Kempe Equivalence of Colourings of Cubic Graphs

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Abstract. Given a graph $G = (V, E)$ and a proper vertex colouring of G, a Kempe chain is a subset of V that induces a maximal connected subgraph of G in which every vertex has one of two colours. To make a Kempe change is to obtain one colouring from another by exchanging the colours of vertices in a Kempe chain. Two colourings are Kempe equivalent if each can be obtained from the other by a series of Kempe changes. A conjecture of Mohar asserts that, for $k \geq 3$, all k-colourings of connected k-regular graphs that are not complete are Kempe equivalent. We address the case $k = 3$ by showing that all 3-colourings of a connected cubic graph G are Kempe equivalent unless G is the complete graph K_4 or the triangular prism.

1 Introduction

Let $G = (V, E)$ denote a simple undirected graph and let k be a positive integer. A k-*colouring* of G is a mapping $\phi : V \to \{1, \dots, k\}$ such that $\phi(u) \neq \phi(v)$ if $uv \in E$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G has a k-colouring.

If a and b are distinct colours of a colouring α, then $G_\alpha(a, b)$ denotes the subgraph of G induced by vertices with colour a or b. An (a, b)-*component under* α of G is a connected component of $G_\alpha(a, b)$ and is known as a *Kempe chain* (we will omit the reference to α when it is unneeded). A *Kempe change* is the operation of interchanging the colours of some (a, b)-component of G. Let $C_k(G)$ be the set of all k-colourings of G. Two colourings $\alpha, \beta \in C_k(G)$ are *Kempe equivalent*, denoted by $\alpha \sim_k \beta$, if each can be obtained from the other by a series of Kempe changes. The equivalence classes $C_k(G)/\sim_k$ are called *Kempe classes*.

Kempe changes were first introduced by Kempe in his well-known failed attempt at proving the Four-Colour Theorem. The Kempe change method has proved to be a powerful tool with applications to several areas such as timetables [17], theoretical physics [21, 22], and Markov chains [20]. The reader is referred to [16, 18] for further details. From a theoretical viewpoint, Kempe equivalence was first addressed by Fisk [11] who proved that all 4-colourings of an Eulerian triangulation of the plane are Kempe equivalent. This result was later extended by Meyniel [14] who showed that all 5-colourings of a planar graph are Kempe equivalent, and by Mohar [16] who proved that all k-colourings, $k > \chi(G)$, of a planar graph G are Kempe equivalent. Las

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Vergnas and Meyniel [19] extended Meyniel's result by proving that all 5-colourings of a K_5-minor free graph are Kempe equivalent. Bertschi [2] showed that all k-colourings of a perfectly contractile graph are Kempe equivalent, and, by further showing that any Meyniel graph is perfectly contractile, answered in the affirmative a conjecture of Meyniel [15]. We note that Kempe equivalence with respect to edge-colourings has also been investigated [1, 13, 16].

Here we are concerned with a conjecture of Mohar [16] on connected *k-regular* graphs, that is, graphs in which every vertex has degree k for some $k \geq 0$. Note that, for every connected 2-regular graph G that is not an odd cycle, it holds that $C_2(G)$ is a Kempe class. Mohar conjectured the following (where K_{k+1} is the complete graph on $k + 1$ vertices).

Conjecture 1 ([16]) *Let $k \geq 3$. If G is a connected k-regular graph that is not K_{k+1}, then $C_k(G)$ is a Kempe class.*

Notice that if $G = K_{k+1}$, then $C_k(G)$ forms an empty Kempe class; so the condition in Conjecture 1 is not necessary but it is neater to exclude this case. Notice also that if $G \neq K_{k+1}$, then $C_k(G)$ is not empty by Brooks' Theorem [7], which states that a graph with maximum degree k has a k-colouring unless it is an odd cycle or a complete graph.

We address Conjecture 1 for the case $k = 3$. For this case the conjecture is known to be false. A counter-example is the 3-prism displayed in Figure 1. The fact that some 3-colourings of the 3-prism are not Kempe equivalent was already observed by van den Heuvel [12]. Our contribution is that the 3-prism is the *only* counter-example for the case $k = 3$, that is, we completely settle the case $k = 3$ by proving the following result for 3-regular graphs also known as *cubic* graphs.

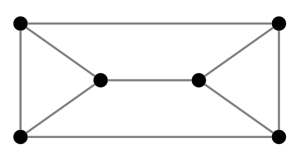

Fig. 1. The 3-prism.

Theorem 1. *If G is a connected cubic graph that is neither K_4 nor the 3-prism, then $C_3(G)$ is a Kempe class.*

We give the proof of our result in the next section. Let us note an immediate corollary of our result. First we need a definition and a lemma. Let d be a positive integer. A graph G is *d-degenerate* if every subgraph of G has a vertex with degree at most d.

