

Modeling Severity Risk under PD-LGD Correlation

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Abstract

In this article, a generic severity risk framework in which loss given default (LGD) is dependent upon probability of default (PD) in an intuitive manner is developed. By modeling the conditional mean of LGD as a function of PD, which also varies with systemic risk factors, this model allows an arbitrary functional relationship between PD and LGD. Based on this framework, several specifications of stochastic LGD are proposed with detailed calibration methods. By combining these models with an extension of CreditRisk+, a versatile mixed Poisson credit risk model that is capable of handling both risk factor correlation and PD-LGD dependency is developed. An efficient simulation algorithm based on importance sampling is also introduced for risk calculation. Empirical studies suggest that ignoring or incorrectly specifying severity risk can significantly underestimate credit risk and a properly defined severity risk model is critical for credit risk measurement as well as downturn LGD estimation.

Keywords: Severity risk; Loss given default; Credit risk; CreditRisk+; Downturn LGD.

1. Introduction

Recent studies have shown evidence that suggests loss given default (LGD) is not only volatile but also positively correlated with default rate or probability of default (PD). One of the reasons behind this relationship is business cycle: During a recession period, default occurs more frequently whilst the value of the firm or collateral tends to fall. The PD-LGD dependency can exacerbate the loss due to default in a recession period when the overall default risk is high. The use of “downturn” LGD in the Basel Capital Accord is a reflection of this phenomenon. Frye (2000b) examines US corporate bonds and finds a significant positive relationship between LGD and default rate. Madan et al. (2006) develop debt models which impose this positive relationship and obtain supporting results when applied to BBB-rated corporate bonds. Carey and Gordy (2004) find that although the relation is less obvious during low default periods, it becomes more apparent when the default rate is high, which is indeed the period of interest. Altman et al. (2005) also find positive correlation between LGD and default rate. However, they are skeptical of the existence of a macroeconomic risk factor that explains this relationship. A comprehensive review of research on PD-LGD relationship and recent developments in severity risk modeling can be found in Altman (2010), which is continuously updated reflecting latest developments and data.

Empirical evidence has led to developments of severity risk models that address the dependency between PD and LGD. Frye (2000a) models the value of collateral, hence LGD, as a linear function of a systemic risk factor, which also governs default rates. Pykhtin (2003), meanwhile, assumes that the value of collateral is an exponential function of a normally distributed systemic risk factor and therefore follows a log-normal distribution. Düllmann and Trapp (2004) build their model based on Frye (2000a) and Pykhtin (2003) but utilize logistic function to ensure that LGD lies in $[0,1]$. Tasche (2004) also takes a similar approach to others but uses a beta distribution for LGD. Giese (2005) proposes a three-parameter function that links PD with LGD. His model is similar to the approach proposed in this paper in the sense that the PD-LGD relationship is directly modeled without any intermediate risk factors. More recently, Van Damme (2011) generalizes Tasche (2004)’s work and develops a generic framework for stochastic LGD modeling. Frye (2014), under certain assumptions, derives a relationship between PD and LGD without any additional parameters, and shows via simulation studies that this parsimonious model works well under different scenarios. However, his model, although it can be considered

a one-parameter model, is not flexible enough to produce different shapes of PD-LGD curve, and no validation using actual data is provided.

While these models differ in the specification of LGD, they are all based on Merton (1974)'s structural model framework. On the other hand, LGD modeling for another popular credit risk framework, CreditRisk+, has received far less attention and it is rare to find a model that incorporates PD-LGD dependency. One of the reasons is perhaps because people are reluctant to abandon the analytic tractability of the model, which is often claimed to be the major benefit of CreditRisk+. It is impossible to allow PD-LGD correlation within the CreditRisk+ framework while maintaining its analytic tractability. Few attempts to incorporate stochastic LGD (but without correlation with PD) into CreditRisk+ have been made: Gordy (2003) incorporates severity risk by assuming that LGD is independently gamma distributed and obtains the loss distribution using saddlepoint approximation. He, however, finds that the increase of risk due to LGD volatility vanishes quickly as the number of obligors grows. Bürgisser et al. (2001) introduce a stochastic variation of loss using two types of factors, obligor specific factors and systemic factors that are independent of each other. The systemic factors induce a pairwise correlation between obligors but does not incorporate a correlation between PD and LGD as these factors are assumed to be independent of the systemic factors that govern PD.

Despite the strong evidence of PD-LGD dependency and the developments of such models from academia, currently available commercial credit risk packages still rely on constant or independent distribution assumption of LGD. For example, CreditMetrics by MSCI, perhaps the most well-known commercial credit risk system based on the structural model, utilizes independent beta distribution to describe recovery rate volatility. However, as evidenced in the empirical studies of this paper and several other studies, these assumptions do not address severity risk adequately. While CreditMetrics is sold as a black box, CreditRisk+ has not been commercialized and those banks which adopt CreditRisk+ or its variants mostly have access to the core engine of their system and are able to modify it. Also, as the Basel IRB risk capital formula is based on Merton (1974)'s model, CreditRisk+ can be adopted as an alternative credit risk model and used for model risk management and validation. More importantly, as demonstrated by Han (2014), CreditRisk+ has more flexibility to accommodate various shapes of loss distribution. These facts together make the development of a proper severity risk model under CreditRisk+ framework particularly appealing.

