Chains of subsemigroups

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Abstract

We investigate the maximum length of a chain of subsemigroups in various classes of semigroups, such as the full transformation semigroups, the general linear semigroups, and the semigroups of order-preserving transformations of finite chains. In some cases, we give lower bounds for the total number of subsemigroups of these semigroups. We give general results for finite completely regular and finite inverse semigroups. Wherever possible, we state our results in the greatest generality; in particular, we include infinite semigroups where the result is true for these.

The length of a subgroup chain in a group is bounded by the logarithm of the group order. This fails for semigroups, but it is perhaps surprising that there is a lower bound for the length of a subsemigroup chain in the full transformation semigroup which is a constant multiple of the semigroup order.

1 The definition

Let S be a semigroup. A collection of subsemigroups of S is called a *chain* if it is totally ordered with respect to inclusion. In this paper we consider the problem of finding the longest chain of subsemigroups in a given semigroup. From among several conflicting candidates for the definition, we define the *length* of a semigroup S to be the largest number of non-empty subsemigroups of S in a chain minus 1; this is denoted $l(S)$. There are several reasons for choosing this definition rather than another, principally: several of the formulae and proofs we will present are more straightforward with this definition (especially that in Proposition 3.1, which is the basis for several of our results); when applied to a group our definition of length coincides with the definition in the literature (for more details see Section 2). There are some negative aspects of this definition too. For example, if S is a null semigroup (the product of every pair of elements equals 0), then $l(S) = |S| - 1$; or if S is empty, then $l(S) = -1$. Our definition of length also differs from the usual order-theoretic definition.

The paper is organised as follows: we review some known results for groups in Section 2; in Section 3 we present some general results about the length of a semigroup and its relationship to the lengths of its ideals; in Section 4 we give a lower bound for the length of the full transformation semigroup on a finite set, and consider the asymptotic behaviour of this bound; in Sections 5 and 6 we perform an analysis similar to that for the full transformation semigroup for the semigroup of all order-preserving transformations, and for the general linear monoid; in Sections 7 and 8 we provide a formula for the length of an arbitrary finite inverse or completely regular semigroup in terms of the lengths of its maximal subgroups, and its numbers of $\mathscr{L}$- and $\mathscr{R}$-classes. In Section 9, as consequences of our results about the full transformation monoid, we give some bounds on the number of subsemigroups, and the maximum rank of a subsemigroup, of the full transformation monoid.

Note that, with the exception of Proposition 3.1, all semigroups we consider are finite.

2 Subgroup chains in groups

In this section we give a brief survey of a well-understood case, that of groups. We will use some of the results in this section later in the paper.

The question of the length $l(G)$ of the longest chain of subgroups in a finite group has been considered for some time. The base and strong generating set algorithm for finite permutation groups, due to Charles Sims, involves constructing a chain of point stabilisers. László Babai [2] pointed out that the length of such a chain in any action of G is bounded above by $l(G)$, so this parameter is important in the complexity analysis of the algorithm. Babai gave a bound, linear in n, for the length of the symmetric group S_n; the precise value of $l(S_n)$ was computed by Cameron, Solomon and Turull [4]: values are given in sequence A007238 in the On-Line Encyclopedia of Integer Sequences [24].

Theorem 2.1. $l(S_n) = \lceil 3n/2 \rceil - b(n) - 1$, *where* $b(n)$ *is the number of ones in the base* 2 *expansion of* n.

It is easy to see that, if N is a normal subgroup of G, then $l(G) = l(N) + l(G/N)$. (In one direction, there is a chain of length $l(N) + l(G/N)$ passing through N; in the other direction, if H and K are subgroups of G with $H < K$, then either $H \cap N < K \cap N$

or $HN/N < KN/N$, so in any subgroup chain in G, each step involves taking a step in either N or G/N.) So, for any group G, we obtain $l(G)$ by summing the lengths of the composition factors of G, and the problem is reduced to evaluating the lengths of the finite simple groups. The result cited in the preceding paragraph deals with the alternating groups. The problem was further considered by Seitz, Solomon and Turull [30, 23]. It is not completely solved for all finite simple groups, but we can say that it is reasonably well understood. In what follows, we will regard a formula containing $l(G)$ for some group G as "known".

We will use a special case of the following (known) result later. The function $\Omega(n)$ gives the number of prime divisors of n, counted with their multiplicities; equivalently, the number of prime power divisors of n.

Proposition 2.2. *For any group G, $l(G) \leq \Omega(|G|)$. Equality holds if and only if each non-abelian composition factors of G is a 2-transitive permutation group of prime degree in which the point stabiliser H also satisfies $l(H) = \Omega(|H|)$. In particular, any soluble group G satisfies $l(G) = \Omega(|G|)$.*

Remark It follows from the Classification of Finite Simple Groups that the non-abelian simple groups with this property are $\mathrm{PSL}(2, 2^a)$ where $2^a + 1$ is a Fermat prime, $\mathrm{PSL}(2, 7)$, $\mathrm{PSL}(2, 11)$, $\mathrm{PSL}(3, 3)$ and $\mathrm{PSL}(3, 5)$.

Proof. It is clear from Lagrange's Theorem that $l(G) \leq \Omega(|G|)$. Equality holds if and only if it holds in every composition factor.

A non-abelian finite simple group with this property has a subgroup of prime index, and so acts as a transitive permutation group of prime degree. By Burnside's theorem, it is 2-transitive. The rest of the proposition is clear. □

Since a subsemigroup of a finite group is a subgroup, these results solve particular cases of our general problem.

3 Generalities about subsemigroup chains

In contrast to groups, where the length of a chain of subgroups is at most the logarithm of the group order, a semigroup may have a chain whose length is equal one less than its order. A null semigroup of any order has this property, as does any semigroup which is not generated by a proper subset (i.e. any semigroup S whose $\mathscr{J}$-classes are semigroups of left or right zeros, $S/\mathscr{J}$ is a chain, and where every element acts as a two sided identity on elements which are lower in the $\mathscr{J}$-order).

We note that, for the sake of the simplicity of several statements in the paper, we will follow the convention that the empty set is a semigroup and that an ideal of a semigroup may be empty.

If S is a semigroup and T is a subsemigroup of S, then $l(T) \leq l(S)$. Similarly, if ρ is a congruence on S, then, since subsemigroups are preserved by homomorphisms, $l(S/\rho) \leq l(S)$.

Let I be an ideal of the semigroup S and let S/I denote the *Rees quotient* of S by I, i.e. the semigroup with the elements of $S \setminus I$ together with an additional zero $0 \notin S$; the multiplication is given by setting $xy = 0$ if the product in S lies in I, and letting it have its value in S otherwise. With this definition the Rees quotient of any semigroup S (even the empty one) by the empty ideal is S with an additional zero $0 \notin S$ adjoined. In particular, the Rees quotient of the empty semigroup by the empty ideal is the trivial semigroup with 1 element. As noted above, in the following result we do not assume any finiteness condition.

Proposition 3.1 (cf. Lemma 1 in [9]). *Let S be a semigroup and let I be an ideal of S. Then $l(S) = l(I) + l(S/I)$.*

Proof. If I is empty, then $l(I) = -1$ and $l(S/I) = l(S) + 1$ since S/I is S with a new zero element adjoined. Hence the conclusion of the lemma holds in this case.

Suppose that I is not empty. We start by showing that $l(S) \geq l(I) + l(S/I)$. Suppose that $\{U_\alpha : \alpha \text{ an ordinal},\ \alpha < l(I)\}$ and $\{V_\alpha : \alpha \text{ an ordinal},\ \alpha < l(S/I)\}$ are chains of non-empty proper subsemigroups of I and S/I, respectively, such that $U_\alpha < U_{\alpha+1}$ and $V_\beta < V_{\beta+1}$ for all $\alpha < l(I)$ and $\beta < l(S/I)$. Then

$$W_\alpha = \begin{cases} U_\alpha & \text{if } \alpha < l(I) \\ (V_\beta \setminus \{0\}) \cup I & \text{if } \alpha = l(I) + \beta < l(I) + l(S/I) \end{cases}$$

is a chain of $l(I) + l(S/I)$ proper subsemigroups of S, and so $l(S) \geq l(I) + l(S/I)$.

