $\mathcal{S}$ of subsets of $\mathcal{U}$ such that:

- the sets A, W, X, Y, Z are disjoint,
- $A = \{a_i : i \in [n]\}$, $W = \{w_i : i \in [m]\}$, $X = \{x_i : i \in [m]\}$, $Y = \{y_i : i \in [m]\}$, $Z = \{z_i : i \in [m]\}$,
- $2n = 3m$,
- for all $t \in [n]$, the element a_t belongs to exactly two sets of $\mathcal{S}$, and
- $\mathcal{S}$ has $5m$ sets; for every $t \in [m]$ there exist integers $1 \leq i < j < k < n$ such that $\mathcal{S}$ contains the sets $\{a_i, w_t\}$, $\{w_t, x_t\}, \{a_j, x_t, y_t\}$, $\{y_t, z_t\}$, $\{a_k, z_t\}$.

Output: A subset $\mathcal{S}_0 \subseteq \mathcal{S}$ of minimum size such that every element in $\mathcal{U}$ belongs to at least one set of $\mathcal{S}_0$.

Theorem 2 Dominating Set *is APX-hard on Multitolerance Graphs.*

Proof. From Theorem 1 it is enough to prove that Special 3-Set Cover is L-reducible to Dominating Set on Multitolerance Graphs.*

Given an instance $\mathcal{I} = (\mathcal{U}, \mathcal{S})$ of Special 3-Set Cover as above we construct a multitolerance graph $f(\mathcal{I}) = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ and $\mathcal{L}$ are the sets of points and line segments in the shadow representation of $f(\mathcal{I})$, as follows. For every element $a_i \in A$, we create the point p_{a_i} of $\mathcal{P}$ on the line $\{(z, -z) : z > 0\}$. Furthermore, for every element $q \in W \cup X \cup Y \cup Z$, we create the point p_q of $\mathcal{P}$ on the line $\{(t, \tan(\frac{\pi}{6})t) : t < 0\}$, such that for every $i \in [m]$ the points that correspond to the elements w_i, x_i, y_i, and z_i appear consecutively on this line (cf. Figure 5). Then, since every set of $\mathcal{S}$ contains at most one element of A and at most two elements of $W \cup X \cup Y \cup Z$, it can be easily verified that we can construct for every set $Q_j \in \mathcal{S}$, $j \in [5m]$, a line segment L_j such that the points of $\mathcal{P}$ that are contained within its shadow S_j are exactly the points of $\mathcal{P}$ that correspond to the elements of Q_j (cf. Figure 5). Furthermore we construct an additional line segment L_{5m+1}, with left endpoint l_{5m+1} and right endpoint r_{5m+1}, respectively, such that l_{5m+1} (resp. r_{5m+1}) lies below and to the left (resp. below and to the right) of every endpoint of $\mathcal{P} \cup \{L_1, L_2, \ldots, L_{5m}\}$. Then note that the line segment L_{5m+1} corresponds to a hovering vertex of every point $p \in \mathcal{P}$ in the multitolerance graph $f(\mathcal{I})$, cf. Lemma 7. Moreover the line segment L_{5m+1} is a neighbor to all other line segments $\{L_1, L_2, \ldots, L_{5m}\}$ in the multitolerance graph $f(\mathcal{I})$, cf. Lemma 5. Finally we add the line segment L_{5m+2} such that L_{5m+1} is its only neighbor, cf. Figure 5. This concludes the construction of the new instance $f(\mathcal{I})$.

Claim 1 *$OPT(f(\mathcal{I})) \leq OPT(\mathcal{I}) + 1$, and thus $OPT(f(\mathcal{I})) \leq 2 \cdot OPT(\mathcal{I})$.*

Proof of Claim 1. Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be an optimum solution of an instance $\mathcal{I}$ to Special 3-Set Cover and let D be the subset of $\mathcal{L}$ in the instance $f(\mathcal{I})$ of Dominating Set, where a line segment L of $f(\mathcal{I})$ belongs to D if and only if the corresponding set of $\mathcal{I}$ belongs to S. Let now $D' = D \cup \{L_{5m+1}\}$. As S is an optimum solution of $\mathcal{I}$ it follows that all the elements of $\mathcal{U}$ belong to some set of S and from the construction of $f(\mathcal{I})$ it follows that all points of $\mathcal{P}$ are contained inside the shadows of the line segments in D. Thus, every point of $\mathcal{P}$ has a neighbor in D. Notice also that from the construction of L_{5m+1} all line segments of $\mathcal{L}$ have L_{5m+1} as a neighbor. Therefore,

*This proof is inspired by the proof of Theorem 1.1(C5) in [5].

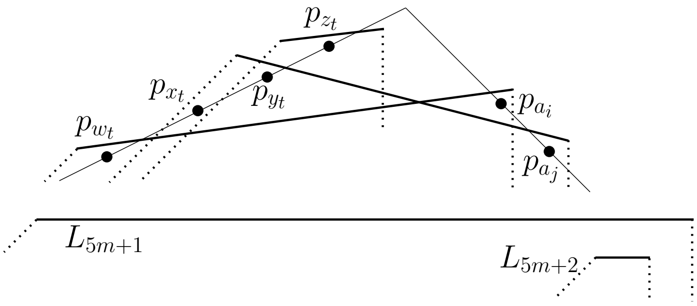

Figure 5: The construction of the shadow representation in Theorem 2.

as $|D| = |S|$ and $L_{5m+1} \notin D$, $D' = D \cup \{L_{5m+1}\}$ is a solution to $f(\mathcal{I})$ of size $\mathrm{OPT}(\mathcal{I}) + 1$. As Dominating Set is a minimization problem we obtain that $\mathrm{OPT}(f(\mathcal{I})) \leq |D'| = \mathrm{OPT}(\mathcal{I}) + 1$. □

We now define the function g which, given a feasible solution D of $f(\mathcal{I})$, returns a feasible solution $g(D)$ of $\mathcal{I}$. Let D be a feasible solution of $f(\mathcal{I})$.

If L_{5m+1} does not belong to D then L_{5m+2} belongs to D, since L_{5m+1} the only neighbor of L_{5m+2}. By replacing L_{5m+2} by L_{5m+1} we obtain a solution of $f(\mathcal{I})$ of the same size. Thus, without loss of generality we may assume that L_{5m+1} belongs to D. Furthermore, by the minimality of D it follows that D does not contain L_{5m+2}. Recall that all line segments $\{L_1, L_2, \ldots, L_{5m}\}$ have L_{5m+1} as a neighbor in D and that every point p of $f(\mathcal{I})$ is contained in the shadow of some line segment $L_p \in \{L_1, L_2, \ldots, L_{5m}\}$ in $f(\mathcal{I})$. Thus, for every point $p \in \mathcal{P} \cap D$, the set $(D \setminus \{p\}) \cup \{L_p\}$ is also a solution of $f(\mathcal{I})$ and has size at most $|D|$. Therefore, without loss of generality we may also assume that D only contains line segments. As $L_{5m+1} \in D$ is not a neighbor of any point of $\mathcal{P}$ in $f(\mathcal{I})$, the set $D \setminus \{L_{5m+1}\}$ contains all neighbors of the points of $f(\mathcal{I})$. Let $\mathcal{S}_0 \subseteq \mathcal{S}$ contain all sets from $\mathcal{S}$ that correspond to the line segments of $D \setminus \{L_{5m+1}\}$. From the construction of $f(\mathcal{I})$ we obtain that each element of $\mathcal{U}$ in $\mathcal{I}$ belongs to at least one set of $\mathcal{S}_0$. We define $g(D)$ to be that set $\mathcal{S}_0$. Finally, notice that $|\mathcal{S}_0| \leq |D| - 1$. This implies the following simple observation.

Observation 2 *If D is a solution of $f(\mathcal{I})$, then $g(D)$ is a solution of $\mathcal{I}$ and $c(g(D)) \leq c(D) - 1$.*

Claim 2 *$OPT(f(\mathcal{I})) = OPT(\mathcal{I}) + 1$.*

Proof of Claim 2. Let D be an optimum solution of $f(\mathcal{I})$. From Observation 2, we obtain that there exists a solution S of $\mathcal{I}$ such that $|S| \leq \mathrm{OPT}(f(\mathcal{I})) - 1$. As Special 3-Set Cover is a minimization problem it follows that $\mathrm{OPT}(\mathcal{I}) \leq |S| \leq \mathrm{OPT}(f(\mathcal{I})) - 1$ and thus, $\mathrm{OPT}(\mathcal{I}) + 1 \leq \mathrm{OPT}(f(\mathcal{I}))$. We now obtain the desired result from Claim 1. □

We finally prove that $c(g(D)) - \mathrm{OPT}(\mathcal{I}) \leq c(D) - \mathrm{OPT}(f(\mathcal{I}))$. Notice that this is enough to prove the reduction for $\alpha = 2$ (Claim 1) and $\beta = 1$. Claim 2 yields that $c(g(D)) - \mathrm{OPT}(\mathcal{I}) = c(g(D)) - \mathrm{OPT}(f(\mathcal{I})) + 1$, and thus it follows by Observation 2 that

$$c(g(D)) - \mathrm{OPT}(f(\mathcal{I})) + 1 \leq c(D) - 1 - \mathrm{OPT}(f(\mathcal{I})) + 1 = c(D) - \mathrm{OPT}(f(\mathcal{I})).$$

This completes the proof of the theorem. ■

5 Bounded dominating set on tolerance graphs

In this section we use the *horizontal shadow representation* of tolerance graphs (cf. Section 3) to provide a polynomial time algorithm for a variation of the minimum dominating set problem on tolerance graphs, namely Bounded Dominating Set, formally defined below. This problem variation may be interesting on its own, but we use our algorithm for Bounded Dominating Set as a subroutine in our algorithm for the minimum dominating set problem on tolerance graphs, cf. Sections 6 and 7. Note that, given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph $G = (V, E)$, the representation $(\mathcal{P}, \mathcal{L})$ defines a partition of the vertex set V into the set V_B of bounded vertices and the set V_U of unbounded vertices. Indeed, every point of $\mathcal{P}$ corresponds to an unbounded vertex in V_U and every line segment of $\mathcal{L}$ corresponds to a bounded vertex of V_B. We denote $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} =\{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, where $|\mathcal{P}| + |\mathcal{L}| = |V_U| + |V_B| = |V|$.

In this section we only deal with tolerance graphs and their horizontal shadow representations. Thus, from now on, all line segments $\{L_i : 1 \leq i \leq |\mathcal{L}|\}$ will be assumed to be *horizontal*. Furthermore, with a slight abuse of notation, for any two elements $x_1, x_2 \in \mathcal{P} \cup \mathcal{L}$, we may say in the following that x_1 is adjacent with x_2 (or x_1 is a neighbor of x_2) if the vertices that correspond to x_1 and x_2 are adjacent in the graph G. Moreover, whenever $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}$ and $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}$, we may say in the following that the set $\mathcal{P}_1 \cup \mathcal{L}_1$ *dominates* $\mathcal{P}_2 \cup \mathcal{L}_2$ if the vertices that correspond to $\mathcal{P}_1 \cup \mathcal{L}_1$ are a dominating set of the subgraph of G induced by the vertices corresponding to $\mathcal{P}_2 \cup \mathcal{L}_2$.

Bounded Dominating Set on Tolerance Graphs
Input: A horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph G.
Output: A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, or the announcement that $\mathcal{L}$ does not dominate $(\mathcal{P}, \mathcal{L})$.

Before we proceed with our polynomial time algorithm for Bounded Dominating Set on tolerance graphs, we first provide some necessary notation and terminology.

5.1 Notation and terminology

For an arbitrary point $t = (t_x, t_y) \in \mathbb{R}^2$ we define two (infinite) lines passing through t:

- the vertical line $\Gamma_t^{\text{vert}} = \{(t_x, s) \in \mathbb{R}^2 : s \in \mathbb{R}\}$, i.e., the line that is parallel to the y-axis, and
- the diagonal line $\Gamma_t^{\text{diag}} = \{(s, s + (t_y - t_x)) \in \mathbb{R}^2 : s \in \mathbb{R}\}$, i.e., the line that is parallel to the main diagonal $\{(s, s) \in \mathbb{R}^2 : s \in \mathbb{R}\}$.

The lines Γ_t^{vert} and Γ_t^{diag} are illustrated in Figure 6(a) (see also Figure 4(a)). For every point $t = (t_x, t_y) \in \mathbb{R}^2$, each of the lines $\Gamma_t^{\text{vert}}, \Gamma_t^{\text{diag}}$ separates $\mathbb{R}^2$ into two regions. With respect to the line Γ_t^{vert} we define the regions $\mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{vert}}) = \{(x, y) \in \mathbb{R}^2 : x \leq t_x\}$ and $\mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{vert}}) = \{(x, y) \in \mathbb{R}^2 : x \geq t_x\}$ of points to the left and to the right of Γ_t^{vert}, respectively. Similarly, with respect to the line Γ_t^{diag}, we define the regions $\mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{diag}}) = \{(x, y) \in \mathbb{R}^2 : y - x \geq t_y - t_x\}$ and $\mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{diag}}) = \{(x, y) \in \mathbb{R}^2 : y - x \leq t_y - t_x\}$ of points to the left and to the right of Γ_t^{diag}, respectively.

Furthermore, for an arbitrary point $t = (t_x, t_y) \in \mathbb{R}^2$ we define the region A_t (resp. B_t) that contains all points that are both to the right (resp. to the left) of Γ_t^{vert} and to the right (resp. to the left) of Γ_t^{diag}. That is,

$$\begin{aligned} A_t &= \mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{vert}}) \cap \mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{diag}}), \\ B_t &= \mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{vert}}) \cap \mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{diag}}). \end{aligned}$$

An example of the regions A_t and B_t is given in Figure 6(a), where A_t (resp. B_t) is the *shaded region* of $\mathbb{R}^2$ that is to the right (resp. to the left) of the point t. Consider an arbitrary horizontal line segment $L_i \in \mathcal{L}$. We denote by l_i and r_i its left and its right endpoint, respectively; note that possibly $l_i = r_i$. Denote by $\mathcal{A} = \{l_i, r_i : 1 \leq i \leq |\mathcal{L}|\}$ the set of all endpoints of all line segments of $\mathcal{L}$. Furthermore denote by $\mathcal{B} = \{\Gamma_t^{\text{diag}} \cap \Gamma_{t'}^{\text{vert}} : t, t' \in \mathcal{A}\}$ the set of all intersection points of the vertical and the diagonal lines that pass from points of $\mathcal{A}$. Note that $\mathcal{A} \subseteq \mathcal{B}$.

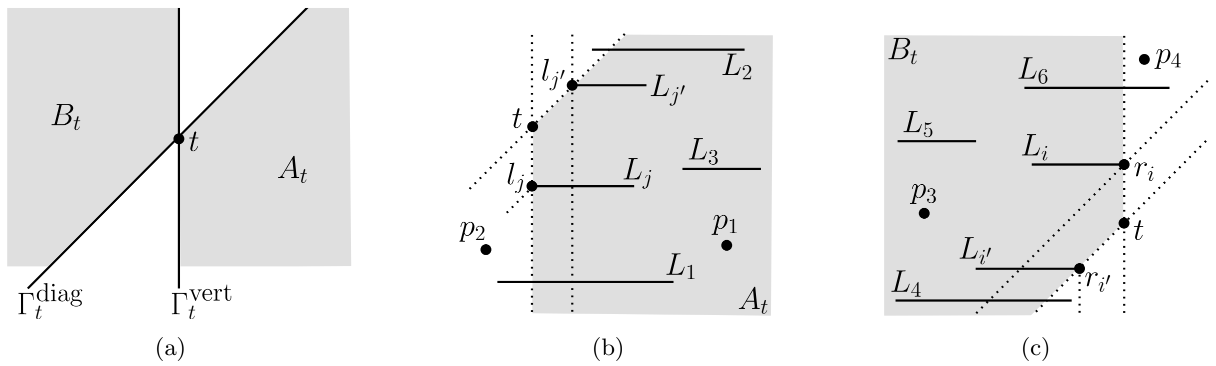

Figure 6: (a) The regions A_t, B_t and the lines $\Gamma_t^{\text{vert}}, \Gamma_t^{\text{diag}}$. (b) A left-crossing pair (j, j'), where $L_3, p_1 \in \mathcal{L}_{j,j'}^{\text{right}}$ and $L_1, L_2, p_2 \notin \mathcal{L}_{j,j'}^{\text{right}}$. (c) A right-crossing pair (i, i'), where $L_5, p_3 \in \mathcal{L}_{i,i'}^{\text{left}}$ and $L_4, L_6, p_4 \notin \mathcal{L}_{i,i'}^{\text{left}}$.

Given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ we always assume that the points $p_1, p_2, \ldots, p_{|\mathcal{P}|}$ are ordered increasingly with respect to their x-coordinates. Similarly we assume that the horizontal line segments $L_1, L_2, \ldots, L_{|\mathcal{L}|}$ are ordered increasingly with respect to the x-coordinates of their endpoint r_i. That is, if $i < j$ then $p_i \in \mathbb{R}^2_{\text{left}}(\Gamma_{p_j}^{\text{vert}})$ and $r_i \in \mathbb{R}^2_{\text{left}}(\Gamma_{r_j}^{\text{vert}})$. Notice that, without loss of generality, we may assume that all points of $\mathcal{P}$ and all endpoints of the horizontal line segments in $\mathcal{L}$ have different x-coordinates.

Definition 7 *Let $L_i, L_{i'}, \in \mathcal{L}$ and let $L_j, L_{j'} \in \mathcal{L}$, where possibly $i' = i$ and possibly $j' = j$. The pair (j, j') is a* left-crossing pair *if $l_j \in S_{l_{j'}}$. Furthermore the pair (i, i') is a* right-crossing pair *if $r_{i'} \in S_{r_i}$. For every left-crossing pair (j, j') we define*

$$\mathcal{L}_{j,j'}^{right} = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq A_t, \text{ where } t = \Gamma_{l_j}^{vert} \cap \Gamma_{l_{j'}}^{diag}\}$$

and for every right-crossing pair (i, i') we define

$$\mathcal{L}_{i,i'}^{left} = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq B_t, \text{ where } t = \Gamma_{r_i}^{vert} \cap \Gamma_{r_{i'}}^{diag}\}.$$

Finally, for every line segment $L_q \in \mathcal{L}$ we define

$$\mathcal{L}_q^{right} = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq \mathbb{R}^2_{right}(\Gamma_{l_q}^{diag})\}.$$

Examples of left-crossing and right-crossing pairs (cf. Definition 7) are illustrated in Figure 6.