The aim of the paper is twofold. Firstly, I develop a framework for severity risk in which the mean of LGD is assumed to vary with PD, which varies

with systemic risk factors. The framework admits nonlinear relationship between PD and LGD, e.g., LGD as a power function of PD, which is found to have the best fit in Altman et al. (2005). Modelling LGD as a direct function of PD instead of systemic risk factors has several advantages: An arbitrary nonlinear PD-LGD relationship can be easily accommodated; as noted by Altman et al. (2005), the existence of a macroeconomic risk factor that explains PD-LGD correlation is not clear; PD and LGD are observable, whereas it is not straightforward to define and observe systemic risk factors. As pointed out by Frye (2014), calibration of a severity risk model using real world data is nontrivial due to rare default events. This is especially problematic for highly rated bonds for which default is extremely rare. While existing papers (except Frye (2014) whose model requires no calibration) are silent on this issue, this paper provides a detailed calibration method that addresses this issue. Based on the framework, several specifications of stochastic LGD are considered and evaluated through empirical studies. Secondly, I combine these models with the common factor CreditRisk+ by Han and Kang (2008) to develop a new credit risk model that is capable of incorporating both risk factor correlation and PD-LGD correlation within CreditRisk+ framework. For this, analytic tractability of the model is sacrificed and a simulation method employing importance sampling is adopted. Equipped with high performance computers, there is no reason to adhere to an analytic solution at the cost of flexibility. As demonstrated later in this article, the simulation method turns out to be not only accurate but also very efficient.

To my knowledge, this is the first paper that accounts for PD-LGD correlation within the CreditRisk+ framework and provides a detailed calibration method considering limited data availability. Frye (2014) addresses this issue by proposing a model without additional parameters (so no need for calibration), but his model is overly restrictive and applicable only to structural models. The severity model and its calibration method proposed here is independent of the credit risk model and can be incorporated into other credit risk models such as the structural model.

The rest of the article is organized as follows. In Section 2, the common factor CreditRisk+ that incorporates risk factor correlation is briefly introduced. Section 3 is devoted to the development of stochastic LGD models. Detailed calibration method of each model is proposed at the end of the section. Applied to a model portfolio, the models are evaluated in several aspects in Section 4. The efficiency of the simulation algorithm is also addressed in this section. Concluding remarks and suggestions are given in Section 5 and the simulation algorithm for loss distribution is illustrated in

Appendix A.

2. The Common Factor CreditRisk+

In the original CreditRisk+ model of CSFP (1997), PD is assumed to be a linear function of the risk factors, i.e., the conditional PD of bond i is assumed to have the form

$$P_i(X) = PD_i \left(w_{0i} + \sum_{k=1}^K w_{ki} X_k \right), \quad (1)$$

where PD_i is the unconditional PD, $X = \{X_1, \dots, X_K\}$ are gamma distributed independent risk factors with mean 1 and variance $\sigma_{X_k}^2$, and the sum of the weights, $\sum_{k=1}^K w_{ki}$, is equal to 1. Even though it is theoretically possible to incorporate asset correlation in CreditRisk+ by appropriately choosing the weights, defining independent risk factors and assigning weights on them is difficult and impractical. For this reason, extensions of the original model that explicitly take the correlation into account have been introduced, and one of the latest developments is the common factor CreditRisk+ model (CreditRisk++) by Han and Kang (2008). Creditrisk++ assumes that a correlated risk factor can be decomposed into a sector specific factor Y_k and a macroeconomic factor \hat{Y} that are independent of each other, i.e.,

$$X_k = \delta_k Y_k + \gamma_k \hat{Y}, \quad k = 1, \dots, K, \quad (2)$$

where

$$Y_k \sim \text{Gamma}(\theta_k, 1), \quad (3)$$

$$\hat{Y} \sim \text{Gamma}(\hat{\theta}, 1). \quad (4)$$

Then, the probability of default can be rewritten as a linear combination of $K + 1$ gamma distributed independent risk factors, \hat{X}_k :

$$P_i = PD_i \left(w_{0i} + \sum_{k=1}^{K+1} w_{ki} \hat{X}_k \right), \quad (5)$$

where

$$\hat{X}_k \sim \text{Gamma}(\theta_k, \delta_k), \quad k = 1, \dots, K, \quad (6)$$

$$\hat{X}_{K+1} \sim \text{Gamma}(\hat{\theta}, 1), \quad \text{and} \quad (7)$$

$$w_{K+1,i} = \sum_{k=1}^K w_{ki} \gamma_k. \quad (8)$$

\hat{X}_k , $k = 1, \dots, K$, are sector specific risk factors and \hat{X}_{K+1} is a macroeconomic risk factor that has influence on all sectors. The degree of influence is determined by δ_k and γ_k . The expected values and covariance matrix of the correlated risk factors, X_k , have the form

$$E(X_k) = \delta_k \theta_k + \gamma_k \hat{\theta}, \quad (9)$$

$$V(X_k) = \delta_k^2 \theta_k + \gamma_k^2 \hat{\theta}, \quad (10)$$

$$COV(X_k, X_l) = \gamma_k \gamma_l \hat{\theta}. \quad (11)$$

Appropriately choosing the parameter values, various covariance structures can be represented by CreditRisk++. The model can be calibrated by minimizing the distance between an observed covariance matrix and the above covariance matrix equation subject to $E(X_k) = 1$. In general, matching the variance terms exactly and then minimizing the distance between covariance terms tends to yield a more stable result. That is, the parameters can be estimated by solving

$$\begin{aligned} \min \quad & \sum_{k=1}^K \sum_{l=k+1}^K (\gamma_k \gamma_l \hat{\theta} - \sigma_{kl})^2 \\ \text{s. t. } \quad & E(X_k) = \delta_k \theta_k + \gamma_k \hat{\theta} = 1, \quad k = 1, \dots, K \\ & V(X_k) = \delta_k^2 \theta_k + \gamma_k^2 \hat{\theta} = \sigma_k^2, \quad k = 1, \dots, K \end{aligned}$$

The main advantage of CreditRisk++ is that it can incorporate risk factor correlations in a very flexible and intuitive manner while maintaining the framework of CreditRisk+. Therefore, most of the numerical algorithms and extensions developed for CreditRisk+ remain valid also for CreditRisk++.

