

NAVIER–STOKES EQUATIONS ON A RAPIDLY ROTATING SPHERE

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(Communicated by the associate editor name)

ABSTRACT. We extend our earlier β -plane results [al-Jaboori and Wirosoetisno, 2011, DCDS-B 16:687–701] to a rotating sphere. Specifically, we show that the solution of the Navier–Stokes equations on a sphere rotating with angular velocity $1/\varepsilon$ becomes zonal in the long time limit, in the sense that the non-zonal component of the energy becomes bounded by εM . Central to our proof is controlling the behaviour of the nonlinear term near resonances. We also show that the global attractor reduces to a single stable steady state when the rotation is fast enough.

1. Introduction. On a sphere of unit radius rotating with angular velocity $1/\varepsilon$ about the z -axis, the non-dimensional Navier–Stokes equations read

$$\begin{aligned} \partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} + \frac{2 \cos \theta}{\varepsilon} \mathbf{v}^\perp + \nabla p &= \mu \Delta \mathbf{v} + f_{\mathbf{v}} \\ \operatorname{div} \mathbf{v} &= 0 \end{aligned} \tag{1.1}$$

where $\nabla_{\mathbf{v}}$ is the covariant derivative and \mathbf{v}^\perp is the velocity vector \mathbf{v} rotated by $+\pi/2$. Here $\theta \in [0, \pi]$ is the (co)latitude and $\phi \in [0, 2\pi)$ is the longitude. Observations of flows on rotating planetary atmospheres and many numerical studies (cf., e.g., [11] and references herein) strongly suggest that as the rotation rate increases, the flow will become more zonal, that is, the velocity will become more aligned with the planetary rotation. The main aim of this article is to prove that the solution of the 2d Navier–Stokes equations (1.1) does indeed become more zonal, in a sense to be precised below, as the planetary rotation rate $\varepsilon^{-1} \rightarrow \infty$. As a corollary, we show that the global attractor reduces to a point for small enough (but finite) ε . A related result for the Euler equations appeared in [3].

It is more convenient to work with the vorticity $\omega := \operatorname{curl} \mathbf{v}$, which by definition has zero integral over the sphere. We thus have Poincaré inequality

$$c_0 \|\omega\|_{L^2}^2 \leq \|\nabla \omega\|_{L^2}^2 := (-\Delta \omega, \omega)_{L^2}; \tag{1.2}$$

here one can show that $c_0 = 2$, but we shall write c_0 to make its origin clear. The evolution equation for ω is

$$\partial_t \omega + \partial(\Delta^{-1} \omega, \omega) + \frac{2}{\varepsilon} \partial_\phi \Delta^{-1} \omega = \mu \Delta \omega + f, \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary: 35B40, 35B41, 35R01, 76D05.

Key words and phrases. Navier–Stokes equations, rotating sphere, global attractor.

I would like to thank the anonymous referees for constructive comments, and particularly for suggesting a simpler proof of Lemma 1.

where here and henceforth Δ^{-1} is uniquely defined by requiring that the result has zero integral over the sphere. In polar coordinates (θ, ϕ) , the Jacobian takes the form

$$\partial(f, g) = \frac{1}{\sin \theta} \left(\frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \phi} - \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \theta} \right) \quad (1.4)$$

and satisfies the so-called *abc* identity: for any a, b and $c \in H^{1+\delta}(S^2)$,

$$(\partial(a, b), c)_{L^2} = (\partial(b, c), a)_{L^2} = (\partial(c, a), b)_{L^2}. \quad (1.5)$$

We shall also make use of the functional form of (1.3),

$$\partial_t \omega + B(\omega, \omega) + \frac{1}{\varepsilon} L\omega + \mu A\omega = f. \quad (1.6)$$

We note that (1.3) was obtained using the Laplace–de Rham operator (Hodge Laplacian) in (1.1a); see, e.g., [8] for more details. Had another vector Laplacian been used, we would have had $\mu(\Delta - c)\omega$ in place of $\mu\Delta\omega$ in (1.3), but this would not affect our main results (although some details in the proof would be different).

The following properties are readily verified: with (\cdot, \cdot) and $|\cdot|$ denoting the L^2 inner product and norm,

$$\begin{aligned} (A\omega, \omega) &= |\nabla\omega|^2 \\ (B(\omega^*, \omega), \omega) &= 0 \\ (L\omega, \omega) &= 0 \end{aligned} \quad (1.7)$$

for all ω and ω^* whenever the expressions make sense. Furthermore, $A = -\Delta$ is self-adjoint, $(A\omega, \omega^*) = (\omega, A\omega^*)$, while $L = \partial_\phi \Delta^{-1}$ is anti-self-adjoint, $(L\omega, \omega^*) = -(\omega, L\omega^*)$, and they commute,

$$AL\omega = LA\omega. \quad (1.8)$$

For functions of vanishing integral on S^2 (all that is relevant in this paper), we define the Sobolev space \dot{H}^s by

$$|u|_{\dot{H}^s}^2 := ((-\Delta)^s u, u)_{L^2} \quad (1.9)$$

for positive integers s . As usual, spaces of non-integral order are then defined by interpolation or in Fourier space.