Lemma 1 ([16, 19]). *Let d and k be integers, $d \geq 0$, $k \geq d + 1$. If G is a d-degenerate graph, then $C_k(G)$ is a Kempe class.*

Corollary 1. *Let G be a connected graph with maximum degree at most 3. Then $C_3(G)$ is a Kempe class unless G is K_4 or the 3-prism.*

Proof. A connected graph with maximum degree 3 is either 3-regular or 2-degenerate (this follows easily from the definition of degenerate, but see also, for example, [10] for a discussion). The corollary follows from Theorem 1 and Lemma 1. □

Recently Conjecture 1 was confirmed for $k \geq 4$ [4]; there were no exceptional cases. This new result and its proof do not imply Theorem 1.

Our result is an example of a type of result that has received much recent attention: that of determining the structure of a *reconfiguration graph*. A reconfiguration graph has as vertex set all solutions to a search problem and an edge relation that describes a transformation of one solution into another. Thus Theorem 1 is concerned with the reconfiguration graph of 3-colourings of a cubic graph with edge relation $\sim_k$ and shows that it is connected except in two cases. To date the structure of reconfiguration graphs of colourings has focussed [3, 5, 6, 8–10] on the case where vertices are joined by an edge only when they differ on just one colour (that is, when one colouring can be transformed into another by changing the colour on a single vertex which is a Kempe change of a Kempe chain that contains only one vertex). For a survey of recent results on reconfiguration graphs see [12].

2 The Proof of Theorem 1

We first give some further definitions and terminology. Let $G = (V, E)$ be a graph. Then G is *H-free* for some graph H if G does not contain an induced subgraph isomorphic to H. A *separator* of G is a set $S \subset V$ such that $G - S$ has more components than G. We say that G is *p-connected* for some integer p if $|V| \geq p + 1$ and every separator of G has size at least p. Some small graphs that we will refer to are defined by their illustrations in Figure 2.

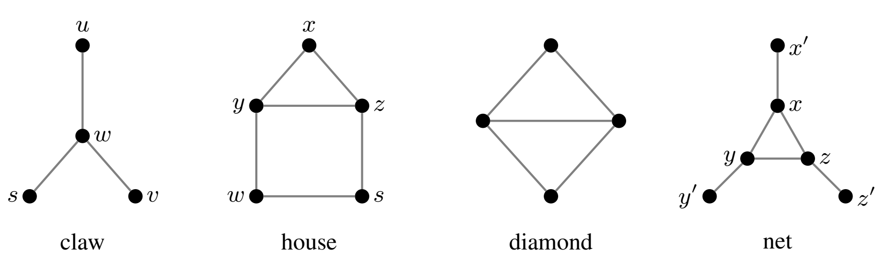

Fig. 2. A number of special graphs used in our paper.

Besides three new lemmas, we will need the aforementioned result of van den Heuvel, which follows from the fact that for the 3-prism T, the subgraphs $T(1, 2)$,

$T(2,3)$ and $T(1,3)$ are connected so that the number of Kempe classes is equal to the number of different 3-colourings of T up to colour permutation, which is two.

Lemma 2 ([12]). *If G is the 3-prism, then $C_3(G)$ consists of two Kempe classes.*

Lemma 3. *If G is a connected cubic graph that is not 3-connected, then $C_3(G)$ is a Kempe class.*

Lemma 4. *If G is a 3-connected cubic graph that is claw-free but that is neither K_4 nor the 3-prism, then $C_3(G)$ is a Kempe class.*

Lemma 5. *If G is a 3-connected cubic graph that is not claw-free, then $C_3(G)$ is a Kempe class.*

Observe that Theorem 1 follows from the above lemmas, which form a case distinction. Hence it suffices to prove Lemmas 3–5. These proofs form the remainder of the paper.

2.1 Proof of Lemma 3

In order to prove Lemma 3, we need two auxiliary results.

Lemma 6 ([19]). *Let $k \geq 1$ be an integer. Let G_1 and G_2 be two graphs such that $G_1 \cap G_2$ is complete. If both $C_k(G_1)$ and $C_k(G_2)$ are Kempe classes, then $C_k(G_1 \cup G_2)$ is a Kempe class.*

Lemma 7 ([16]). *Let $k \geq 1$ be an integer and let G be a subgraph of a graph G'. Let α' and β' be the restrictions, to G, of two k-colourings α and β of G'. If α and β are Kempe equivalent, then α' and β' are Kempe equivalent.*

For convenience we restate Lemma 3 before we present its proof.

Lemma 3 (restated). *If G is a connected cubic graph that is not 3-connected, then $C_3(G)$ is a Kempe class.*

Proof. As G is cubic, G has at least four vertices. Because G is not 3-connected, G has a separator S of size at most 2. Let S be a minimum separator of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = S$. As every vertex in S has degree at most 2 in each G_i and G is cubic, G_1 and G_2 are 2-degenerate. Hence, by Lemma 1, $C_3(G_1)$ and $C_3(G_2)$ are Kempe classes. If S is a clique, we apply Lemma 6. Thus we can assume that S, and any other minimum separator of G, is not a clique. Then $S = \{x, y\}$ for two distinct vertices x and y with $xy \notin E(G)$.