3. Modeling Severity Risk

In the CreditRisk+, LGD is assumed constant. However, as mentioned earlier, there is strong evidence that LGD is not only stochastic, but more importantly, correlated with PD. To take this into consideration, LGD is assumed to have a beta distribution whose mean is dependent upon the risk factors. More specifically, the following specification for LGD of bond i , U_i , is considered:

$$U_i|X = \text{Beta}(a_i, b_i) \quad (12)$$

with

$$CLGD_i := E(U_i|X) = LGD_i \left(c_{0i} + \sum_{k=1}^K c_{ki} X_k \right), \quad \sum_{k=0}^K c_{ki} = 1, \quad (13)$$

where LGD_i is the unconditional mean of bond i 's LGD.¹ Since the conditional PD is also a linear function of the risk factors, PD and LGD become correlated through the risk factors. It is often the case that the risk factors are not observable, and it is convenient to write LGD as a function of PD. If we assume $c_{ki} = C_i w_{ki}$ for some constant C_i , we have

$$CLGD_i = \phi_{0i} + \phi_{1i}P_i \quad (14)$$

where

$$\phi_{0i} = \frac{c_{0i} - w_{0i}}{1 - w_{0i}} LGD_i, \quad \phi_{1i} = \frac{c_{0i} - w_{0i}}{1 - w_{0i}} \frac{LGD_i}{PD_i}.$$

If $\phi_{1i} = 0$, LGD becomes an independent beta random variable as assumed in CreditMetrics. Equation (14) can be extended by allowing a nonlinear relationship between PD and LGD. Based on the regression results in Altman et al. (2005), with which the results in this paper are also in line, three LGD specifications are considered:

$$CLGD_i = f(P_i) \quad (15)$$

$$f(P_i) = \phi_{0i} + \phi_{1i}P_i \quad (\text{Linear}) \quad (16)$$

$$f(P_i) = \phi_{0i}P_i^{\phi_{1i}} \quad (\text{Power}) \quad (17)$$

$$f(P_i) = \frac{1}{1 + \exp(-\phi_{0i} - \phi_{1i}P_i)} \quad (\text{Logistic}) \quad (18)$$

The advantage of the logistic function is that $CLGD_i$ is guaranteed to be within $[0,1]$, whereas the other functions require an ad hoc boundary condition that caps LGD at 1. It turns out in the empirical studies that the boundary condition in the linear specification has a significant impact on risk measurement while LGD rarely exceeds the boundaries in the power specification.

3.1. Model Calibration

There are four parameters, ϕ_{0i} , ϕ_{1i} , a_i , and b_i to be estimated for bond i in each of the LGD models. ϕ_{0i} and ϕ_{1i} are associated with the systemic component of the severity risk, and a_i and b_i are associated with the idiosyncratic component of the severity risk.

¹Since K correlated risk factors can be represented by $K + 1$ independent risk factors under CreditRisk++, K independent risk factors, $X_k \sim \text{Gamma}(\alpha_k, \beta_k)$ with $\alpha_k \beta_k = 1$, are assumed without loss of generality for the rest of the article.

3.1.1. Calibration of ϕ_{0i} and ϕ_{1i}

Suppose that we can define a group of homogeneous bonds to which bond i belongs, and we can observe its historic default rates and loss rates. Then, one obvious choice of estimation method is to minimize the sum of the squared errors:

$$\begin{aligned} \min_{\phi_{0i}, \phi_{1i}} \sum_{t=1}^T (CLGD_{it} - f(P_{it}; \phi_{0i}, \phi_{1i}))^2 \\ \text{subject to } E(f(P_i)) = LGD_i, \end{aligned} \quad (19)$$

where $CLGD_{it}$ and P_{it} are the loss rate and the default rate of the group at time t . The constraint is to ensure that the expectation of $f(P_i)$ is equal to the unconditional expectation of bond i 's LGD, LGD_i . For the Linear model, $E(f(P_i))$ can be immediately obtained from

$$E(f(P_i)) = \phi_{0i} + \phi_{1i}PD_i. \quad (20)$$

For the other two models, however, the following integration with respect to the risk factors, X , needs to be computed numerically.

$$E(f(P_i)) = \int_x f(P_i(x))f_X(x)dx, \quad P_i(x) = PD_i \left(w_{0i} + \sum_{k=1}^K w_{ki}x_k \right), \quad (21)$$

where $f_X(x)$ is the joint probability density function of X .

A practical approach to define homogeneous groups is to partition the bonds according to their rating, seniority, and collateral type. A problem of this method is that some groups may not contain enough default events to get reliable estimation results. In this case, the variation of the default rates can be dominated by a few default events. This is a well known problem for the high ranking groups which consist of a small number of bonds whose defaults are extremely rare. Also, grouping bonds with the same rating will result in a narrow range of default rate variation.

Alternatively, the parameters are estimated using the default rates of a pool that consists of all bonds. This will produce a more stable default rate time series at the cost of homogeneity. A drawback of this approach is that, while the calibration result will be valid only for the PD range of the all bond pool, the level of the PD of an individual bond can be significantly different from that of the pool. Therefore, the following procedure is adopted to obtain an individual bond's conditional LGD.

1. Estimate ϕ_0 and ϕ_1 from the all bond pool by solving

$$\min \sum_{t=1}^T (CLGD_t - f(P_t; \phi_0, \phi_1))^2,$$

where $CLGD_t$ and PD_t are respectively the loss rate and the default rate of all bonds at time t .

2. Define the adjusted conditional PD of bond i as follows:

$$P'_i := \frac{\overline{PD}}{PD_i} P_i,$$

where \overline{PD} is the mean of P_t during the sample period. This adjustment makes the conditional PD of an individual bond centered at \overline{PD} .

