

Minimum Connected Transversals in Graphs: New Hardness Results and Tractable Cases Using the Price of Connectivity*

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Abstract. We perform a systematic study in the computational complexity of the connected variant of three related transversal problems: VERTEX COVER, FEEDBACK VERTEX SET, and ODD CYCLE TRANSVERSAL. Just like their original counterparts, these variants are NP-complete for general graphs. A graph G is H -free for some graph H if G contains no induced subgraph isomorphic to H . It is known that CONNECTED VERTEX COVER is NP-complete even for H -free graphs if H contains a claw or a cycle. We show that the two other connected variants also remain NP-complete if H contains a cycle or claw. In the remaining case H is a linear forest. We show that CONNECTED VERTEX COVER, CONNECTED FEEDBACK VERTEX SET, and CONNECTED ODD CYCLE TRANSVERSAL are polynomial-time solvable for sP_2 -free graphs for every constant $s \geq 1$. For proving these results we use known results on the price of connectivity for vertex cover, feedback vertex set, and odd cycle transversal. This is the first application of the price of connectivity that results in polynomial-time algorithms.

Keywords: vertex cover, feedback vertex set, odd cycle transversal, price of connectivity

1 Introduction

We consider graph modification problems, where the aim is to test if a given graph can be modified to have some desired property using a small number of graph operations. Many such problems can be formulated as a vertex transversal problem. That is, the modified graph, obtained after deleting a number of vertices of the original graph, must not contain any graph of a prescribed family \mathcal{F} . More formally, we are looking for a small \mathcal{F} -transversal of a graph $G = (V, E)$, where an \mathcal{F} -transversal of G is a subset $S \subseteq V$ such that $G - S$ is \mathcal{F} -free. In other words, S intersects every subset of V that induces a subgraph isomorphic to a graph in \mathcal{F} . For instance, let P_k denote the path on k vertices. Then a set S is a *vertex cover* if S is a $\{P_2\}$ -transversal. Note that $V - S$ is an independent set. A set S is a *feedback vertex set* if S is an \mathcal{F} -transversal for the infinite family $\mathcal{F} = \{C_3, C_4, C_5, \dots\}$ (where C_k is the cycle on k vertices). In this case, the graph $G - S$ is a forest. To give another example, a set S is an *odd cycle*

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transversal if S is an \mathcal{F} -transversal for the infinite family $\mathcal{F} = \{C_3, C_5, C_7, \dots\}$. In this case, the graph $G - S$ is bipartite.

Vertex covers, feedback vertex sets, and odd cycle transversals are well-studied concepts. The corresponding decision problems that ask whether a given graph has a vertex cover, feedback vertex set, or odd cycle transversal of size at most k for some given integer k are called VERTEX COVER, FEEDBACK VERTEX SET and ODD CYCLE TRANSVERSAL, respectively. These problems are well known to be NP-complete [14,20]. In this paper we focus on the connected variants of these problems. We say that a vertex cover S of a graph G is connected if S induces a connected subgraph of G . The notions of a connected feedback vertex set and connected odd cycle transversal are defined in a similar way.

CONNECTED VERTEX COVER

Instance: a graph G and an integer k .

Question: does G have a connected vertex cover S with $|S| \leq k$?

CONNECTED FEEDBACK VERTEX SET

Instance: a graph G and an integer k .

Question: does G have a connected feedback vertex set S with $|S| \leq k$?

CONNECTED ODD CYCLE TRANSVERSAL

Instance: a graph G and an integer k .

Question: does G have a connected odd cycle transversal S with $|S| \leq k$?

1.1 Related Work

The three connected problem variants are known to be NP-complete. It is therefore a natural question whether there exist special graph classes for which they become polynomial-time solvable. Below we briefly survey known hard and tractable results for each of the three problems.

Already in the seventies, Garey and Johnson [15] proved that CONNECTED VERTEX COVER is NP-complete even for planar graphs of maximum degree 4. Later, Priyadarsini and Hemalatha [25] strengthened this result to 2-connected planar graphs of maximum degree 4, and Fernau and Manlove [13] strengthened it to planar bipartite graphs of maximum degree 4. Wanatabe, Kajita, and Onaga [32] proved NP-completeness of CONNECTED VERTEX COVER for 3-connected graphs. More recently, Munaro [23] established NP-completeness of CONNECTED VERTEX COVER for line graphs of planar cubic bipartite graphs and for planar bipartite graphs of arbitrarily large girth. Ueno, Kajitani, and Gotoh [31] proved that CONNECTED VERTEX COVER can be solved in polynomial time for graphs of maximum degree at most 3 and for trees. Escoffier, Gourvès, and Monnot [12] improved the latter result by showing that CONNECTED VERTEX COVER is polynomial-time solvable for chordal graphs. It is known that VERTEX COVER is polynomial-time solvable for chordal graphs as well. The same authors proposed to study the complexity of CONNECTED VERTEX COVER for other graph classes for which VERTEX COVER is polynomial-time solvable.