Definition 8 *Let $S \subseteq \mathcal{P} \cup \mathcal{L}$ be an arbitrary set. Let (i, i') be a right-crossing pair and (j, j') be a left-crossing pair. If $L_i, L_{i'} \in S$ and $S \subseteq \mathcal{L}_{i,i'}^{left}$, then (i, i') is the* end-pair *of the set S. If $L_j, L_{j'} \in S$ and $S \subseteq \mathcal{L}_{j,j'}^{right}$, then (j, j') is the* start-pair *of the set S.*

Definition 9 *Let $S \subseteq \mathcal{P} \cup \mathcal{L}$ be an arbitrary set. The line segment $L_q \in S$ is the* diagonally leftmost line segment *in S if there exists a line segment $L_j \in \mathcal{L} \cap S$ such that (j, q) is the start-pair of S.*

Observation 3 *Every non-empty set $S \subseteq \mathcal{L}$ has a unique end-pair, a unique start-pair, and a unique diagonally leftmost line segment.*

5.2 The algorithm

In this section we present our algorithm for Bounded Dominating Set on tolerance graphs, cf. Algorithm 1. Given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph G, we first add two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ (with endpoints l_0, r_0 and $l_{|\mathcal{L}|+1}, r_{|\mathcal{L}|+1}$, respectively) such that all elements of $\mathcal{P} \cup \mathcal{L}$ are contained in A_{r_0} and in $B_{l_{|\mathcal{L}|+1}}$. Let $\mathcal{L}' = \mathcal{L} \cup \{L_0, L_{|\mathcal{L}|+1}\}$. Note that $(\mathcal{P}, \mathcal{L}')$ is a horizontal shadow representation of some tolerance graph G', where the bounded vertices V'_B of G' correspond to the line segments of $\mathcal{L}'$ and the unbounded vertices V'_U of G' correspond to the points of $\mathcal{P}$. Furthermore note that $V'_B = V_B \cup \{v_0, v_{|\mathcal{L}|+1}\}$ and $V'_U = V_U$, where v_0 and $v_{|\mathcal{L}|+1}$ are the (isolated) bounded vertices of G' that correspond to the line segments L_0 and $L_{|\mathcal{L}|+1}$, respectively. Finally observe now that the set V'_B dominates the augmented graph G' if and only if the set V_B dominates the graph G; moreover, a set $S \subseteq V_B$ dominates G if and only if $S \cup \{v_0, v_{|\mathcal{L}|+1}\}$ dominates G'.

For simplicity of the presentation, we refer in the following to the augmented set $\mathcal{L}'$ of horizontal line segments by $\mathcal{L}$. In the remainder of this section we will write $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$ with the understanding that the first and the last line segments L_1 and $L_{|\mathcal{L}|}$ of $\mathcal{L}$ are dummy. Furthermore, we will refer to the augmented tolerance graph G' by G.

For every pair of points $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$, define $X(a, b)$ to be the set of all points of $\mathcal{P}$ and all line segments of $\mathcal{L}$ that are contained in the region $B_b \setminus \Gamma_b^{\text{vert}}$ and to the right of the line Γ_a^{diag}, cf. Figure 7. That is,

$$R(a,b) = \left(B_b \setminus \Gamma_b^{\text{vert}}\right) \cap \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}}), \tag{1}$$
$$X(a,b) = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq R(a,b)\}. \tag{2}$$

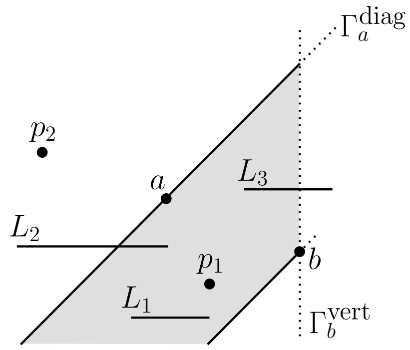

Figure 7: The shaded region contains the points of $R(a, b) \subseteq \mathbb{R}^2$, where $(a, b) \in \mathcal{A} \times \mathcal{B}$. The set $X(a, b)$ contains all elements of $\mathcal{P} \cup \mathcal{L}$ that lie within $R(a, b)$. In this example, $L_1, p_1 \in X(a, b)$ and $L_2, L_3, p_2 \notin X(a, b)$.

Now we present the main definition of this section, namely the quantity $BD_{(\mathcal{P},\mathcal{L})}(a, b, q, i, i')$ for the Bounded Dominating Set problem on tolerance graphs.

Definition 10 *Let $(a, b) \in \mathcal{A} \times \mathcal{B}$ be a pair of points such that $b \in \mathbb{R}^2_{right}(\Gamma_a^{diag})$. Let (i, i') be a right-crossing pair and L_q be a line segment such that $L_q \in \mathcal{L}^{left}_{i,i'}$ and $L_i, L_{i'} \in \mathcal{L}^{right}_q$. Furthermore let $b \in \mathbb{R}^2_{left}(\Gamma_{r_i}^{vert})$. Then $BD_{(\mathcal{P},\mathcal{L})}(a, b, q, i, i')$ is a dominating set $Z \subseteq \mathcal{L}$ of $X(a, b)$ with the* smallest *size, such that:*

- *(i, i') is the end-pair of Z and*
- *L_q is the diagonally leftmost line segment of Z.*

If such a dominating set $Z \subseteq \mathcal{L}$ of $X(a, b)$ does not exist, we define $BD_{(\mathcal{P},\mathcal{L})}(a, b, q, i, i') = \bot$ and $|BD_{(\mathcal{P},\mathcal{L})}(a, b, q, i, i')| = \infty$.

Note that always $L_q, L_i, L_{i'} \in BD_{(\mathcal{P},\mathcal{L})}(a,b,q,i,i')$. Furthermore some of the line segments $L_q, L_i, L_{i'}$ may coincide, i.e., the set $\{L_q, L_i, L_{i'}\}$ may have one, two, or three distinct elements. However, since $b \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{r_i})$ in Definition 10, it follows that $L_i \nsubseteq B_b \setminus \Gamma^{\text{vert}}_b$, and thus $L_i \notin X(a,b)$. For simplicity of the presentation we may refer to the set $BD_{(\mathcal{P},\mathcal{L})}(a,b,q,i,i')$ as $BD_G(a,b,q,i,i')$, where $(\mathcal{P},\mathcal{L})$ is the horizontal shadow representation of the tolerance graph G, or just as $BD(a,b,q,i,i')$ whenever the horizontal shadow representation $(\mathcal{P},\mathcal{L})$ is clear from the context.

Observation 4 *$BD(a,b,q,i,i') \neq \bot$ if and only if $\mathcal{L} \cap \mathcal{L}^{right}_q \cap \mathcal{L}^{left}_{i,i'}$ is a dominating set of $X(a,b)$.*

Observation 5 *$BD(a,b,q,i,i') = \{L_q, L_i, L_{i'}\}$ if and only if $\{L_q, L_i, L_{i'}\}$ dominates $X(a,b)$.*

Observation 6 *If $R(a,b) \subseteq S_i$ then $BD(a,b,q,i,i') = \{L_q, L_i, L_{i'}\}$.*

Due to Observations 4-6, without loss of generality we assume below (in Lemmas 8-13) that $BD(a,b,q,i,i') \neq \bot$ and that $BD(a,b,q,i,i') \neq \{L_q, L_i, L_{i'}\}$, and thus also $R(a,b) \nsubseteq S_i$ (cf. Observation 6). We provide our recursive computations for $BD(a,b,q,i,i')$ in Lemmas 8, 10, and 13. In Lemma 8 we consider the case where $b \in S_{l_i}$ and in Lemmas 10 and 13 we consider the case where $b \notin S_{l_i}$.

Lemma 8 *Suppose that $BD(a,b,q,i,i') \neq \bot$ and that $BD(a,b,q,i,i') \neq \{L_q, L_i, L_{i'}\}$, where $R(a,b) \nsubseteq S_i$. If $b \in S_{l_i}$ then*

$$BD(a,b,q,i,i') = BD(a,b^*,q,i,i'), \tag{3}$$

where $b^* = \Gamma^{vert}_b \cap \Gamma^{diag}_{l_i}$.

Proof. Define the point $b^* = \Gamma^{\text{vert}}_b \cap \Gamma^{\text{diag}}_{l_i}$ of the plane. If $a \in S_{l_i}$ then $R(a,b) \subseteq S_i$, which is a contradiction. Thus $a \notin S_{l_i}$, and therefore $R(a,b^*) \subseteq R(a,b)$. Consider now an element $x \in X(a,b) \setminus X(a,b^*)$. Then $x \cap S_i \neq \emptyset$, and thus x is dominated by the line segment L_i. Therefore, for every set Z of line segments such that $L_i \in Z$, we have that Z dominates the set $X(a,b)$ if and only if Z dominates the set $X(a,b^*)$. Therefore $BD(a,b,q,i,i') = BD(a,b^*,q,i,i')$. ■

Due to Lemma 8, without loss of generality we may assume in the following (in Lemmas 9-13) that $b \notin S_{l_i}$. In order to provide our second recursive computation for $BD(a,b,q,i,i')$ in Lemma 10 (cf. Eq. (4)), we first prove in the next lemma that the set at the right hand side of Eq. (4) is indeed a dominating set of $X(a,b)$, in which L_q is the diagonally leftmost line segment and (i,i') is the end-pair.

Lemma 9 *Suppose that $BD(a,b,q,i,i') \neq \bot$ and that $BD(a,b,q,i,i') \neq \{L_q, L_i, L_{i'}\}$, where $R(a,b) \nsubseteq S_i$ and $b \notin S_{l_i}$. Let $c \in \mathbb{R}^2$ and $L_{q'}, L_j, L_{j'} \in \mathcal{L}$ such that:*

1. *$L_{q'} \in \left(\mathcal{L}^{right}_q \cap \mathcal{L}^{left}_{i,i'}\right) \setminus \{L_i\}$,*
2. *(j,j') is a right-crossing pair of $\left(\mathcal{L}^{right}_q \cap \mathcal{L}^{left}_{i,i'}\right) \setminus \{L_i\}$, where $j' = i'$ whenever $i \neq i'$,*
3. *$L_{q'} \in \mathcal{L}^{left}_{j,j'}$ and $L_j, L_{j'} \in \mathcal{L}^{right}_{q'}$,*
4. *$c = \Gamma^{vert}_{r_j} \cap \Gamma^{diag}_b$ if $r_j \in \mathbb{R}^2_{left}(\Gamma^{vert}_b)$, and $c = b$ otherwise, and*
5. *the set $X(a,b) \setminus X(a,c)$ is dominated by $\{L_j, L_{j'}\}$.*

If $BD(a,c,q',j,j') \neq \bot$ then $\{L_q, L_i\} \cup BD(a,c,q',j,j')$ is a dominating set of $X(a,b)$, in which L_q is the diagonally leftmost line segment and (i,i') is the end-pair.

Proof. Assume that $BD(a,c,q',j,j') \neq \perp$. Since $X(a,b) \setminus X(a,c)$ is dominated by $\{L_j, L_{j'}\}$ by the assumptions of the lemma, it follows that $\{L_q, L_i\} \cup BD(a,c,q',j,j')$ is a dominating set of $X(a,b)$.

We now prove that (i,i') is the end-pair of $\{L_q, L_i\} \cup BD(a,c,q',j,j')$. First recall by the assumptions of the lemma that $L_j, L_{j'} \in \mathcal{L} \cap \mathcal{L}_{i,i'}^{\text{left}}$ and note that $\mathcal{L}_{j,j'}^{\text{left}} \subseteq \mathcal{L}_{i,i'}^{\text{left}}$. Therefore, since $BD(a,c,q',j,j') \subseteq \mathcal{L} \cap \mathcal{L}_{j,j'}^{\text{left}}$ by definition, it follows that $BD(a,c,q',j,j') \subseteq \mathcal{L} \cap \mathcal{L}_{i,i'}^{\text{left}}$. Let first $i' = i$. Then clearly $L_i = L_{i'} \in \{L_q, L_i\} \cup BD(a,c,q',j,j') \subseteq \mathcal{L} \cap \mathcal{L}_{i,i}^{\text{left}}$, and thus in this case $(i,i') = (i,i)$ is the end-pair of $\{L_q, L_i\} \cup BD(a,c,q',j,j')$. Let now $i' \neq i$. Then $j' = i'$ by the assumptions of the lemma, and thus $BD(a,c,q',j,j') = BD(a,c,q',j,i')$. Then $L_i, L_{i'} \in \{L_q, L_i\} \cup BD(a,c,q',j,j') \subseteq \mathcal{L} \cap \mathcal{L}_{i,i'}^{\text{left}}$, and thus again (i,i') is the end-pair of $\{L_q, L_i\} \cup BD(a,c,q',j,j')$.

Finally, since $L_{q'} \in \left(\mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}\right) \setminus \{L_i\}$ by the assumptions of the lemma, it follows that $L_{q'} \subseteq \mathbb{R}^2_{\text{right}}(\Gamma_{l_q}^{\text{diag}})$, cf. Definition 7. Therefore, since $L_{q'}$ is by definition the diagonally leftmost line segment of $BD(a,c,q',j,j')$, it follows that L_q is the diagonally leftmost line segment of $\{L_q, L_i\} \cup BD(a,c,q',j,j')$. This completes the proof of the lemma. ∎

Given the statement of Lemma 9, we are now ready to provide our second recursive computation for $BD(a,b,q,i,i')$ in the next lemma.

Lemma 10 *Suppose that $BD(a,b,q,i,i') \neq \perp$ and that $BD(a,b,q,i,i') \neq \{L_q, L_i, L_{i'}\}$, where $R(a,b) \nsubseteq S_i$ and $b \notin S_{l_i}$. If $BD(a,b,q,i,i') \setminus L_i$ dominates all elements of $\{x \in X(a,b) : x \cap (S_i \cup F_i) \neq \emptyset\}$ then*

$$BD(a,b,q,i,i') = \{L_q, L_i\} \cup \min_{c,q',j,j'} \{BD(a,c,q',j,j')\}, \tag{4}$$

where the minimum is taken over all c, q', j, j' that satisfy the Conditions 1-5 of Lemma 9.

Proof. Let $Z \subseteq \mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$ be a dominating set of $X(a,b)$ such that L_q is the diagonally leftmost line segment of Z and (i,i') is the end-pair of Z. Suppose that $|Z| = |BD(a,b,q,i,i')|$ and that all elements of $\{x \in X(a,b) : x \cap (S_i \cup F_i) \neq \emptyset\}$ are dominated by $Z \setminus L_i$. Recall that $L_i \notin X(a,b)$. Thus, $Z \setminus \{L_i\}$ is a dominating set of $X(a,b)$. Let (j,j') denote the end-pair of $Z \setminus \{L_i\}$. Then all elements of $X(a,b)$ that are contained in $\mathbb{R}^2_{\text{right}}(\Gamma_{r_j}^{\text{vert}})$ must be dominated by $\{L_j, L_{j'}\}$. Define

$$c = \begin{cases} \Gamma_{r_j}^{\text{vert}} \cap \Gamma_b^{\text{diag}}, & \text{if } r_j \in \mathbb{R}^2_{\text{left}}(\Gamma_b^{\text{vert}}) \\ b, & \text{otherwise} \end{cases}.$$

That is, the set $X(a,b) \setminus X(a,c)$ is dominated by $\{L_j, L_{j'}\}$. Let $L_{q'}$ denote the diagonally leftmost line segment of $Z \setminus \{L_i\}$. Note that, if $L_q \neq L_i$ then $L_{q'} = L_q$. Furthermore note that $L_{q'} \in \mathcal{L}_{j,j'}^{\text{left}}$ and $L_j, L_{j'} \in \mathcal{L}_{q'}^{\text{right}}$. Since $Z \subseteq \mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$, it follows that (j,j') is a right-crossing pair of $\left(\mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}\right) \setminus \{L_i\}$ and that $L_{q'} \in \left(\mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}\right) \setminus \{L_i\}$. Furthermore, if $i \neq i'$ then $L_{i'} \in Z \setminus \{L_i\}$, and thus, by the choice of the right-crossing pair (j,j') as the end-pair of $Z \setminus \{L_i\}$, it follows that $j' = i'$.

Since $L_j, L_{j'} \in \mathcal{L}_{i,i'}^{\text{left}} \setminus \{L_i\}$, note that $L_i \notin BD(a,b,q',j,j')$. Moreover note that $X(a,c) \subseteq X(a,b)$, and thus $Z \setminus \{L_i\}$ is also a dominating set of $X(a,c)$. Therefore, since (j,j') is the end-pair of $Z \setminus \{L_i\}$, it follows that

$$|\{L_q\} \cup BD(a,c,q',j,j')| = |BD(a,c,q',j,j')| \leq |Z \setminus \{L_i\}|, \text{ if } L_q \neq L_i$$

and that

$$|BD(a,c,q',j,j')| \leq |Z \setminus \{L_i\}|, \text{ if } L_q = L_i.$$

That is, in both cases where $L_q \neq L_i$ or $L_q = L_i$, we have that

$$\begin{aligned} |\{L_q, L_i\} \cup BD(a,b,q',j,j')| &= 1 + \left|\left(\{L_q\} \cup BD(a,c,q',j,j')\right) \setminus \{L_i\}\right| \\ &= 1 + \left|BD(a,c,q',j,j')\right| \\ &\leq 1 + |Z \setminus \{L_i\}| = |Z| = |BD(a,b,q,i,i')|. \end{aligned} \tag{5}$$

Finally Lemma 9 implies that, if $BD(a,c,q',j,j') \neq \perp$, then $\{L_q, L_i\} \cup BD(a,c,q',j,j')$ is a dominating set of $X(a,b)$, in which L_q is the diagonally leftmost line segment and (i,i') is the end-pair. Therefore $|BD(a,b,q,i,i')| \leq |\{L_q, L_i\} \cup BD(a,b,q',j,j')|$, and thus it follows by Eq. (5) that $|BD(a,b,q,i,i')| = |\{L_q, L_i\} \cup BD(a,b,q',j,j')|$ ∎

In order to provide our third recursive computation for $BD(a,b,q,i,i')$ in Lemma 13 (cf. Eq. (6)), we first prove in Lemmas 11 and 12 that the set at the right hand side of Eq. (6) is indeed a dominating set of $X(a,b)$, in which L_q is the diagonally leftmost line segment and (i,i') is the end-pair.

Lemma 11 *Suppose that $BD(a,b,q,i,i') \neq \perp$ and that $BD(a,b,q,i,i') \neq \{L_q, L_i, L_{i'}\}$, where $R(a,b) \nsubseteq S_i$ and $b \notin S_{l_i}$. Let $c \in \mathbb{R}^2$ such that:*

1. *$c \in \mathcal{B} \cap R(a,b)$ and $c \in \mathbb{R}^2_{right}(\Gamma^{vert}_{l_i}) \setminus F_{l_i}$,*
2. *$\mathcal{P} \cap X(a,b) \cap F_c \cap F_i = \emptyset$.*

If $BD(a,c,q,i,i') \neq \perp$ and $BD(c,b,q,i,i') \neq \perp$, then $BD(a,c,q,i,i') \cup BD(c,b,q,i,i')$ is a dominating set of $X(a,b)$, in which L_q is the diagonally leftmost line segment and (i,i') is the end-pair.