As with periodic boundary conditions, one can show that with $f \in L^\infty(\mathbb{R}_+, H^{s-1})$ the solution ω of the NSE (1.3) belongs to $L^\infty((1, \infty), H^s) \cap L_{\text{loc}}^2((1, \infty), H^{s+1})$, regardless of the initial data $\omega(0)$ (assumed to be in L^2). Moreover, for each $s \in \{0, 1, \dots\}$, there exist $N_s(f; \mu)$ and $\tau_s(\omega(0), f; \mu)$ such that, for all $t \geq \tau_s$,

$$|\nabla^s \omega(t)|^2 + \int_t^{t+1} |\nabla^{s+1} \omega(\tau)|^2 d\tau \leq N_s(|f|_{L_t^\infty H_x^{s-1}}; \mu). \quad (1.10)$$

We note that, thanks essentially to (1.8), τ_s are independent of the rotation rate ε . The proof is essentially identical to that in the β -plane case [1] (see also [6] who obtained a closely related result for $s = 1$), so we shall not repeat it here. Regularity in Gevrey spaces has been established [2], although in this case the result is slightly weaker than in the doubly-periodic case [5].

The planetary rotation, represented by the antisymmetric operator L in (1.6), breaks the symmetry of the sphere and defines a special direction. We therefore introduce the averaging operator \bar{P} ,

$$\bar{P}u(\theta, t) := \frac{1}{2\pi} \int_0^{2\pi} u(\theta, \phi, t) d\phi, \quad (1.11)$$

and split the vorticity ω into its zonal part $\bar{\omega} := \bar{P}\omega$, which is independent of ϕ , and the remainder $\tilde{\omega} := (1 - \bar{P})\omega$. Similarly, we split $f = \tilde{f} + \bar{f}$ with $\tilde{f} := \bar{P}f$ and $\bar{f} := (1 - \bar{P})f$. The following orthogonality properties follow from the definitions:

$$\begin{aligned} L\bar{\omega} &= 0 \\ B(\bar{\omega}, \bar{\omega}) &= 0 \\ (\bar{\omega}, \tilde{\omega})_{H^s} &= 0. \end{aligned} \tag{1.12}$$

2. L^2 Estimate for the Non-zonal Part. We do the initial stage of the computation here in order to motivate the crucial “non-resonance” Lemma 1. We start by multiplying (1.6) by $\tilde{\omega}$ in L^2 . Using (1.7) and (1.12), we find

$$\frac{1}{2} \frac{d}{dt} |\tilde{\omega}|^2 + \mu |\nabla \tilde{\omega}|^2 + (B(\omega, \omega), \tilde{\omega}) = (f, \tilde{\omega}). \tag{2.1}$$

Using (1.7b) and (1.12b), we rewrite the nonlinear term as

$$\begin{aligned} (B(\omega, \omega), \tilde{\omega}) &= (B(\omega, \tilde{\omega}), \tilde{\omega}) + (B(\omega, \bar{\omega}), \tilde{\omega}) \\ &= (B(\tilde{\omega}, \bar{\omega}), \tilde{\omega}) + (B(\tilde{\omega}, \tilde{\omega}), \tilde{\omega}) \\ &= -(B(\tilde{\omega}, \tilde{\omega}), \bar{\omega}), \end{aligned} \tag{2.2}$$

giving us

$$\frac{1}{2} \frac{d}{dt} |\tilde{\omega}|^2 + \mu |\nabla \tilde{\omega}|^2 = (B(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) + (\tilde{f}, \tilde{\omega}). \tag{2.3}$$

Let $\nu := \mu c_0$. Using Poincaré inequality (1.2) on half of $\mu |\nabla \tilde{\omega}|^2$ in (2.3) and multiplying by $e^{\nu t}$, we find

$$\begin{aligned} \frac{d}{dt} |\tilde{\omega}|^2 + \nu |\tilde{\omega}|^2 + \mu |\nabla \tilde{\omega}|^2 &\leq 2(B(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) + 2(\tilde{f}, \tilde{\omega}) \\ \Leftrightarrow \frac{d}{dt} (e^{\nu t} |\tilde{\omega}|^2) + \mu e^{\nu t} |\nabla \tilde{\omega}|^2 &\leq 2e^{\nu t} (B(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) + 2e^{\nu t} (\tilde{f}, \tilde{\omega}). \end{aligned} \tag{2.4}$$

Integrating, this gives

$$\begin{aligned} |\tilde{\omega}(t)|^2 + \mu \int_0^t e^{\nu(\tau-t)} |\nabla \tilde{\omega}(\tau)|^2 d\tau \\ \leq e^{-\nu t} |\tilde{\omega}(0)|^2 + 2 \int_0^t e^{\nu(\tau-t)} \{ (B(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) + (\tilde{f}, \tilde{\omega}) \} d\tau. \end{aligned} \tag{2.5}$$

As will be shown below, the integrand on the right-hand side is rapidly oscillating, so the integral will be of order ε , giving an order- ε bound on $|\tilde{\omega}(t)|^2$ for large t .