Grigoriev and Sitters [17] proved that CONNECTED FEEDBACK VERTEX SET is NP-complete for planar graphs with maximum degree 9. Besides this result not much is known on the computational complexity of this problem except that it is fixed parameter tractable when parameterized by k [21]. The CONNECTED ODD CYCLE TRANSVERSAL has been mainly studied from parameterized point of view [9,10]. It is implicit in these works that CONNECTED ODD CYCLE TRANSVERSAL is NP-complete,

though we were not able to find any proof of this (simple) result. For the sake of completeness, we note now that it is implied by the stronger results we present in Theorems 2 and 4.

1.2 Our Results.

We continue the investigations on the complexity of CONNECTED VERTEX COVER and also initiate a complexity study of CONNECTED FEEDBACK VERTEX SET and CONNECTED ODD CYCLE TRANSVERSAL. Our paper consists of two parts. In the first part (Section 2) we prove new hardness results and in the second part (Section 3) new tractable results. As we explain below we try to do so in a systematic way. We refer to Table 1 for a survey of our results together with some related known results. This table leads to some natural open problems, which we discuss in Section 4, together with some other directions for future work.

The *girth* of a graph is the length of a shortest cycle in it. We prove that CONNECTED FEEDBACK VERTEX SET and CONNECTED ODD CYCLE TRANSVERSAL are NP-complete on graphs of girth at least p for any fixed constant $p \geq 3$. In order to obtain this result for CONNECTED ODD CYCLE TRANSVERSAL we first prove NP-completeness of ODD CYCLE TRANSVERSAL for graphs of girth at least p for any fixed constant $p \geq 3$ (analogous results were already known for VERTEX COVER and FEEDBACK VERTEX SET). In the same section we also show that CONNECTED FEEDBACK VERTEX SET and CONNECTED ODD CYCLE TRANSVERSAL are NP-complete for line graphs.

The above hardness results imply that CONNECTED FEEDBACK VERTEX SET and CONNECTED ODD CYCLE TRANSVERSAL are NP-complete on H -free graphs whenever H is a graph that contains a cycle or a claw. Due to the results of Munaro [23] the same holds for CONNECTED VERTEX COVER. Moreover, the same holds for the original variants FEEDBACK VERTEX SET and ODD CYCLE TRANSVERSAL but not for VERTEX COVER, which is polynomial-time solvable for claw-free graphs. Due to the hardness results we can restrict ourselves to H -free graphs where H is a *linear forest*, that is, the disjoint union of one or more paths.

We prove that CONNECTED VERTEX COVER, CONNECTED FEEDBACK VERTEX SET, and CONNECTED ODD CYCLE TRANSVERSAL are polynomial-time solvable for sP_2 -free graphs for every constant $s \geq 1$. Since VERTEX COVER is polynomial-time solvable for sP_2 -free graphs for any $s \geq 1$,³ our result for CONNECTED VERTEX COVER restricted to sP_2 -free graphs can be seen as a new step in the aforementioned research direction of Escoffier, Gourvès, and Monnot [12].

To prove our tractability results we first focus on the original problems VERTEX COVER, FEEDBACK VERTEX SET, and ODD CYCLE TRANSVERSAL. In particular we will use the enumeration algorithms of Tsukiyama, Ide, Ariyoshi and Shirakawa [30] and Schwikowski and Speckenmeyer [28] for listing all maximal independent sets and all maximal feedback vertex sets, respectively. We will also use the fact that the number of maximal independent sets is polynomially bounded for sP_2 -free graphs [1]. However, we need to prove analogous results for feedback vertex sets and odd cycle transversals. This leads to polynomial-time enumeration algorithms for minimal feedback vertex sets of an sP_2 -free graph and minimal odd cycle transversals of an sP_2 -free graph. What remains to do is to relate these results to the connected problem variants and below we describe how to do this.