Proof. Assume that $BD(a,c,q,i,i') \neq \perp$ and $BD(c,b,q,i,i') \neq \perp$. First note that, since $c \in R(a,b)$ by assumption, it follows that $X(a,c) \cup X(c,b) \subseteq X(a,b)$, cf. Eq. (2). Furthermore, since $c \in R(a,b) \subseteq B_b$ and $c \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i}) \setminus F_{l_i}$ by the assumption, it follows that also $b \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i}) \setminus F_{l_i}$. Now recall that $b \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{r_i})$ by Definition 10, and thus also $c \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{r_i})$. Therefore, since $c \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i}) \setminus F_{l_i}$ by the assumption, it follows that $S_c \cap \Gamma^{\text{diag}}_c \subseteq S_i \cup F_i$. Moreover, since $c \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$ and $b \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{r_i})$, it follows that $F_c \cap R(a,b) \subseteq S_i \cup F_i$.

The line segments of $\mathcal{L} \cap X(a,b)$ can be partitioned into the following sets:

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{L} \cap X(a,c) \\
\mathcal{L}_2 &= \mathcal{L} \cap X(c,b) \\
\mathcal{L}_3 &= \{L_k \in \mathcal{L} \cap X(a,b) : L_k \cap F_c \neq \emptyset\} \\
\mathcal{L}_4 &= \{L_k \in \mathcal{L} \cap X(a,b) : L_k \cap S_c \cap \Gamma^{\text{diag}}_c \neq \emptyset\}
\end{aligned}$$

Since $BD(a,c,q,i,i') \neq \perp$ and $BD(c,b,q,i,i') \neq \perp$ by assumption, it follows that the line segments of $\mathcal{L}_1$ are all dominated by $BD(a,c,q,i,i')$ and the line segments of $\mathcal{L}_2$ are all dominated by $BD(c,b,q,i,i')$. Furthermore, since $F_c \cap R(a,b) \subseteq S_i \cup F_i$ as we proved above, it follows that all line segments of $\mathcal{L}_3$ are dominated by the line segment L_i. Moreover, since $S_c \cap \Gamma^{\text{diag}}_c \subseteq S_i \cup F_i$ as we proved above, it follows that all line segments of $\mathcal{L}_4$ are dominated also by the line segment L_i. That is, all line segments of $\mathcal{L} \cap X(a,b) = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ are dominated by $BD(a,c,q,i,i') \cup BD(c,b,q,i,i')$.

Since $\mathcal{P} \cap X(a,b) \cap F_c \cap F_i = \emptyset$ by the assumption, the points of $\mathcal{P} \cap X(a,b)$ can be partitioned into the following sets:

$$\begin{aligned}
\mathcal{P}_1 &= \mathcal{P} \cap X(a,c) \\
\mathcal{P}_2 &= \mathcal{P} \cap X(c,b) \\
\mathcal{P}_3 &= \mathcal{P} \cap X(a,b) \cap F_c \cap S_i
\end{aligned}$$

It is easy to see that the points of $\mathcal{P}_1$ are all dominated by $BD(a,c,q,i,i')$ and that the points of $\mathcal{P}_2$ are all dominated by $BD(c,b,q,i,i')$. Furthermore the points of $\mathcal{P}_3$ are dominated by the line segment L_i. Thus all points of $\mathcal{P} \cap X(a,b) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ are dominated by $BD(a,c,q,i,i') \cup BD(c,b,q,i,i')$. Summarizing, $BD(a,c,q,i,i') \cup BD(c,b,q,i,i')$ is a dominating set of $X(a,b)$.

Furthermore, since (i, i') is the end-pair of both $BD(a, c, q, i, i')$ and $BD(c, b, q, i, i')$, it follows that (i, i') is also the end-pair of $BD(a, c, q, i, i') \cup BD(c, b, q, i, i')$. Similarly, since L_q is the diagonally leftmost line segment of both $BD(a, c, q, i, i')$ and $BD(c, b, q, i, i')$, it follows that L_q is also the diagonally leftmost line segment of $BD(a, c, q, i, i') \cup BD(c, b, q, i, i')$. This completes the proof of the lemma. ■

Lemma 12 *Suppose that* $BD(a, b, q, i, i') \neq \perp$ *and that* $BD(a, b, q, i, i') \neq \{L_q, L_i, L_{i'}\}$, *where* $R(a, b) \nsubseteq S_i$ *and* $b \notin S_{l_i}$. *Let* $c' \in \mathbb{R}^2$ *and* $L_{q'} \in \mathcal{L}$ *such that:*

1. $c' \in \mathcal{B} \cap R(a, b)$ *and* $c' \in F_{l_i}$,
2. $L_i, L_{i'} \in \mathcal{L}_{q'}^{right}$,
3. $L_{q'} \in \mathcal{L}_q^{right} \cap \mathcal{L}_{i,i'}^{left}$ *and* $l_{q'} \in F_{l_i}$,
4. $c' \in \Gamma_{l_{q'}}^{diag}$ *or* $c' \in \Gamma_b^{diag}$, *and*
5. $\mathcal{P} \cap X(a, b) \cap F_{c'} = \emptyset$.

If $BD(a, c', q, i, i') \neq \perp$ *and* $BD(c', b, q', i, i') \neq \perp$ *then* $BD(a, c', q, i, i') \cup BD(c', b, q', i, i')$ *is a dominating set of* $X(a, b)$, *in which* L_q *is the diagonally leftmost line segment and* (i, i') *is the end-pair.*

Proof. Assume that $BD(a, c', q, i, i') \neq \perp$ and $BD(c', b, q', i, i') \neq \perp$. First note that, since $c' \in R(a, b)$ by assumption, it follows that $X(a, c') \cup X(c', b) \subseteq X(a, b)$, cf. Eq. (2). Since $c' \in F_{l_i}$ by assumption, it follows that $F_{c'} \subseteq F_{l_i} \subseteq S_i \cup F_i$. Moreover, if $c' \in \Gamma_{l_{q'}}^{\text{diag}}$ then $S_{c'} \cap \Gamma_{c'}^{\text{diag}} \subseteq \Gamma_{l_{q'}}^{\text{diag}}$, and thus $S_{c'} \cap \Gamma_{c'}^{\text{diag}} \subseteq S_{q'} \cup F_{q'}$.

Similarly to the proof of Lemma 11, the line segments of $\mathcal{L} \cap X(a, b)$ can be partitioned into the following sets:

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{L} \cap X(a, c'), \\
\mathcal{L}_2 &= \mathcal{L} \cap X(c', b), \\
\mathcal{L}_3 &= \{L_k \in \mathcal{L} \cap X(a, b) : L_k \cap F_{c'} \neq \emptyset\}, \\
\mathcal{L}_4 &= \{L_k \in \mathcal{L} \cap X(a, b) : L_k \cap S_{c'} \cap \Gamma_{c'}^{\text{diag}} \neq \emptyset\}.
\end{aligned}$$

Since $BD(a, c', q, i, i') \neq \perp$ and $BD(c', b, q', i, i') \neq \perp$ by assumption, it follows that the line segments of $\mathcal{L}_1$ are all dominated by $BD(a, c', q, i, i')$ and that the line segments of $\mathcal{L}_2$ are all dominated by $BD(c', b, q', i, i')$. Furthermore, since $F_{c'} \subseteq S_i \cup F_i$ as we proved above, it follows that all line segments of $\mathcal{L}_3$ are dominated by the line segment L_i. If $c' \in \Gamma_b^{\text{diag}}$ then $\mathcal{L}_4 = \emptyset$. Suppose that $c' \in \Gamma_{l_{q'}}^{\text{diag}}$. Then, since $S_{c'} \cap \Gamma_{c'}^{\text{diag}} \subseteq S_{q'} \cup F_{q'}$ as we proved above, it follows that all line segments of $\mathcal{L}_4$ are dominated by the line segment $L_{q'}$. That is, in both cases where $c' \in \Gamma_b^{\text{diag}}$ or $c' \in \Gamma_{l_{q'}}^{\text{diag}}$, all line segments of $\mathcal{L} \cap X(a, b) = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ are dominated by $BD(a, c', q, i, i') \cup BD(c', b, q', i, i')$.

Since $c' \in F_{l_i}$ and $\mathcal{P} \cap X(a, b) \cap F_{c'} = \emptyset$ by the assumption, it follows that the points of $\mathcal{P} \cap X(a, b)$ can be partitioned into the following sets:

$$\begin{aligned}
\mathcal{P}_1 &= \mathcal{P} \cap X(a, c'), \\
\mathcal{P}_2 &= \mathcal{P} \cap X(c', b).
\end{aligned}$$

It is easy to see that the points of $\mathcal{P}_1$ are all dominated by $BD(a, c', q, i, i')$ and that the points of $\mathcal{P}_2$ are all dominated by $BD(c', b, q', i, i')$. Summarizing, $BD(a, c', q, i, i') \cup BD(c', b, q', i, i')$ is a dominating set of $X(a, b)$.

Since (i, i') is the end-pair of both $BD(a, c', q, i, i')$ and $BD(c', b, q', i, i')$, it follows that (i, i') is also the end-pair of $BD(a, c', q, i, i') \cup BD(c', b, q', i, i')$. Now note that L_q is the diagonally leftmost line segment of $BD(a, c', q, i, i')$ and $L_{q'}$ is the diagonally leftmost line segment of $BD(c', b, q', i, i')$. Therefore, since $L_{q'} \in \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$ by assumption, it follows that L_q remains the diagonally leftmost line segment of $BD(a, c', q, i, i') \cup BD(c', b, q', i, i')$. This completes the proof of the lemma. ■

Given the statements of Lemmas 11 and 12, we are now ready to provide our third recursive computation for $BD(a, b, q, i, i')$ in the next lemma.

Lemma 13 *Suppose that $BD(a, b, q, i, i') \neq \perp$ and that $BD(a, b, q, i, i') \neq \{L_q, L_i, L_{i'}\}$, where $R(a, b) \nsubseteq S_i$ and $b \notin S_{l_i}$. If $BD(a, b, q, i, i') \setminus L_i$ does not dominate all elements of $\{x \in X(a, b) : x \cap (S_i \cup F_i) \neq \emptyset\}$ then*

$$BD(a, b, q, i, i') = \min_{c, c', q'} \begin{cases} BD(a, c, q, i, i') \cup BD(c, b, q, i, i') \\ BD(a, c', q, i, i') \cup BD(c', b, q', i, i') \end{cases} , \tag{6}$$

where the minimum is taken over all c, c', q' that satisfy the Conditions of Lemmas 11 and 12, i.e.,

1. $c, c' \in \mathcal{B} \cap R(a, b)$,
2. $c \in \mathbb{R}^2_{right}(\Gamma^{vert}_{l_i}) \setminus F_{l_i}$ *and* $c' \in F_{l_i}$,
3. $L_i, L_{i'} \in \mathcal{L}^{right}_{q'}$,
4. $L_{q'} \in \mathcal{L}^{right}_q \cap \mathcal{L}^{left}_{i,i'}$ *and* $l_{q'} \in F_{l_i}$,
5. $c' \in \Gamma^{diag}_{l_{q'}}$ *or* $c' \in \Gamma^{diag}_b$, *and*
6. $\mathcal{P} \cap X(a, b) \cap F_c \cap F_i = \emptyset$ *and* $\mathcal{P} \cap X(a, b) \cap F_{c'} = \emptyset$.

Proof. Assume that $BD(a, b, q, i, i') \setminus L_i$ does not dominate all elements of $\{x \in X(a, b) : x \cap (S_i \cup F_i) \neq \emptyset\}$. Recall that $b \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{r_i})$ by Definition 10. First we prove that also $b \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$. Assume otherwise that $b \notin \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$. Then, since $b \notin S_{l_i}$ by the assumption of the lemma, it follows that $b \in B_{l_i}$. Thus $(S_i \cup F_i) \cap B_b = \emptyset$, i.e., L_i does not dominate any element of $X(a, b)$, cf. Eq. (2). Therefore, since $BD(a, b, q, i, i') \setminus L_i$ does not dominate all elements of $\{x \in X(a, b) : x \cap (S_i \cup F_i) \neq \emptyset\}$ by assumption, it follows that $BD(a, b, q, i, i')$ does also not dominate all elements of $X(a, b)$, which is a contradiction to the assumption that $BD(a, b, q, i, i') \neq \perp$. Therefore $b \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$.

Let $x_0 \in X(a, b)$ be such that $x_0 \cap (S_i \cup F_i) \neq \emptyset$ and x_0 is not dominated by $BD(a, b, q, i, i') \setminus L_i$. Let also $Z \subseteq \mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$ be an arbitrary dominating set of $X(a, b)$ such that L_q is the diagonally leftmost line segment of Z and (i, i') is the end-pair of Z. Suppose that $|Z| = |BD(a, b, q, i, i')|$ and that x_0 is dominated by L_i but not by $Z \setminus L_i$. Note that such a dominating set Z always exists due to our assumption on $BD(a, b, q, i, i')$. We distinguish now two cases.

Case 1. $x_0 \cap \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{l_i}) \neq \emptyset$. Let $t \in \mathbb{R}^2$ be an arbitrary point of $x_0 \cap \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{l_i})$. Since $x_0 \in X(a, b)$ and $b \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{r_i})$ by Definition 10, it follows that $t \in S_i \cup F_i$. If $t \in S_i$ then let $t^* \in R(a, b)$ be an arbitrary point on the intersection of the line segment L_i with the reverse shadow F_t of the point t, i.e., $t^* \in R(a, b) \cap L_i \cap F_t$. Note that t^* always exists, since $x_0 \in X(a, b)$, $R(a, b) \nsubseteq S_i$ by the assumption of the lemma, and $b \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$ as we proved above. Otherwise, if $t \in F_i$, then we define $t^* = t$. Since $t \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{l_i})$ by assumption, note that in both cases where $t \in S_i$ and $t \in F_i$, we have that $t \in S_{t^*}$ and that either $t^* \in L_i$ or $t^* \in F_i \setminus L_i$.

Suppose that there exists a line segment $L_k \in Z \setminus L_i$ such that $t^* \in S_k$. Then, since $t \in S_{t^*}$, it follows that also $t \in S_k$. Thus the element $x_0 \in X(a,b)$ is dominated by $L_k \in Z \setminus L_i$, which is a contradiction. Therefore $t^* \notin S_k$ for every line segment $L_k \in Z \setminus L_i$.

Let j be the greatest index such that for the line segment $L_j \in Z \setminus L_i$ we have $r_j \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$. That is, for every other line segment $L_s \in Z \setminus L_i$ with $r_s \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$, we have $r_s \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{l_j})$. If $r_j \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$ then we define $t_1 = r_j$. If $r_j \notin \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$ then we define $t_1 = l_i$. Furthermore, if such a line segment L_j does not exist in $Z \setminus L_i$ (i.e., if $r_s \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$ for every $L_s \in Z \setminus L_i$), then we define again $t_1 = l_i$.

Let $L_{j'} \in Z \setminus L_i$ be a line segment such that $l_{j'} \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$ and that, for every other line segment $L_s \in Z \setminus L_i$ with $l_s \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$, we have $l_s \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{l_{j'}})$. If $l_{j'} \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{b})$ then we define $t_2 = l_{j'}$. If $l_{j'} \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{b})$ then we define $t_2 = b$. Furthermore, if such a line segment $L_{j'}$ does not exist in $Z \setminus L_i$ (i.e., if $l_s \notin \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$ for every $L_s \in Z \setminus L_i$), then we define again $t_2 = b$.

Now we define
$$c = \Gamma^{\text{vert}}_{t_1} \cap \Gamma^{\text{diag}}_{t_2}.$$
It is easy to check by the above definition of t_1 and t_2 that $c \in \mathcal{B} \cap R(a,b)$ and that $c \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i}) \setminus F_{l_i}$.