We briefly recall the properties of spherical harmonics. We denote by $\mathbf{k} := (k, \hat{k})$ a *wavevector*, with $k \in \{0, 1, \dots\}$ and $\hat{k} \in \{-k, -k+1, \dots, k\}$. Writing

$$|\mathbf{k}| := \sqrt{k(k+1)}, \tag{2.6}$$

the Laplacian Δ has eigenvalues $-|\mathbf{k}|^2$, with each eigenspace having dimension $2k+1$ and spanned by the spherical harmonics $Y_{\mathbf{k}}(\theta, \phi)$,

$$\Delta Y_{\mathbf{k}} = -|\mathbf{k}|^2 Y_{\mathbf{k}}. \tag{2.7}$$

The spherical harmonics are orthonormal,

$$(Y_j, Y_{\mathbf{k}})_{L^2} = \delta_{j\mathbf{k}}, \tag{2.8}$$

where $\delta_{j\mathbf{k}} = 1$ when $j = k$ and $\hat{j} = \hat{k}$, and $\delta_{j\mathbf{k}} = 0$ otherwise. Their explicit expressions (including phase and normalisation) can be found in [10, §14.30], whose

convention we follow; we note that [9] used the same convention for spherical harmonics. For our purpose here, we only note that

$$Y_{(k, \hat{k})}(\theta, \phi) = C_{\mathbf{k}} e^{i\hat{k}\phi} P_k^{\hat{k}}(\cos \theta) \quad (2.9)$$

where $C_{\mathbf{k}}$ is real and the Legendre polynomial $P_k^{\hat{k}}(\cdot)$ has real coefficients and is of degree (exactly) k . Now it is clear from (2.9) that $Y_{\mathbf{k}}$ is also an eigenfunction of L , and we use this fact to define the frequency $\Omega_{\mathbf{k}}$ (ignoring the case $k = 0$),

$$LY_{\mathbf{k}} = 2 \partial_{\phi} \Delta^{-1} Y_{\mathbf{k}} = -\frac{2i\hat{k}}{|\mathbf{k}|^2} Y_{\mathbf{k}} =: i\Omega_{\mathbf{k}} Y_{\mathbf{k}}. \quad (2.10)$$

Assuming sufficient regularity (true for positive time due to the assumption on f in Theorem 1), we Fourier-expand the vorticity as

$$\omega(\theta, \phi, t) = \sum_{\mathbf{k}} \omega_{\mathbf{k}}(t) e^{-i\Omega_{\mathbf{k}} t/\varepsilon} Y_{\mathbf{k}}(\theta, \phi), \quad (2.11)$$

where the factor $e^{-i\Omega_{\mathbf{k}} t/\varepsilon}$ has been included since we expect $\tilde{\omega}$ to oscillate rapidly. Consistent with our definitions of $\bar{\omega}$ and $\tilde{\omega}$, we write $\tilde{\omega}_{\mathbf{k}} = \omega_{\mathbf{k}}$ when $\hat{k} \neq 0$ and $\tilde{\omega}_{\mathbf{k}} = 0$ when $\hat{k} = 0$, and $\bar{\omega}_{\mathbf{k}} = \omega_{\mathbf{k}}$ when $\hat{k} = 0$ and $\bar{\omega}_{\mathbf{k}} = 0$ otherwise. Similar notations are understood for \tilde{f} and \bar{f} . Here and henceforth, the sum over wavevector is understood to be

$$\sum_{\mathbf{k}} := \sum_{k=0}^{\infty} \sum_{\hat{k}=-k}^k \quad (2.12)$$

although the $k = 0$ term is often zero. Similarly, we expand the forcing f as

$$f(\theta, \phi, t) = \sum_{\mathbf{k}} f_{\mathbf{k}}(t) Y_{\mathbf{k}}(\theta, \phi) \quad (2.13)$$

without the rapidly oscillating exponential since later we will demand that f vary slowly in time. Writing

$$B_{j\mathbf{k}\mathbf{l}} := (\partial(\Delta^{-1} Y_j, Y_{\mathbf{k}}), Y_{\mathbf{l}})_{L^2}, \quad (2.14)$$

the nonlinear term in (2.5) is

$$\begin{aligned} (B(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) &= \sum_{j\mathbf{k}\mathbf{l}} B_{j\mathbf{k}\mathbf{l}} \tilde{\omega}_j \tilde{\omega}_{\mathbf{k}} \bar{\omega}_{\mathbf{l}} e^{-i(\Omega_j + \Omega_{\mathbf{k}})/\varepsilon} \\ &= \frac{1}{2} \sum_{j\mathbf{k}\mathbf{l}} (B_{j\mathbf{k}\mathbf{l}} + B_{\mathbf{k}j\mathbf{l}}) \tilde{\omega}_j \tilde{\omega}_{\mathbf{k}} \bar{\omega}_{\mathbf{l}} e^{-i(\Omega_j + \Omega_{\mathbf{k}})/\varepsilon} \end{aligned} \quad (2.15)$$

where $\bar{\omega}_{\mathbf{l}}$ denotes the complex conjugate of $\omega_{\mathbf{l}}$ (here and elsewhere, small overbar denotes ϕ -average and full overline denotes complex conjugate). As will be seen below, our main difficulty is the resonances in (2.15), which occur when $\Omega_j + \Omega_{\mathbf{k}} = 0$. It has long been known that there are “no fast-fast-slow resonance” in many geophysical fluid systems, including (1.3). However, in order to carry out our estimates, we also need control not only *at* resonances, but also *near* them.