Because S is a minimum separator, x and y are non-adjacent and G is cubic, we have that x has either one neighbour in G_1 and two in G_2, or the other way around; the same holds for y. For $i = 1, 2$, let $N_i(x)$ and $N_i(y)$ be the set of neighbours of x and y, respectively, in G_i.

Suppose that $|N_1(x)| = |N_1(y)|$. Then $|N_2(x)| = |N_2(y)|$ also and we can suppose without loss of generality that $|N_1(x)| = |N_1(y)| = 1$. Let x_1 be the unique neighbour of x in G_1. Then $x_1 y \notin E(G)$ else $\{x_1\}$ is a separator smaller than S. So $\{x_1, y\}$ is a minimal separator that separates $G_1 \setminus \{x_1\}$ from $G_2 \cup \{x\}$ and x_1 has two neighbours in $G_1 \setminus \{x_1\}$ and y only has one. So, relabelling if necessary, we can assume that $|N_1(x)| \neq |N_1(y)|$. Moreover, we can let $N_1(x) = \{x_1\}$ and $N_2(y) = \{y_1\}$, where y_1 is the unique neighbour of y in G_2.

It now suffices to prove the following two claims.

Claim 1. All colourings α such that $\alpha(x) \neq \alpha(y)$ are Kempe equivalent in $C_3(G)$.

We prove Claim 1 as follows. We add an edge e between x and y. This results in graphs $G_1 + e$, $G_2 + e$ and $G + e$. We first prove that $C_3(G + e)$ is a Kempe class. Because x and y have degree 1 in G_1 and G_2, respectively, and G is cubic, we find that the graphs $G_1 + e$ and $G_2 + e$ are 2-degenerate. Hence, by Lemma 1, $C_3(G_1 + e)$ and $C_3(G_2 + e)$ are Kempe classes. By Lemma 6, it holds that $C_3(G + e)$ is a Kempe class. Applying Lemma 7 completes the proof of Claim 1.

Claim 2. For every colouring α such that $\alpha(x) = \alpha(y)$, there exists a colouring β with $\beta(x) \neq \beta(y)$ such that α and β are Kempe equivalent in $C_3(G)$.

We assume without loss of generality that $\alpha(x) = \alpha(y) = 1$ and $\alpha(y_1) = 2$. If $\alpha(x_1) = 2$, then we apply a Kempe change on the $(1, 3)$-component of G that contains x. Note that y does not belong to this component. Hence afterwards we obtain the desired colouring γ. If $\alpha(x_1) = 3$, then we first apply a Kempe change on the $(2, 3)$-component of G that contains x_1. Note that this does not affect the colours of x, y and y_1 as they do not belong to this component. Afterwards we proceed as before. This completes the proof of Claim 2 (and the lemma). □

2.2 Proof of Lemma 4

We require some further terminology and three lemmas. We *identify* two vertices x and y in a graph G if we replace them by a new vertex adjacent to all neighbours of x and y in G. Two colourings α and β of a graph G *match* if there exists two vertices x, y with a common neighbour in G such that $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$.

We highlight that the following lemma is a statement about *any* graph and might prove to be of wider use.

Lemma 8. *Let $k \geq 1$ and G' be the graph obtained from a graph G by identifying two non-adjacent vertices x and y. If $C_k(G')$ is a Kempe class, then all k-colourings γ of G with $\gamma(x) = \gamma(y)$ are Kempe equivalent.*

Proof. Let α and β be two k-colourings of G with $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$. Let z be the vertex of G' that is obtained after identifying x and y. Let α' and β' be the k-colourings of G' that agree with α and β, respectively, on $V(G) \setminus \{x, y\}$ and for which $\alpha'(z) = \alpha(x)(= \alpha(y))$ and $\beta'(z) = \beta(x)(= \beta(y))$. By our assumption, there exists a Kempe chain from α' to β' in G'. We mimic this Kempe chain in G. Note that any (a, b)-component in G' that contains z corresponds to at most two (a, b)-components in G, as x and y may get separated. Hence, every Kempe change on an (a, b)-component corresponds to either one or two Kempe changes in G (if x and y are in different (a, b)-components, then we apply the corresponding Kempe change in G' on each of these two components). In this way we obtain a Kempe chain from α to β as required. □

Lemma 9. *Let $k \geq 3$. If α and β are matching k-colourings of a 3-connected graph G of maximum degree k, then $\alpha \sim_k \beta$.*

Proof. If G is $(k-1)$-degenerate, then $\alpha \sim_k \beta$ by Lemma 1. Assume that G is not $(k-1)$-degenerate. Then G is k-regular. Since α and β match, there exist two vertices u and v of G that have a common neighbour w such that $\alpha(u) = \alpha(v)$ and $\beta(u) = \beta(v)$. Let x denote the vertex of G' obtained by identifying u and v.

Let S be a separator of G'. If S does not contain x, then S is a separator of G. Then $|S| \geq 3$ as G is 3-connected. If S contains x, then S must contain another vertex as well; otherwise $\{u, v\}$ is a separator of size 2 of G, which is not possible. Hence, $|S| \geq 2$ in this case. We conclude that G' is 2-connected.