Assume that there exists at least one point $p_k \in \mathcal{P} \cap X(a,b) \cap F_c \cap F_i$. Then, since $BD(a,b,q,i,i') \neq \bot$ by assumption, there must be a line segment $L_{k'} \in Z \setminus L_i$ such that $L_{k'}$ dominates p_k. Since $p_k \in F_c$ by assumption, it follows that $L_{k'} \cap F_c \neq \emptyset$. If $r_{k'} \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$ then $r_{k'} \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{c})$ by the above definition of c, and thus the line segment $L_{k'}$ does not dominate the point p_k, which is a contradiction. Therefore $r_{k'} \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$. If $l_{k'} \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$ then $l_{k'} \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{c})$ by the above definition of c, and thus the line segment $L_{k'}$ does not dominate the point p_k, which is a contradiction. Therefore $l_{k'} \notin \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$. Summarizing, $r_{k'} \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$ and $l_{k'} \notin \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$, and thus $L_{k'} \cap F_{t^*} \neq \emptyset$. That is, $t^* \in S_{k'}$ for some $L_{k'} \in Z \setminus L_i$, which is a contradiction as we proved above. Thus there does not exist such a point p_k, i.e.,
$$\mathcal{P} \cap X(a,b) \cap F_c \cap F_i = \emptyset.$$

Assume that $t^* \in L_i$. Then, since $t^* \notin S_k$ for every line segment $L_k \in Z \setminus L_i$ as we proved above, we can partition the set $Z \setminus \{L_q, L_i, L_{i'}\}$ into the sets Z_{below}, Z_{left}, and Z_{right} as follows:
$$\begin{aligned} Z_{\text{below}} &= \{L_k \in Z \setminus \{L_q, L_i, L_{i'}\} : L_k \cap S_i \neq \emptyset\}, \\ Z_{\text{left}} &= \{L_k \in Z \setminus \{L_q, L_i, L_{i'}\} : L_k \cap S_i = \emptyset, L_k \subseteq \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})\}, \\ Z_{\text{right}} &= \{L_k \in Z \setminus \{L_q, L_i, L_{i'}\} : L_k \cap S_i = \emptyset, L_k \subseteq \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})\}. \end{aligned} \tag{7}$$

Assume now that $t^* \in F_i \setminus L_i$; then $t^* = t$ is a point of x_0. Note that all points of $\mathcal{P} \cap X(a,b) \cap F_i$ are dominated by $Z \setminus L_i$, since they are not dominated by L_i and $BD(a,b,q,i,i') \neq \bot$ by assumption. Therefore x_0 is a line segment, i.e., $x_0 \in \mathcal{L}$. Assume that there exists a line segment $L_k \in Z \setminus L_i$ such that $L_k \cap (S_{t^*} \cup F_{t^*}) \neq \emptyset$. Then x_0 is dominated by $L_k \in Z \setminus L_i$, which is a contradiction. Therefore $L_k \cap (S_{t^*} \cup F_{t^*}) = \emptyset$ for every line segment $L_k \in Z \setminus L_i$. That is, for every $L_k \in Z \setminus L_i$ we have that either $L_k \subseteq B_{t^*}$ or $L_k \subseteq A_{t^*}$. Therefore, in the case where $t^* \in F_i \setminus L_i$, we can partition the set $Z \setminus \{L_q, L_i, L_{i'}\}$ into the sets Z_{below}, Z_{left}, and Z_{right} as follows:
$$\begin{aligned} Z_{\text{below}} &= \emptyset, \\ Z_{\text{left}} &= \{L_k \in Z \setminus \{L_q, L_i, L_{i'}\} : L_k \subseteq B_{t^*}\}, \\ Z_{\text{right}} &= \{L_k \in Z \setminus \{L_q, L_i, L_{i'}\} : L_k \subseteq A_{t^*}\}. \end{aligned} \tag{8}$$

Notice that, in both cases where $t^* \in L_i$ and $t^* \in F_i \setminus L_i$, the set $Z_1 = Z_{\text{below}} \cup Z_{\text{left}} \cup \{L_q, L_i, L_{i'}\}$ is a dominating set of $X(a,c)$. Furthermore the set $Z_2 = Z_{\text{right}} \cup \{L_q, L_i, L_{i'}\}$ is a dominating set of $X(c,b)$. Moreover, L_q is the diagonally leftmost line segment and (i,i') is the end-pair of both Z_1 and Z_2. Therefore $|BD(a,c,q,i,i')| \leq |Z_1|$ and $|BD(c,b,q,i,i')| \leq |Z_2|$. Now, since $\{L_q, L_i, L_{i'}\} \subseteq BD(a,c,q,i,i') \cap BD(c,b,q,i,i')$, we have that

$$\begin{aligned}
|BD(a,c,q,i,i') \cup BD(c,b,q,i,i')| &\leq |BD(a,c,q,i,i')| + |BD(c,b,q,i,i')| - |\{L_q, L_i, L_{i'}\}| \\
&\leq |Z_1| + |Z_2| - |\{L_q, L_i, L_{i'}\}| \\
&= |Z_{\text{below}} \cup Z_{\text{left}} \cup \{L_q, L_i, L_{i'}\}| \\
&\quad + |Z_{\text{right}} \cup \{L_q, L_i, L_{i'}\}| - |\{L_q, L_i, L_{i'}\}| \\
&= |Z_{\text{below}}| + |Z_{\text{left}}| + |Z_{\text{right}}| + |\{L_q, L_i, L_{i'}\}| \\
&= |Z| = |BD(a,b,q,i,i')|.
\end{aligned}$$

Finally Lemma 11 implies that, if $BD(a,c,q,i,i') \neq \perp$ and $BD(c,b,q,i,i') \neq \perp$, then $BD(a,c,q,i,i') \cup BD(c,b,q,i,i')$ is a dominating set of $X(a,b)$, in which L_q is the diagonally leftmost line segment and (i,i') is the end-pair. Therefore

$$|BD(a,b,q,i,i')| \leq |BD(a,c,q,i,i') \cup BD(c,b,q,i,i')|.$$

It follows that $|BD(a,b,q,i,i')| = |BD(a,c,q,i,i') \cup BD(c,b,q,i,i')|$.

Case 2. $x_0 \cap \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{l_i}) = \emptyset$. Then, since $x_0 \cap (S_i \cup F_i) \neq \emptyset$ by the initial assumption on x_0, it follows that $x_0 \cap F_i \neq \emptyset$. Note that all points in $\mathcal{P} \cap X(a,b) \cap F_i$ are dominated by $Z \setminus \{L_i\}$, since they are not dominated by L_i and $BD(a,b,q,i,i') \neq \perp$ by assumption. Therefore $x_0 \in \mathcal{L}$. Let $t^* \in \mathbb{R}^2$ be an arbitrary point of $x_0 \cap F_i$.

If $i' \neq i$ and $l_{i'} \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_i})$, then $L_{i'} \in Z \setminus \{L_i\}$ and $L_{i'}$ dominates x_0, which is a contradiction. Therefore, if $i' \neq i$ then $l_{i'} \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_i})$. Furthermore, it follows that if $L_q \neq L_i$ then also $L_q \neq L_{i'}$.

Assume that there exists a line segment $L_k \in Z \setminus L_i$ such that $L_k \cap (S_{t^*} \cup F_{t^*}) \neq \emptyset$. Then x_0 is dominated by $L_k \in Z \setminus L_i$, which is a contradiction. Therefore $L_k \cap (S_{t^*} \cup F_{t^*}) = \emptyset$ for every line segment $L_k \in Z \setminus L_i$. That is, for every $L_k \in Z \setminus L_i$ we have that either $L_k \subseteq B_{t^*}$ or $L_k \subseteq A_{t^*}$. Therefore, similarly to Eq. (8) in Case 1, we can partition the set $Z \setminus \{L_q, L_i, L_{i'}\}$ into the sets Z_{left} and Z_{right} as follows:

$$\begin{aligned}
Z_{\text{left}} &= \{L_k \in Z \setminus \{L_q, L_i, L_{i'}\} : L_k \subseteq B_{t^*}\}, \\
Z_{\text{right}} &= \{L_k \in Z \setminus \{L_q, L_i, L_{i'}\} : L_k \subseteq A_{t^*}\}. \qquad (9)
\end{aligned}$$

Similarly to Case 1, let j be the greatest index such that for the line segment $L_j \in Z \setminus L_i$ we have $r_j \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$. That is, for every other line segment $L_s \in Z \setminus L_i$ with $r_s \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$, we have $r_s \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{l_j})$. If $r_j \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$ then we define $t_1 = r_j$. If $r_j \notin \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_i})$ then we define $t_1 = l_i$. Furthermore, if such a line segment L_j does not exist in $Z \setminus L_i$ (i.e., if $r_s \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{t^*})$ for every $L_s \in Z \setminus L_i$), then we define again $t_1 = l_i$.

Let $L_{j'} \in Z \setminus L_i$ be a line segment such that $l_{j'} \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$ and that, for every other line segment $L_s \in Z \setminus L_i$ with $l_s \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$, we have $l_s \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{l_{j'}})$. If $l_{j'} \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_i})$ then we define $L_{q'} = L_{j'}$. If $l_{j'} \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_i})$ then we define $L_{q'} = L_i$. Furthermore, if such a line segment $L_{j'}$ does not exist in $Z \setminus L_i$ (i.e., if $l_s \notin \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{t^*})$ for every $L_s \in Z \setminus L_i$), then we define again $L_{q'} = L_i$.

Thus, in both cases where $L_{q'} = L_{j'}$ and $L_{q'} = L_i$, it follows that $L_{q'} \in \mathcal{L}^{\text{right}}_q \cap \mathcal{L}^{\text{left}}_{i,i'}$ and that $l_{q'} \in F_{l_i}$. Note that it can be either $L_{q'} \neq L_q$ or $L_{q'} = L_q$. Furthermore recall that, if $i' \neq i$, then $l_{i'} \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_i})$ as we proved above. Therefore $L_i, L_{i'} \in \mathcal{L}^{\text{right}}_{q'}$.

Now we define the point t_2 as follows. If $l_{q'} \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_b)$ then we define $t_2 = l_{q'}$. Otherwise, if $l_{q'} \notin \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_b)$ then we define $t_2 = b$. Furthermore we define

$$c' = \Gamma^{\text{vert}}_{t_1} \cap \Gamma^{\text{diag}}_{t_2}.$$

Therefore, due to the above definition of t_1 and t_2, it follows that $c' \in \Gamma_{l_{q'}}^{\text{diag}}$ or $c' \in \Gamma_{b}^{\text{diag}}$. Furthermore note that $c' \in S_{t^*}$. It is easy to check by the definition of t_1 and t_2 that $c' \in \mathcal{B} \cap R(a,b)$ and that $c' \in F_{l_i}$. Since $c' \in F_{l_i}$, note that $F_{c'} \subseteq F_i$, and thus $F_{c'} \cap F_i = F_{c'}$. Thus, similarly to Case 1, we can prove that

$$\mathcal{P} \cap X(a,b) \cap F_{c'} = \emptyset.$$

Now recall the partition of the set $Z \setminus \{L_q, L_i, L_{i'}\}$ into the sets Z_{left} and Z_{right}, cf. Eq. (9). Notice that the set $Z_1 = Z_{\text{left}} \cup \{L_q, L_i, L_{i'}\}$ is a dominating set of $X(a,c')$ and that the set $Z_2 = Z_{\text{right}} \cup \{L_{q'}, L_i, L_{i'}\}$ is a dominating set of $X(c',b)$. Furthermore, L_q is the diagonally leftmost line segment of Z_1 and (i,i') is the end-pair of Z_1. Similarly, $L_{q'}$ is the diagonally leftmost line segment of Z_2 and (i,i') is the end-pair of Z_2. Therefore $|BD(a,c',q,i,i')| \leq |Z_1|$ and $|BD(c',b,q',i,i')| \leq |Z_2|$.

Let first $L_q = L_{q'}$. Then, since $\{L_q, L_i, L_{i'}\} \subseteq BD(a,c',q,i,i') \cup BD(c',b,q',i,i')$, it follows that

$$\begin{aligned}
|BD(a,c',q,i,i') \cup BD(c',b,q',i,i')| &\leq |BD(a,c',q,i,i')| + |BD(c',b,q',i,i')| - |\{L_q, L_i, L_{i'}\}| \\
&\leq |Z_1| + |Z_2| - |\{L_q, L_i, L_{i'}\}| \\
&= |Z_{\text{left}} \cup \{L_q, L_i, L_{i'}\}| + |Z_{\text{right}} \cup \{L_{q'}, L_i, L_{i'}\}| - |\{L_q, L_i, L_{i'}\}| \\
&= |Z_{\text{left}}| + |Z_{\text{right}}| + |\{L_q, L_i, L_{i'}\}| \\
&= |Z| = |BD(a,b,q,i,i')|.
\end{aligned}$$

Let now $L_q \neq L_{q'}$. Then $l_{q'} \in \mathbb{R}^2_{\text{right}}(\Gamma_{l_q}^{\text{diag}})$, since $L_{q'} \in \mathcal{L}_q^{\text{right}}$ as we proved above. Furthermore, since $l_{q'} \in \mathbb{R}^2_{\text{left}}(\Gamma_{l_i}^{\text{diag}})$ by definition of q', it follows that $L_q \neq L_i$. Therefore also $L_q \neq L_{i'}$, as we proved above. Moreover, if $L_{q'} \neq L_i$ then $L_{q'} = L_{j'}$ by the above definition of q', and thus $L_{q'} \in Z_{\text{right}}$. Therefore, in both cases where $L_{q'} = L_i$ and $L_{q'} \neq L_i$, we have $Z_2 = Z_{\text{right}} \cup \{L_{q'}, L_i, L_{i'}\} = Z_{\text{right}} \cup \{L_i, L_{i'}\}$. Thus, since $\{L_i, L_{i'}\} \subseteq BD(a,c',q,i,i') \cap BD(c',b,q',i,i')$, it follows that

$$\begin{aligned}
|BD(a,c',q,i,i') \cup BD(c',b,q',i,i')| &\leq |BD(a,c',q,i,i')| + |BD(c',b,q',i,i')| - |\{L_i, L_{i'}\}| \\
&\leq |Z_1| + |Z_2| - |\{L_i, L_{i'}\}| \\
&= |Z_{\text{left}} \cup \{L_q, L_i, L_{i'}\}| + |Z_{\text{right}} \cup \{L_i, L_{i'}\}| - |\{L_i, L_{i'}\}| \\
&= |Z_{\text{left}} \cup \{L_q\}| + |Z_{\text{right}}| + |\{L_i, L_{i'}\}| \\
&= |Z_{\text{left}}| + |Z_{\text{right}}| + |\{L_q\}| + |\{L_i, L_{i'}\}| \\
&= |Z| = |BD(a,b,q,i,i')|.
\end{aligned}$$

Finally Lemma 12 implies that, if $BD(a,c',q,i,i') \neq \perp$ and $BD(c',b,q',i,i') \neq \perp$ then $BD(a,c',q,i,i') \cup BD(c',b,q',i,i')$ is a dominating set of $X(a,b)$, in which L_q is the diagonally leftmost line segment and (i,i') is the end-pair. Therefore

$$|BD(a,b,q,i,i')| \leq |BD(a,c',q,i,i') \cup BD(c',b,q',i,i')|.$$

It follows that $|BD(a,b,q,i,i')| = |BD(a,c',q,i,i') \cup BD(c',b,q',i,i')|$.

Summarizing Case 1 and Case 2, it follows that the value of $BD(a,b,q,i,i')$ can be computed by Eq. (6), where the minimum is taken over all values of c, c', q', as stated in the lemma. ∎

Using the recursive computations of Lemmas 8, 10, and 13, we are now ready to present Algorithm 1 for computing Bounded Dominating Set on tolerance graphs in polynomial time.

Theorem 3 *Given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph G with n vertices, Algorithm 1 solves* Bounded Dominating Set *in $O(n^9)$ time.*

Algorithm 1 Bounded Dominating Set on Tolerance Graphs

Input: A horizontal shadow representation $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$

Output: A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, or the announcement that $\mathcal{L}$ does not dominate $(\mathcal{P}, \mathcal{L})$

1: Add two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ completely to the left and to the right of $\mathcal{P} \cup \mathcal{L}$, respectively
2: $\mathcal{L} \leftarrow \mathcal{L} \cup \{L_0, L_{|\mathcal{L}|+1}\}$; denote $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, where now L_1 and $L_{|\mathcal{L}|}$ are dummy
3: $\mathcal{A} \leftarrow \{l_i, r_i : 1 \leq i \leq |\mathcal{L}|\}$; $\mathcal{B} \leftarrow \{\Gamma_t^{\text{diag}} \cap \Gamma_{t'}^{\text{vert}} : t, t' \in \mathcal{A}\}$
4: **for** every pair of points $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$ **do** {initialization}
5: $\quad X(a, b) \leftarrow \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq \left(B_b \setminus \Gamma_b^{\text{vert}}\right) \cap \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})\}$
6: $\quad$**for** every $q, i, i' \in \{1, 2, \ldots, |\mathcal{L}|\}$ **do**
7: $\quad\quad$**if** $r_{i'} \in S_{r_i}$ **then** {(i, i') is a right-crossing pair}
8: $\quad\quad\quad$**if** $L_q \in \mathcal{L}_{i,i'}^{\text{left}}$, $L_i, L_{i'} \in \mathcal{L}_q^{\text{right}}$, and $b \in \mathbb{R}^2_{\text{left}}(\Gamma_{r_i}^{\text{vert}})$ **then**
9: $\quad\quad\quad\quad\mathcal{L}_{i,i'}^{\text{left}} \leftarrow \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq B_t$, where $t = \Gamma_{r_i}^{\text{vert}} \cap \Gamma_{r_{i'}}^{\text{diag}}\}$
10: $\quad\quad\quad\quad\mathcal{L}_q^{\text{right}} \leftarrow \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq \mathbb{R}^2_{\text{right}}(\Gamma_{l_q}^{\text{diag}})\}$
11: $\quad\quad\quad\quad$**if** $\mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$ does not dominate all elements of $X(a, b)$ **then**
12: $\quad\quad\quad\quad\quad BD(a, b, q, i, i') \leftarrow \perp$
13: $\quad\quad\quad\quad$**else if** $\{L_q, L_i, L_{i'}\}$ dominates all elements of $X(a, b)$ **then**
14: $\quad\quad\quad\quad\quad BD(a, b, q, i, i') \leftarrow \{L_q, L_i, L_{i'}\}$
15: $\quad\quad\quad\quad$**else**
16: $\quad\quad\quad\quad\quad BD(a, b, q, i, i') \leftarrow \mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$ {initialization}
17: **for** every pair of points $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$ **do**
18: $\quad$**for** every $q, i, i' \in \{1, 2, \ldots, |\mathcal{L}|\}$ **do**
19: $\quad\quad$**if** $r_{i'} \in S_{r_i}$ **then** {(i, i') is a right-crossing pair}
20: $\quad\quad\quad$**if** $L_q \in \mathcal{L}_{i,i'}^{\text{left}}$, $L_i, L_{i'} \in \mathcal{L}_q^{\text{right}}$, and $b \in \mathbb{R}^2_{\text{left}}(\Gamma_{r_i}^{\text{vert}})$ **then**
21: $\quad\quad\quad\quad$Compute the solutions Z_1, Z_2, Z_3 by Lemmas 8, 10, and 13, respectively
22: $\quad\quad\quad\quad$**for** $k = 1$ to 3 **do**
23: $\quad\quad\quad\quad\quad$**if** $|Z_k| < |BD(a, b, q, i, i')|$ **then** $BD(a, b, q, i, i') \leftarrow Z_k$
24: **if** $BD(l_1, r_{\mathcal{L}}, 1, |\mathcal{L}|, |\mathcal{L}|) = \perp$ **then return** $\mathcal{L}$ does not dominate $(\mathcal{P}, \mathcal{L})$
25: $\quad$**else return** $BD(l_1, r_{\mathcal{L}}, 1, |\mathcal{L}|, |\mathcal{L}|) \setminus \{L_1, L_{|\mathcal{L}|}\}$

Proof. In the first line, Algorithm 1 augments the horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ by adding to $\mathcal{L}$ the two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ (with endpoints l_0, r_0 and $l_{|\mathcal{L}|+1}, r_{|\mathcal{L}|+1}$, respectively) such that all elements of $\mathcal{P} \cup \mathcal{L}$ are contained in A_{r_0} and in $B_{l_{|\mathcal{L}|+1}}$. In the second line the algorithm renumbers the elements of the set $\mathcal{L}$ such that $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, where in this new enumeration the line segments L_1 and $L_{|\mathcal{L}|}$ are dummy. Furthermore, in line 3, the algorithm computes the point sets $\mathcal{A}$ and $\mathcal{B}$ (cf. Section 5.1).

In lines 4-16 the algorithm performs all initializations. In particular, first in line 5 the algorithm computes the sets $X(a, b) \subseteq \mathcal{P} \cup \mathcal{L}$ for all feasible pairs $(a, b) \in \mathcal{A} \times \mathcal{B}$ (cf. Eq. (2)). Then the algorithm iteratively executes lines 9-16 for all values of $q, i, i' \in \{1, 2, \ldots, |\mathcal{L}|\}$ for which $BD(a, b, q, i, i')$ can be defined (these conditions on q, i, i' are tested in lines 6-8, cf. Definition 10). For all such values of q, i, i', the algorithm computes an initial value for $BD(a, b, q, i, i')$ in lines 9-16. In particular, in lines 12 and 14 it computes the values of $BD(a, b, q, i, i')$ which can be computed directly (cf. Observations 4 and 5). In the case where $BD(a, b, q, i, i') \neq \perp$ and $BD(a, b, q, i, i') \neq \{L_q, L_i, L_{i'}\}$, the set $\mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$ is a feasible (but not necessarily optimal) solution (cf. Definition 10), therefore in this case the algorithm initializes in line 16 the value of $BD(a, b, q, i, i')$ to $\mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$.