Denoting the coupling coefficients of the Jacobian by

$$\mathcal{J}_{j\mathbf{k}\mathbf{l}} := (\partial(Y_j, Y_{\mathbf{k}}), Y_{\mathbf{l}})_{L^2}, \quad (2.16)$$

we handle the resonances with:

Lemma 1. *For all wavevectors j , \mathbf{k} and \mathbf{l} with $\hat{j}\hat{k} \neq 0$ and $\hat{l} = 0$,*

$$B_{j\mathbf{k}\mathbf{l}} + B_{\mathbf{k}j\mathbf{l}} = -\frac{1}{2\hat{j}} \mathcal{J}_{j\mathbf{k}\mathbf{l}} (\Omega_j + \Omega_{\mathbf{k}}). \quad (2.17)$$

We note that the right-hand side is symmetric in j and \mathbf{k} as expected: $\mathcal{J}_{\mathbf{k}j\mathbf{l}} = -\mathcal{J}_{j\mathbf{k}\mathbf{l}}$ while $\hat{l} = 0$ implies that $\hat{k} = -\hat{j}$.

Proof. Using (2.7) in (2.16), we have

$$B_{jkl} = (\partial(-|j|^{-2}Y_j, Y_k), Y_l) = -|j|^{-2}(\partial(Y_j, Y_k), Y_l) = -|j|^{-2}\mathcal{J}_{jkl}. \quad (2.18)$$

Using the fact that $\partial(g, f) = -\partial(f, g)$, we find similarly

$$B_{kjl} = (\partial(-|k|^{-2}Y_k, Y_j), Y_l) = |k|^{-2}(\partial(Y_j, Y_k), Y_l) = |k|^{-2}\mathcal{J}_{jkl}. \quad (2.19)$$

Adding these and using (2.10),

$$B_{jkl} + B_{kjl} = \mathcal{J}_{jkl} \left(\frac{1}{|k|^2} - \frac{1}{|j|^2} \right) = \mathcal{J}_{jkl} \left(\frac{\Omega_j}{2\hat{j}} - \frac{\Omega_k}{2\hat{k}} \right), \quad (2.20)$$

which gives us (2.17) using the fact that $0 = \hat{l} = \hat{j} + \hat{k}$. \square

Theorem 1. *Let the forcing f in (1.3) be bounded as*

$$|\nabla^2 f(t)|_{L^2}^2 + \int_t^{t+1} |\partial_\tau f|_{L^2}^2 d\tau =: K_0 < \infty \quad (2.21)$$

for all $t \geq T$ and some $T \geq 0$. Then there exist $T_0(f, \omega(0); \mu)$ and $M_0(K_0; \mu)$ such that, for $t \geq T_0$,

$$\sup_{t \geq T_0} |\tilde{\omega}(t)|_{L^2}^2 + \mu \int_t^{t+1} |\nabla \tilde{\omega}(\tau)|_{L^2}^2 d\tau \leq \varepsilon M_0(K_0; \mu). \quad (2.22)$$

Proof. Except for some estimates in the nonlinear term, the proof essentially follows that in [1]. Here and in what follows, $x \leq_c y$ means $x \leq cy$ for some positive absolute constant c which may assume different values in different inequalities. We note that the hypothesis (2.21) and (1.10) imply that our solution ω is bounded uniformly in $L_t^\infty H_x^3 \cap L_{t,1}^2 H_x^4$.

We bound the integral on the right-hand side in (2.5). Starting with the last term (omitting the $e^{-\nu t}$ factor for now), we integrate it by parts in time to bring out a factor of ε :

$$\begin{aligned} \int_0^t e^{\nu\tau} (\tilde{f}, \tilde{\omega})_{L^2} d\tau &= \int_0^t \sum_{\mathbf{k}}' e^{\nu\tau + i\Omega_{\mathbf{k}}t/\varepsilon} \tilde{f}_{\mathbf{k}} \overline{\tilde{\omega}_{\mathbf{k}}} d\tau \\ &= \varepsilon \left[\sum_{\mathbf{k}}' \frac{e^{\nu\tau + i\Omega_{\mathbf{k}}t/\varepsilon}}{i\Omega_{\mathbf{k}}} \tilde{f}_{\mathbf{k}} \overline{\tilde{\omega}_{\mathbf{k}}} \right]_0^t - \varepsilon \int_0^t \left\{ \sum_{\mathbf{k}}' \frac{e^{i\Omega_{\mathbf{k}}t/\varepsilon}}{i\Omega_{\mathbf{k}}} \frac{d}{d\tau} (e^{\nu\tau} \tilde{f}_{\mathbf{k}} \overline{\tilde{\omega}_{\mathbf{k}}}) \right\} d\tau. \end{aligned} \quad (2.23)$$