Suppose that $\mathcal{C} = \{U_\alpha : \alpha \text{ an ordinal},\ \alpha < l(S)\}$ is a chain of non-empty proper subsemigroups such that $U_\alpha < U_{\alpha+1}$ for all $\alpha < l(S)$. We will show that we may assume, without loss of generality, that for all $\alpha < l(S)$ either:

$$\big(U_{\alpha+1} \setminus U_\alpha\big) \cap I = \emptyset \qquad \text{or} \qquad U_{\alpha+1} \setminus U_\alpha \subseteq I. \tag{1}$$

Since the union of a subsemigroup and an ideal is a semigroup, it follows that $U_\alpha \cup (U_{\alpha+1} \cap I)$ is a subsemigroup of S. Hence

$$U_\alpha \leq U_\alpha \cup (U_{\alpha+1} \cap I) \leq U_{\alpha+1}.$$

If $l(S)$ is finite, then either $U_\alpha \cup (U_{\alpha+1} \cap I) = U_\alpha$ or $U_\alpha \cup (U_{\alpha+1} \cap I) = U_{\alpha+1}$. Therefore $\big(U_{\alpha+1} \setminus U_\alpha\big) \cap I = \emptyset$ or $U_{\alpha+1} \setminus U_\alpha \subseteq I$, respectively.

Suppose that $l(S)$ is infinite. Then replacing any subchain $U_\alpha < U_{\alpha+1}$ of $\mathcal{C}$ which fails (1) by $U_\alpha < U_\alpha \cup (U_{\alpha+1} \cap I) < U_{\alpha+1}$ we obtain another chain of length $l(S)$. Furthermore, $U_\alpha \cup (U_{\alpha+1} \cap I) < U_{\alpha+1}$ and $U_\alpha < U_\alpha \cup (U_{\alpha+1} \cap I)$ satisfy (1).

Assume without loss of generality that $\mathcal{C}$ satisfies (1). Note that $\{U_\alpha \cap I : \alpha < l(S)\}$ is a chain of non-empty subsemigroups of I (although $U_\alpha \cap I$ may equal $U_{\alpha+1} \cap I$ for some $\alpha < l(S)$) and $\{U_\alpha/I : \alpha < l(S)\}$ is a chain of non-empty proper subsemigroups of S/I. By (1), for all $\alpha < l(S)$ either

$$U_\alpha \cap I = U_{\alpha+1} \cap I \quad \text{and} \quad U_\alpha/I < U_{\alpha+1}/I$$

or

$$U_\alpha/I = U_{\alpha+1}/I \quad \text{and} \quad U_\alpha \cap I < U_{\alpha+1} \cap I$$

Therefore $l(S) \leq l(I) + l(S/I)$. □

If S is a semigroup and $x, y \in S$, then we write $x \mathscr{J} y$ if the principal (two-sided) ideal S^1xS^1 generated by x equals the ideal S^1yS^1 generated by y. The relation $\mathscr{J}$ is an equivalence relation called *Green's $\mathscr{J}$-relation*, and the equivalence classes are called *$\mathscr{J}$-classes*. If J_1 and J_2 are $\mathscr{J}$-classes of a semigroup, then we write $J_1 \leq_{\mathscr{J}} J_2$ if $S^1xS^1 \subseteq S^1yS^1$ for any $x \in J_1$ and $y \in J_2$. It is straightforward to verify that $\leq_{\mathscr{J}}$ is a partial order on the $\mathscr{J}$-classes of S.

If J is a $\mathscr{J}$-class of a finite semigroup S, then its *principal factor* J^* is the semigroup with elements $J \cup \{0\}$ ($0 \notin J$) and the product xy of $x, y \in J$ defined to be its value in S if $x, y, xy \in J$ and 0 otherwise. In other words, if J is not minimal, then J^* is the Rees quotient of the principal ideal generated by any element of J by the ideal consisting of those elements in S whose $\mathscr{J}$-classes are not greater than J under $\leq_{\mathscr{J}}$. If J is minimal, then J is a subsemigroup of S, and J^* is not isomorphic to the quotient in the previous sentence (which is isomorphic to J), since J^* has one more element. Sometimes, in the literature, the principal factor of the minimal $\mathscr{J}$-class J of a finite semigroup is defined to be J itself, but we do not follow this convention.

A semigroup S is *regular* if for every $x \in S$ there is $y \in S$ such that $xyx = x$.

Lemma 3.2. *Let S be a finite regular semigroup and let $J_1, J_2, \ldots, J_m$ be the $\mathscr{J}$-classes of S. Then $l(S) = l(J_1^*) + l(J_2^*) + \cdots + l(J_m^*) - 1$.*

Proof. Assume without loss of generality that J_1 is maximal in the partial order of $\mathscr{J}$-classes on S. It follows that $I = S \setminus J_1$ is an ideal. Hence by Proposition 3.1 it follows that $l(S) = l(I) + l(S/I)$. If $I = \emptyset$, then $S = J_1$, and so $l(S) = l(J_1) = l(J_1^*) - 1$, in which case we are finished.

Suppose that $I \neq \emptyset$. Then S/I is isomorphic to J_1^* and so $l(S) = l(I) + l(J_1^*)$. Since S is regular and I is an ideal, it follows by Proposition A.1.16 in [22] that I is regular and the $\mathscr{J}$-classes of I are $J_2, J_3, \ldots, J_m$. Therefore repeating the argument in the previous paragraph a further $m-2$ times, we obtain

$$l(S) = l(J_1^*) + l(J_2^*) + \cdots + l(J_m^*) - 1,$$

as required. □

We conclude this section with a simple application of the results in this, and the previous, sections.

Proposition 3.3. *Let S be a semigroup generated by a single element s and let $m, n \in \mathbb{N}$ be the least numbers such that $s^{m+n} = s^m$. Then $l(S) = m + \Omega(n) - 1$, where $\Omega(n)$ is the number of prime power divisors of n.*

Proof. The $\mathscr{J}$-classes of S are

$$\{\{s\},\{s^2\},\ldots,\{s^{m-1}\},\{s^m,s^{m+1},\ldots,s^{m+n-1}\}\},$$

where the non-singleton class is the cyclic group C_n with n elements. By repeatedly applying Proposition 3.1,

$$l(S) = m + l(C_n) - 1,$$

and $l(C_n) = \Omega(n)$ by Proposition 2.2. □

4 The full transformation semigroup

4.1 Long chains

The full transformation semigroup, denoted T_n, consists of all functions with domain and codomain $\{1,\ldots,n\}$ under the usual composition of functions. Clearly $|T_n| = n^n$. In this section, we will prove the following theorem.

Theorem 4.1.

$$l(T_n) \geq \mathrm{e}^{-2}n^n - 2\mathrm{e}^{-2}(1-\mathrm{e}^{-1})n^{n-1/3} - o(n^{n-1/3}).$$

The *rank* of an element of T_n is the cardinality of its image. The $\mathscr{J}$-classes of T_n are the sets J_k of all elements of rank k. Since T_n is regular, Lemma 3.2 implies that $l(T_n)$ is the sum of the lengths of the principal factors J_k^* of its $\mathscr{J}$-classes, minus 1.

A element f of rank k in T_n has a kernel, which is the partition of $\{1,\ldots,n\}$ into its pre-images (hence with k parts), and an image, a k-subset of $\{1,\ldots,n\}$. The set of all maps with given kernel Q and given image A is an $\mathscr{H}$-class in the semigroup T_n, and has cardinality $|A|!$. So the number of maps of rank k is

$$N(n,k) = S(n,k)\binom{n}{k}k!,$$

where $S(n,k)$ is the Stirling number of the second kind.

If f_1 and f_2 are two maps of rank k with kernels Q_1, Q_2 and images A_1, A_2 respectively, then f_1f_2 has rank k if A_1 is a transversal for the partition Q_2, and smaller rank otherwise. So, if P is a set of k-partitions of $\{1,\ldots,n\}$ (partitions with k parts), and S a set of k-subsets, with the property that no element of S is a transversal for any element of P, then the set of maps with kernel in P and image in S is a null semigroup in J_k^*. We call a set (P,S) with this property a *league*, and define its *content* to be $|P|\cdot|S|$.