We now prove that G' is $(k-1)$-degenerate. Note that, in G', w has degree $k-1$, x has degree at least k and all other vertices have degree k. Let $u_1, \ldots, u_r$ for some $r \geq k-1$ be the neighbours of x not equal to w. Since G' is 2-connected, the graph $G'' = G' \backslash x$ is connected. This means that every u_i is connected to w via a path in G'', which corresponds to a path in G' that does not contain x. Since w has degree $k-1$ and every vertex not equal to x has degree k, we successively delete vertices of these paths starting from w towards u_i so that each time we delete a vertex of degree at most $k-1$. Afterwards we can delete x as x has degree 0. The remaining vertices form an induced subgraph of G' whose components each have maximum degree at least k and at least one vertex of degree at most $k-1$. Hence, we can continue deleting vertices of degree at most $k-1$ and thus find that G' is $(k-1)$-degenerate. Then, by Lemma 1, $C_k(G')$ is a Kempe class. Hence, by Lemma 8, we find that $\alpha \sim_k \beta$ as required. This completes the proof. □

Lemma 10. *Every 3-connected cubic claw-free graph G that is neither K_4 nor the 3-prism is house-free, diamond-free and contains an induced net (see also Figure 2).*

Proof. First suppose that G contains an induced diamond D. Then, since G is cubic, the two non-adjacent vertices in D form a separator and G is not 3-connected, a contradiction. Consequently, G is diamond-free.

Now suppose that G contains an induced house H. We use the vertex labels of Figure 2. So, s, w, x are the vertices that have degree 2 in H, and s and w are adjacent. As G is cubic, w has a neighbour $t \in V(G) \setminus V(H)$. Since G is cubic and claw-free, t must be adjacent to s. If $tx \in E$, then G is the 3-prism. If $tx \notin E$, then t and x form a separator of size 2. In either case we have a contradiction. Consequently, G is house-free.

We now prove that G has an induced net. As G is cubic and claw-free, it has a triangle and each vertex of the triangle has one neighbour in G outside the triangle. Because G is not K_4 and diamond-free, these neighbours are distinct. Then, because G is house-free, no two of them are adjacent. Hence, together with the vertices of the triangle, they induce a net. □

We restate Lemma 4 before we present its proof.

Lemma 4 (restated). *If G is a 3-connected cubic graph that is claw-free but that is neither K_4 nor the 3-prism, then $C_3(G)$ is a Kempe class.*

Proof. By Lemma 10, G contains an induced net N. For the vertices of N we use the labels of Figure 2. In particular, we refer to x, y and z as the *t-vertices* of N, and x', y' and z' as the *p-vertices*. Let α and β be two 3-colourings of G. In order to show that $\alpha \sim_3 \beta$ we distinguish two cases.

Case 1. There are two p-vertices with identical colours under α or β.
Assume that $\alpha(x') = \alpha(y') = 1$. Then $\alpha(z) = 1$ as the t-vertices form a triangle, so colour 1 must be used on one of them. Assume without loss of generality that $\alpha(z') = \alpha(x) = 2$ and so $\alpha(y) = 3$. If $\beta(z') = \beta(x)$, then α and β match (as x and z' have z as a common neighbour). Then, by Lemma 9, we find that $\alpha \sim_3 \beta$. Otherwise $\beta(z') = \beta(y)$, since the colour of z' must appear on one of x and y. Note that the $(2, 3)$-component containing x under α consists only of x and y. Then a Kempe exchange applied to this component yields a colouring α' such that $\alpha'(z') = \alpha'(y)$. As y and z' have z as a common neighbour as well, this means that α' and β match. Hence, it holds that $\alpha \sim_3 \alpha' \sim_3 \beta$, where the second equivalence follows from Lemma 9.

Case 2. All three p-vertices have distinct colours under both α and β.
Assume without loss of generality that $\alpha(x) = \alpha(z') = 1$, $\alpha(y) = \alpha(x') = 2$, and $\alpha(z) = \alpha(y') = 3$. Note that Kempe chains of G are paths or cycles, as no vertex in a chain can have degree 3 since all its neighbours in a chain are coloured alike and G is claw-free. So, we will refer to (a, b)*-paths* rather than (a, b)-components.

We will now prove that there exists a colouring α' with $\alpha \sim_3 \alpha'$ that assigns the same colour to two p-vertices of N. This suffices to complete the proof of the lemma, as afterwards we can apply Case 1.

Consider the $(1, 2)$-path P that contains x'. If P does not contain z', then a Kempe exchange on P gives us a desired colouring α' (with x' and z' coloured alike). So we can assume that x' and z' are joined by a $(1, 2)$-path P_{12}, and, similarly, x' and y' by a $(2, 3)$-path P_{23}, and y' and z' by a $(1, 3)$-path P_{13}.