3. Then, the conditional mean of the LGD of bond i is given by

$$CLGD_i = \frac{LGD_i}{E(f(P'_i))} f(P'_i).$$

This satisfies $E(CLGD_i) = LGD_i$.

3.1.2. Calibration of a_i and b_i

The parameters of the beta distribution can be estimated from the variance of LGD. As the mean of the beta distribution is $CLGD_i$, which is known from the previous step, it is convenient to reparametrize the beta distribution with respect to the mean $\mu_{\beta i}$ and the sample size $\nu_{\beta i}$:

$$\mu_{\beta i} = \frac{a_i}{a_i + b_i}, \quad \nu_{\beta i} = a_i + b_i. \quad (22)$$

The variance is given by

$$\sigma_{\beta i}^2 = \frac{a_i b_i}{(a_i + b_i)^2 (a_i + b_i + 1)} = \frac{\mu_{\beta i} (1 - \mu_{\beta i})}{1 + \nu_{\beta i}}. \quad (23)$$

Therefore, we have

$$E(U_i | X) = \mu_{\beta i} = CLGD_i, \quad (24)$$

$$V(U_i | X) = \sigma_{\beta i}^2 = \frac{\mu_{\beta i} (1 - \mu_{\beta i})}{1 + \nu_{\beta i}}. \quad (25)$$

By the law of total variance, the unconditional variance of LGD, $VLGD_i$, can be decomposed into two parts.

$$\begin{aligned}
VLGD_i &= E[V(U_i|X)] + V[E(U_i|X)] \\
&= E\left(\frac{\mu_{\beta i}(1 - \mu_{\beta i})}{1 + \nu_{\beta i}}\right) + V(\mu_{\beta i}) \\
&= \frac{LGD_i - LGD_i^2}{1 + \nu_{\beta i}} + \frac{\nu_{\beta i}}{1 + \nu_{\beta i}}V(CLGD_i).
\end{aligned} \tag{26}$$

The last row is from $E(\mu_{\beta i}) = LGD_i$. $V(CLGD_i)$ can be obtained from the integration

$$V(CLGD_i) = \int_x (f(P_i(x)) - LGD_i)^2 f_X(x) dx. \tag{27}$$

This has a closed form solution for the linear model, but needs to be solved numerically for other models. Solving (26) for $\nu_{\beta i}$ yields

$$\nu_{\beta i} = -\frac{VLGD - LGD + LGD^2}{VLGD - V(CLGD_i)}. \tag{28}$$

Then, a and b are given by

$$a_i = \mu_{\beta i}\nu_{\beta i}, \quad b_i = (1 - \mu_{\beta i})\nu_{\beta i}. \tag{29}$$

4. Empirical Studies

4.1. The Portfolio

A model portfolio of senior unsecured bonds is constructed based on the actual market data; bond characteristics are set by referring to Moody's (2012) annual report. PD and its standard deviation of each rating are assumed to be based on the annual issuer-weighted corporate default rates reported in Exhibit 30 of the report. LGD of each rating is based on the average senior unsecured bond recovery rates in Exhibit 21.² The portfolio is assumed to consist of 1,000 bonds which have exposure at default (EaD) of 100 million dollars. The distribution of the bond exposure across ratings is assumed based on the debts outstanding in the market. Aaa rated bonds

²Secured or collateralized bonds are fully recovered except some extreme circumstances and bears little severity risk. As our focus is the effect of severity risk on credit loss, only unsecured bonds are assumed.

are excluded from the investment set as there is no default record of Aaa firms since 1920. The characteristics of the portfolio are summarized in Table 1. The table shows that PD is strongly correlated with the rating, whereas LGD is rather constant at around 60% across the ratings. This does not imply no correlation between PD and LGD. The PD-LGD correlation of our concern is not cross-sectional correlation but time-series correlation, which is evident as shown in Section 4.2.

Rating	PD (%)	SPD (%)	LGD (%)	EaD/ Bond	Num. Bonds
Aa	0.06	0.19	63.00	100	50
A	0.10	0.27	68.00	100	350
Baa	0.27	0.47	59.00	100	350
Ba	1.07	1.65	53.00	100	100
B	3.42	4.03	62.00	100	100
Caa-C	13.77	17.03	64.00	100	50

Table 1: The model portfolio. The portfolio consists of 1,000 bonds distributed evenly across 10 industry groups in Table 2 and according to the composition in the last column across ratings. Each bond is assumed to have a notional value of 100 million dollars. PD, SPD, and LGD are respectively probability of default, its standard deviation, and loss given default all in percentage values, and they are based on Exhibit 20, 21, and 30 of Moody’s (2012).

Among Moody’s 11 broad industry groups, the portfolio consists of bonds from 10 industry groups except ‘Government Related Issuers’ since default in this group is extremely rare and reliable estimates of PD and LGD cannot be obtained. The bonds are assumed to be evenly distributed across industries. That is, there are 100 bonds in each of 10 industry groups which are allocated to the ratings according to the proportions in the last column of Table 1. The 10 industry groups and their descriptive statistics are reported in Table 2. Finally, each bond is assumed to be issued by different issuers, i.e., defaults of the bonds occur independently of each other.

4.2. PD-LGD Correlation

The LGD models are estimated from the default time series of ‘All Bonds’ during the period 1982-2011, which appear in Exhibit 20 and 30 of the Moody’s report. The descriptive statistics are reported in Table 3. The correlation between PD and LGD is very high with the correlation coefficient 0.71 during the sample period. This is in line with the high correlation of 0.75 during 1982-2001 reported by Altman et al. (2005).