If a league (P,S) of rank k has content m, then the set of all maps f with kernel in P and image in S has the property that the product of any two of its elements has rank smaller than k; so this set, together with zero, forms a null subsemigroup of the principal factor J_k^* of order $1 + m\cdot k!$. This semigroup has a chain of subsemigroups of length equal to one less than its order. Combining these observations with Lemma 3.2, we obtain the following result.

Proposition 4.2. *Let $F(n,k)$ be the largest content of a league of rank k on $\{1,\ldots,n\}$. Then*

$$l(T_n) \geq \sum_{k=1}^{n} F(n,k)k! - 1.$$

We prove Theorem 4.1 by a suitable choice of leagues, as follows. Choose one element of the set $\{1,\ldots,n\}$, say n; let P be the set of all k-partitions having n as a singleton part, and S the set of all k-subsets not containing n. Then clearly (P,S) is a league, and its content is

$$\binom{n-1}{k} S(n-1,k-1).$$

Lemma 4.3. *The expected rank $E(n)$ of a transformation in T_n chosen uniformly at random satisfies*

$$E(n) = (1-\mathrm{e}^{-1})n + O(1).$$

Moreover, the standard deviation $\sigma(n)$ of the rank satisfies

$$\sigma(n) \leq \sqrt{\mathrm{e}^{-1} - 2\mathrm{e}^{-2}}\sqrt{n+1} \tag{2}$$

for n large enough.

Proof. The exact values of the expectation $E(n)$ and of the variance $V(n)$ are given in [16], where their asymptotic estimates are also given. The expected rank is given by

$$E(n) = n\left[1 - \left(1 - \frac{1}{n}\right)^n\right] = (1-\mathrm{e}^{-1})n + O(1).$$

For the variance, we have

$$\begin{aligned}
V(n) &= n\left[\left(1-\frac{1}{n}\right)^n - \left(1-\frac{2}{n}\right)^n\right] + n^2\left[\left(1-\frac{2}{n}\right)^n - \left(1-\frac{1}{n}\right)^{2n}\right] \\
&= n\left[\mathrm{e}^{-1}\left(1 - \frac{1}{2n} + o(n^{-1})\right) - \mathrm{e}^{-2}\left(1 - \frac{2}{n} + o(n^{-1})\right)\right] \\
&\quad + n^2\left[\mathrm{e}^{-2}\left(1 - \frac{2}{n} - \frac{2}{3n^2} + o(n^{-2})\right) - \mathrm{e}^{-2}\left(1 - \frac{1}{n} - \frac{1}{6n^2} + o(n^{-2})\right)\right] \\
&= n(\mathrm{e}^{-1} - 2\mathrm{e}^{-2}) + \frac{3\mathrm{e}^{-2} - \mathrm{e}^{-1}}{2} + o(1).
\end{aligned}$$

Since $\frac{3\mathrm{e}^{-2}-\mathrm{e}^{-1}}{2} < \mathrm{e}^{-1} - 2\mathrm{e}^{-2}$, we have $V(n) \leq (\mathrm{e}^{-1} - 2\mathrm{e}^{-2})(n+1)$ for n large enough. ☐

We now return to the proof of Theorem 4.1. Let $\tau = \sqrt{\mathrm{e}^{-1} - 2\mathrm{e}^{-2}}$ and $K = \{k : |k - E(n-1)| < n^{1/6}\tau n^{1/2}\}$; we then have

$$n - k \geq n - E(n-1) - \tau n^{2/3} = \mathrm{e}^{-1}n - \tau n^{2/3} - o(n^{2/3})$$

for any $k \in K$. Also, for all n large enough, K contains all k such that $|k - E(n-1)| < n^{1/6}\sigma(n-1)$. Chebyshev's inequality then yields

$$\sum_{k \in K} N(n-1, k-1) \geq (n-1)^{n-1}\left(1 - n^{-1/3}\right) \geq \mathrm{e}^{-1} n^{n-1}(1 - n^{-1/3}).$$

Therefore, we obtain an overall chain of length at least

$$\begin{aligned} \sum_{k \in K} \binom{n-1}{k} S(n-1, k-1) k! &= \sum_{k \in K} (n-k) N(n-1, k-1) \\ &\geq (\mathrm{e}^{-1} n - \tau n^{2/3} - o(n^{2/3}))\mathrm{e}^{-1} n^{n-1}\left(1 - n^{-1/3}\right) \\ &= \mathrm{e}^{-2} n^n - 2\mathrm{e}^{-2}(1 - \mathrm{e}^{-1})(n^{n-1/3}) - o(n^{n-1/3}). \end{aligned}$$

4.2 Combinatorial results

The question of finding $F(n,k)$, the largest possible content of a league (P, S), where P is a set of k-partitions and S a set of k-subsets of $\{1, \ldots, n\}$, is purely combinatorial, and maybe of some interest. We give here some general bounds and some exact values.

We showed above that

$$F(n,k) \geq \binom{n-1}{k} S(n-1, k-1). \tag{3}$$

Another strategy gives a different bound, which is better for small k: for $n \geq 2$, we have

$$F(n,k) \geq \binom{n-2}{k-2} S(n-1, k). \tag{4}$$

This is proved by letting S consist of all k-sets containing 1 and 2, and P the set of all k-partitions not separating 1 and 2. Further improvements are possible.

In the extreme cases, we can evaluate $F(n,k)$ precisely, as follows.

Proposition 4.4. (a) *$F(n, 1) = 0$.*

(b) *For $n > 3$, $F(n, 2) = 3(2^{n-3} - 1)$, and a pair (P, S) meets the bound if and only if S is the set of edges of a triangle T and P is the set of 2-partitions with T contained in a part.*

(c) *$F(n, n-1) = s^2(2s-1)$, $s^2(2s+1)$, or $s(s+1)(2s+1)$ when $n = 3s$, $3s+1$, or $3s+2$, respectively, with $s \geq 1$.*

(d) *$F(n, n) = 0$.*

Proof. In the first and last case, the proof is trivial and ommited.

(b) Consider the case $k = 2$. Then S is the set of edges of a graph, and the partitions in P do not cross edges, so each part of such a partition is a union of connected components of a graph. We are going to make moves which will all increase $|S| \cdot |P|$.

First, by including edges so that each component is a complete graph, and by including all partitions whose parts are unions of components, we do not decrease $|S| \cdot |P|$. So we may assume that this is the case. Thus, if the components have sizes $a_1, \ldots, a_r$, then

$$|S| \cdot |P| = \left(\sum_{i=1}^{r} \binom{a_i}{2} \right) (2^{r-1} - 1),$$

where $\sum_{i=1}^{r} a_i = n$.

Next we claim that, if $a_i \geq 4$, then we can increase $|S| \cdot |P|$ by replacing the part of size a_i by two parts of sizes 1 and $a_i - 1$. For we increase r by 1, so the second factor more than doubles; so it will suffice to show that

$$\binom{a_i - 1}{2} \geq \frac{1}{2} \binom{a_i}{2},$$

since then the first factor will be at least half of its previous value. Now the displayed inequality is equivalent to $a_i \geq 4$.

Also, splitting a part of size 2 into two parts of size 1 more than doubles the second factor and reduces the first factor by 1. So this is also an improvement (except in the case where the resulting partition has all $a_i = 1$, when the product is zero).

So we can continue the process, increasing the objective function, until all a_i are equal to 1 or 3.

If two a_i are equal to 3, then replacing them by 5 and 1 improves the sum, since

$$2\binom{3}{2} < \binom{5}{2}.$$

Then we can replace the 5 by three parts of sizes 3, 1 and 1, by the preceding argument.

So we end with a part of size 3 and $n - 3$ parts of size 1, giving the value $3(2^{n-3} - 1)$ claimed, and also the extremal configuration described.

(c) Now consider the case $k = n - 1$. Let (P, S) satisfy the conditions. Identify each element of S by the single point it omits, and each element of P by the pair of points (or edge) in the same class; then the condition asserts that no point of P is on an edge of S. So to optimise we want P to be a complete graph on, say, m points, and S to consist of the remaining $n - m$ points. Then $|S| \cdot |P| = m(m-1)(n-m)/2$.