We need to consider the effect of adding the connectivity constraint on the minimum size of an \mathcal{F} -transversal for a graph family \mathcal{F} . This effect is in fact measured

³ This follows from combining results of [1,30], as explained in more detail in Section 3.

	girth p	line graphs	sP_2 -free
CON. VERTEX COVER	NP-c [23]	NP-c [23]	P
CON. FEEDBACK VERTEX SET	NP-c	NP-c	P
CON. ODD CYCLE TRANSVERSAL	NP-c	NP-c	P
VERTEX COVER	NP-c [24]	P [27]	P [1,30]
FEEDBACK VERTEX SET	NP-c [24]	NP-c [29]	P
ODD CYCLE TRANSVERSAL	NP-c	NP-c	P

Table 1. The complexities of the three connected transversal problems together and the original transversal problems for graphs of girth at least p for every constant $p \geq 3$, line graphs, or sP_2 -free graphs for any $s \geq 1$. The results for VERTEX COVER and FEEDBACK VERTEX SET on graphs of arbitrarily large girth follow from Poljak’s construction [24] (see also Section 2). All unreferenced results are new results from this paper.

by a known concept called the *price of connectivity*. This concept was coined by Cardinal and Levy [8] for vertex cover, but can be defined for any graph property for which a connected variant is meaningful: for a class of graphs \mathcal{G} and graph property π , the price of connectivity is the worst-case ratio $\pi'(G)/\pi(G)$ over all connected graphs $G \in \mathcal{G}$, where $\pi(G)$ and $\pi'(G)$ denote the smallest subset and smallest connected subset, respectively, of the vertices of G satisfying π . Apart from further results on vertex cover [4,5,6], the price of connectivity has been studied for dominating set [4,5,7,11,33], face hitting set [17,26], and feedback vertex set [2], until in [18] known results for vertex cover and feedback vertex set for H -free graphs were extended in a larger framework of \mathcal{F} -transversals. In particular, it is known that for sP_2 -free graphs demanding connectivity results only in an additive constant with respect to the size of a smallest vertex cover, feedback vertex set, or odd cycle transversal [2,18]. This is exactly what we need to finalize our proofs.

2 Hardness Results

In this section we prove our hardness results (see also Table 1). We first need to introduce some terminology. The *line graph* $L(G)$ of a graph $G = (V, E)$ has the edges of E as its vertices, and two vertices e_1 and e_2 of $L(G)$ are adjacent if and only if e_1 and e_2 have an end-vertex in common in G . The *claw* is the graph with vertices s, t_1, t_2, t_3 and edges st_1, st_2, st_3 . It is well-known and readily seen that every line graph is claw-free.

Our proofs for the ODD CYCLE TRANSVERSAL and CONNECTED ODD CYCLE TRANSVERSAL problems can be found in Section 2.1 and for the CONNECTED FEEDBACK VERTEX SET problem in Section 2.2.

2.1 Odd Cycle Transversal and Connected Odd Cycle Transversal

We will first prove that ODD CYCLE TRANSVERSAL and CONNECTED ODD CYCLE TRANSVERSAL are NP-complete for line graphs.

A *2-factor* of a graph G is a set of cycles such that every vertex of G belongs to exactly one of them. Let EVEN CYCLE FACTOR denote the problem of deciding if a graph has an *even cycle factor*, that is, a 2-factor in which each cycle is even.

We need the following lemmas due to Hell, Kirkpatrick, Kratchovíl, and Kříž.

Lemma 1 ([19]). EVEN CYCLE FACTOR is NP-complete.

We also need the next lemma. Here, we let $G[S]$ denote the subgraph of a graph G induced by a subset $S \subseteq V(G)$.

Lemma 2. *Let G be a graph on n vertices and m edges. If $L(G)$ has an odd cycle transversal of size at most $m - n$, then G has an even cycle factor.*

Proof. Let T be the vertices of $L(G)$ that do not belong to the odd cycle transversal. As $L(G)$ has m vertices, T contains at least n vertices. By definition, $L(G)[T]$ is bipartite which means in particular that the neighbours of each vertex are independent. Thus, as $L(G)$ is a line graph and so claw-free, no vertex in $L(G)[T]$ has degree greater than 2. Hence $L(G)[T]$ is the disjoint union of a set of paths and even cycles. Each path on r vertices in $L(G)$ corresponds to a path on $r + 1$ vertices in G , and each cycle on r vertices in $L(G)$ corresponds to a cycle on r vertices in G . Thus, if p of the components of $G[T]$ are paths, then the number of vertices of G incident with T is $|T| + p$. As T has at least n vertices and G has exactly n vertices, we must have $p = 0$, and so T is the disjoint union of even cycles that corresponds to an even cycle factor of G . \square

Theorem 1. ODD CYCLE TRANSVERSAL is NP-complete for line graphs.