Corr.	I1	I2	I3	I4	I5	I6	I7	I8	I9	I10
I1	1.00									
I2	0.47	1.00								
I3	0.62	0.84	1.00							
I4	-0.04	0.26	0.13	1.00						
I5	0.45	0.65	0.59	-0.01	1.00					
I6	0.51	0.76	0.61	0.12	0.60	1.00				
I7	0.35	0.55	0.57	0.02	0.19	0.31	1.00			
I8	0.19	0.67	0.50	0.28	0.32	0.59	0.51	1.00		
I9	0.45	0.59	0.52	0.28	0.24	0.43	0.58	0.50	1.00	
I10	0.44	0.36	0.34	0.17	0.01	0.29	0.53	0.42	0.51	1.00
Mean	0.41	1.69	1.97	1.35	0.86	2.67	2.30	1.32	2.49	0.17
Std	0.72	2.03	2.09	1.70	2.62	3.86	2.30	1.90	2.98	0.27

Table 2: Descriptive statistics of the default rates of 10 broad industry groups. The statistics are calculated from the industry default time series during 1970-2011, excerpted from Exhibit 38 of Moody's (2012). I1: Banking, I2: Capital Industries, I3: Consumer Industries, I4: Energy & Environment, I5: Non-Bank Finance, I6: Media & Publishing, I7: Retail & Distribution, I8: Technology, I9: Transportation, I10: Utilities.

	Mean	Median	Stdev	Min	Max	Corr.
PD	1.67	1.28	1.20	0.37	5.45	0.71
LGD	58.46	56.80	9.53	41.50	78.40	

Table 3: Descriptive statistics of PD and LGD of all bonds. The statistics are calculated from the default rates and the recovery rates of all bonds during 1982-2011, excerpted from Exhibit 20 and 30 of Moody's (2012).

Regression results of the three LGD specifications in (16)-(18) are reported in Table 4 and displayed in Figure 1. All three models show similar fitting capabilities with R-squared values larger than 0.5: The power function has a marginally higher R-squared value (0.56) compared to the linear function (0.54) and the logistic function (0.55). Altman et al. (2005) also find that power function slightly outperforms other functional forms in terms of R-squared. ϕ_1 's of all three models are significant at 99% level.

The impact of each LGD specification on the credit risk can be anticipated from the regression results. First note that, with the calibration procedure in Section 3.1.1, the PD-LGD relationship around the mean of PD, 1.67%, plays a dominant role for the estimation of the expected loss and the unexpected loss at a low confidence level whilst the relationship at higher PD values becomes more important for the unexpected loss at a higher confidence level. From the figure, it can be seen that the linear regression line and the logistic regression line are very similar and lie below the power regression line around the mean of PD. This suggests that the Power model will yield the highest expected loss and the highest unexpected loss at a low confidence level. However, LGD increases with PD most rapidly under the Linear model followed by the Logistic model. This means that the unexpected loss from the Linear model will become the highest and the unexpected loss from the Power model will become the lowest as the confidence level increases. Comparison of Value-at-Risk (VaR) among the models is less obvious as VaR is the sum of the expected loss and the unexpected loss.

	Linear	Power	Logistic
ϕ_0	0.487* (0.000)	1.291 (0.070)	-0.067 (0.455)
ϕ_1	5.851* (0.000)	0.187* (0.000)	25.434* (0.000)
R ²	0.539	0.563	0.550
F-value	32.679*	36.144*	34.174*

Table 4: Regression results of the models in (16)-(18). The parameters are estimated from the default rates and the recovery rates of all bonds during 1982-2011, excerpted from Exhibit 20 and 30 of Moody's (2012). The figures in parenthesis are p -values and * refers to 99% significance.

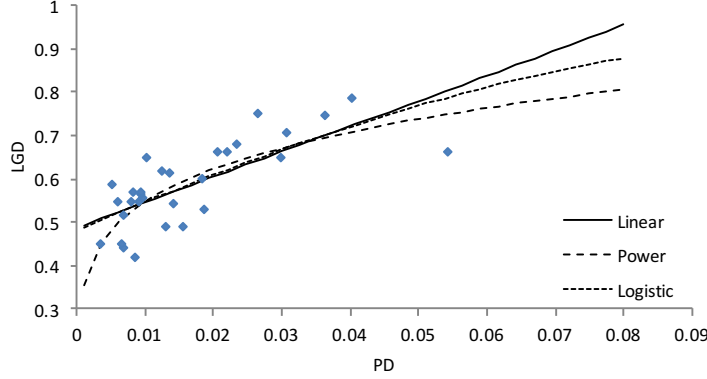


Figure 1: Regression of LGD on PD. Refer to Table 4 for details.

4.3. Credit Risk Calculation

Credit risk of the portfolio is computed using CreditRisk++ with five different LGD models. Besides the three LGD models proposed in Section 3, a constant LGD model is chosen as a benchmark. A beta-distributed independent LGD model is also included for comparison.

The risk factors are defined as the PDs of the 10 industry sectors, from which 11 independent systemic risk factors—10 industry specific and 1 macroeconomic—are derived. To estimate the idiosyncratic risk parameters, a_i and b_i , the standard deviation of LGD is assumed to be 25%, which was estimated by Schuermann (2004) from the US corporate exposures. The loss distribution is generated via the Monte Carlo simulation method described in Appendix A which incorporates importance sampling in order to reduce the simulation error.