Here and henceforth, the prime on the sum indicates that the resonant terms (i.e. those with $\Omega_{\mathbf{k}} = 0$) are omitted. Taking note of (2.10), we rewrite (2.23) as

$$\begin{aligned} \int_0^t e^{\nu\tau} (\tilde{f}, \tilde{\omega}) d\tau &= \varepsilon [e^{\nu\tau} (L^{-1} \tilde{f}, \tilde{\omega})]_0^t \\ &\quad - \varepsilon \int_0^t e^{\nu\tau} \{ \nu (L^{-1} \tilde{f}, \tilde{\omega}) + (L^{-1} \partial_\tau \tilde{f}, \tilde{\omega}) + (L^{-1} \tilde{f}, \partial_\tau^* \tilde{\omega}) \} d\tau, \end{aligned} \quad (2.24)$$

which is well defined since L^{-1} acts only on functions not in $\ker L$, and where

$$\partial_t^* \tilde{\omega} := e^{-tL/\varepsilon} \partial_t (e^{tL/\varepsilon} \tilde{\omega}) = -\tilde{B}(\omega, \omega) - \mu A \tilde{\omega} + \tilde{f}. \quad (2.25)$$

where $\tilde{B} := (1 - \bar{P})B$. We estimate the terms in (2.24) using the standard Sobolev and interpolation inequalities, keeping in mind that functions have zero average over the sphere. For notational brevity, all unadorned norms are L^2 . We bound the endpoints terms as

$$|(L^{-1} \tilde{f}, \tilde{\omega})| \leq_c |\nabla \tilde{f}| |\nabla \tilde{\omega}|. \quad (2.26)$$

In the integral, we bound the first term as above and the middle term as

$$|(L^{-1}\partial_t\tilde{f}, \tilde{\omega})| \leq c |\partial_t\tilde{f}| |\Delta\tilde{\omega}|. \quad (2.27)$$

Now the last term is, noting that $(L^{-1}\tilde{f}, \tilde{f}) = 0$,

$$(L^{-1}\tilde{f}, -B(\omega, \omega) + \mu\Delta\tilde{\omega} + \tilde{f}) = -(L^{-1}\tilde{f}, B(\omega, \omega)) + \mu(L^{-1}\tilde{f}, \Delta\tilde{\omega}), \quad (2.28)$$

which we bound as

$$|(L^{-1}\tilde{f}, \Delta\tilde{\omega})| \leq c |\nabla^3\tilde{\omega}| |\nabla\tilde{f}|. \quad (2.29)$$

This is the worst term (in the sense of requiring the most regularity on f) in the entire L^2 estimate. For the nonlinear term in (2.28), we write $\psi := \Delta^{-1}\omega$, extend ψ and ω to functions independent of the radius r in a thin shell in \mathbb{R}^3 around the unit sphere, and write the Jacobian as

$$\partial(\psi, \omega) = (\nabla\psi \times \nabla\omega) \cdot \mathbf{e}_r \quad (2.30)$$

where the gradients and cross product are in \mathbb{R}^3 and \mathbf{e}_r is the unit vector in the radial direction. Working in \mathbb{R}^3 , we obtain after a little computation

$$\begin{aligned} \Delta\partial(\psi, \omega) &= \partial(\Delta\psi, \omega) + \partial(\psi, \Delta\omega) + 2\partial(\nabla\psi, \nabla\omega) \\ &\quad + 2\sum_i (\nabla(\partial_i\psi) \times \nabla\omega) \cdot (\partial_i\mathbf{e}_r) + 2\sum_i (\nabla\psi \times \nabla(\partial_i\omega)) \cdot (\partial_i\mathbf{e}_r). \end{aligned} \quad (2.31)$$

Noting that $\nabla\mathbf{e}_r$ is smooth in the shell, we then restrict back to the sphere and estimate (with ‘‘l.o.t.’’ denoting lower-order terms majorisable by those already present)

$$\begin{aligned} |(\partial(\psi, \omega), L^{-1}\tilde{f})| &\leq |(\partial(\psi, \Delta\omega), \partial_\phi^{-1}\tilde{f})| + 2|(\partial(\nabla\psi, \nabla\omega), \partial_\phi^{-1}\tilde{f})| + \text{l.o.t.} \\ &\leq c |\nabla\psi|_{L^\infty} |\Delta\omega|_{L^2} |\nabla\tilde{f}|_{L^2} + |\nabla^2\psi|_{L^4} |\nabla\tilde{f}|_{L^2} |\nabla\omega|_{L^4} \\ &\leq c |\nabla\Delta\psi| |\Delta\omega| |\nabla\tilde{f}| \end{aligned} \quad (2.32)$$

where all norms are L^2 on the last line. We conclude that

$$\begin{aligned} \left| \int_0^t e^{\nu\tau} (\tilde{f}, \tilde{\omega}) \, d\tau \right| &\leq \varepsilon c |\nabla\tilde{f}(0)| |\nabla\tilde{\omega}(0)| + \varepsilon c e^{\nu t} |\nabla\tilde{f}(t)| |\nabla\tilde{\omega}(t)| \\ &\quad + \varepsilon c \int_0^t e^{\nu\tau} \{ |\partial_t\tilde{f}| |\Delta\tilde{\omega}| + \mu |\nabla^3\tilde{\omega}| |\nabla\tilde{f}| + (1 + \mu + |\Delta\omega|) |\nabla\omega| |\nabla\tilde{f}| \} \, d\tau. \end{aligned} \quad (2.33)$$