Let G' be the subgraph of G induced by the three paths. Note that P_{12} has end-vertices y and z', P_{23} has end-vertices z and x' and P_{13} has end-vertices x and y'. Hence, G' contains the vertices of N and every vertex in $G' - N$ is an internal vertex of one of the three paths. As G is cubic, this means that each vertex in $G' - N$ belongs to exactly one path. Moreover, as G is claw-free and cubic, two vertices in $G' - N$ that are on different paths are adjacent if and only if they have a p-vertex as a common neighbour.

In Figure 3 are illustrations of G' and the colourings of this proof that we are about to discuss. Let $x'' \neq x$ be the vertex in P_{12} adjacent to x'. From the above it follows that x'' is adjacent to the neighbour of x' on P_{23} and that no other vertex of P_{12} (apart from x') is adjacent to a vertex of P_{23}. As G is cubic, this also means that x'' has no neighbour outside G'. Apply a Kempe exchange on P_{12} and call the resulting colouring γ. By the arguments above, the new $(2, 3)$-path Q_{23} (under γ) that contains y' has vertex set $(V(P_{23}) \cup \{x''\}) \backslash \{x', y, z\}$. Apply a Kempe exchange on Q_{23}. This results in a colouring α' with $\alpha'(y') = \alpha'(z') = 2$, hence α' is a desired colouring. This completes the proof of Case 2 and thus of the lemma. □

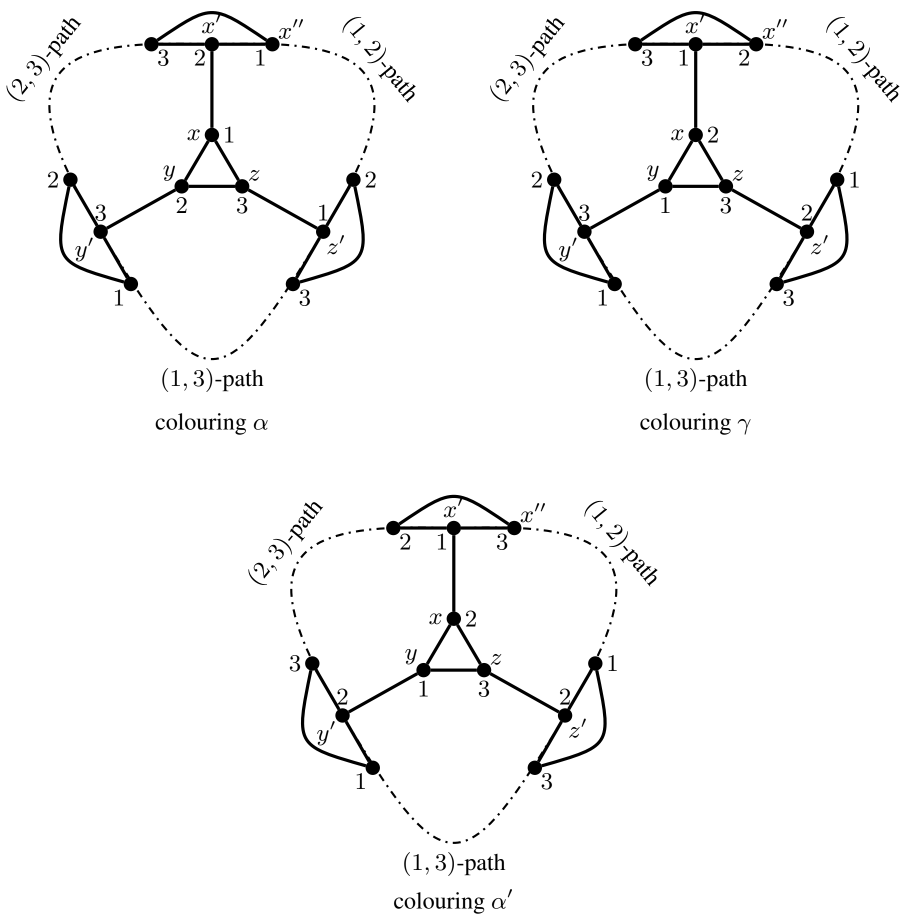

Fig. 3. Colourings of G' in the proof of Lemma 4. The dotted lines indicate paths of arbitrary length.

2.3 Proof of Lemma 5

We first need another lemma.

Lemma 11. *Let W be a set of three vertices with a common neighbour in a 3-connected cubic graph G. Suppose that every 3-colouring γ of G that colours alike exactly one pair of W is Kempe equivalent to a 3-colouring γ' such that γ' colours alike a different pair of W. Then $C_3(G)$ is a Kempe class.*

Proof. Let α and β be two 3-colourings of G. To prove the lemma we show that $\alpha \sim_3 \beta$. By Lemma 9, it is sufficient to find a matching pair of colourings that are Kempe equivalent to α and β respectively (this lemma will be applied repeatedly).