4.4. Simulation Efficiency

Before analyzing the PD-LGD correlation effect on credit risk, the efficiency of the simulation method is examined by comparing it with an analytic method. The credit risk of the portfolio is calculated using both methods under the constant LGD assumption. The analytic solution is obtained using the numerical algorithm proposed by Haaf et al. (2004) and the simulation results are obtained from 10,000 iterations. Simulation is repeated 100 times to calculate the simulation error. For importance sampling, the predetermined portfolio loss is set to 6,000, which is approximately the loss at 99.9% confidence level. The results are reported in Table 5. The error reduction achieved by importance sampling is truly remarkable. While

the root mean squared error (RMSE) of the usual Monte Carlo simulation increases with the confidence level reaching almost 14% at 99.99%, the error remains stable around 1% across confidence levels when importance sampling technique is employed. Considering the fact that the predetermined loss level is set around 99.9% loss, this stable performance over a wide range of confidence level including the expected loss is very encouraging. In terms of computation cost, the simulation method is not more expensive than the analytic method at least for the portfolio and the numerical algorithm under consideration. Indeed, the naive simulation takes a shorter time (0.86 seconds) than the analytic method (1.02 seconds). Of course, the computation cost of the simulation method will increase linearly with the size of the portfolio, while that of the analytic method will increase at a slower pace. Nevertheless, considering the simulation is programmed with Matlab and run in a desktop environment, computational burden should by no means be a barrier for adopting a simulation-based algorithm even for a large size portfolio.

	Analytic	Simulation			Simulation (IS)		
		Mean	RMSE	MAE	Mean	RMSE	MAE
EL	790.64	791.33	0.87	0.70	790.31	0.72	0.61
VaR99	3739.47	3737.75	3.24	2.53	3742.91	1.10	0.89
VaR99.9	6383.79	6315.85	5.43	4.18	6377.37	0.94	0.76
VaR99.99	9194.96	8695.86	13.65	11.48	9195.91	0.67	0.55
Elapsed	1.02	0.86			1.04		

Table 5: Simulation errors of risk measures. Analytic, Simulation, and Simulation (IS) respectively refer to analytic method, naive Monte-Carlo simulation and Monte-Carlo simulation with importance sampling. The predetermined loss for important sampling is set to 6,000. Simulation errors are computed from 100 simulation runs with 10,000 iterations in each simulation. RMSE: Root mean squared error (%); MAE: Mean absolute error (%); Elapsed: elapsed time (seconds).

The convergence of the simulation error is assessed by repeating the simulation with different numbers of iterations. The results are displayed in Figure 2, in which RMSEs of 99.9% loss are plotted against the number of iterations. The figure reveals the effect of importance sampling clearly. The simulation error of importance sampling is surprisingly small even when the number of iterations is of the same order as the number of bonds in the portfolio. For example, the simulation with 5,000 iterations for the portfolio of 1,000 bonds has RMSE of only 1.16%. This suggests that even for a large

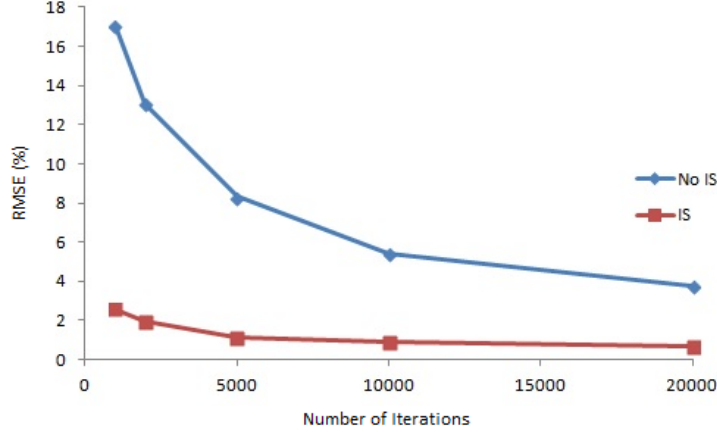


Figure 2: Convergence of the simulation error. x -axis represents the number of iterations and y -axis RMSE(%) of VaR99.9. No IS: Naive Monte-Carlo simulation; IS: Monte-Carlo simulation with importance sampling.

portfolio, a moderate number of iterations should be sufficient to achieve a reasonably high precision.

4.5. Severity Risk and Its Contribution to Credit Risk

Table 6 reports the credit risk measures of the portfolio estimated by CreditRisk++ with the five LGD models. The results are also illustrated in Figure 3. EL and VaR x in the first column respectively refer to the expected loss and the loss at x probability level. For each risk measure, Mean is the average of 100 simulations and RMSE is the root mean squared error measured as the percentage relative to the mean. RMSE values confirm that importance sampling works equally well even when the PD-LGD dependency is taken into account. Diff is the percentage difference from the constant LGD model. To assess the effect of idiosyncratic risk, systemic risk measures (Sys) are also reported beside the total risk measures (Total) for each model. Systemic risk is calculated by setting the variance of the beta distribution to 0.

Comparing the risk measures from the independent LGD model with those from the constant LGD model, it can be seen that the idiosyncratic severity risk is mostly diversified away and its contribution to the credit risk is negligible. This has been reported in other studies; for example, Gordy (2003) shows that idiosyncratic severity risk vanishes very quickly and it becomes trivial even for a small size portfolio with a few hundreds of

bonds. This confirms that a simple stochastic LGD model does not reflect the severity risk adequately.