This is maximised when m is roughly $2n/3$; a detailed but elementary calculation gives the stated result. □

Table 1 gives some further exact values, computed with the GAP [12] package GRAPE [29], except the value of $F(7, 4)$, which was computed by Chris Jefferson using the Minion [11] constraint satisfaction solver. Each table entry also gives a lower bound, which is the maximum of the values in (3) and (4). The column headed "Total" multiplies $F(n, k)$ (or the lower bound) by $k!$, and sums over k. The entries in columns $k = 1$ and $k = n$ are zero and have been omitted.

A kind of dual problem, which is also connected to the theory of transformation semigroups (though not to the questions considered here) is the following:

n	Total	$k=2$	3	4	5	6
2	$0,0$					
3	$2,2$	$1,1$				
4	$24,18$	$3,3$	$3,2$			
5	$330,326$	$9,7$	$28,28$	$6,6$		
6	$5382,5130$	$21,15$	$150,150$	$125,125$	$12,10$	
7	$98250,93782$	$45,31$	$760,620$	$1350,1350$	$390,390$	$20,15$

Table 1: Values and bounds for $F(n,k)$

> Given n and k, what is the smallest size of a collection of k-subsets of $\{1,\dots,n\}$ which contains a transversal to every k-partition of $\{1,\dots,n\}$?

For some asymptotic results about this question, see [3]; for an application to semigroups, in the special case where there is a permutation group G such that every orbit of G on k-sets has this property, see [1].

5 Order-preserving transformations

A transformation $f \in T_n$ is *order-preserving* if $(i)f < (j)f$ whenever $i < j$. In this section we consider O_n, the semigroup of all order-preserving transformations of $\{1,\dots,n\}$. It is shown in [17], for example, that $|O_n| = \binom{2n-1}{n}$.

We will denote by $F^*(n,k)$ denote the maximum content of a league (P,S) where P consists of k-partitions corresponding to kernels of order-preserving transformations, and S is an k-element subset of $\{1,2,\dots,n\}$.

Proposition 5.1.

$$l(O_n) \geq \binom{2n-3}{n} - 1 = \frac{(n-1)(n-2)}{(2n-1)(2n-2)}|O_n| - 1.$$

Note that this lower bound is asymptotically $|O_n|/4$.

Proof. It is well known that each $\mathscr{H}$-class in O_n is a singleton. For any given value of the rank k, there are $\binom{n}{k}$ choices for the image of a transformation in O_n, and $\binom{n-1}{k-1}$ choices for its kernel, which must be a partition of $\{1,\dots,n\}$ into k intervals [10] – this is because we specify such a partition by giving the $k-1$ points which divide the interval appropriately. Therefore, the number of transformations of rank k in O_n is given by

$$N^*(n,k) = \binom{n}{k}\binom{n-1}{k-1}.$$

We can apply the same strategy as in T_n in order to obtain long chains of subsemigroups in O_n. Let S be the set of all k-subsets of $\{1,\dots,n\}$ not containing n and let P be the set of partitions of $\{1, \ldots, n\}$ into k intervals such that the last interval is $\{n\}$. We then have that $|S| = \binom{n-1}{k}$ and $|P| = \binom{n-2}{k-2} = \binom{n-2}{n-k}$ and so

$$F^*(n,k) \geq \binom{n-1}{k}\binom{n-2}{n-k}. \tag{5}$$

Hence we have a chain of length

$$\sum_{k=1}^{n} \binom{n-1}{k}\binom{n-2}{n-k} - 1 = \binom{2n-3}{n} - 1,$$

using the Vandermonde convolution: $\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$. □

As we did for the full transformation monoid, in the extreme cases, we can evaluate $F^*(n,k)$ precisely, as follows.

Proposition 5.2. (a) *$F^*(n,1) = 0$.*

(b)

$$F^*(n,2) = \max\left\{ \begin{array}{l} \frac{1}{2}(n - \lfloor r^* \rfloor + 1)(n - \lfloor r^* \rfloor)(\lfloor r^* \rfloor - 1), \\ \frac{1}{2}(n - \lceil r^* \rceil + 1)(n - \lceil r^* \rceil)(\lceil r^* \rceil - 1) \end{array} \right\}$$

where $r^* = \left(2(n+1) - \sqrt{(n+1)^2 - 3n}\right)/3$.

(c) $F^*(n, n-1) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil$.

(d) $F^*(n,n) = 0$.

The bound in (b) is asymptotically $(2/27)n^3$; that in (c), $n^2/4$.

Proof. The proofs are very similar to the case of arbitrary leagues; as such, we shall use a similar notation.

(b) Again, we can represent S as a graph and each part of any partition in P is a union of connected components of that graph. We can still assume that S forms a union of r cliques, of cardinalities $a_1, \ldots, a_r$. However, for a graph with r connected components, there are at most $r-1$ possible choices for a partition in P, with equality if and only if the vertex set of each connected component is an interval. Therefore, the maximum content of a league with partitions into intervals is given by

$$\max_{1 \leq r \leq n} \max_{a_1, \ldots, a_r} \left\{ (r-1) \sum_{i=1}^{r} \frac{a_i(a_i - 1)}{2} \right\},$$

where the inner maximum is taken over all $a_1, \ldots, a_r$ such that $a_i \geq 1$ for all i and $\sum_{i=1}^{r} a_i = n$. This inner maximum is achieved for $a_1 = \ldots = a_{r-1} = 1$, $a_r = n - r + 1$ and is equal to $\frac{1}{2}(n - r + 1)(n - r)(r - 1)$. Maximising this polynomial gives the result.

(c) Again, we can represent S as a set of m points and P as a graph on the remaining $n-m$ points. This time, P can only contain edges of the form $\{i, i-1\}$ for any i such that neither i nor $i-1$ are amongst the m points of S. Hence P is a disjoint union of paths with at most $n-m-1$ edges, which is achieved if the points of S are 1 up to m and P is the path from $m+1$ to n. Together, we obtain a content of $m(n-m-1)$, maximised for $m = \lfloor (n-1)/2 \rfloor$ or $m = \lceil (n-1)/2 \rceil$. $\square$

Table 2 gives some values for the function $F^*(n,k)$ giving the maximum content of a league where the parts of the partitions are intervals, together with the lower bound in (5). Again, the zeros for $k=1$ and $k=n$ are omitted.

n	Total	$k=2$	3	4	5	6
2	$0,0$					
3	$1,1$	$1,1$				
4	$5,5$	$3,3$	$2,2$			
5	$22,21$	$6,6$	$12,12$	$4,3$		
6	$88,84$	$12,10$	$40,40$	$30,30$	$6,4$	
7	$345,330$	$20,15$	$100,100$	$150,150$	$66,60$	$9,5$

Table 2: Values and bounds for $F^*(n,k)$ in the monoid of order-preserving transformations.

6 The general linear semigroup

For q a prime power and n a positive integer, let $\mathrm{GLS}(n,q)$ denote the semigroup of all linear maps on the n-dimensional vector space V over the Galois field $\mathrm{GF}(q)$ of order q. We have $|\,\mathrm{GLS}(n,q)| = q^{n^2}$, since the linear maps are representable as $n \times n$ matrices.

Our technique here resembles that in the case of the full transformation semigroup. For $1 \le k \le n$, the set of linear maps of rank at most k forms an ideal, so we can analyse the principal factors.

One important difference is that the structure is far more top-heavy. Indeed, the group $\mathrm{GL}(n,q)$ of maps of full rank n contains a non-zero proportion of the whole semigroup.