Turning to the nonlinear term in (2.5), in Fourier components it reads

$$\begin{aligned} \int_0^t e^{\nu\tau} (B(\tilde{\omega}, \tilde{\omega}), \tilde{\omega})_{L^2} \, d\tau &= \int_0^t \sum_{\mathbf{jkl}} e^{\nu\tau - i(\Omega_j + \Omega_k)\tau/\varepsilon} B_{\mathbf{jkl}} \tilde{\omega}_j \tilde{\omega}_k \overline{\tilde{\omega}_l} \, d\tau \\ &= \frac{1}{2} \int_0^t \sum'_{\mathbf{jkl}} e^{\nu\tau - i(\Omega_j + \Omega_k)\tau/\varepsilon} (B_{\mathbf{jkl}} + B_{\mathbf{kjl}}) \tilde{\omega}_j \tilde{\omega}_k \overline{\tilde{\omega}_l} \, d\tau, \end{aligned} \quad (2.34)$$

where the prime on the sum again indicates that resonant terms, i.e. those with $\Omega_j + \Omega_k = 0$, are omitted (since then $B_{\mathbf{jkl}} + B_{\mathbf{kjl}} = 0$ by Lemma 1). As in (2.23), we integrate this by parts to bring out a factor of ε . Defining the symmetric bilinear operator $B_\Omega(\cdot, \cdot)$ by

$$(B_\Omega(\tilde{\omega}^\sharp, \tilde{\omega}^\flat), \tilde{\omega})_{L^2} := \frac{i}{2} \sum'_{\mathbf{jkl}} \frac{\bar{B}_{\mathbf{jkl}} + \bar{B}_{\mathbf{kjl}}}{\Omega_j + \Omega_k} \tilde{\omega}_j^\sharp \tilde{\omega}_k^\flat \overline{\tilde{\omega}_l}, \quad (2.35)$$

where \bar{B}_{jkl} is the coefficient of $\bar{P}B$, $\bar{B}_{jkl} := (\bar{P}\partial(\Delta^{-1}Y_j, Y_k), Y_l)$, we have [cf. (2.24)]

$$\begin{aligned} \int_0^t e^{\nu\tau} (B(\tilde{\omega}, \tilde{\omega}), \bar{\omega})_{L^2} d\tau &= \varepsilon [e^{\nu\tau} (B_\Omega(\tilde{\omega}, \tilde{\omega}), \bar{\omega})(\tau)]_0^t \\ &- \varepsilon \int_0^t e^{\nu\tau} \{ \nu (B_\Omega(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) + 2 (B_\Omega(\partial_\tau^* \tilde{\omega}, \tilde{\omega}), \bar{\omega}) + (B_\Omega(\tilde{\omega}, \tilde{\omega}), \partial_\tau \bar{\omega}) \} d\tau. \end{aligned} \quad (2.36)$$

As before, we bound each term in the last integral. Now Lemma 1 implies

$$(B_\Omega(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) = \frac{1}{4} (\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), \bar{\omega}), \quad (2.37)$$

so we can bound the first term as

$$|(B_\Omega(\tilde{\omega}, \tilde{\omega}), \bar{\omega})| \leq c |\nabla \tilde{\omega}|_{L^2}^2 |\bar{\omega}|_{L^\infty} \leq c |\nabla \tilde{\omega}|^2 |\nabla \bar{\omega}|. \quad (2.38)$$

Next, the last term in (2.36) reads (omitting the factor of 4)

$$\begin{aligned} &(\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), -\bar{B}(\omega, \omega) + \mu \Delta \bar{\omega} + \tilde{f}) \\ &= -(\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), \bar{\partial}(\psi, \omega)) + \mu (\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), \Delta \bar{\omega}) + (\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), \tilde{f}). \end{aligned} \quad (2.39)$$

We estimate each term in turn, keeping in mind that $|\bar{u}|_{L^\infty} \leq c |\nabla \bar{u}|$,

$$|(\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), \tilde{f})| \leq c |\nabla \tilde{\omega}|_{L^2}^2 |\tilde{f}|_{L^\infty} \leq c |\nabla \tilde{\omega}|^2 |\nabla \tilde{f}| \quad (2.40)$$

$$|(\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), \Delta \bar{\omega})| \leq c |\nabla \tilde{\omega}|_{L^4}^2 |\Delta \bar{\omega}|_{L^2} \leq c |\nabla \tilde{\omega}| |\Delta \tilde{\omega}| |\Delta \bar{\omega}| \quad (2.41)$$

$$\begin{aligned} |(\partial(\partial_\phi^{-1} \tilde{\omega}, \tilde{\omega}), \bar{\partial}(\psi, \omega))| &\leq c |\nabla \tilde{\omega}|_{L^4}^2 |\bar{\partial}(\psi, \omega)|_{L^2} \\ &\leq c |\nabla \tilde{\omega}| |\Delta \tilde{\omega}| |\nabla \psi|_{L^\infty} |\nabla \omega| \leq c |\nabla \omega|^3 |\Delta \tilde{\omega}|. \end{aligned} \quad (2.42)$$