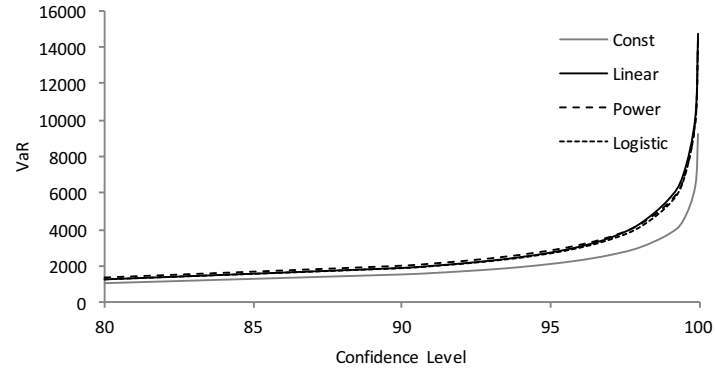
When PD-LGD dependency is accounted for, the systemic component of the severity risk is incorporated and the contribution of the severity risk to the credit risk becomes nontrivial. For example, 99% VaR experiences an increase of 51.2%, 45.4%, and 43.2% in Linear, Power, and Logistic model, respectively. From Figure 3, it can be seen that the increase in VaR becomes more prominent at higher confidence levels. This is because both PD and LGD increase with the confidence level. As expected from the PD-LGD regression results, the Power model produces the highest risk at a low confidence level where the LGD predicted by the Power model is the highest. However, it is the Linear model that produces the highest risk in the tail region. This is due to the linear dependency of LGD on PD in the Linear model which causes LGD to increase faster than that in other models. In fact, if LGD is not bounded at 100%, the risk measured by the Linear model becomes almost 80% higher compared to the constant LGD model (not reported here). Note that not only VaR but also expected loss is, though at a lesser degree, significantly increased: 17.7% (Linear), 23.0% (Power), and 16.1% (Logistic). This is because when PD and LGD are correlated, the usual expected loss equation $EL = PD \cdot LGD \cdot EaD$ no longer holds. This is obvious but often overlooked.

Comparison between the total risk and the systemic risk confirms that most of the increase in risk is caused by the systemic component. This implies that the idiosyncratic component of LGD can be safely ignored, and the LGD model can be simplified by discarding the beta distribution assumption. Without the idiosyncratic component, simulation results will become more robust due to reduced randomness.

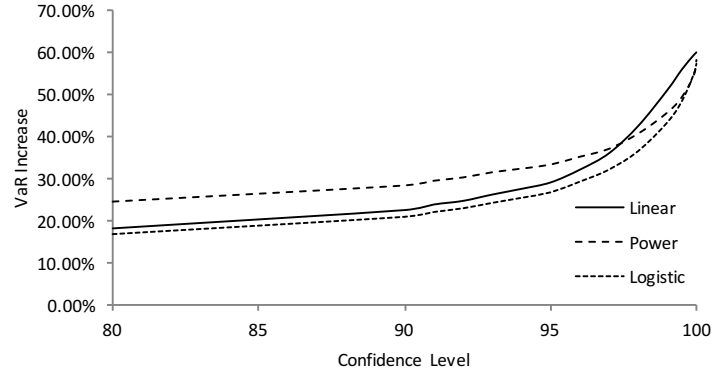
The findings in this section suggest that assuming LGD as a constant or a stochastic but independent variable can underestimate risk significantly. The severity risk can increase the credit risk over 50% in the particular example considered here. Many other researches also find nontrivial increase of credit risk when PD-LGD correlation is accounted for. It is also important to note that the increase in credit risk depends on the confidence level: Credit risk increases more when the confidence level is higher. The results can be used as a guidance to determine downturn LGD. For example, 55% increase in 99.9% VaR under the Logistic model suggests that the downturn LGD should be set to 1.55 times the normal LGD. Downturn LGD should not be a mere reflection of the expected LGD in the market downturn. Rather, it should be determined by taking the correlation effect into account so that the total risk is addressed adequately.

	Const	Indep	Linear		Power		Logistic	
			Total	Sys	Total	Sys	Total	Sys
EL								
Mean	791	792	931	931	972	972	918	918
Diff		0.12	17.73	17.73	22.96	22.95	16.07	16.08
RMSE		0.82	0.93	0.90	0.86	0.87	0.86	0.87
VaR99								
Mean	2949	2955	5652	5648	5438	5435	5353	5349
Diff		0.21	51.15	51.04	45.43	45.34	43.16	43.03
RMSE		1.30	1.32	1.30	1.30	1.25	1.11	1.08
VaR99.9								
Mean	5593	5596	10171	10171	9865	9858	9905	9902
Diff		0.06	59.33	59.32	54.53	54.42	55.15	55.11
RMSE		0.96	0.94	0.95	0.85	0.83	0.86	0.88
VaR99.99								
Mean	8404	8407	14722	14722	14545	14545	14471	14472
Diff		0.03	60.11	60.11	58.18	58.18	57.39	57.39
RMSE		0.86	0.65	0.66	0.92	0.91	0.71	0.72

Table 6: Risk measures by five different severity risk models: Constant (Const); Independent beta distributed (Indep); Linear function of PD (Linear); Power function of PD (Power); Logistic function of PD (Logistic). Total is the total credit risk and Sys is the systemic risk only. EL and VaR x in the first column respectively refer to the expected loss and the loss at x probability level. For each risk measure, Mean is the average of 100 simulations and RMSE is the root mean squared error measured as the percentage relative to the mean. Diff is the percentage difference from the constant LGD model.



(a) Credit risk measures (million dollars)



(b) Credit risk increase relative to the Const model

Figure 3: Credit risk measures by different severity risk models. x -axis is the confidence level. The Indep model is not displayed here as it is almost identical to the Const model.

5. Concluding Remarks

In this article, a generic severity risk model in which LGD is dependent upon PD in an intuitive manner is developed. By modeling the conditional mean of LGD as a function of PD, which also varies with systemic risk factors, this model allows an arbitrary functional relationship between PD and LGD including nonlinear forms such as power function that is found to offer the best fit for PD-LGD covariation. Based on this generic framework, three specifications of stochastic LGD, i.e., LGD as a linear function of PD, LGD as a power function of PD, and LGD as a logistic function of PD, are proposed with detailed calibration methods that rely on easily obtainable data. By combining these models with a generalized CreditRisk+ model, a versatile mixed Poisson credit risk model is developed. This model is capable of handling various forms of PD-LGD dependency as well as risk factor correlation. This added capability comes at a cost of analytic tractability and a simulation method based on importance sampling is introduced for risk calculation. The simulation method turns out to be very accurate and efficient without any notable disadvantage.