Proposition 6.1. *Given q, there is a constant $c(q)$, with $0 < c(q) < 1$, so that*

$$\lim_{n\to\infty} \frac{\mathrm{GL}(n,q)}{\mathrm{GLS}(n,q)} = c(q).$$

Proof.

$$\begin{aligned} |\operatorname{GL}(n,q)| &= \prod_{k=1}^{n}(q^n - q^{n-k}) \\ &= q^{n^2}\prod_{k=1}^{n}(1-q^{-k}) \\ &\geq |\operatorname{GLS}(n,q)|\prod_{k\geq 1}(1-q^{-k}). \end{aligned}$$

The infinite product converges to a limit $c(q) > 0$. Euler's Pentagonal Numbers Theorem [15, Theorem 4.1.3] gives

$$c(q) = \sum_{k\in\mathbb{Z}}(-1)^k q^{-k(3k-1)/2} = 1 - q^{-1} - q^{-2} + q^{-5} + q^{-7} - q^{-12} - \cdots,$$

a form handy for calculation. For example, $c(2) = 0.288788095\ldots$. In fact, $c(q)$ is an evaluation of Jacobi's theta-function. □

The other main difference here is that the kernel of a linear map of rank k is the partition of the vector space into cosets of a $(n-k)$-dimensional subspace U (the "kernel" of the map in the usual sense of linear algebra), and a k-dimensional subspace W is a transversal for the kernel partition if and only if $U \cap W = \{0\}$. So the linear analogue of a league is a pair (P, S), where P is a set of $(n-k)$-dimensional subspaces and S a set of k-dimensional subspaces such that, for all $U \in P$ and $W \in S$, we have $U \cap W \neq \{0\}$. The simplest construction of a league is to take an $(n-1)$-dimensional subspace H of V, and to take S and P to consist of all subspaces of the appropriate dimension contained in H; or dually, take a 1-dimensional subspace K of V, and to take S and P to be all the subspaces of the appropriate dimension containing K.

For $1 \leq k \leq n$, the number of maps of rank k is

$$\left(\begin{bmatrix} n \\ k \end{bmatrix}_q\right)^2 |\operatorname{GL}(k,q)|.$$

Here

$$\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the Gaussian coefficient, the number of k-dimensional subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$. This coefficient is a monic polynomial in q of degree $k(n-k)$ with non-negative integer coefficients, so is at least $q^{k(n-k)}$. Using the fact that $|\operatorname{GL}(k,q)| \geq c(q)q^{k^2}$, we see that the number of maps of rank $k = n - d$ is at least $c(q)q^{n^2-d^2}$. So the largest principal factors are at the top.

The league just described in the principal factor of rank k contains

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

pairs. We have

$$\begin{aligned} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \Big/ \left(\begin{bmatrix} n \\ k \end{bmatrix}_q \right)^2 &= \frac{(q^k-1)(q^{n-k}-1)}{(q^n-1)^2} \\ &\geq (1-1/q)^2 q^{-n}. \end{aligned}$$

Altogether, we obtain a chain of length at least

$$\begin{aligned} l(\mathrm{GLS}(n,q)) \quad &\geq (1-1/q)^2 q^{-n} \textstyle\sum_{k=0}^{n-1} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q \right)^2 |\,\mathrm{GL}(k,q)| - 1 \\ &= (1-1/q)^2 q^{-n} (|\,\mathrm{GLS}(n,q)| - |\,\mathrm{GL}(n,q)|) - 1. \end{aligned}$$

By Proposition 6.1, we have

$$|\,\mathrm{GLS}(n,q)| - |\,\mathrm{GL}(n,q)| \geq q^{n^2}(1 - c(q) - o(1)),$$

where the $o(1)$ is for fixed q as $n \to \infty$. We obtain:

Theorem 6.2. $l(\mathrm{GLS}(n,q)) \geq (1 - c(q) - o(1))(1-1/q)^2 q^{-n} |\,\mathrm{GLS}(n,q)|$.

7 Inverse semigroups

An *inverse semigroup* is a semigroup S such that for all $x \in S$, there exists a unique $x^{-1} \in S$ where $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x$. The *symmetric inverse monoid* consists of the injective functions between subsets of a fixed set X. It is the analogue of the symmetric group in the context of inverse semigroups i.e. every inverse semigroup is isomorphic to an inverse subsemigroup of some symmetric inverse monoid.

The length of the symmetric inverse monoid on any finite set was determined in [9]. However, the main theorem of [9] holds for arbitrary finite inverse semigroups, and the proof is essentially that given in [9]. We state the theorem in its full generality, and give a slightly different proof from that in [9], which makes use of the description of the maximal subsemigroups of a Rees matrix semigroup given in [13].

Let G be a group and let $n \in \mathbb{N}$. Then the *Brandt semigroup* $B(G,n)$ has elements $(\{1,\dots,n\} \times G \times \{1,\dots,n\}) \cup \{0\}$ with multiplication defined by

$$(i,g,j)(k,h,l) = \begin{cases} (i,gh,l) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and $0x = x0 = 0$ for all $x \in B(G,n)$.

It follows from the Rees Theorem [18, Theorem 5.1.8] that the principal factor of a $\mathscr{J}$-class J of a finite inverse semigroup S is isomorphic to $B(G,n)$ where G is any maximal subgroup of S contained in J and n is the number of $\mathscr{L}$- and $\mathscr{R}$-classes of J. Every inverse semigroup is regular, and so, to calculate the length of an inverse semigroup, it suffices, by Lemma 3.2, to work out the length of a Brandt semigroup.

Proposition 7.1. *Let G be a group and let $n \in \mathbb{N}$. Then:*

$$\begin{aligned} l(B(G,n)) &= n(l(G)+1) + \frac{n(n-1)}{2}|G| + n - 1 \\ &= n(l(G)+2) + \frac{n(n-1)}{2}|G| - 1. \end{aligned} \tag{6}$$

Proof. We proceed by induction on n and $|G|$. If $n=1$, then $l(B(G,n)) = l(G)+1$ and (6) holds.

Let $n \in \mathbb{N}$, $n > 1$, and let G be a finite group. Suppose that if either: ($m < n$ and $|H| = |G|$) or ($m = n$ and $|H| < |G|$), then

$$l(B(H,m)) = m(l(H)+1) + \frac{m(m-1)}{2}|H| + m - 1.$$

We will show that (6) holds for n and G.

Remark 1 of [13] implies that a maximal subsemigroup of $B(G,n) = (I \times G \times I) \cup \{0\}$ is isomorphic to either:

(i) $B(H,n)$ where H is a maximal subgroup of G; or

(ii) $B(G,n) \setminus (J \times G \times K)$ where J and K partition I.

(This is also shown directly in Theorem 6 of [9].) The semigroups of type (ii) are always maximal, while the ones in part (i) may or may not be. It follows that either

$$l(B(G,n)) = 1 + l(B(H,n))$$

for some maximal subgroup H of G, or

$$l(B(G,n)) = 1 + l(B(G,n) \setminus (J \times G \times K))$$

where J and K partition I.

In the latter case,

$$B(G,n) \setminus (J \times G \times K) = (J \times G \times J) \cup (K \times G \times K) \cup (K \times G \times J) \cup \{0\}.$$

It is routine to verify that

$$(J \times G \times J) \cup \{0\} \cong B(G,|J|) \quad \text{and} \quad (K \times G \times K) \cup \{0\} \cong B(G,|K|)$$

and that
$$(K \times G \times J) \cup \{0\}$$
is a null ideal of $B(G,n) \setminus (J \times G \times K)$. Thus, by applying Proposition 3.1 to the null ideal and Lemma 3.2 to the (regular!) quotient of $B(G,n) \setminus (J \times G \times K)$ by the ideal, it follows that

$$l(B(G,n) \setminus (J \times G \times K)) = l(B(G,|J|)) + l(B(G,|K|)) + l(K \times G \times J \cup \{0\}).$$

Since every non-empty subset of $(K \times G \times J) \cup \{0\}$ is a subsemigroup, it follows that

$$l(K \times G \times J) \cup \{0\}) = |J||K||G|$$

and so by induction that

$$l(B(G,n) \setminus (J \times G \times K)) = n(l(G)+1) + \frac{n(n-1)}{2}|G| + n - 2.$$

By the second part of the inductive hypothesis

$$\begin{aligned} l(B(H,n)) &= n(l(H)+1) + \frac{n(n-1)}{2}|H| + n - 1 \\ &\leq n(l(G)+1) + \frac{n(n-1)}{2}|G| + n - 2 \\ &= l(B(G,n) \setminus (J \times G \times K)). \end{aligned}$$

Thus, when we are constructing a chain of semigroups, if we have a choice between semigroups of types (i) or (ii), we should choose type (ii) to obtain the longest possible chain. We conclude that
$$l(B(G,n)) = 1 + l(B(G,n) \setminus (J \times G \times K))$$
and (6) holds. □

The following result for inverse semigroups now follows immediately from Lemma 3.2 and Proposition 7.1.