For the penultimate term in (2.36), we note

$$\begin{aligned} 4 (B_\Omega(\partial_t^* \tilde{\omega}, \tilde{\omega}), \bar{\omega}) &= -(\partial(\partial_t^* \tilde{\omega}, \partial_\phi^{-1} \tilde{\omega}), \bar{\omega}) = (\partial(\partial_\phi^{-1} \tilde{\omega}, \bar{\omega}), \partial_t^* \tilde{\omega}) \\ &= (\partial(\partial_\phi^{-1} \tilde{\omega}, \bar{\omega}), -\tilde{B}(\omega, \omega) + \mu \Delta \tilde{\omega} + \tilde{f}) \end{aligned} \quad (2.43)$$

and estimate each term as

$$|(\partial(\partial_\phi^{-1} \tilde{\omega}, \bar{\omega}), \tilde{f})| \leq c |\nabla \tilde{\omega}|_{L^2} |\nabla \tilde{f}|_{L^2} |\bar{\omega}|_{L^\infty} \leq c |\nabla \tilde{\omega}| |\nabla \bar{\omega}| |\nabla \tilde{f}| \quad (2.44)$$

$$|(\partial(\partial_\phi^{-1} \tilde{\omega}, \bar{\omega}), \Delta \tilde{\omega})| \leq c |\nabla \tilde{\omega}|_{L^2} |\nabla \bar{\omega}|_{L^\infty} |\Delta \tilde{\omega}|_{L^2} \leq c |\nabla \tilde{\omega}| |\Delta \tilde{\omega}| |\Delta \bar{\omega}| \quad (2.45)$$

$$\begin{aligned} |(\partial(\partial_\phi^{-1} \tilde{\omega}, \bar{\omega}), \tilde{\partial}(\psi, \omega))| &\leq c |\partial(\partial_\phi^{-1} \tilde{\omega}, \bar{\omega})|_{L^2} |\partial(\psi, \omega)|_{L^2} \\ &\leq c |\nabla \tilde{\omega}|_{L^2} |\nabla \bar{\omega}|_{L^\infty} |\nabla \psi|_{L^\infty} |\nabla \omega|_{L^2} \\ &\leq c |\nabla \omega|^3 |\Delta \omega|. \end{aligned} \quad (2.46)$$

Bounding the endpoints in (2.36) by (2.37) and collecting, we have

$$\begin{aligned} \left| \int_0^t e^{\nu\tau} (B(\tilde{\omega}, \tilde{\omega}), \bar{\omega}) d\tau \right| &\leq \varepsilon |\nabla \omega(0)|^3 + \varepsilon e^{\nu t} |\nabla \omega(t)|^3 \\ &+ \varepsilon c \int_0^t e^{\nu\tau} \{ (\mu + |\Delta \omega|) |\nabla \omega|^3 + \mu |\Delta \omega|^2 |\nabla \omega| + |\nabla \omega|^2 |\nabla \tilde{f}| \} d\tau. \end{aligned} \quad (2.47)$$

With (2.33) and (2.47), and shifting the origin of t , (2.5) then implies

$$\begin{aligned}
|\tilde{\omega}(t)|^2 + \mu \int_{t_0}^t e^{\nu(\tau-t)} |\nabla \tilde{\omega}(\tau)|^2 d\tau &\leq e^{\nu(t_0-t)} |\tilde{\omega}(t_0)|^2 \\
&+ c\varepsilon \sup_{\tau \in \{t_0, t\}} (|\nabla \omega|^3 + |\nabla \omega| |\nabla \tilde{f}|)(\tau) \\
&+ c\varepsilon \int_{t_0}^t e^{\nu(\tau-t)} \{(\mu + |\Delta \omega|) |\nabla \omega|^3 + \mu |\Delta \omega|^2 |\nabla \omega| \\
&\quad + |\nabla \omega| |\partial_\tau \tilde{f}| + \mu |\nabla^3 \omega| |\nabla \tilde{f}| + (1 + \mu + |\Delta \omega|) |\nabla \omega| |\nabla \tilde{f}|\} d\tau.
\end{aligned} \tag{2.48}$$

To use (1.10) here, we note that

$$\int_t^{t+1} |u(\tau)|^2 d\tau \leq M \quad \forall t \quad \Rightarrow \quad \int_{t_0}^t e^{\nu(\tau-t)} |u(\tau)|^2 d\tau \leq M/(1 - e^{-\nu}). \tag{2.49}$$

Taking $t_0 = \tau_3$, this and (1.10) bound the integral on right-hand side of (2.48). The theorem follows by taking $t - t_0$ sufficiently large. \square

3. H^s Estimates and Dimension of the Global Attractor. As in [1], with $f(t) \in H^{s+2}$, one can obtain similar bounds for $\tilde{\omega}$ in H^s . Since the proof is similar to that in [1], we only sketch briefly here the case $s = 1$.