As the three vertices of W have a common neighbour, in any 3-colouring at least two of them are coloured alike. Let $W = \{x, y, z\}$. We can assume that $\alpha(x) = \alpha(y)$. If $\beta(x) = \beta(y)$, then α and β match and we are done. So we can instead assume that $\beta(x) \neq \beta(y)$ and thus $\beta(y) = \beta(z)$. If $\alpha(y) = \alpha(z)$, then, again, α and β match. Otherwise α colours alike exactly one pair of W and, by the premise of the lemma, we can find a 3-colouring α' that is Kempe equivalent to α and colours alike a different pair of W. If $\alpha'(y) = \alpha'(z)$, then α' and β match. Otherwise we must have that $\alpha'(x) = \alpha'(z)$. As $\beta(x) \neq \beta(y)$ and $\beta(y) = \beta(z)$, there exists a 3-colouring β' that is Kempe equivalent to β and that colours alike a different pair of W than β. So $\beta'(x) \in \{\beta'(y), \beta'(z)\}$ and β' matches either α or α'. In both cases we are done. □

We restate Lemma 5 before we present its proof.

Lemma 5 (restated). *If G is a 3-connected cubic graph that is not claw-free, then $C_3(G)$ is a Kempe class.*

Proof. Note that if a vertex has three neighbours coloured alike it is a single-vertex Kempe chain. We will write that such a vertex can be recoloured to refer to the exchange of such a chain.

We make repeated use of Lemma 9: two colourings are Kempe equivalent if they match.

Let C be a claw in G with vertex labels as in Figure 2. Note that in every 3-colouring of G, two of s, u and v are coloured alike, since s, u and v have a common neighbour. If some fixed pair of s, u and v is coloured alike by every 3-colouring of G, then every pair of colourings matches and we are done. So let α be a 3-colouring of G and assume that $\alpha(u) = \alpha(v) = 1$ and that there are colourings for which u and v have distinct colours, or, equivalently, colourings for which s has the same colour as either u or v. By Lemma 11, it is sufficient to find such a 3-colouring that is Kempe equivalent to α. Our approach is to divide the proof into a number of cases, and, in each case, start from α and make a number of Kempe changes until a colouring in which s agrees with either u or v is obtained. We will denote such a colouring ω to indicate a case is complete.

First some simple observations. If $\alpha(s) = 1$, then let $\omega = \alpha$ and we are done. So we can assume instead that $\alpha(s) = 2$ (and so, of course, $\alpha(w) = 3$). If it is possible to recolour one of u, v or s, then we can let ω be the colouring obtained. Thus we can assume now that each vertex of u, v and s has two neighbours that are not coloured alike.

For a colouring γ, vertex x, and colours a and b let $F^{ab}_{\gamma,x}$ denote the (a,b)-component at s under γ. We can assume that $F^{12}_{\alpha,s}$ contains both u and v as otherwise exchanging $F^{12}_{\alpha,s}$ results in a colouring in which s agrees with either u or v.

Let $N(u) = \{w, u_1, u_2\}$, $N(v) = \{w, v_1, v_2\}$, and $N(s) = \{w, s_1, s_2\}$. Note that the vertices $u_1, u_2, v_1, v_2, s_1, s_2$ are not necessarily distinct.

Case 1. $\alpha(u_1) \neq \alpha(u_2)$, $\alpha(v_1) \neq \alpha(v_2)$ and $\alpha(s_1) \neq \alpha(s_2)$.
So each of u, v and s has degree 1 in $F^{12}_{\alpha,s}$ and therefore $F^{12}_{\alpha,s}$ has at least one vertex of degree 3. Let x be the vertex of degree 3 in $F^{12}_{\alpha,s}$ that is closest to u and let α' be the colouring obtaining by recolouring x. Then u is not in $F^{12}_{\alpha',s}$ which can be exchanged to obtain ω.

Case 2. $\alpha(s_1) = \alpha(s_2)$.
Then $\alpha(s_1) = \alpha(s_2) = 1$ else ω can be obtained by recolouring s.

Subcase 2.1: $\alpha(u_1) = \alpha(u_2)$ or $\alpha(v_1) = \alpha(v_2)$.
The two cases are equivalent so we consider only the first. We have $\alpha(u_1) = \alpha(u_2) = 2$ else u is not in $F^{12}_{\alpha,s}$. Note that $F^{23}_{\alpha,s}$ consists only of s and w. If $F^{23}_{\alpha,s}$ is exchanged, u has three neighbours coloured 2, and can be recoloured to obtain ω (as u and s are both now coloured 3).

Subcase 2.2: $\alpha(u_1) \neq \alpha(u_2)$ and $\alpha(v_1) \neq \alpha(v_2)$.
We can assume that $\alpha(u_1) = \alpha(v_1) = 2$, and $\alpha(u_2) = \alpha(v_2) = 3$.

In this case, we take a slightly different approach. Let ω now be some fixed 3-colouring with $\omega(s) \in \{\omega(u), \omega(v)\}$.We show that $\alpha \sim_3 \omega$ by making Kempe changes from α until a colouring that matches ω (or a colouring obtained from ω by a Kempe change) is reached.