Theorem 7.2 (cf. Theorem 7 in [9])**.** *Let S be a finite inverse semigroup with $\mathscr{J}$-classes $J_1, \ldots, J_m$. If $n_i \in \mathbb{N}$ denotes the number of $\mathscr{L}$- and $\mathscr{R}$-classes in J_i, and G_i is any maximal subgroup of S contained in J_i, then*

$$\begin{aligned} l(S) &= -1 + \sum_{i=1}^{m} l(B(G_i, n_i)) \\ &= -1 + \sum_{i=1}^{m} n_i(l(G_i)+1) + \frac{n_i(n_i-1)}{2}|G_i| + (n_i - 1). \end{aligned}$$

Given a specific inverse semigroup S, Theorem 7.2 gives a formula for $l(S)$ in terms of the numbers n_i of $\mathscr{L}$- and $\mathscr{R}$-classes and the lengths of the maximal subgroups G of the $\mathscr{J}$-classes of S. Thus to determine the length of a particular semigroup, it suffices to determine these values.

For example, if I_n denotes the symmetric inverse monoid on an n-element set and $x, y \in I_n$, then $x \mathscr{J} y$ if and only if the size of the domain of x is equal to the size of the domain of y; [18, Exercise 5.11.2]. Hence the number of $\mathscr{J}$-classes in I_n is $n+1$, corresponding to the possible sizes of subsets of $\{1, \ldots, n\}$. If J is the $\mathscr{J}$-class of I_n, consisting of partial permutations defined on i points, then the number of $\mathscr{L}$- and $\mathscr{R}$-classes in J is $\binom{n}{i}$ and every maximal subgroup of J is isomorphic to the symmetric group S_i on i points. So, in the formula in Theorem 7.2, $m = n+1$, $n_i = \binom{n}{i-1}$ and $G_i = S_{i-1}$, so we have

$$l(I_n) = -1 + \sum_{i=1}^{n+1} \binom{n}{i-1} (l(S_{i-1}) + 2) + \binom{n}{i-1} \left(\binom{n}{i-1} - 1 \right) \frac{(i-1)!}{2} - 1,$$

where the values of $l(S_{i-1})$ for $i > 1$ are given by Theorem 2.1 and $l(S_0) = 0$. The first few values of $l(I_n)$ are given in Table 3, for further terms see [25].

Three further examples are: the dual symmetric inverse monoid I_n^* where $m = n$, n_i is the Stirling number of the second kind $S(n, i)$, and $G_i = S_i$; see [7, Theorem 2.2], Table 3, and [26]; the partial injective order-preserving mappings POI_n on an n-element chain where $m = n+1$, $n_i = \binom{n}{i-1}$, and G_i is trivial; see [6], Table 3, and [27]; the partial injective orientation-preserving mappings $POPI_n$ on an n-element chain where $m = n+1$, $n_i = \binom{n}{i-1}$, and G_i is the cyclic group with i elements when $i > 0$ and the trivial group when $i = 0$; see [5], Table 3, and [28].

n	1	2	3	4	5	6	7	8	9
$l(I_n)$	1	6	25	116	722	5956	59243	667500	8296060
$l(I_n^*)$	0	2	17	180	3298	88431	3064050	130905678	6732227475
$l(POI_n)$	1	5	17	53	167	550	1899	6809	25067
$l(POPI_n)$	1	6	24	92	363	1483	6191	26077	109987

Table 3: The length of the longest chain of non-empty proper subsemigroups of some well-known inverse semigroups.

We consider the asymptotic value of $l(I_n)$ compared to $|I_n|$.

Theorem 7.3. *If S is any of the symmetric inverse monoid I_n, the dual symmetric inverse monoid I_n^*, the partial order-preserving injective mappings POI_n, the partial orientation-preserving injective mappings $POPI_n$, then*

$$\lim_{n \to \infty} \frac{l(S)}{|S|} = \frac{1}{2}.$$

Proof. We present the proof in the case that $S = I_n$, the other proofs are similar.

It is routine to check that

$$|I_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$$

(see also [18, Exercise 5.11.3]). By Theorem 7.2,

$$\begin{aligned}
l(I_n) &= -1 + \sum_{i=0}^{n} \left[\binom{n}{i} (l(S_i)+1) + \binom{n}{i} \left(\binom{n}{i} - 1 \right) \frac{i!}{2} + \binom{n}{i} - 1 \right] \\
&= \frac{|I_n|}{2} - 1 + \sum_{i=0}^{n} \left[\binom{n}{i} (l(S_i)+1) - \binom{n}{i} \frac{i!}{2} + \binom{n}{i} - 1 \right] \\
&= \frac{|I_n|}{2} - n - 2 + \sum_{i=0}^{n} \binom{n}{i} \left[l(S_i) + 2 - \frac{i!}{2} \right] \\
&= \frac{|I_n|}{2} + \frac{n-1}{2} + \sum_{i=2}^{n} \binom{n}{i} \left[l(S_i) + 2 - \frac{i!}{2} \right].
\end{aligned}$$

Note that, for $n \geq 1$,

$$|I_n| \geq \binom{n}{n-1}^2 (n-1)! = n \cdot n! \tag{7}$$

and so to show that $l(I_n)$ is asymptotically $\frac{|I(n)|}{2}$ it suffices to show that the ratio of

$$\sum_{i=2}^{n} \binom{n}{i} \left[l(S_i) + 2 - \frac{i!}{2} \right]$$

to $|I_n|$ tends to 0 as $n \to \infty$. By Theorem 2.1

$$\left| \sum_{i=2}^{n} \binom{n}{i} \left[l(S_i) + 2 - \frac{i!}{2} \right] \right| \leq \sum_{i=2}^{n} \binom{n}{i} \left[\frac{3i}{2} + 2 + \frac{i!}{2} \right] \leq \sum_{i=2}^{n} \binom{n}{i} i!$$

Using the inequalities (7) and

$$|I_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i! \geq n \sum_{i=2}^{n-1} \binom{n}{i} i!$$

it follows that

$$\frac{\sum_{i=2}^{n} \binom{n}{i} i!}{|I_n|} = \frac{n!}{|I_n|} + \frac{\sum_{i=2}^{n-1} \binom{n}{i} i!}{|I_n|} \leq \frac{2}{n} \to 0$$

as $n \to \infty$ and the proof is complete. □

7.1 Longest chains of inverse subsemigroups

In this section we consider the question of determining the longest chains of *inverse* subsemigroups of a finite inverse semigroup. We define the *inverse subsemigroup length* of an inverse semigroup S to be the largest number of non-empty inverse subsemigroups of S in a chain minus 1; this is denoted $l^*(S)$. Since every group is an inverse semigroup, and every subsemigroup of a finite group is a subgroup, if G is a finite group, then $l(G) = l^*(G)$.

We will prove the following theorem.

Theorem 7.4. *Let S be a finite inverse semigroup with $\mathscr{J}$-classes $J_1, \ldots, J_m$. If $n_i \in \mathbb{N}$ denotes the number of $\mathscr{L}$- and $\mathscr{R}$-classes in J_i, and G_i is any maximal subgroup of S contained in J_i, then*

$$\begin{aligned} l^*(S) &= -1 + \sum_{i=1}^{m} l^*(B(G_i, n_i)) \\ &= -1 + \sum_{i=1}^{m} n_i(l(G_i) + 1) + n_i - 1. \end{aligned}$$

The proof is similar to the proof of Theorem 7.2. We start by proving analogues of Proposition 3.1 and Lemma 3.2 for the inverse subsemigroup length, rather than length, of an inverse semigroup.

To prove the analogue of Proposition 3.1, we require the following facts about inverse semigroups. Let S be an inverse semigroup, let T and U be inverse subsemigroups, and let I be an ideal in S. Then the following are inverse semigroups: the ideal I, the quotient S/I, the intersection $T \cap U$, and the union $T \cup I$. If V is an inverse subsemigroup of S/I, then $V \setminus \{0\} \cup I$ is an inverse subsemigroup of S.