The main computations of the algorithm are performed in lines 17-23. In particular, the algorithm iteratively executes lines 21-23 for all values of a, b, q, i, i' for which $BD(a, b, q, i, i')$ can be defined (these conditions on a, b, q, i, i' are tested in lines 17-20, cf. Definition 10). In line 21 the algorithm computes all the necessary values that are the candidates for the value $BD(a, b, q, i, i')$ and in lines 22-23 the algorithm computes $BD(a, b, q, i, i')$ from these candidate values. The correctness of this computation of $BD(a, b, q, i, i')$ follows by Lemmas 8, 10, and 13, respectively.

Finally, the algorithm computes the final output in lines 24-25. Indeed, since in the (augmented) horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ the two dummy horizontal line segments are isolated (i.e., the line segments L_1 and $L_{|\mathcal{L}|}$ in the augmented representation, cf. lines 1-2 of the algorithm), they must be included in every minimum bounded dominating set of the (augmented) tolerance graph. Therefore the algorithm correctly returns in line 25 the computed set $BD(l_1, r_{|\mathcal{L}|}, 1, |\mathcal{L}|, |\mathcal{L}|) \setminus \{L_1, L_{|\mathcal{L}|}\}$, as long as $BD(l_1, r_{|\mathcal{L}|}, 1, |\mathcal{L}|, |\mathcal{L}|) \neq \bot$. Furthermore, if $BD(l_1, r_{|\mathcal{L}|}, 1, |\mathcal{L}|, |\mathcal{L}|) = \bot$ then the whole (augmented) set $\mathcal{L}$ does not dominate all elements of the (augmented) set $\mathcal{P} \cup \mathcal{L}$, and thus in this case the algorithm correctly returns a negative announcement in line 24.

Regarding the running time of Algorithm 1, first recall that the sets $\mathcal{A}$ and $\mathcal{B}$ have $O(n)$ and $O(n^2)$ elements, respectively. Thus the first three lines of the algorithm can be implemented in $O(n^2)$ time. Due to the for-loop of line 4, the lines 5-16 are executed at most $O(n^3)$ times. Recall by Eq. (1) and (2) that, for every pair $(a, b) \in \mathcal{A} \times \mathcal{B}$, the region $R(a, b)$ can be specified in constant time (cf. the shaded region in Figure 7) and the vertex set $X(a, b)$ can be computed in $O(n)$ time. That is, line 5 of the algorithm can be executed in $O(n)$ time. For every fixed pair (a, b), the lines 7-16 are executed at most $O(n^3)$ times, due to the for-loop of line 6. Furthermore the if-statements of lines 7 and 8 can be executed in constant time, while the computations of $\mathcal{L}_{i,i'}^{\text{left}}$ and $\mathcal{L}_q^{\text{right}}$ in lines 9 and 10 can be computed in $O(n)$ time each. The if-statement of line 11 can be executed in $O(n^2)$ time, since in the worst case we check adjacency between each element of $\mathcal{L} \cap \mathcal{L}_q^{\text{right}} \cap \mathcal{L}_{i,i'}^{\text{left}}$ and each element of $X(a, b)$. Moreover, each of the lines 12-16 can be trivially executed in at most $O(n)$ time. Therefore the total execution time of lines 4-16 is $O(n^8)$.

Due to the for-loop of lines 17 and 18, the lines 19-23 are executed at most $O(n^6)$ times, since there exist at most $O(n^3)$ pairs (a, b) and at most $O(n^3)$ triples $\{q, i, i'\}$. Furthermore, since each of the lines 19 and 20 can be executed in constant time, the execution time of the lines 19-23 is dominated by the execution time of line 21, i.e., by the recursive computation of the set $BD(a, b, q, i, i')$ from Lemmas 8, 10, and 13. Note that we have already computed in lines 12 and 14 of the algorithm whether $BD(a, b, q, i, i') \neq \bot$ and $BD(a, b, q, i, i') \neq \{L_q, L_i, L_{i'}\}$. Moreover it can also be checked in constant time whether $R(a, b) \nsubseteq S_i$ and whether $b \in S_{l_i}$, and thus we can decide in constant time in line 21 whether Lemmas 8, 10, and 13 can be applied. If Lemma 8 can be applied, the corresponding candidate for $BD(a, b, q, i, i')$ can be computed in constant time by a previously computed value (cf. Eq. (3)).

Assume now that Lemma 10 can be applied. Then the corresponding candidate for $BD(a, b, q, i, i')$ is computed by the right-hand side of Eq. (4), for all values of c, q', j, j' that satisfy the conditions of Lemma 9. Note by Condition 2 of Lemma 9 that, if $i \neq i'$, then $j' = i'$. Therefore every feasible quadruple (i, i', j, j') is either (i, i, j, j') or (i, i', j, i'), i.e., there exist at most $O(n^3)$ feasible quadruples (i, i', j, j'). Thus, since we already considered $O(n^2)$ iterations for all pairs (i, i') in line 18 of the algorithm, we only need to consider another $O(n)$ iterations (multiplicatively) in line 21 for all feasible pairs (j, j') in the execution of Lemma 10. Furthermore there are at most $O(n)$ feasible values of q' by Conditions 1 and 3 of Lemma 9. Moreover the value of c is uniquely determined (in constant time) by the values of j and b (cf. Condition 4 of Lemma 9); once c has been computed, we also need $O(n)$ additional time to check Condition 5 of Lemma 9. Therefore, Lemma 10 can be applied in $O(n^3)$ time in line 21 of the algorithm.

Assume finally that Lemma 13 can be applied. Then the corresponding candidate for $BD(a,b,q,i,i')$ is computed by the right-hand side of Eq. (6), for all values of c, c', q' that satisfy the conditions of Lemma 13. Note that there exist $O(n^2)$ feasible values for c, cf. Conditions 1 and 2 of Lemma 13. Furthermore, once the value of c has been chosen, we need $O(n)$ additional time to check Condition 6 of Lemma 13. Thus, the upper part of the right-hand side of Eq. (6) can be computed in $O(n^3)$ time. On the other hand, there exist $O(n)$ feasible values for q', cf. Conditions 3 and 4 of Lemma 13. For every value of q' there exist $O(n)$ feasible values for c', cf. Condition 5 of Lemma 13; once the value of c' has been chosen, we need $O(n)$ additional time to check Condition 6 of Lemma 13. Thus, the lower part of the right-hand side of Eq. (6) can be also computed in $O(n^3)$ time. That is, Lemma 13 can be applied in $O(n^3)$ time in line 21 of the algorithm.

Summarizing, the total execution time of the lines 17-23 is $O(n^9)$. Therefore, since the execution time of lines 4-16 is $O(n^8)$, the total running time of Algorithm 1 is $O(n^9)$. ■

6 Restricted bounded dominating set on tolerance graphs

In this section we use Algorithm 1 of Section 5 to provide a polynomial time algorithm (cf. Algorithm 2) for a slightly modified version of Bounded Dominating Set on tolerance graphs, which we call Restricted Bounded Dominating Set, formally defined below.

> Restricted Bounded Dominating Set on Tolerance Graphs
> **Input:** A 6-tuple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$, where $(\mathcal{P}, \mathcal{L})$ is a horizontal shadow representation of a tolerance graph G, (j, j') is a left-crossing pair of G, and (i, i') is a right-crossing pair of G.
> **Output:** A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, where (j, j') is the start-pair and (i, i') is the end-pair of Z, or the announcement that $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$.

In order to present Algorithm 2 for Restricted Bounded Dominating Set on tolerance graphs, we first reduce this problem to Bounded Dominating Set on tolerance graphs, cf. Lemma 20. Before we present this reduction to Bounded Dominating Set, we first need to prove some properties in the following auxiliary Lemmas 14-18. These properties will motivate the definition of bad and irrelevant points $p \in \mathcal{P}$ and of bad and irrelevant line segments $L_t \in \mathcal{L}$, cf. Definition 11. The main idea behind Definition 11 is the following. If an instance contains a bad point $p \in \mathcal{P}$ or a bad line segment $L_t \in \mathcal{L}$, then $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$. On the other hand, if an instance contains an irrelevant point $p \in \mathcal{P}$ or an irrelevant line segment $L_t \in \mathcal{L}$, we can safely ignore p (resp. L_t).

Lemma 14 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs. Let $l = \Gamma^{vert}_{l_j} \cap \Gamma^{diag}_{l_{j'}}$ and $r = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}}$. If there exists a point $p \in \mathcal{P}$ such that $p \in \mathbb{R}^2_{left}(\Gamma^{diag}_l)$ or $p \in \mathbb{R}^2_{right}(\Gamma^{vert}_r)$, then $\mathcal{L} \cap \mathcal{L}^{right}_{j,j'} \cap \mathcal{L}^{left}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$.*

Proof. Assume otherwise that $Z \subseteq \mathcal{L}$ is a solution of $\mathcal{I}$. First suppose that there exists a point $p \in \mathcal{P}$ such that $p \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_l)$, where $l = \Gamma^{\text{vert}}_{l_j} \cap \Gamma^{\text{diag}}_{l_{j'}}$. Then, by Lemma 6, there must exist a line segment $L_k \in Z$ such that $p \in S_k$. Thus $l_k \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_{j'}})$, which is a contradiction to the fact that (j, j') is the start-pair of Z.

Now suppose that there exists a point $p \in \mathcal{P}$ such that $p \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_r)$, where $r = \Gamma^{\text{vert}}_{r_i} \cap \Gamma^{\text{diag}}_{r_{i'}}$. Then, by Lemma 6, there must exist a line segment $L_k \in Z$ such that $p \in S_k$. Thus $r_k \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{r_i})$, which is a contradiction to the fact that (i, i') is the end-pair of Z. ■

Lemma 15 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs. Let $l = \Gamma^{vert}_{l_j} \cap \Gamma^{diag}_{l_{j'}}$ and $r = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}}$. If there exists a point $p \in \mathcal{P}$ such that $p \in S_l \cup S_r$ then at least one of the line segments $\{L_{j'}, L_i\}$ is a neighbor of p.*

Proof. Recall by Definition 7 in Section 5.1 that $l_j \in S_{l_{j'}}$ and $r_{i'} \in S_{r_i}$, since (j, j') is a left-crossing pair and (i, i') is a right-crossing pair. Therefore, since $l = \Gamma^{\text{vert}}_{l_j} \cap \Gamma^{\text{diag}}_{l_{j'}}$ and $r = \Gamma^{\text{vert}}_{r_i} \cap \Gamma^{\text{diag}}_{r_{i'}}$ by the assumptions of the lemma, it follows that $l \in S_{l_{j'}}$ and $r \in S_{r_i}$.

If $p \in S_l$ then also $p \in S_{l_{j'}}$ (since $l \in S_{l_{j'}}$ as we proved above), and thus $L_{j'}$ is a neighbor of p by Lemma 6. Similarly, if $p \in S_r$ then also $p \in S_{r_i}$ (since $r \in S_{r_i}$ as we proved above), and thus L_i is a neighbor of p by Lemma 6. ■

Lemma 16 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs. Let $l = \Gamma^{vert}_{l_j} \cap \Gamma^{diag}_{l_{j'}}$ and $r = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}}$. If there exists a line segment $L_t \in \mathcal{L}$ such that $L_t \subseteq B_l$ or $L_t \subseteq A_r$, then $\mathcal{L} \cap \mathcal{L}^{right}_{j,j'} \cap \mathcal{L}^{left}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$.*

Proof. Assume otherwise that $Z \subseteq \mathcal{L}$ is a solution of $\mathcal{I}$. First suppose that there exists a line segment $L_t \in \mathcal{L}$ such that $L_t \subseteq B_l$, where $l = \Gamma^{\text{vert}}_{l_j} \cap \Gamma^{\text{diag}}_{l_{j'}}$. Then, by Lemma 5, there must exist a line segment $L_k \in Z$ such that $L_t \cap S_k \neq \emptyset$ or $L_k \cap S_t \neq \emptyset$. If $L_t \cap S_k \neq \emptyset$ then $l_k \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_{j'}})$, which is a contradiction to the fact that (j, j') is the start-pair of Z. If $L_k \cap S_t \neq \emptyset$ then $l_k \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{l_j})$, which is again a contradiction to the fact that (j, j') is the start-pair of Z.

Now suppose that there exists a line segment $L_t \in \mathcal{L}$ such that $L_t \subseteq A_r$, where $r = \Gamma^{\text{vert}}_{r_i} \cap \Gamma^{\text{diag}}_{r_{i'}}$. Then, by Lemma 5, there exists a line segment $L_k \in Z$ such that $L_t \cap S_k \neq \emptyset$ or $L_k \cap S_t \neq \emptyset$. If $L_t \cap S_k \neq \emptyset$ then $r_k \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{r_i})$, which is a contradiction to the fact that (i, i') is the end-pair of Z. If $L_k \cap S_t \neq \emptyset$ then $r_k \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{r_{i'}})$, which is again a contradiction to the fact that (i, i') is the end-pair of Z. ■

Lemma 17 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs. Let $l = \Gamma^{vert}_{l_j} \cap \Gamma^{diag}_{l_{j'}}$ and $r = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}}$. If there exists a line segment $L_t \in \mathcal{L}$ with one of its endpoints in $B_l \cup A_r$ and one point (not necessarily an endpoint) in $\overline{B_l} \cap \overline{A_r}$, then at least one of the line segments $\{L_j, L_{j'}, L_i, L_{i'}\}$ is a neighbor of L_t. Moreover, L_t does not belong to any optimum solution Z of* Restricted Bounded Dominating Set.

Proof. Let Z be an optimum solution of Restricted Bounded Dominating Set. Let $L_t \in \mathcal{L}$ be a line segment with one of its endpoints in $B_l \cup A_r$ and one point (not necessarily an endpoint) in $\overline{B_l} \cap \overline{A_r}$. Notice that $r_t \in A_r$ or $l_t \in B_l$. Let first $r_t \in A_r$. Since L_t has also a point in $\overline{B_l} \cap \overline{A_r}$, it follows that L_t has a point in $(S_i \cup F_i) \cup (S_{i'} \cup F_{i'})$. Therefore L_t is a neighbor of L_i or $L_{i'}$ by Lemma 5. Let now $l_t \in B_l$. Since L_t has also a point in $\overline{B_l} \cap \overline{A_r}$, it follows that L_t has a point in $(S_j \cup F_j) \cup (S_{j'} \cup F_{j'})$. Therefore L_t is a neighbor of L_i or $L_{i'}$ by Lemma 5. Finally, since $r_t \in A_r$ or $l_t \in B_l$, it follows that $r_t \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{r_i})$ or $l_t \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{l_j})$. Therefore $L_t \notin \mathcal{L}^{\text{right}}_{j,j'}$ or $L_t \notin \mathcal{L}^{\text{left}}_{i,i'}$. Thus, since $Z \subseteq \mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$, it follows that $L_t \notin Z$. ■

Lemma 18 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs. Let $l = \Gamma^{vert}_{l_j} \cap \Gamma^{diag}_{l_{j'}}$ and $r = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}}$. If there exists a line segment $L_t \in \mathcal{L}$ such that $L_t \subseteq \overline{B_l} \cap \overline{A_r}$ and $L_t \notin \mathcal{L}^{right}_{j,j'} \cap \mathcal{L}^{left}_{i,i'}$ then at least one of the line segments $\{L_j, L_{j'}, L_i, L_{i'}\}$ is a neighbor of L_t. Moreover, L_t does not belong to any optimum solution Z of* Restricted Bounded Dominating Set.

Proof. Suppose first that $L_t \notin \mathcal{L}^{\text{right}}_{j,j'}$. Then $l_t \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{l_j})$ or $l_t \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_{j'}})$. We first consider the case where $l_t \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{l_j})$. Then, since $l_t \in \overline{B_l} \cap \overline{A_r}$ by assumption, it follows that $l_t \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{l_{j'}})$. This implies that $l_t \in S_{j'}$, and thus $L_{j'}$ is a neighbor of L_t. We now consider the case where $l_t \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_{j'}})$. Then, since $l_t \in \overline{B_l} \cap \overline{A_r}$ by assumption, it follows that $l_t \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_j})$. This implies that $l_t \in F_j$, and thus L_j is a neighbor of L_t.

The case where $L_t \notin \mathcal{L}^{\text{left}}_{i,i'}$ can be dealt with in exactly the same way, implying that, in this case, L_i or $L_{i'}$ is a neighbor of L_t. ■

From Lemmas 14 and 16 we define now the notions of a *bad point* $p \in \mathcal{P}$ and a *bad line segment* $L_t \in \mathcal{L}$, respectively. Moreover, from Lemmas 15, 17, and 18 we define the notions of an *irrelevant point* $p \in \mathcal{P}$ and of an *irrelevant line segment* $L_t \in \mathcal{L}$, as follows.

Definition 11 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs. Let $l = \Gamma^{vert}_{l_j} \cap \Gamma^{diag}_{l_{j'}}$ and $r = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}}$. A point $p \in \mathcal{P}$ is a* bad point *if $p \in \mathbb{R}^2_{left}(\Gamma^{diag}_{l})$ or $p \in \mathbb{R}^2_{right}(\Gamma^{vert}_{r})$. A point $p \in \mathcal{P}$ is an* irrelevant point *if $p \in S_l \cup S_r$. A line segment $L_t \in \mathcal{L}$ is a* bad line segment *if $L_t \subseteq B_l$ or $L_t \subseteq A_r$. Finally a line segment $L_t \in \mathcal{L}$ is an* irrelevant line segment *if either $L_t \subseteq \overline{B_l} \cap \overline{A_r}$ and $L_t \notin \mathcal{L}^{right}_{j,j'} \cap \mathcal{L}^{left}_{i,i'}$, or L_t has an endpoint in $B_l \cup A_r$ and another point in $\overline{B_l} \cap \overline{A_r}$.*

The next lemma will enable us to reduce Restricted Bounded Dominating Set to Bounded Dominating Set on tolerance graphs, cf. Lemma 20.