Multiplying (1.3) by $-\Delta \tilde{\omega}$ in L^2 , we find

$$\frac{d}{dt} (e^{\nu t} |\nabla \tilde{\omega}|^2) + \mu e^{\nu t} |\Delta \tilde{\omega}|^2 \leq 2e^{\nu t} (\partial(\psi, \omega), \Delta \tilde{\omega}) - 2e^{\nu t} (f, \Delta \tilde{\omega}). \tag{3.1}$$

The last term is integrated by parts as in the L^2 case, while for the nonlinear term we use the fact that $B(\bar{\omega}, \bar{\omega}) = \partial(\bar{\psi}, \bar{\omega}) = 0$ to write

$$(\partial(\psi, \omega), \Delta \tilde{\omega}) = (\partial(\bar{\psi}, \tilde{\omega}), \Delta \tilde{\omega}) + (\partial(\tilde{\psi}, \bar{\omega}), \Delta \tilde{\omega}) + (\partial(\tilde{\psi}, \tilde{\omega}), \Delta \tilde{\omega}). \tag{3.2}$$

Bounding the terms as

$$\begin{aligned}
|(\partial(\tilde{\psi}, \tilde{\omega}), \Delta \tilde{\omega})| &\leq |(\partial(\nabla \tilde{\psi}, \tilde{\omega}), \nabla \tilde{\omega})| + \text{l.o.t.} \leq c |\nabla^2 \tilde{\psi}|_{L^2} |\nabla \tilde{\omega}|_{L^4}^2 \\
&\leq \frac{\mu}{8} |\Delta \tilde{\omega}|^2 + \frac{c}{\mu} |\tilde{\omega}|^2 |\nabla \tilde{\omega}|^2 \\
|(\partial(\bar{\psi}, \tilde{\omega}), \Delta \tilde{\omega})| &\leq c |\nabla \bar{\psi}|_{L^\infty} |\nabla \tilde{\omega}|_{L^2} |\Delta \tilde{\omega}|_{L^2} \leq \frac{\mu}{8} |\Delta \tilde{\omega}|^2 + \frac{c}{\mu} |\nabla \tilde{\omega}|^2 |\bar{\omega}|^2 \\
|(\partial(\tilde{\psi}, \bar{\omega}), \Delta \tilde{\omega})| &\leq c |\nabla \tilde{\psi}|_{L^\infty} |\nabla \bar{\omega}|_{L^2} |\Delta \tilde{\omega}|_{L^2} \leq \frac{\mu}{8} |\Delta \tilde{\omega}|^2 + \frac{c}{\mu} |\nabla \tilde{\omega}|^2 |\nabla \bar{\omega}|^2,
\end{aligned} \tag{3.3}$$

we find [cf. (2.47)]

$$\left| \int_0^t 2e^{\nu\tau} (B(\omega, \omega), \Delta \tilde{\omega}) d\tau \right| \leq \frac{3\mu}{8} \int_0^t e^{\nu\tau} |\Delta \tilde{\omega}|^2 d\tau + \frac{cN_2}{\mu} \int_0^t e^{\nu\tau} |\nabla \tilde{\omega}|^2 d\tau. \tag{3.4}$$

Moving the first term to the l.h.s. and using (2.22) along with (2.49) to obtain an $O(\varepsilon)$ bound for the integral on the second term, we conclude that there exist $T_1(f, \omega(0); \mu)$ and $M_1(|\nabla^3 f|_{L_t^\infty L_x^2}^2 + |\nabla \partial_t f|_{L_t^\infty L_x^2}^2; \mu)$, such that for all $t \geq T_1$,

$$|\nabla \tilde{\omega}(t)|^2 + \mu \int_t^{t+1} |\Delta \tilde{\omega}(\tau)|^2 d\tau \leq \varepsilon M_1. \tag{3.5}$$

Proceeding along similar lines (see [1] for more details), for $s = 2, 3, \dots$, we have $T_s(f, \omega(0); \mu)$ and $M_s(|\nabla^{s+2} f|_{L_t^\infty L_x^2}^2 + |\nabla^s \partial_t f|_{L_t^\infty L_x^2}^2; \mu)$ such that

$$|\nabla^s \tilde{\omega}(t)|^2 + \mu \int_t^{t+1} |\nabla^{s+1} \tilde{\omega}(\tau)|^2 d\tau \leq \varepsilon M_s \quad \forall t \geq T_s. \quad (3.6)$$

When the forcing is independent of time, $\partial_t f = 0$, the regularity results of the NSE allow one to conclude that there exists a global attractor \mathcal{A} whose Hausdorff dimension is bounded as [4, 7]

$$\dim_H \mathcal{A} \leq c_S G^{2/3} (1 + \log G)^{1/3} \quad (3.7)$$

where $G := |\nabla^{-1} f|_{L^2} / \mu^2$ is the Grashof number and c_S is an absolute constant. It has long been known (and easily shown by some computation) that $\dim_H \mathcal{A} = 0$ for $G \leq G_0$, that is, \mathcal{A} reduces to a single stable steady state for sufficiently small Grashof number. Using (3.6) for $s = 3$ and proceeding as in [1], one can show that there exists $\varepsilon_*(|\nabla^2 f|; \mu)$ such that

$$\dim_H \mathcal{A} = 0 \quad \text{for all } \varepsilon \leq \varepsilon_*. \quad (3.8)$$

The proof is essentially identical to that in [1] and we shall not repeat it here.

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Received xxxx 20xx; revised xxxx 20xx.

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