Proposition 7.5. *Let S be an inverse semigroup and let I be an ideal of S. Then $l^*(S) = l^*(I) + l^*(S/I)$.*

Proof. From the comments preceding the proposition, it is straightforward to verify that, the proof of this proposition follows by an argument analogous to that used to prove Proposition 3.1. □

The analogue of Lemma 3.2, follows as a corollary of Proposition 7.5 using the analogue of the proof of Lemma 3.2.

Corollary 7.6. *Let S be a finite inverse semigroup and let $J_1, J_2, \ldots, J_m$ be the $\mathscr{J}$-classes of S. Then $l^*(S) = l^*(J_1^*) + l^*(J_2^*) + \cdots + l^*(J_m^*) - 1$.*

As in the previous subsection, to calculate the inverse subsemigroup length of an inverse semigroup, it suffices, by Corollary 7.6, to find the inverse subsemigroup length of a Brandt semigroup.

Proposition 7.7. *Let G be a group and let $n \in \mathbb{N}$. Then:*

$$l^*(B(G, n)) = n(l(G) + 1) + n - 1 = n(l(G) + 2) - 1 \tag{8}$$

Proof. We proceed by induction on n and $|G|$. If $n = 1$, then $l^*(B(G, n)) = l(G) + 1$ and (8) holds.

Let $n \in \mathbb{N}$, $n > 1$, and let G be a finite group. Suppose that if either: ($m < n$ and $|H| = |G|$) or ($m = n$ and $|H| < |G|$), then

$$l^*(B(H, m)) = m(l(H) + 1) + m - 1.$$

We will show that (8) holds for n and G.

As in the proof of Proposition 7.1, a maximal subsemigroup of $B(G, n) = (I \times G \times I) \cup \{0\}$ is isomorphic to either:

(i) $B(H, n)$ where H is a maximal subgroup of G; or

(ii) $B(G, n) \setminus (J \times G \times K)$ where J and K partition I.

The subsemigroups of type (i) are inverse subsemigroups, and hence maximal inverse subsemigroups. The subsemigroups of type (ii) are not regular semigroups, since

$$B(G, n) \setminus (J \times G \times K) = (J \times G \times J) \cup (K \times G \times K) \cup (K \times G \times J) \cup \{0\},$$

and $(K \times G \times J) \cup \{0\}$ is a null subsemigroup. It follows that

$$U := (J \times G \times J) \cup (K \times G \times K) \cup \{0\}$$

is a maximal inverse subsemigroup of $B(G, n) \setminus (J \times G \times K)$, and hence of $B(G, n)$. Since the $\mathscr{J}$-classes of U are $J \times G \times J$, $K \times G \times K$, and $\{0\}$, by Corollary 7.6,

$$l^*(U) = l^*(B(G, |J|)) + l^*(B(G, |K|)).$$

Therefore either:

$$l^*(B(G, n)) = 1 + l^*(B(H, n))$$

for some maximal subgroup H of G, or

$$l^*(B(G, n)) = 1 + l^*(B(G, m)) + l^*(B(G, r))$$

where $m + r = n$. By induction, and since $n > 1$,

$$\begin{aligned} 1 + l^*(B(G, m)) + l^*(B(G, r)) &= n(l(G) + 1) + n - 1 \\ &> n\, l(G) + n \\ &= n(l(H) + 1) + n \\ &= 1 + l^*(B(H, n)), \end{aligned}$$

and the result follows. □

In particular, we see that

$$l^*(I_n) = -1 + \sum_{i=1}^{n+1} \binom{n}{i-1} (l(S_{i-1} + 2) - 1.$$

Some small values of the inverse subsemigroup lengths of the four examples of inverse semigroups from the previous section can be seen in Table 4.

n	1	2	3	4	5	6	7	8	9
$l^*(I_n)$	1	5	15	39	96	229	533	1217	2742
$l^*(I_n^*)$	0	2	11	49	223	1065	5337	28231	158939
$l^*(POI_n)$	1	4	11	26	57	120	247	502	1013
$l^*(POPI_n)$	1	6	17	44	97	208	429	884	1814

Table 4: The length of the longest chain of non-empty proper inverse subsemigroups of some well-known inverse semigroups.

8 Completely regular semigroups

In this section, we consider a special type of semigroup, which does not have any leagues in any of its $\mathscr{J}$-classes. A semigroup is *completely regular* if every element belongs to a subgroup.

It follows by the Rees Theorem [18, Theorems 3.2.3 and 4.1.3] that the principal factor of a $\mathscr{J}$-class J of a finite completely regular semigroup S is isomorphic to a Rees 0-matrix semigroup $\mathcal{M}^0[I, G, J; P]$ where G is a finite group and P is a $|J| \times |I|$ matrix with entries in G.

Theorem 8.1. *Let S be a completely regular semigroup where the numbers of $\mathscr{L}$- and $\mathscr{R}$-classes are m and n, and where the $\mathscr{J}$-classes of S are $J_1, \ldots, J_r$. If G_i is a maximal subgroup of S contained in J_i, then*

$$l(S) = m + n - r - 1 + \sum_{i=1}^{r} l(G_i).$$

Proof. By Lemma 3.2, it suffices to show that

$$l(\mathcal{M}^0[I, G, J; P]) = |I| + |J| + l(G) - 1$$

where $\mathcal{M}^0[I, G, J; P]$ is a Rees 0-matrix semigroup over the group G and P is a $|J| \times |I|$ matrix with entries in G (i.e. there are no entries equal 0). Furthermore, since the length of a semigroup S with zero adjoined is 1 more than the length of S, it suffices to show that

$$l(\mathcal{M}[I, G, J; P]) = |I| + |J| + l(G) - 2.$$

where $R = \mathcal{M}[I, G, J; P]$ is a Rees matrix semigroup without zero.

We proceed by induction on $|R| = |I| \times |G| \times |J|$. If $|I| = |J| = |G| = 1$, then $|R| = 1$ and so $l(R) = 0$ and $|I| + |J| + l(G) - 2 = 1 + 1 + 0 - 2 = 0$.

As in the proof of Proposition 7.1, the length of R is the length of one of its maximal subsemigroups plus 1. Remark 1 of [13] implies that a maximal subsemigroup of R is isomorphic to one of:

(i) $I \setminus \{i\} \times G \times J$ for some $i \in I$;

(ii) $I \times G \times J \setminus \{j\}$ for some $j \in J$;

(iii) $\mathcal{M}[I, H, J; Q]$ where H is a maximal subgroup of G and Q is a $|J| \times |I|$ matrix with entries in H.

Thus every maximal subsemigroup T of R is isomorphic to a completely regular Rees matrix semigroup. In any case, by induction, $l(T) = |I| + |J| + l(G) - 3$, the result follows. $\square$

A semigroup S is a *band* if every element is an idempotent, i.e. $x^2 = x$ for all $x \in S$. Every band is a completely regular semigroup where the maximal subgroups are trivial, and so Theorem 8.1 tells us that

$$l(S) = m + n - r - 1$$

where m, n, and r are the numbers of $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{J}$-classes of S, respectively.

The n-generated *free band* B_n is the free object in the category of bands, and, as it turns out, it is finite; see [18, Section 4.5] for more details. The $\mathscr{J}$-classes in B_n are in 1-1 correspondence with the non-empty subsets of $\{1, 2, \ldots, n\}$, and the number of $\mathscr{L}$- and $\mathscr{R}$-classes in any $\mathscr{J}$-class corresponding to a subset of size k is:

$$k \prod_{i=1}^{k-2} (k - i)^{2^i}.$$

The following is an immediate corollary of these observations and Theorem 8.1.

Corollary 8.2. *The length of the free band B_n with n generators is:*

$$2 \sum_{k=1}^{n} \left[\binom{n}{k} k \prod_{i=1}^{k-2} (k - i)^{2^i} \right] - 2^n.$$

Since every band with n generators is a homomorphic image of the free band B_n, it follows that $l(B_n)$ is an upper bound for $l(S)$ for every n generated band S.

n	1	2	3	4	5	6
$l(B_n)$	0	4	34	1264	3323778	33022614177128

Table 5: The length of the longest chain of non-empty proper subsemigroups in the free band B_n.

9 Numbers of subsemigroups

Our technique for producing long chains also gives lower bounds for the number of subsemigroups of certain semigroups.