Let $\{a, b, c\} = \{1, 2, 3\}$. If $\omega(s_1) = \omega(s_2)$, then ω matches α (recall $\alpha(s_1) = \alpha(s_2)$ in this case). So assume that $\omega(s_1) = a$ and $\omega(s_2) = b$. Then $\omega(s) = c$, and we can assume, without loss of generality, that $\omega(w) = a$. Note that we can assume that $\omega(u) \neq \omega(v)$ else α and ω match and we are done. So, as u and v are symmetric under α, we can assume that $\omega(u) = b$ and $\omega(v) = c$. If $\omega(u_2) = a$ or $\omega(v_2) = a$, then, again, α and ω match (recall that $\alpha(w) = \alpha(u_2) = \alpha(v_2)$) so we assume otherwise (noting that this implies $\omega(u_2) = c$ and $\omega(v_2) = b$) and consider two cases. For convenience, we first illustrate our current knowledge of α and ω in Figure 4. (Though it is not pertinent in this case, we again observe that the six vertices of degree 1 in the illustration might not, in fact, be distinct.)

Subcase 2.2.1: $\omega(w) = a \in \{\omega(u_1), \omega(v_1)\}$.
Notice that $F^{23}_{\alpha,s}$ contains only s and w. If it is exchanged, then a colouring is obtained where w, u_1 and v_1 are coloured alike and this colouring matches ω.

Subcase 2.2.2: $\omega(w) = a \notin \{\omega(u_1), \omega(v_1)\}$.
So $\omega(u_1) = c$ and $\omega(v_1) = b$. Thus $F^{ab}_{\omega,w}$ contains only u and w, and the colouring obtained by its exchange matches α as w and v_1 are both coloured b.

Case 3. $\alpha(u_1) = \alpha(u_2)$, $\alpha(v_1) \neq \alpha(v_2)$, and $\alpha(s_1) \neq \alpha(s_2)$.
If $\alpha(u_1) = \alpha(w)$, then the three neighbours of u are coloured alike and it can be recoloured to obtain ω. So suppose $\alpha(u_1) = \alpha(u_2) = 2$. We may assume that $\alpha(s_1) = 1$, $\alpha(s_2) = \alpha(v_2) = 3$, and $\alpha(v_1) = 2$; see the illustration of Figure 5.

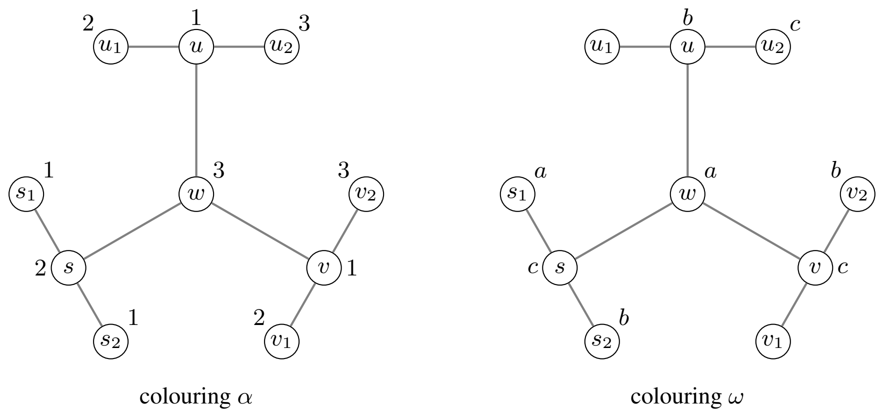

Fig. 4. The colourings of Subcase 2.2 of Lemma 5.

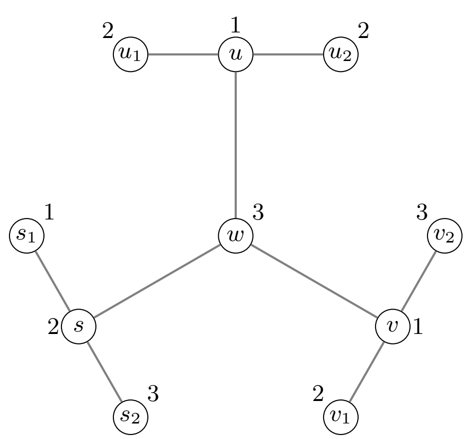

Fig. 5. The colouring α of Case 3 of Lemma 5.

We continue to assume that $F^{12}_{\alpha,s}$ contains u and v and note that s and v have degree 1 therein.

Subcase 3.1: $F^{12}_{\alpha,s}$ is not a path.
Let t be vertex of degree 3 closest to s in $F^{12}_{\alpha,s}$. Then t can be recoloured to obtain a colouring α' such that $F^{12}_{\alpha',s}$ does not contain v. Exchanging $F^{12}_{\alpha',s}$, we obtain ω.

Subcase 3.2: $F^{12}_{\alpha,s}$ is a path.
Note that $F^{12}_{\alpha,s}$ is a path from s to v through s_1 and u.