Lemma 19 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs, which has no bad or irrelevant points $p \in \mathcal{P}$ and no bad or irrelevant line segments $L \in \mathcal{L}$. Then we can add a new line segment $L_{j,1}$ to the set $\mathcal{P} \cup \mathcal{L}$ such that L_j is the only neighbor of $L_{j,1}$.*

Proof. Since there are no bad or irrelevant points $p \in \mathcal{P}$ and no bad or irrelevant line segments $L \in \mathcal{L}$ by assumption, there exists a point $x \in \mathbb{R}^2$ such that, for every $p \in \mathcal{P}$ and for every $L_t \in \mathcal{L} \setminus \{L_j\}$, we have that $p, L_t \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{x})$. That is, no element of $\mathcal{P} \cup (\mathcal{L} \setminus \{L_j\})$ has any point in the interior of the region $R_1 = \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{l_j}) \cap \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{x})$. Furthermore we define the region $R'_1 \subseteq R_1$, where $R'_1 = R_1 \cap \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{l_{j'}})$. This region R'_1 is illustrated in Figure 8 for the case where $j' \neq j$; the case where $j' = j$ is similar. Now we add to $\mathcal{L}$ a new line segment $L_{j,1}$ arbitrarily within the interior of the region R'_1, cf. Figure 8. By the definition of R'_1 it is easy to verify that $L_{j,1}$ is adjacent only to L_j. ■

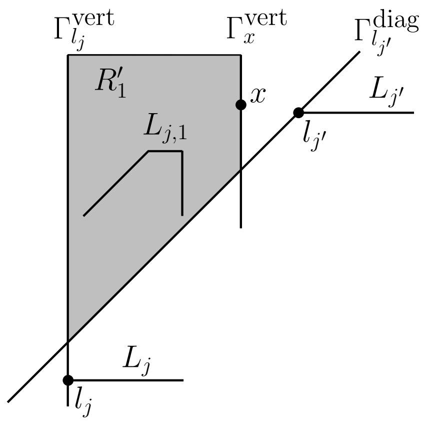

Figure 8: The addition of the line segment $L_{j,1}$, in the case where $j' \neq j$.

In the following we denote by $l_{j,1}$ the left endpoint of the new line segment $L_{j,1}$. Similarly to Definition 10 in Section 5.2, we present in the next definition the quantity $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i')$ for the Restricted Bounded Dominating Set problem on tolerance graphs.

Definition 12 *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* Restricted Bounded Dominating Set *on tolerance graphs. Then $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i')$ is a dominating set $Z \subseteq \mathcal{L} \cap \mathcal{L}^{right}_{j,j'} \cap \mathcal{L}^{left}_{i,i'}$*

of $(\mathcal{P}, \mathcal{L})$ *with the smallest size, in which* (j, j') *and* (i, i') *are the start-pair and the end-pair, respectively. If such a dominating set* Z *does not exist, we define* $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') = \bot$ *and* $|RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i')| = \infty$.

Observation 7 $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') \neq \bot$ *if and only if* $L_j, L_{j'} \in \mathcal{L}^{left}_{i,i'}$, $L_i, L_{i'} \in \mathcal{L}^{right}_{j,j'}$, *and* $\mathcal{L} \cap \mathcal{L}^{right}_{j,j'} \cap \mathcal{L}^{left}_{i,i'}$ *is a dominating set of* $(\mathcal{P}, \mathcal{L})$.

For simplicity of the presentation we may refer to the set $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i')$ as $RD_G(j, j', i, i')$, where $(\mathcal{P}, \mathcal{L})$ is the horizontal shadow representation of the tolerance graph G. In the next lemma we reduce the computation of $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i')$ to the computation of an appropriate value for the bounded dominating set problem (cf. Section 5).

Lemma 20 *Let* $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ *be an instance of* Restricted Bounded Dominating Set *on tolerance graphs, which has no bad or irrelevant points* $p \in \mathcal{P}$ *and no bad or irrelevant line segments* $L \in \mathcal{L}$. *Let* $(\mathcal{P}, \widehat{\mathcal{L}})$ *be the augmented representation that is obtained from* $(\mathcal{P}, \mathcal{L})$ *by adding the line segment* $L_{j,1}$ *as in Lemma 19. Furthermore let* $r = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}}$. *If* $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') \neq \bot$ *then* $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') = BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$.

Proof. Let $l = \Gamma^{\text{vert}}_{l_j} \cap \Gamma^{\text{diag}}_{l_{j'}}$ and $r = \Gamma^{\text{vert}}_{r_i} \cap \Gamma^{\text{diag}}_{r_{i'}}$. Then, since by assumption there are no bad or irrelevant points $p \in \mathcal{P}$ or line segments $L \in \mathcal{L}$ in the instance $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$, it follows that all elements of $\mathcal{P} \cup \mathcal{L}$ are entirely contained in the region $A_l \cap B_r$ of $\mathbb{R}^2$, cf. Definition 11. Therefore all elements of $\mathcal{P} \cup \mathcal{L}$ belong to the set $\{L_i\} \cup X(l, r)$, cf. Eq. (2) in Section 5.2. Now recall from the construction of the augmented representation $(\mathcal{P}, \widehat{\mathcal{L}})$ from $(\mathcal{P}, \mathcal{L})$ in the proof of Lemma 19 that $L_{j,1}$ is the only element of $\mathcal{P} \cup \widehat{\mathcal{L}}$ that does not belong to the set $\{L_i\} \cup X(l, r)$, cf. Figure 8. Furthermore, it is easy to check that the set of elements of $\mathcal{P} \cup \widehat{\mathcal{L}}$ is exactly the set $\{L_i\} \cup X(l_{j,1}, r)$.

Since $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') \neq \bot$ by assumption, it follows by Observation 7 that $L_j, L_{j'} \in \mathcal{L}^{\text{left}}_{i,i'}$ and $L_i, L_{i'} \in \mathcal{L}^{\text{right}}_{j,j'}$ as well as that $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ is a dominating set of $(\mathcal{P}, \mathcal{L})$. Furthermore, since L_j is the only neighbor of $L_{j,1}$ in the augmented representation $(\mathcal{P}, \widehat{\mathcal{L}})$, it follows that $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ is also a dominating set of $(\mathcal{P}, \widehat{\mathcal{L}})$. Moreover, since $\mathcal{L}^{\text{right}}_{j,j'} \subseteq \mathcal{L}^{\text{right}}_{j'}$ (cf. Definition 7 in Section 5.1), it follows that also $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ is a dominating set of $(\mathcal{P}, \widehat{\mathcal{L}})$. Therefore $BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i') \neq \bot$ by Observation 4. That is, $BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$ is a dominating set $Z \subseteq \widehat{\mathcal{L}}$ of $X(l_{j,1}, r)$ with the smallest size, in which (i, i') is its end-pair and $L_{j'}$ is its diagonally leftmost line segment (cf. Definition 10 in Section 5.2). Since $L_{j'}$ is the diagonally leftmost line segment of $BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j'i, i')$, it follows that $L_{j,1} \notin BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$. Therefore $L_j \in BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$, since L_j is the only neighbor of $L_{j,1}$ in $(\mathcal{P}, \widehat{\mathcal{L}})$. Thus (j, j') is the start-pair of $BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$. Finally, since also $\mathcal{P} \cup \widehat{\mathcal{L}} = \{L_i\} \cup X(l_{j,1}, r)$ as we proved above, it follows that $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') = BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$. ■

We are now ready to present Algorithm 2 which, given an instance $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ of Restricted Bounded Dominating Set on tolerance graphs, either outputs a set $Z \subseteq \mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ of minimum size that dominates all elements of $(\mathcal{P}, \mathcal{L})$, or it announces that such a set Z does not exist. Algorithm 2 uses Algorithm 1 (which solves Bounded Dominating Set on tolerance graphs, cf. Section 5) as a subroutine.

Theorem 4 *Given a 6-tuple* $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$, *where* $(\mathcal{P}, \mathcal{L})$ *is a horizontal shadow representation of a tolerance graph* G *with* n *vertices,* (j, j') *is a left-crossing pair and* (i, i') *is a right-crossing pair of* $(\mathcal{P}, \mathcal{L})$, *Algorithm 2 computes* Restricted Bounded Dominating Set *in* $O(n^9)$ *time.*

Algorithm 2 Restricted Bounded Dominating Set on Tolerance Graphs

Input: A 6-tuple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$, where $(\mathcal{P}, \mathcal{L})$ is a horizontal shadow representation of a tolerance graph G, (j, j') is a left-crossing pair and (i, i') is a right-crossing pair of $(\mathcal{P}, \mathcal{L})$.

Output: A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, where (j, j') is the start-pair and (i, i') is the end-pair of Z, or the value $\perp$.

1: **if** $(\mathcal{P}, \mathcal{L})$ contains a bad point $p \in \mathcal{P}$ or a bad line segment $L_k \in \mathcal{L}$ (cf. Definition 11) **then**
2:     **return** $\perp$
3: **if** $L_j, L_{j'} \in \mathcal{L}^{\text{left}}_{i,i'}$, $L_i, L_{i'} \in \mathcal{L}^{\text{right}}_{j,j'}$, and $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ is a dominating set of $(\mathcal{P}, \mathcal{L})$ **then**
4:     Compute the sets $\mathcal{P}_1 \subseteq \mathcal{P}$ and $\mathcal{L}_1 \subseteq \mathcal{L}$ of irrelevant points and line segments (cf. Definition 11)
5:     $\mathcal{P} \leftarrow \mathcal{P} \setminus \mathcal{P}_1$; $\mathcal{L} \leftarrow \mathcal{L} \setminus \mathcal{L}_1$; $r \leftarrow \Gamma^{\text{vert}}_{r_i} \cap \Gamma^{\text{diag}}_{r_{i'}}$
6:     $\widehat{\mathcal{L}} \leftarrow \mathcal{L} \cup \{L_{j,1}\}$ (cf. Lemma 19)
7:     **return** $BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$ {by calling Algorithm 1}
8: **else** **return** $\perp$

Proof. If the horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ contains at least one bad point $p \in \mathcal{P}$ or at least one bad line segment $L_k \in \mathcal{L}$ (cf. Definition 11) then $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$ by Lemmas 14 and 16. Thus, in the case where such a bad point or bad line segment exists in $(\mathcal{P}, \mathcal{L})$, Algorithm 2 correctly returns $\perp$, cf. lines 1-2. Furthermore, due to Observation 7, the algorithm correctly returns $\perp$ in line 8 if at least one of the conditions checked in line 3 is not satisfied.

Assume now that all conditions that are checked in line 3 are satisfied. Then $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') \neq \perp$ by Observation 7. Let $\mathcal{P}_1 \subseteq \mathcal{P}$ and $\mathcal{L}_1 \subseteq \mathcal{L}$ be the set of all irrelevant points and line segments, respectively (cf. Definition 11). Then, by Lemmas 15, 17, and 18, every point $p \in \mathcal{P}_1$ and every line segment $L_t \in \mathcal{L}_1$ is dominated by at least one of the line segments $\{L_j, L_{j'}, L_i, L_{i'}\}$. Furthermore, by Lemmas 17 and 18, no line segment $L_t \in \mathcal{L}_1$ is contained in any optimum solution Z of Restricted Bounded Dominating Set. Thus Algorithm 2 correctly removes the sets $\mathcal{P}_1$ and $\mathcal{L}_1$ of the irrelevant points and line segments from the instance, cf. lines 4-5 of the algorithm.

In line 6 the algorithm augments the set $\mathcal{L}$ of line segments to the set $\widehat{\mathcal{L}}$ by adding to it the line segment $L_{j,1}$ as in Lemma 19. Then the algorithm returns in line 7 the value $BD_{(\mathcal{P},\widehat{\mathcal{L}})}(l_{j,1}, r, j', i, i')$ by calling Algorithm 1 as a subroutine (cf. Section 5). The correctness of this computation in line 7 follows immediately by Lemma 20.

Regarding the running time of Algorithm 2, note by Definition 11 that we can check in constant time whether a given point $p \in \mathcal{P}$ (resp. a given line segment $L_t \in \mathcal{L}$) is bad or irrelevant. Therefore each of the lines 1, 2, and 4 of the algorithm can be executed in $O(n)$ time. The execution time of the if-statement of line 3 is dominated by the $O(n^2)$ time that is needed to check whether $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ is a dominating set of $(\mathcal{P}, \mathcal{L})$. Furthermore lines 5-6 can be executed trivially in total $O(n)$ time. Finally, line 7 can be executed in $O(n^9)$ time by Theorem 3, and thus the total running time of Algorithm 2 is $O(n^9)$. ∎

7 Dominating set on tolerance graphs

In this section we present our main algorithm of the paper (cf. Algorithm 3) which computes in polynomial time a minimum dominating set of a tolerance graph G, given by a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$. Algorithm 3 uses as subroutines Algorithms 1 and 2, which solve Bounded Dominating Set and Restricted Bounded Dominating Set on tolerance graphs, respectively (cf. Sections 5 and 6). Throughout this section we assume without loss of generality that the given tolerance graph G is connected and that G is given with a *canonical* horizontal shadow representation $(\mathcal{P}, \mathcal{L})$. It is important to note here that, in contrast to Algorithms 1 and 2, the minimum dominating set D that is computed by Algorithm 3 can also contain unbounded vertices. Thus always $D \neq \perp$, since in the worst case D contains the whole set $\mathcal{P} \cup \mathcal{L}$.

For every $p \in \mathcal{P}$ we denote by $N(p) = \{L_k \in \mathcal{L} : p \in S_k\}$ and $H(p) = \{x \in \mathcal{P} \cup \mathcal{L} : x \cap S_p \neq \emptyset\}$. Note that, due to Lemmas 6 and 7, $N(p)$ is the set of neighbors of p and $H(p)$ is the set of hovering vertices of p. Furthermore, for every $L_k \in \mathcal{L}$ we denote by $N(L_k) = \{p \in \mathcal{P} : p \in S_k\} \cup \{L_t \in \mathcal{L} : L_t \cap S_k \neq \emptyset \text{ or } L_k \cap S_t \neq \emptyset\}$. Note that, due to Lemmas 5 and 6, $N(L_k)$ is the set of neighbors of L_k.

Observation 8 *Let $(\mathcal{P}, \mathcal{L})$ be a canonical representation of a connected tolerance graph G, and let $p \in \mathcal{P}$. Then $N(p) \subseteq N(x)$ for every $x \in H(p)$ by Lemma 1. Furthermore $H(p) \cap \mathcal{L} \neq \emptyset$ by Lemma 2.*

Lemma 21 *Let $(\mathcal{P}, \mathcal{L})$ be a canonical horizontal shadow representation of a connected tolerance graph G and let D be a minimum dominating set of $(\mathcal{P}, \mathcal{L})$. If there exists a point $p \in \mathcal{P}$ such that $p \in D$ and $(N(p) \cup H(p)) \cap D \neq \emptyset$, then there exists a dominating set D' of $(\mathcal{P}, \mathcal{L})$ such that $|D'| = |D|$ and $|D' \cap \mathcal{P}| = |D \cap \mathcal{P}| - 1$.*

Proof. We may assume without loss of generality that $\mathcal{P} \neq \emptyset$ and $\mathcal{L} \neq \emptyset$. Indeed, if $\mathcal{P} = \emptyset$ then we can just solve the problem Bounded Dominating Set (see Section 5); furthermore, if $\mathcal{L} = \emptyset$, then the graph G is an independent set. Consider a point $p \in \mathcal{P}$ such that $p \in D$. Suppose first that $x \in D$ for some $x \in N(p)$, i.e., $N(p) \cap D \neq \emptyset$. Recall by Observation 8 that $H(p) \cap \mathcal{L} \neq \emptyset$ and consider a line segment $L_k \in H(p) \cap \mathcal{L}$. We will prove that the set $D' = (D \setminus \{p\}) \cup \{L_k\}$ is a minimum dominating set of G. First note that p is dominated by $x \in D \setminus \{p\} \subseteq D'$. Furthermore $N(p) \subseteq N(L_k)$ by Observation 8, since $L_k \in H(p)$. This implies that $N(p)$ is dominated by L_k in D'. Thus, since $|D'| = |D|$, it follows that D' is a minimum dominating set of G.

Suppose now that $x \in D$ for some $x \in H(p)$, i.e., $H(p) \cap D \neq \emptyset$. Since G is assumed to be connected, it follows that $N(p) \neq \emptyset$. Let $L_k \in N(p)$. We will prove that the set $D' = (D \setminus \{p\}) \cup \{L_k\}$ is a minimum dominating set of G. First note that p is dominated by $L_k \in D'$. Recall by Observation 8 that $N(p) \subseteq N(x)$. This implies that $N(p)$ is dominated by x in D'. Thus, since $|D'| = |D|$, it follows that D' is a minimum dominating set of G.

To finish the proof of the lemma, note that $|D' \cap \mathcal{P}| = |D \cap \mathcal{P}| - 1$ follows from the construction of D', as we always replace in D' the point $p \in \mathcal{P}$ by a line segment $L_k \in \mathcal{L}$. ■

Define now the subset $\mathcal{P}^* \subseteq \mathcal{P}$ of points as follows:

$$\mathcal{P}^* = \{p \in \mathcal{P} : p \notin H(p') \text{ for every point } p' \in \mathcal{P} \setminus \{p\}\}. \tag{10}$$

Equivalently, $\mathcal{P}^*$ contains all points $p \in \mathcal{P}$ such that $p \notin S_{p'}$ for every other point $p' \in \mathcal{P} \setminus \{p\}$. Note by the definition of the set $\mathcal{P}^*$ that for every $p_1, p_2 \in \mathcal{P}^*$ we have $p_1 \notin S_{p_2} \cup F_{p_2}$. Furthermore recall that the points of $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ have been assumed to be ordered increasingly with respect to their x-coordinates. Therefore, since $\mathcal{P}^* \subseteq \mathcal{P}$, the points of $\mathcal{P}^*$ are also ordered increasingly with respect to their x-coordinates.

Definition 13 *Let $(\mathcal{P}, \mathcal{L})$ be a horizontal shadow representation. A dominating set D of $(\mathcal{P}, \mathcal{L})$ is* normalized *if:*

1. *$(N(p) \cup H(p)) \cap D = \emptyset$ whenever $p \in D \cap \mathcal{P}$, and*

2. *$D \cap \mathcal{P} \subseteq \mathcal{P}^*$.*

Lemma 22 *Let $(\mathcal{P}, \mathcal{L})$ be a canonical horizontal shadow representation of a connected tolerance graph G. Then there exists a minimum dominating set D of $(\mathcal{P}, \mathcal{L})$ that is normalized.*

Proof. Let D be a minimum dominating set of G that contains the smallest possible number of points from the set $\mathcal{P}$. That is, $|D \cap \mathcal{P}| \leq |D' \cap \mathcal{P}|$ for every minimum dominating set D' of G. Let $p \in D \cap \mathcal{P}$.

First assume that $(N(p) \cup H(p)) \cap D \neq \emptyset$. Then Lemma 21 implies that there exists another minimum dominating set D' of G such that $|D' \cap \mathcal{P}| = |D \cap \mathcal{P}| - 1 < |D \cap \mathcal{P}|$, which is a contradiction to the choice of D. Therefore $(N(p) \cup H(p)) \cap D = \emptyset$ for every $p \in D \cap \mathcal{P}$.