We note that some results are known for groups. The number of subgroups of S_n is bounded below by roughly $2^{n^2/16}$. For this group contains an elementary abelian subgroup of order $2^{\lfloor n/2 \rfloor}$ generated by $\lfloor n/2 \rfloor$ disjoint transpositions; and an elementary abelian group of order 2^m has

$$\begin{bmatrix} m \\ k \end{bmatrix}_2$$

subgroups of order 2^k, this number being greater than $2^{k(m-k)}$, and so at least $2^{\lfloor m^2/4 \rfloor}$ when $k = \lfloor m/2 \rfloor$. Remarkably, Pyber [21] found an upper bound for the number of subgroups, also of the form 2^{cn^2} for constant c.

If a null semigroup has n non-zero elements, then it has 2^n subsemigroups, since the zero together with any set of non-zero elements forms a subsemigroup. So the existence of large null semigroups in principal factors of T_n, for example, gives lower bounds for the number of subsemigroups, and on the number of generators required.

Theorem 9.1. *Let*

$$c = \frac{\mathrm{e}^{-2}}{3\sqrt{\mathrm{e}^{-1} - 2\mathrm{e}^{-2}}\sqrt{3}}.$$

Then

(a) *the number of subsemigroups of T_n is at least $2^{(c-o(1))n^{n-1/2}}$;*

(b) *the smallest number $d(n)$ for which any subsemigroup of T_n can be generated by $d(n)$ elements is at least $(c - o(1))n^{n-1/2}$.*

Proof. The reader is reminded of the notation used in the proof of Theorem 4.1. We have exhibited then a null subsemigroup of T_n of order $(n-k)N(n-1, k-1)$ for all $k \in \{1, \ldots, n\}$. We shall give a lower bound on the the order of the largest of those semigroups. In particular, we restrict ourselves to the set $J = \{k : |k - E(n-1)| < d\tau n^{1/2}\}$ where $E(n-1)$ is the expected rank of a transformation in T_{n-1}, $\tau = \sqrt{\mathrm{e}^{-1} - 2\mathrm{e}^{-2}}$ and d is a constant which we will specify later. Using similar arguments as before, we can then prove that for all $k \in J$,

$$\begin{aligned} n - k &\geq \mathrm{e}^{-1} n - o(n) \\ \sum_{k \in J} N(n-1, k-1) &\geq \mathrm{e}^{-1} \frac{d^2 - 1}{d^2} n^{n-1} \\ \sum_{k \in J} (n-k) N(n-1, k-1) &\geq \mathrm{e}^{-1} \frac{d^2 - 1}{d^2} n^n - o(n^n), \end{aligned}$$

and hence the largest semigroup for $k \in J$ has order at least

$$\frac{\mathrm{e}^{-2}}{2\tau} \frac{d^2 - 1}{d^3} n^{n-1/2} - o(n^{n-1/2}).$$

The fraction is maximised for $d = \sqrt{3}$. $\square$

Remark Part (b) answers a question of Brendan McKay to the first author a few years ago and gives a partial answer to Open Problem 1 in [14]. The analogous number for S_n (the smallest d such that any subgroup can be generated by at most d elements) is only $\lfloor n/2 \rfloor$ for $n > 3$, as shown by McIver and Neumann [20]. Jerrum [19] gave a weaker bound $n - 1$, but with a constructive (and computationally efficient) proof.

10 Open problems

Problem 1 Does the ratio $l(T_n)/|T_n|$ tend to a limit as $n \to \infty$? If so, what is this limit? Is it possible to improve on the constant e^{-2} by either more careful analysis, or counting the extra steps available in a principal factor?

Problem 2 Evaluate the function $F(k, n)$ giving the largest content of a league of rank k on $\{1, \ldots, n\}$, and the function $F^*(n, k)$ giving the largest content involving partitions into intervals.

Problem 3 In most cases, our results are not strong enough to show that the number of subsemigroups of a semigroup S is at least $c^{|S|}$ for some $c > 1$. Does such a result hold in the case $S = T_n$, for example?

Problem 4 What can be said about the number of inverse subsemigroups of an inverse semigroup, for example the symmetric inverse semigroup I_n?

References

[1] João Araújo and Peter J. Cameron, Two generalizations of homogeneity in groups with applications to regular semigroups, *Trans. Amer. Math. Soc.*, in press.

[2] L. Babai, On the length of subgroup chains in the symmetric group, *Comm. Algebra* **14** (1986), 1729–1736.

[3] Csilla Bújtas and Zsolt Tuza, Smallest set transversals of k-partitions, *Graphs and Combinatorics* **25** (2009), 807–816.

[4] P. J. Cameron, R. Solomon and A. Turull, Chains of subgroups in symmetric groups, *J. Algebra* **127** (1989), 340–352.

[5] V. H. Fernandes, The monoid of all injective orientation preserving partial transformations on a finite chain, *Comm. Algebra*, **28** (2000), 3401–3426.

[6] V. H. Fernandes, The monoid of all injective order preserving partial transformations on a finite chain, *Semigroup Forum*, **62** (2001), 178–204.

[7] D. G. Fitzgerald and J. Leech, Dual symmetric inverse monoids and representation theory, *J. Austral. Math. Soc. A*, **64** (1998), 345–367.

[8] M. Gadouleau and Z. Yan, Packing and covering properties of rank metric codes, *IEEE Trans. Inform. Theory* **54** (2008), 3873–3883.

[9] Olexandr Ganyushkin and Ivan Livinsky, Length of the inverse symmetric semigroup, *Algebra Discrete Math.*, **12** (2011), 64–71.

[10] Olexandr Ganyushkin and Volodymyr Mazorchuk, *Classical Finite Transformation Semigroups*, Springer-Verlag, London, 2009.

[11] Ian P. Gent, Chris Jefferson, and Ian Miguel, MINION: A Fast, Scalable, Constraint Solver, in *Proceedings of the 2006 Conference on ECAI 2006: 17th European Conference on Artificial Intelligence August 29 – September 1, 2006, Riva Del Garda, Italy*, 98–102, (2006).

[12] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.5; 2014, `http://www.gap-system.org`.

[13] N. Graham, R. Graham, and J. Rhodes. Maximal subsemigroups of finite semigroups. *J. Combinatorial Theory*, 4:203–209, 1968.

[14] Robert D. Gray, The minimal number of generators of a finite semigroup, *Semigroup Forum*, **89** 135–154, (2014).

[15] Marshall Hall, Jr., *Combinatorial Theorey*, Blaisdell, Waltham, Mass., 1967.

[16] Peter M. Higgins, *Techniques of Semigroup Theory*, Oxford Science Publications, Oxford, 1992.

[17] Peter M. Higgins, Combinatorial results for semigroups of order-preserving mappings. *Math. Proc. Camb. Phil. Soc.* **113** (1993), 281–296.

[18] John M. Howie, *Fundamentals of semigroup theory*, The Clarendon Press Oxford University Press, London Mathematical Society Monographs, Volume 12, 1995.

[19] M. R. Jerrum, A compact representation for permutation groups, *J. Algorithms* **7** (1986), 60–78.

[20] A. McIver and P. M. Neumann, Enumerating finite groups, *Quart. J. Math.* (2) **38** (1987), 473–488.

[21] L. Pyber, Asymptotic results for permutation groups, pp. 197–219 in *Groups and Computation*, (Larry Finkelstein and William M. Kantor, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science **11**, American Mathematical Society, Providence, RI, 1993.

[22] John Rhodes and Benjamin Steinberg, The q-theory of finite semigroups, Springer Monographs in Mathematics, 2009.

[23] G. M. Seitz, R. Solomon, and A. Turull, Chains of subgroups in groups of Lie type II, *J. London Math. Soc.* (2), 42 (1990), 93–100.

[24] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A007238`.

[25] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A227914`.

[26] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A242428`.

[27] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A242429`.

[28] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A242432`.

[29] L. H. Soicher, The GRAPE package for GAP, Version 4.6.1, 2012, `http://www.maths.qmul.ac.uk/~leonard/grape/`.

[30] R. Solomon and A. Turull, Chains of subgroups in groups of Lie type, I, *J. Algebra* **132** (1990), pp. 174184; III, *J. London Math. Soc.* (2), **44** (1991), pp. 437–444.