The severity risk models are applied to a model portfolio and evaluated. The model portfolio is artificially constructed based on the actual market data. The empirical studies suggest that ignoring or incorrectly specifying, e.g., by assuming independence, severity risk can significantly underestimate credit risk: In this study, risk increases over 50% from the case of constant LGD when the confidence level is 99.9% or higher. Banks should recognize this and adopt a proper severity risk model to assess their credit risk adequately. It is also important to consider the effect of PD-LGD correlation when determining downturn LGD. All models yield similar risk measures and no particular model offers a significantly better performance. Still, the logistic model could be a preferred choice in the sense that LGD is implicitly bounded within $[0, 1]$ without any ad hoc boundary condition. In fact, the framework behind the models is very flexible and can be easily generalized to accommodate other types of PD-LGD relationship. It can also be extended to other credit risk models such as the structural models based on Merton (1974).

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Appendix A. Importance Sampling for Loss Distribution

When severity risk is incorporated, CreditRisk+ is no longer analytically tractable and loss distribution can only be obtained via a simulation method. Glasserman and Li (2003) develop an importance sampling technique for a mixed Poisson credit risk model and Han and Kang (2008) show that the method indeed produces a very accurate result at a low computational cost. Extension of Glasserman and Li (2003)’s work for CreditRisk+ with severity risk is straightforward and is illustrated below. As demonstrated in Section 4.4, this technique is very efficient and generates sufficiently accurate results for all severity risk models.

Suppose there are N assets in the portfolio and K systemic risk factors $X_k \sim \text{Gamma}(\alpha_k, \beta_k)$ which are independent of each other. Define θ as the exponential twisting parameter for importance sampling. The cumulant generating function of the portfolio loss, $L = \sum_{i=1}^N L_i$ under the framework of CreditRisk+ is given by

$$\psi(\theta) = \psi^{(1)}(\theta) + \psi^{(2)}(\theta) \quad (\text{A.1})$$

where

$$\psi^{(1)}(\theta) = \sum_{i=1}^N P_i w_{0i} \left(e^{V_i \theta} - 1 \right), \quad (\text{A.2})$$

$$\psi^{(2)}(\theta) = - \sum_{k=1}^K \alpha_k \log \left(1 - \beta_k \sum_{i=1}^N P_i w_{ki} \left(e^{V_i \theta} - 1 \right) \right) \quad (\text{A.3})$$

where P_i is PD conditional on the risk factors and $V_i = LGD_i \cdot EAD_i$ is expected amount of loss given default. The first order derivative of $\psi(\theta)$ with respect to θ is

$$\psi'(\theta) = \psi^{(1)'}(\theta) + \psi^{(2)'}(\theta) \quad (\text{A.4})$$

where

$$\psi^{(1)'}(\theta) = \sum_{i=1}^N P_i w_{0i} V_i e^{V_i \theta}, \quad (\text{A.5})$$

$$\psi^{(2)'}(\theta) = \sum_{k=1}^K \frac{\alpha_k \beta_k \sum_{i=1}^N P_i w_{ki} V_i e^{V_i \theta}}{1 - \beta_k \sum_{i=1}^N P_i w_{ki} (e^{V_i \theta} - 1)}. \quad (\text{A.6})$$

Portfolio loss simulation is performed by following the procedure.

1. Solve

$$\psi'(\theta) = L_p \quad (\text{A.7})$$

$$\theta = \max(0, \theta) \quad (\text{A.8})$$

for θ . L_p is a predetermined portfolio loss, e.g., Value-at-Risk, around which samples are to be drawn. For CreditRisk+, L_p can be determined from an analytic solution or by running a simulation without importance sampling.

2. Compute τ_k , $k = 1, \dots, K$ from

$$\tau_k = \sum_{i=1}^N P_i w_{ki} (e^{V_i \theta} - 1)$$

3. Draw samples of risk factors

$$X_k \sim \text{Gamma} \left(\alpha_k, \frac{\beta_k}{1 - \beta_k \tau_k} \right), \quad k = 1, \dots, K.$$

4. Compute the conditional default probabilities

$$P_i = PD_i \left(w_{i0} + \sum_{k=1}^K w_{ki} X_k \right), \quad i = 1, \dots, N$$

5. Draw samples of default events

$$D_i \sim \text{Poisson} \left(P_i e^{V_i \theta} \right), \quad i = 1, \dots, N.$$

6. Compute the conditional mean of LGD using either of the equations.

$$CLGD_i = \phi_{0i} + \phi_{1i} P_i \quad (\text{Linear})$$

$$CLGD_i = \phi_{0i} P_i^{\phi_{1i}} \quad (\text{Power})$$

$$CLGD_i = \frac{1}{1 + \exp(-\phi_{0i} - \phi_{1i} P_i)} \quad (\text{Logistic})$$

for $i = 1, \dots, N$. Set $\mu_{\beta,i} = CLGD_i$ and compute beta distribution parameters from

$$a_i = \mu_{\beta,i} \nu_{\beta,i}, \quad b_i = (1 - \mu_{\beta,i}) \nu_{\beta,i}.$$

Skip this step for Beta Indep model.

7. Draw samples of loss given default.

$$U_i \sim \text{Beta}(a_i, b_i), \quad i = 1, \dots, N$$

Skip this step for Linear model and set $U_i = E(U_i|X)$ with the first equation of step 6.

8. Portfolio loss is given by

$$L = \sum_{i=1}^N L_i = \sum_{i=1}^N D_i \cdot U_i \cdot EaD_i$$

And the probability associated with this loss is given by the likelihood ratio,

$$LR = \exp(-\theta L' + \psi(\theta))$$

where $L' = \sum_{i=1}^N D_i \cdot LGD_i \cdot EaD_i$ is portfolio loss under constant LGD assumption.

9. Repeat from step 3 until the desired number of iterations is reached.