Now assume that $p \in (\mathcal{P} \setminus \mathcal{P}^*) \cap D$. Then, by the definition of the set $\mathcal{P}^*$, there exists a point $p' \in \mathcal{P}$ such that $p \in H(p')$. Note by Observation 8 that $N(p') \subseteq N(p)$. Suppose that $p' \in D$. Then, since $p \in H(p')$, Lemma 21 implies that there exists a minimum dominating set D' such that $|D' \cap \mathcal{P}| = |D \cap \mathcal{P}| - 1 < |D \cap \mathcal{P}|$, which is a contradiction to the choice of D. Therefore $p' \notin D$. Thus, since D is a dominating set of G and $p' \notin D$, there must exist an $L_k \in N(p')$ such that $L_k \in D$. Therefore, since $N(p') \subseteq N(p)$, it follows that $L_k \in N(p) \cap D$. Then Lemma 21 implies that there exists a minimum dominating set D' of G such that $|D' \cap \mathcal{P}| = |D \cap \mathcal{P}| - 1 < |D \cap \mathcal{P}|$, which is again a contradiction to the choice of D. This implies that $(\mathcal{P} \setminus \mathcal{P}^*) \cap D = \emptyset$ and therefore $D \cap \mathcal{P} \subseteq \mathcal{P}^*$. Thus the dominating set D is normalized. ∎

In the remainder of this section, whenever we refer to a minimum dominating set D of a connected tolerance graph G that is given by a canonical horizontal shadow representation $(\mathcal{P}, \mathcal{L})$, we will always assume (due to Lemma 22) that D is *normalized*. Moreover, given such a canonical horizontal shadow representation $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, we add two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ (with endpoints l_0, r_0 and $l_{|\mathcal{L}|+1}, r_{|\mathcal{L}|+1}$, respectively) such that all elements of $\mathcal{P} \cup \mathcal{L}$ are contained in A_{r_0} and in $B_{l_{|\mathcal{L}|+1}}$. Denote $\mathcal{L}' = \mathcal{L} \cup \{L_0, L_{|\mathcal{L}|+1}\}$. Furthermore we add one dummy point $p_{|\mathcal{P}|+1}$ such that all elements of $\mathcal{P} \cup \mathcal{L}'$ are contained in $B_{p_{|\mathcal{P}|+1}}$. Denote $\mathcal{P}' = \mathcal{P} \cup \{p_{|\mathcal{P}|+1}\}$.

Note that $(\mathcal{P}', \mathcal{L}')$ is a horizontal shadow representation of some tolerance graph G', where the bounded vertices V'_B of G' correspond to the line segments of $\mathcal{L}'$ and the unbounded vertices V'_U of G' correspond to the points of $\mathcal{P}'$. Furthermore note that, although G is connected, G' is not connected as it contains the three isolated vertices that correspond to L_0, $L_{|\mathcal{L}|+1}$, and $p_{|\mathcal{P}|+1}$. However, since there exists by Lemma 22 a minimum dominating set D of G that is normalized, it is easy to verify that G' also admits a normalized minimum dominating set. Therefore, whenever we refer to a minimum dominating set D' of the augmented tolerance graph G', we will always assume that D' is normalized.

For simplicity of the presentation, we refer in the following to the augmented sets $\mathcal{P}'$ and $\mathcal{L}'$ of points and horizontal line segments by $\mathcal{P}$ and $\mathcal{L}$, respectively. In the remainder of this section we will write $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$ with the understanding that the last point $p_{|\mathcal{P}|}$ of $\mathcal{P}$, as well as the first and the last line segments L_1 and $L_{|\mathcal{L}|}$ of $\mathcal{L}$, are dummy. Note that the last point $p_{|\mathcal{P}|}$ (i.e., the new dummy point) belongs to the set $\mathcal{P}^*$. Furthermore, we will refer to the augmented tolerance graph G' by G. For every $p_i, p_j \in \mathcal{P}^*$ with $i < j$, we denote by

$$G_j = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq B_{p_j} \setminus \Gamma^{\text{vert}}_{p_j}\}, \quad (11)$$

$$G(i,j) = \{x \in G_j : x \subseteq A_{p_i}\}. \quad (12)$$

that is, G_j is set of elements of $\mathcal{P} \cup \mathcal{L}$ that are entirely contained in the region $B_{p_j} \setminus \Gamma^{\text{vert}}_{p_j}$, and $G(i,j)$ is the subset of G_j that contains the elements of $\mathcal{P} \cup \mathcal{L}$ that are entirely contained in the region A_{p_i}. Note that $p_j \notin G_j$ and $p_j \notin G(i,j)$.

Definition 14 *Let $p_j \in \mathcal{P}^*$ and (i,i') be a right-crossing pair in G_j. Then $D(j,i,i')$ is a* minimum normalized dominating set *of G_j whose end-pair is (i,i'). If there exists no dominating set Z of G_j whose end-pair is (i,i'), we define $D(j,i,i') = \perp$.*

Observation 9 *$D(j,i,i') \neq \perp$ if and only if $\mathcal{L}^{left}_{i,i'}$ is a dominating set of G_j.*

Observation 10 *If $X(r_{i'}, p_j)$ is not dominated by the set $\{L_i, L_{i'}\}$ then $D(j,i,i') = \bot$. Furthermore, if there exists a point $p \in \mathcal{P} \cap G_j$ such that $p \in \mathbb{R}^2_{right}(\Gamma^{vert}_{r_i})$ then $D(j,i,i') = \bot$.*

Due to Observation 9, without loss of generality we assume below (in Lemmas 23 and 24) that $D(j,i,i') \neq \bot$. Before we provide our recursive computation for $D(j,i,i')$ in Lemma 24 (cf. Eq. (14)), we first prove in the next lemma that the upper part of the right hand side of Eq. (14) is indeed a normalized dominating set of G_j, in which (i,i') is its end-pair.

Lemma 23 *Let G be a tolerance graph, $(\mathcal{P}, \mathcal{L})$ be a canonical representation of G, $p_j \in \mathcal{P}^*$, and (i,i') be a a right-crossing pair of G_j. Assume that $D(j,i,i') \neq \bot$. Let q, q', z, z', w, w' such that:*

1. *$p_{q'} \in \mathcal{P}^*$, where $1 \leq q' < j$,*
2. *$L_i, L_{i'} \notin N(p_{q'}) \cup H(p_{q'})$,*
3. *(w,w') is a left-crossing pair of $G(q',j)$,*
4. *(z,z') is a right-crossing pair of $G_{q'}$,*
5. *$q = \min\{1 \leq k \leq q' : p_k \in \mathcal{P}^*,\ p_k \in A_\zeta\}$, where $\zeta = \Gamma^{vert}_{r_z} \cap \Gamma^{diag}_{r_{z'}}$,*
6. *$\big(H(p_q) \cup H(p_{q'})\big) \setminus \Big(\bigcup_{q \leq k \leq q'} N(p_k)\Big)$ are dominated by the line segments $\{L_z, L_{z'}, L_w, L_{w'}\}$,*
7. *$G(q,q')$ is dominated by $\{p_k \in \mathcal{P}^* : q \leq k \leq q'\}$.*

If $D(q,z,z') \neq \bot$ and $RD_{G(q',j)}(w,w',i,i') \neq \bot$ then the set

$$D(q,z,z') \cup \big\{p_k \in \mathcal{P}^* : q \leq k \leq q'\big\} \cup RD_{G(q',j)}(w,w',i,i')$$

is a normalized dominating set of G_j, in which (i,i') is its end-pair.

Proof. The choices of $q, q', z, z', w, w', i, i'$, as described in the assumptions of the lemma, are illustrated in Figure 9. Assume that $D(q,z,z') \neq \bot$ and that $RD_{G(q',j)}(w,w',i,i') \neq \bot$. We denote for simplicity $D = D_1 \cup D_2 \cup D_3$, where

$$\begin{aligned} D_1 &= D(q,z,z'), \\ D_2 &= \big\{p_k \in \mathcal{P}^* : q \leq k \leq q'\big\}, \\ D_3 &= RD_{G(q',j)}(w,w',i,i'). \end{aligned} \tag{13}$$

First we prove that D is a dominating set of G_j and that (i,i') is the end-pair of D. Since $D_1 \neq \bot$ and $D_3 \neq \bot$, note that the set G_q is dominated by D_1 and that the set $G(q',j)$ is dominated by D_3. Furthermore, by Condition 7 of the lemma, the set $G(q,q')$ is dominated by D_2. It remains to prove that, if $x \notin D$ is an element of G_j such that $x \cap F_{p_q} \neq \emptyset$, or $x \cap F_{p_{q'}} \neq \emptyset$, or $x \cap S_{p_q} \neq \emptyset$, or $x \cap S_{p_{q'}} \neq \emptyset$, then x is dominated by some element of D.

Assume that $x \notin D$ is an element of G_j such that $x \cap S_{p_q} \neq \emptyset$ or $x \cap S_{p_{q'}} \neq \emptyset$. Then $x \in H(p_q) \cup H(p_{q'})$ by Lemma 7. If $x \in \bigcup_{q \leq k \leq q'} N(p_k)$ then x is clearly dominated by D_2, cf. Eq. (13). Otherwise $x \in \big(H(p_q) \cup H(p_{q'})\big) \setminus \Big(\bigcup_{q \leq k \leq q'} N(p_k)\Big)$, and thus x is dominated by the line segments $\{L_z, L_{z'}, L_w, L_{w'}\}$ by Condition 6 of the lemma.

Now assume that $x \notin D$ is an element of G_j such that $x \cap F_{p_q} \neq \emptyset$ or $x \cap F_{p_{q'}} \neq \emptyset$. Suppose that $x \in \mathcal{P}$, i.e., $x \in F_{p_q}$ or $x \in F_{p_{q'}}$. If $x \in F_{p_q}$ then $p_q \in S_x$, and thus $p_q \in H(x)$ by Lemma 7. This is a contradiction, since $p_q \in \mathcal{P}^*$ by Condition 5 of the lemma, cf. the definition of $\mathcal{P}^*$ in Eq. (10). Similarly, if $x \in F_{p_{q'}}$ then we arrive again to a contradiction, since $p_{q'} \in \mathcal{P}^*$ by Condition 1 of the lemma. Therefore $x \notin \mathcal{P}$, i.e., $x \in \mathcal{L}$. Let $x = L_k$. Since $L_k \cap F_{p_q} \neq \emptyset$ or $L_k \cap F_{p_{q'}} \neq \emptyset$, it follows that $p_q \in S_k$ or $p_{q'} \in S_k$, and thus $x = L_k \in N(p_q) \cup N(p_{q'})$. That is, x is dominated by $\{p_q, p_{q'}\}$. Therefore D is a dominating set of G_j. Furthermore, since (i, i') is the end-pair of D_3, it follows that (i, i') is also the end-pair of $D = D_1 \cup D_2 \cup D_3$.

We now prove that D is normalized. First note that $D_1 = D(q, z, z')$ is normalized by Definition 14 and that D_2 is normalized as it only contains elements of $\mathcal{P}^*$, cf. Definition 13. Moreover, due to Definition 13, D_3 is normalized as it contains only elements of $\mathcal{L}$, cf. Definition 12 in Section 6. That is, each of D_1, D_2, and D_3 is normalized. Furthermore note that, due to the Conditions 2, 3, and 4 of the lemma, for any two elements x, x' that belong to different sets among D_1, D_2, D_3, no point of x belongs to the shadow of x'. Therefore the whole set D is normalized. Summarizing, D is a normalized dominating set of G_j whose end-pair is (i, i'). ■

Given the statement of Lemma 23, we are now ready to provide our recursive computation of the sets $D(j, i, i')$.

Lemma 24 *Let G be a tolerance graph, $(\mathcal{P}, \mathcal{L})$ be a canonical representation of G, $p_j \in \mathcal{P}^*$, and (i, i') be a a right-crossing pair of G_j such that $D(j, i, i') \neq \bot$. Then*

$$D(j,i,i') = \min_{q',z,z',w,w'} \begin{cases} D(q,z,z') \cup \{p_k \in \mathcal{P}^* : q \leq k \leq q'\} \cup RD_{G(q',j)}(w,w',i,i') \\ BD_{G_j}(l_1,b,1,i,i'), \text{ where } b = \Gamma_{r_i}^{vert} \cap \Gamma_{r_{i'}}^{diag} \end{cases} . \tag{14}$$

where the minimum is taken over all q', z, z', w, w' that satisfy the Conditions 1-7 of Lemma 23.*

Proof. Let Z be a normalized dominating set of G_j such that (i, i') is its end-pair and $Z = |D(j, i, i')|$. We distinguish the following two cases.

Case 1. $Z \cap \mathcal{P}^* = \emptyset$, i.e., $Z \subseteq \mathcal{L}$. Denote $b = \Gamma_{r_i}^{\text{vert}} \cap \Gamma_{r_{i'}}^{\text{diag}}$ and observe that $X(l_1, b) \subseteq G_j$. Therefore, since Z is a dominating set of G_j, it follows that Z is also a dominating set of $X(l_1, b)$. Moreover recall that L_1 is a dummy isolated line segment, and thus $L_1 \in Z$. In particular, L_1 is the diagonally leftmost line segment of Z. Therefore $|BD_{G_j}(l_1, b, 1, i, i')| \leq |Z|$, since $Z \subseteq \mathcal{L}$ and (i, i') is the end-pair of Z by assumption.

Since $D(j, i, i') \neq \bot$ by assumption, it follows by Observation 10 that there are no points $p \in \mathcal{P} \cap G_j$ such that $p \in \mathbb{R}^2_{\text{right}}(\Gamma_{r_i}^{\text{vert}})$, and that $X(r_{i'}, p_j)$ is dominated by L_i and $L_{i'}$. Therefore $BD_{G_j}(l_1, b, 1, i, i')$ is a dominating set of G_j that has (i, i') as its end-pair. Moreover, due to Definition 13, $BD_{G_j}(l_1, b, 1, i, i')$ is normalized as it contains only elements of $\mathcal{L}$ (cf. Definition 10 in Section 5.2). Thus $|Z| \leq |BD_{G_j}(l_1, b, 1, i, i')|$. That is, $|Z| = |BD_{G_j}(l_1, b, 1, i, i')|$.

Case 2. $Z \cap \mathcal{P}^* \neq \emptyset$. Let $q' = \max\{k < j : p_k \in \mathcal{P}^* \cap Z\}$, cf. Figure 9. From the assumption that Z is normalized, it follows that for every line segment $L_k \in Z \cap \mathcal{L}$, either $L_k \subseteq B_{p_{q'}}$ or $L_k \subseteq A_{p_{q'}}$. Therefore the set $Z \cap \mathcal{L}$ can be partitioned into two sets $Z_{\mathcal{L},1}$ and $Z_{\mathcal{L},2}$, where

$$\begin{aligned} Z_{\mathcal{L},1} &= \{L_k \in Z \cap \mathcal{L} : L_k \subseteq B_{p_{q'}}\}, \\ Z_{\mathcal{L},2} &= \{L_k \in Z \cap \mathcal{L} : L_k \subseteq A_{p_{q'}}\}. \end{aligned}$$

In particular, note that $L_i, L_{i'} \notin N(p_{q'}) \cup H(p_{q'})$. Now we prove that $L_i, L_{i'} \in Z_{\mathcal{L},2}$. Assume otherwise $L_i \in Z_{\mathcal{L},1}$, i.e., $L_i \subseteq B_{p_{q'}}$. Then $r_i \in B_{p_{q'}}$, and thus $p_{q'} \in \mathbb{R}^2_{\text{right}}(\Gamma_{r_i}^{\text{vert}})$. This is a contradiction by Observation 10, since $D(j, i, i') \neq \bot$ by assumption. Now assume that $L_{i'} \in Z_{\mathcal{L},1}$, i.e., $L_{i'} \subseteq B_{p_{q'}}$. Then $r_{i'} \in B_{p_{q'}}$, and thus $p_{q'} \in \mathbb{R}^2_{\text{right}}(\Gamma_{r_{i'}}^{\text{diag}})$. This is a contradiction to the assumption that (i, i') is the end-pair of $D(j, i, i')$. Summarizing, $L_i, L_{i'} \in Z_{\mathcal{L},2}$.

Notice that $Z_{\mathcal{L},2} \subseteq \mathcal{L}$ is a bounded dominating set of $G(q', j)$ with (i, i') as its end-pair, and thus $Z_{\mathcal{L},2} \neq \emptyset$. Since $Z_{\mathcal{L},2} \subseteq \mathcal{L}$, Observation 3 implies that $Z_{\mathcal{L},2}$ contains a unique start-pair. Let (w, w') be the left-crossing pair of $G(q', j)$ which is the start-pair of $Z_{\mathcal{L},2}$. Then

$$|RD_{G(q',j)}(w, w', i, i')| \leq |Z_{\mathcal{L},2}|, \tag{15}$$

*Note that the value of q is uniquely determined by the value of q' and by the pair (z, z'), cf. Condition 5 of Lemma 23.

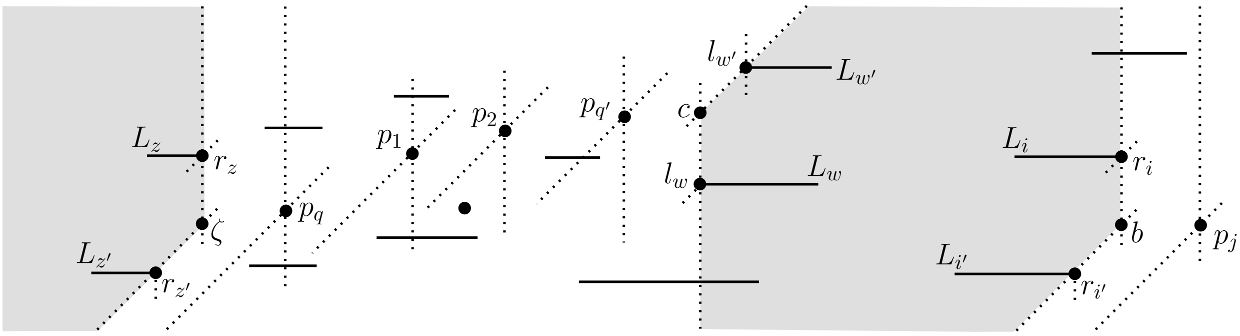

Figure 9: The recursion for Case 2 of Lemma 24, where $p_q, p_1, p_2, p_{q'} \in P^*$.

and thus $RD_{G(q',j)}(w, w', i, i') \neq \perp$.