Subcase 3.2.1: F^{13}_{α,s_2} is a path from s_1 to s_2.
Note that $F^{13}_{\alpha,u} \neq F^{13}_{\alpha,s_2}$, since if $F^{13}_{\alpha,u}$ is a path, then u would be an endvertex coloured 1, implying $u = s_1$ and thus contradicting that C is a claw. As G is cubic, a vertex can belong to both $F^{12}_{\alpha,s}$ and F^{13}_{α,s_2} if it is an endvertex of one of them, and we note that s_1 is the only such vertex.

Let α' be the colouring obtained from α by the exchange of F^{13}_{α,s_2}. If $s \notin F^{12}_{\alpha',v}$, then let ω be the colouring obtained by the further exchange of $F^{12}_{\alpha',v}$.

Otherwise, $F^{12}_{\alpha',v} = F^{12}_{\alpha',s}$, s and v each have degree 1 therein, and we can assume it is a path (else, as in Subcase 3.1, there is a vertex of degree 3 that can be recoloured to obtain α'' and $F^{12}_{\alpha'',s}$ does not contain v and can be exchanged to obtain ω). We can also assume that $F^{12}_{\alpha',s}$ contains $F^{12}_{\alpha,s} \setminus \{s_1\}$: if not, then $F^{13}_{\alpha,s_2} \setminus \{s_1, s_2\} \cap F^{12}_{\alpha,v} \neq \emptyset$ (recall that $F^{12}_{\alpha,s}$ is a path from s to v through s_1 and u) but their common vertices would have degree 4. Thus, in particular, $F^{12}_{\alpha',s}$ contains u and the vertex t at distance 2 from s in $F^{12}_{\alpha,s}$.

As t is not an endvertex in $F^{12}_{\alpha',s}$, s_1 is its only neighbour coloured 3 under α'. So $F^{23}_{\alpha',w}$ contains four vertices: w, s, s_1 and t. Let α'' be the colouring obtained from α' by the exchange of $F^{23}_{\alpha',w}$. If $t \notin \{u_1, u_2\}$, then u has three neighbours with colour 2 with α'' and so can be recoloured to obtain ω. Otherwise the conditions of Case 1 are now met.

Subcase 3.2.2: F^{13}_{α,s_2} is not a path from s_1 to s_2.
If $s_1 \notin F^{13}_{\alpha,s_2}$, then the exchange of F^{13}_{α,s_2} gives a colouring in which s_1 and s_2 are coloured alike (the colour of s is not affected by the exchange and either both or neither of u and v change colour). Thus Case 2 can now be used.

So we can assume that $s_1 \in F^{13}_{\alpha,s_2}$ has degree 1 in F^{13}_{α,s_2} (recall that s_1 has degree 2 in $F^{12}_{\alpha,s}$). If s_2 has degree 2 in F^{13}_{α,s_2}, then $F^{23}_{\alpha,s}$ contains only w, s and s_2. If it is exchanged, u has three neighbours with colour 2 and can be recoloured to ω.

Thus s_1 and s_2 both have degree 1 in F^{13}_{α,s_2}. Let x be the vertex of F^{13}_{α,s_2} closest to s_2. Then x can be recoloured to obtain a colouring α' such that F^{13}_{α',s_2} does not contain s_1. Exchanging F^{13}_{α',s_2} again takes us to Case 2. This completes Case 3.

By symmetry, we are left to consider the following case to complete the proof of the lemma.

Case 4. $\alpha(u_1) = \alpha(u_2)$, $\alpha(v_1) = \alpha(v_2)$, and $\alpha(s_1) \neq \alpha(s_2)$.
If $\alpha(v_1) = \alpha(v_2) = 3$, then v can be recoloured to obtain ω. So we can assume that $\alpha(v_1) = \alpha(v_2) = 2$, and, similarly, that $\alpha(u_1) = \alpha(u_2) = 2$. We can also assume that $F^{23}_{\alpha,s}$ is a path since otherwise the vertex of degree 3 closest to s can be recoloured. Define $S = \{u_1, u_2, v_1, v_2\}$. We distinguish two cases.

Subcase 4.1: $|S \cap F^{23}_{\alpha,s}| \geq 2$.
As $F^{23}_{\alpha,s}$ is a path and w is an endvertex, one vertex of S, say v_1, has degree 2 in $F^{23}_{\alpha,s}$. Consider $F^{13}_{\alpha,w}$: it consists only of vertices w, u, and v. After it is exchanged, v_1 has three neighbours with colour 3 and recolouring v_1 allows us to apply Case 3.

Subcase 4.2: $|S \cap F^{23}_{\alpha,s}| \leq 1$.
It follows, without loss of generality, that $\{u_1, u_2\} \cap F^{23}_{\alpha,s} = \emptyset$. Exchange F^{23}_{α,u_1} and F^{23}_{α,u_2} (which might be two distinct components or just one) to obtain a colouring α'. As $w \in F^{23}_{\alpha',s}$ (and hence $w \notin F^{23}_{\alpha,u_1} \cup F^{23}_{\alpha,u_2}$), every neighbour of u is coloured 3 and it can be recoloured to obtain ω. This completes Case 4 and the proof of Lemma 5. □

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