Recall that G_j contains the isolated (dummy) line segment L_1, and thus $L_1 \in Z_{\mathcal{L},1}$. Therefore $Z_{\mathcal{L},1} \neq \emptyset$. Since $Z_{\mathcal{L},1} \subseteq \mathcal{L}$, Observation 3 implies that $Z_{\mathcal{L},1}$ contains a unique end-pair. Let (z, z') be the right-crossing pair of $G_{q'}$ which is the end-pair of $Z_{\mathcal{L},1}$. Denote $\zeta = \Gamma^{\text{vert}}_{r_z} \cap \Gamma^{\text{diag}}_{r_{z'}}$, cf. Figure 9.

Consider now an arbitrary point $p \in \mathcal{P}^* \cap Z$. We will prove that $p \notin F_\zeta \cup S_\zeta$. Assume otherwise that $p \in F_\zeta$. Then $p \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{r_z})$, and thus also $p \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_{r_{z'}})$. Moreover $p \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{diag}}_{r_{z'}})$. This implies that $p \in F_{r_{z'}}$. That is, $r_{z'} \in S_p$, and thus Lemma 7 implies that $L_{z'} \in H(p)$. This is a contradiction to the assumption that Z is normalized, since both $p, L_{z'} \in Z$. Thus $p \notin F_\zeta$. Now assume that $p \in S_\zeta$. Then $p \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{r_{z'}})$, and thus also $p \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{diag}}_{r_z})$. Furthermore $p \in \mathbb{R}^2_{\text{left}}(\Gamma^{\text{vert}}_{r_z})$. This implies that $p \in S_{r_z}$, and thus $L_z \in N(p)$. This is again a contradiction to the assumption that Z is normalized, since both $p, L_z \in Z$. Thus $p \notin S_\zeta$. Summarizing, for every $p \in P^* \cap Z$ we have that $p \notin F_\zeta \cup S_\zeta$, i.e., either $p \in A_\zeta$ or $p \in B_\zeta$. Therefore the set $P^* \cap Z$ can be partitioned into two sets $Z_{\mathcal{P}^*,1}$ and $Z_{\mathcal{P}^*,2}$, where

$$\begin{aligned} Z_{\mathcal{P}^*,1} &= \{p \in P^* \cap Z : p \in B_\zeta\}, \\ Z_{\mathcal{P}^*,2} &= \{p \in P^* \cap Z : p \in A_\zeta\}. \end{aligned}$$

Note that $p_q \in Z_{\mathcal{P}^*,2}$. Furthermore, since (z, z') is the end-pair of $Z_{\mathcal{L},1}$, note that all line segments of $Z_{\mathcal{L},1}$ are contained in B_ζ. Therefore all elements of the set $Z_1 = Z_{\mathcal{L},1} \cup Z_{\mathcal{P}^*,1}$ are contained in B_ζ, and thus (z, z') is the end-pair of Z_1. Define now $q = \min\{1 \leq k \leq q' : p_k \in \mathcal{P}^*,\ p_k \in A_\zeta\}$, cf. Figure 9. Recall that $p_q \notin G_q$, cf. Eq. (11). It is easy to check that no line segment of $Z_{\mathcal{L},2}$ dominates any element of G_q, cf. Figure 9. Similarly, no point of $Z_{\mathcal{P}^*,2}$ dominates any element of G_q. Thus the set Z_1 is a dominating set of G_q. Furthermore Z_1 is normalized, since $Z_1 \subseteq Z$ and Z is normalized by assumption. That is, Z_1 is a normalized dominating set of G_q with (z, z') as its end-pair. Therefore,

$$|D(q, z, z')| \leq |Z_1|, \tag{16}$$

and thus $D(q, z, z') \neq \perp$.

We now prove that $Z_{\mathcal{P}^*,2} = \{p_k \in \mathcal{P}^* : q \leq k \leq q'\}$. Clearly $Z_{\mathcal{P}^*,2} \subseteq \{p_k \in \mathcal{P}^* : q \leq k \leq q'\}$ by the definition of the index q and of the set $Z_{\mathcal{P}^*,2}$. Recall that for every line segment $L_t \in Z$, either $L_t \in Z_{\mathcal{L},1}$ or $L_t \in Z_{\mathcal{L},2}$. If $L_t \in Z_{\mathcal{L},1}$ then $L_t \subseteq B_\zeta \subseteq B_{p_q}$. Denote $c = \Gamma^{\text{vert}}_{l_w} \cap \Gamma^{\text{diag}}_{l_{w'}}$, cf. Figure 9. If $L_t \in Z_{\mathcal{L},2}$ then $L_t \subseteq A_c \subseteq A_{p_{q'}}$, since (w, w') is the start-pair of $Z_{\mathcal{L},2}$. Thus, for every line segment $L_t \in Z$, either $L_t \subseteq B_{p_q}$ or $L_t \subseteq A_{p_{q'}}$. Therefore $N(p_k) \cap Z = \emptyset$, for every $k \in \{q, q+1, \ldots, q'\}$, and thus all points $p_k \in \mathcal{P}^*$, where $q \leq k \leq q'$, must belong to Z. That is, $\{p_k \in \mathcal{P}^* : q \leq k \leq q'\} \subseteq Z_{\mathcal{P}^*,2}$. Therefore,

$$Z_{\mathcal{P}^*,2} = \{p_k \in \mathcal{P}^* : q \leq k \leq q'\}. \tag{17}$$

Recall that for every line segment $L_k \in Z$, either $L_k \subseteq B_{p_q}$ or $L_k \subseteq A_{p_{q'}}$, as we proved above. Therefore $G(q, q')$ must be dominated by $Z_{\mathcal{P}^*,2}$. Furthermore, due to Eq. (17), $Z_{\mathcal{P}^*,2}$

Algorithm 3 Dominating Set on Tolerance Graphs

Input: A canonical horizontal shadow representation $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$.

Output: A set $D \subseteq \mathcal{L} \cup \mathcal{P}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$.

1: Add two dummy line segments L_0 (resp. $L_{|\mathcal{L}|+1}$) completely to the left (resp. right) of $\mathcal{P} \cup \mathcal{L}$
2: Add a dummy point $p_{|\mathcal{P}|+1}$ completely to the right of $L_{|\mathcal{L}|+1}$
3: $\mathcal{P} \leftarrow \mathcal{P} \cup \{p_{|\mathcal{P}|+1}\}$; $\mathcal{L} \leftarrow \mathcal{L} \cup \{L_0, L_{|\mathcal{L}|+1}\}$
4: Denote $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, where now $p_{|\mathcal{P}|}$, L_1, and $L_{|\mathcal{L}|}$ are dummy
5: $\mathcal{P}^* = \{p \in \mathcal{P} : p \notin H(p') \text{ for every point } p' \in \mathcal{P} \setminus \{p\}\}$
6: **for** every pair of points $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$ **do**
7:     $X(a, b) \leftarrow \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq (B_b \setminus \Gamma_b^{\text{vert}}) \cap \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})\}$
8: **for** every $p_j \in \mathcal{P}^*$ **do**
9:     $G_j \leftarrow \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq B_{p_j} \setminus \Gamma_{p_j}^{\text{vert}}\}$
10:     **for** every $i, i' \in \{1, 2, \ldots, |\mathcal{L}|\}$ **do**
11:         **if** $L_i, L_{i'} \in G_j$ and $r_{i'} \in S_{r_i}$ **then** $\{(i, i')$ is a right-crossing pair of $G_j\}$
12:             **if** $\mathcal{L}_{i,i'}^{\text{left}}$ does not dominate all elements of G_j **then** $D(j, i, i') \leftarrow \perp$
13:                 **else** Compute $D(j, i, i')$ by Lemma 24 {by calling Algorithms 1 and 2}
14: **return** $D(|\mathcal{P}|, |\mathcal{L}|, |\mathcal{L}|) \setminus \{L_1, L_{\mathcal{L}}\}$

clearly dominates the set $\bigcup_{q \leq k \leq q'} N(p_k)$. Moreover every hovering vertex of p_q and of $p_{q'}$ must be dominated by $Z_{\mathcal{P}^*,2}$ or by the set $\{L_z, L_{z'}, L_w, L_{w'}\}$. Therefore $\{L_z, L_{z'}, L_w, L_{w'}\}$ must dominate the set $(H(p_q) \cup H(p_{q'})) \setminus \left(\bigcup_{q \leq k \leq q'} N(p_k)\right)$.

Now note that the sets $D(q, z, z')$, $Z_{\mathcal{P}^*,2}$, and $RD_{G(q',j)}(w, w', i, i')$ are mutually disjoint. Furthermore, it follows by Eq. (15) and (16) that

$$\begin{aligned} \left|D(q, z, z')\right| + |Z_{\mathcal{P}^*,2}| + \left|RD_{G(q',j)}(w, w', i, i')\right| &\leq |Z_1| + |Z_{\mathcal{P}^*,2}| + |Z_{\mathcal{L},2}| \\ &= |Z_{\mathcal{L},1} \cup Z_{\mathcal{P}^*,1}| + |Z_{\mathcal{P}^*,2}| + |Z_{\mathcal{L},2}| \\ &= |Z| = |D(j, i, i')|. \end{aligned} \tag{18}$$

Therefore $\left|D(q, z, z') \cup Z_{\mathcal{P}^*,2} \cup RD_{G(q',j)}(w, w', i, i')\right| \leq |D(j, i, i')|$. On the other hand, since $Z_{\mathcal{P}^*,2} = \{p_k \in \mathcal{P}^* : q \leq k \leq q'\}$ by Eq. (17), Lemma 23 implies that, if $D(q, z, z') \neq \perp$ and $RD_{G(q',j)}(w, w', i, i') \neq \perp$, then $D(q, z, z') \cup Z_{\mathcal{P}^*,2} \cup RD_{G(q',j)}(w, w', i, i')$ is a normalized dominating set of G_j, in which (i, i') is its end-pair. Therefore

$$|D(j, i, i')| \leq \left|D(q, z, z') \cup Z_{\mathcal{P}^*,2} \cup RD_{G(q',j)}(w, w', i, i')\right|. \tag{19}$$

The lemma follows by Eq. (18) and (19). ■

We are now ready to present Algorithm 3 which, given a canonical horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a connected tolerance graph G, computes a (normalized) minimum dominating set D of G. The correctness of Algorithm 3 is proved in Theorem 5.

Theorem 5 *Given a canonical horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a connected tolerance graph G with n vertices, Algorithm 3 computes in $O(n^{15})$ time a (normalized) minimum dominating set D of G.*

Proof. In the first line, Algorithm 3 augments the given canonical horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ by adding to $\mathcal{L}$ the dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ (with endpoints l_0, r_0 and

$l_{|\mathcal{L}|+1}, r_{|\mathcal{L}|+1}$, respectively) such that all elements of $\mathcal{P} \cup \mathcal{L}$ are contained in A_{r_0} and in $B_{l_{|\mathcal{L}|+1}}$. Furthermore, in the second line, the algorithm further augments the set of points $\mathcal{P}$ by adding to it the dummy point $p_{|\mathcal{P}|+1}$ such that all elements of $\mathcal{P} \cup \mathcal{L}'$ are contained in $B_{p_{|\mathcal{P}|+1}}$. In lines 3 and 4 the algorithm renumbers the elements of the sets $\mathcal{P}$ and $\mathcal{L}$ such that $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, where in this new enumeration the point $p_{|\mathcal{P}|}$ is dummy and the line segments L_1 and $L_{|\mathcal{L}|}$ are dummy as well. In lines 5-9 the algorithm computes the subset $\mathcal{P}^* \subseteq \mathcal{P}$ (cf. Eq. (10)), all feasible subsets $X(a,b) \subseteq \mathcal{P} \cup \mathcal{L}$ (cf. Eq. (2) in Section 5.2), and all sets G_j, where $p_j \in \mathcal{P}^*$ (cf. Eq. (11)).

The main computations of the algorithm are performed in lines 12-13, which are executed for every point $p_j \in \mathcal{P}^*$ and for every right-crossing pair (i,i') of the set G_j. In line 12 the algorithm checks whether $\mathcal{L}^{\text{left}}_{i,i'}$ dominates all elements of G_j. If it is not the case, it correctly computes $D(j,i,i') = \perp$ by Observation 9. Otherwise, if $\mathcal{L}^{\text{left}}_{i,i'}$ is a dominating set of G_j, then the algorithm computes in line 13 the value of $D(j,i,i')$ with the recursive formula of Lemma 24. Note that, to compute all the necessary values for this recursive formula, Algorithm 3 needs to call Algorithms 1 and 2 as subroutines, cf. Lemma 24.

Once all values $D(j,i,i')$ have been computed, the set $D(|\mathcal{P}|, |\mathcal{L}|, |\mathcal{L}|)$ is a minimum normalized dominating set of $G_{|\mathcal{P}|}$ whose end-pair is $(|\mathcal{L}|, |\mathcal{L}|)$, cf. Definition 14. Recall that $p_{|\mathcal{P}|} \notin G_{|\mathcal{P}|}$, i.e., $G_{|\mathcal{P}|} = (\mathcal{P} \setminus \{p_{|\mathcal{P}|}\}) \cup \mathcal{L}$. Therefore, since the two dummy line segments are isolated, they must belong to the dominating set $D(|\mathcal{P}|, |\mathcal{L}|, |\mathcal{L}|)$ of $G_{|\mathcal{P}|}$. Thus the algorithm correctly returns in line 14 the value $D(|\mathcal{P}|, |\mathcal{L}|, |\mathcal{L}|) \setminus \{L_1, L_{|\mathcal{L}|}\}$ as a minimum normalized dominating set for the input tolerance graph G.

Regarding the running time of Algorithm 3, first note that the execution time of lines 1-5 is dominated by the computation of the set $\mathcal{P}^*$ in line 5; this can be done in at most $O(n^2)$ time, since we check in the worst case for every two points $p, p' \in \mathcal{P}$ whether $p \in H(p')$. Due to the for-loop of line 6, line 7 is executed at most $O(n^3)$ times. Furthermore recall by Eq. (1) and (2) that, for every pair $(a,b) \in \mathcal{A} \times \mathcal{B}$, the vertex set $X(a,b)$ can be computed in $O(n)$ time. Therefore, lines 6-7 are executed in $O(n^4)$ time. Due to the for-loop of line 8, the lines 9-13 are executed $O(n)$ times, since there are at most $O(n)$ points in the set $\mathcal{P}^*$. For every fixed $p_j \in \mathcal{P}^*$, line 9 can be trivially executed in $O(n)$ time. For every fixed $p_j \in \mathcal{P}^*$, the lines 11-13 are executed $O(n^2)$ times, due to the for-loop of line 10. Furthermore, for every fixed triple (j,i,i'), line 11 can be executed in constant time and line 12 can be easily executed in $O(n^2)$ time.

It remains to upper bound the execution time of line 13 using Lemma 24. Before we execute line 13 for the first time, we perform two preprocessing steps. In the first preprocessing step we compute, for each of the $O(n)$ possible values for j, the graph G_j in $O(n)$ time (cf. Eq. (11)) and then we compute by Algorithm 1 in $O(n^9)$ time the values $BD_{G_j}(l_1, b, 1, i, i')$ for every feasible pair (i,i'), cf. Theorem 3 in Section 5. That is, we compute in the first preprocessing step the values $BD_{G_j}(l_1, b, 1, i, i')$ for every triple (j,i,i') in $O(n^{10})$ time. In the second preprocessing step we compute, for each of the $O(n^6)$ possible values for q', j, w, w', i, i', the graph $G(q', j)$ in $O(n)$ time (cf. Eq. (12)) and then we compute by Algorithm 2 in $O(n^9)$ time the values $RD_{G(q',j)}(w, w', i, i')$, cf. Theorem 4 in Section 6. That is, we compute in the second preprocessing step all values $RD_{G(q',j)}(w, w', i, i')$ in $O(n^{15})$ time.

Consider a fixed value for the triple (j,i,i'). Then there exist $O(n)$ feasible values for q', cf. Conditions 1 and 2 of Lemma 23. Furthermore there exist $O(n^2)$ feasible values for the pair (z,z'), cf. Condition 4 of Lemma 23. Once the values of q, z, z' have been chosen, we can compute in $O(n)$ time the value of q, cf. Conditions 5 and 6 of Lemma 23. Furthermore, once the values of q' and q have been chosen, we can check Condition 7 of Lemma 23 in $O(n^2)$ time. Thus, given a fixed value for the triple (j,i,i'), we can compute in $O(n^5)$ time the sets $D(q,z,z') \cup \{p_k \in \mathcal{P}^* : q \leq k \leq q'\}$, for all feasible values of the triples (q,z,z'). Moreover, for each of the $O(n^2)$ feasible pairs (w,w') (cf. Condition 3 of Lemma 23) we can compute in $O(n)$ time the set $D(q,z,z') \cup \{p_k \in \mathcal{P}^* : q \leq k \leq q'\} \cup RD_{G(q',j)}(w,w',i,i')$, cf. Lemma 23. That is, for a fixed value of the triple (j, i, i'), we can compute all these sets in $O(n^8)$ time, and thus we can compute all values of $D(j, i, i')$ in $O(n^{11})$ time.

Summarizing, the running time of the algorithm is dominated by the two preprocessing steps for computing in advance all values $BD_{G_j}(l_1, b, 1, i, i')$ and $RD_{G(q',j)}(w, w', i, i')$, and thus the running time of Algorithm 3 is $O(n^{15})$. ■

8 Concluding Remarks

In this paper we introduced two new geometric representations for tolerance and multitolerance graphs, called the *horizontal shadow representation* and the *shadow representation*, respectively. Using these new representations we first proved that the dominating set problem is APX-hard on multitolerance graphs and then we provided a polynomial time algorithm for this problem on tolerance graphs, thus answering to a longstanding open question. Therefore, given the (seemingly) small difference between the definition of tolerance and multitolerance graphs, this dichotomy result appears to be surprising.

The two new representations have the potential for further exploitation via sweep line algorithms. For example, using the shadow representation, it is not very difficult to design a polynomial sweep line algorithm for the independent dominating set problem, even on the larger class of multitolerance graphs. In particular, although the complexity of the dominating set problem has been established in this paper for both tolerance and multitolerance graphs, an interesting research direction would be to use these new representations also for other related problems, e.g., for the connected dominating set problem. A major open problem in tolerance and multitolerance graphs is to establish the computational complexity of the *Hamiltonicity problems*. We hope that the two new geometric representations can provide new insights also for these problems.

Our algorithm for tolerance graphs is highly non-trivial and its running time is upper-bounded by $O(n^{15})$, where n is the number of vertices in the input tolerance graph. Using more sophisticated data structures our algorithm could run slightly faster. As our main aim in this paper was to establish the *first* polynomial-time algorithm for this problem, rather than finding an optimized efficient algorithm, an interesting research direction is to explore to what extend the running time can be reduced. The existence of a *practically efficient* polynomial-time algorithm for the dominating set problem on tolerance graphs remains widely open.

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