Proof. We reduce from EVEN CYCLE FACTOR, which is NP-complete by Lemma 1. Let G be an instance of this problem with n vertices and m edges. We claim that G has an even cycle factor if and only if its line graph $L(G)$ has an odd cycle transversal of size at most $m - n$.

If G has an even cycle factor, then $L(G)$ contains a set T of n vertices that induce a disjoint union of cycles. Hence $S = V(L(G)) \setminus T$ is an odd cycle transversal of $L(G)$ that has size $|L(G)| - n = m - n$. To complete the proof, we apply Lemma 2. \square

Theorem 2. CONNECTED ODD CYCLE TRANSVERSAL is NP-complete for line graphs.

Proof. As in the proof of Theorem 1 we reduce from EVEN CYCLE FACTOR. Let G be an instance of this problem with n vertices and m edges. We add a new vertex x to G and an edge between x and each vertex of G . We also add three vertices y_1, y_2, y_3 with edges xy_1, y_1y_2, y_2y_3 and y_3x . We call the resulting graph G' . We observe that G has an even cycle factor if and only if G' has an even cycle factor. Note that G' has $n + 4$ vertices and $m + n + 4$ edges. We construct the line graph $L(G')$ of G' and claim that G' has an even cycle factor if and only if $L(G')$ has a connected odd cycle transversal of size at most m .

First suppose that G' has an even cycle factor F . By construction, F contains the cycle on vertices x, y_1, y_2, y_3 , which we denote by C . As a consequence, no edge xu for $u \in V(G)$ belongs to a cycle of F . Then $L(G')$ contains a set T of $n + 4$ vertices that induce a disjoint union of cycles that do not contain any vertex xu for $u \in V(G)$. Hence $S = V(L(G')) \setminus T$ is an odd cycle transversal of $L(G')$ that has size $|L(G')| - |V(F)| = m + n + 4 - (n + 4) = m$ and that contains all vertices of the form xu . We need to show that S induces a connected subgraph of $L(G')$. This follows from the fact that every vertex $e = uv \in S \cap E(G)$ is adjacent to xu (and xv), which belongs to S , and the fact that the set $\{xu \mid u \in V(G)\}$ form a clique in $L(G')$.

Now suppose that $L(G')$ has a connected odd cycle transversal of size at most m . Then we can apply Lemma 2 (noting again that G' has $n + 4$ vertices and $m + n + 4$ edges) to find that G' has an even cycle factor. \square

For our next hardness result on CONNECTED ODD CYCLE TRANSVERSAL we first need to show the same result for ODD CYCLE TRANSVERSAL. For that we need the following lemma.

Lemma 3. *Let G' be a graph obtained from a graph G after subdividing an edge of G twice. Then the size of a minimum odd cycle transversal of G is equal to the size of a minimum odd cycle transversal of G' .*

Proof. Let uv be the edge of G that is subdivided twice. Let a, b be the two new vertices obtained after the subdividing so that G' contains the path $uabv$.

First suppose that S is a minimum odd cycle transversal of G . If one or two of u, v are in S , then S is an odd cycle transversal of G' (as we can safely place a, b in $G' - S$). The same is true if $u, v \notin S$, as any even cycle in $G - S$ is still even after subdividing e twice.

Now suppose that S' is a minimum odd cycle transversal of G' . If one or two of u, v are in S' , then a, b are not in S' by minimality of S' , meaning that S' is also an odd cycle transversal of G . Suppose that u, v are not in S' . If a or b is in S' , then we can safely replace a or b by u or v , respectively, to obtain a new odd cycle transversal of G' with the same size as S' . If none of u, a, b, v is in $G' - S'$, then, as contracting the path $uabv$ back to uv does not result in an odd cycle, we again find that S' is an odd cycle transversal of G . \square

Repeatedly applying Lemma 3 in combination with the fact that ODD CYCLE TRANSVERSAL is NP-complete (due to Theorem 1) implies the following theorem.

Theorem 3. *For every constant $p \geq 3$, ODD CYCLE TRANSVERSAL is NP-complete for graphs of girth at least p .*

We now show that we also have hardness when the problem asks for a connected transversal.

Theorem 4. *For every constant $p \geq 3$, CONNECTED ODD CYCLE TRANSVERSAL is NP-complete for graphs of girth at least p .*

Proof. We reduce from CONNECTED VERTEX COVER restricted to graphs of girth at least p . Recall that Munaro [23] proved that this problem is NP-complete. Let G be a graph of girth at least p . We may assume without loss of generality that G has at least two edges. Between each pair of adjacent vertices of G add a path with $2\lfloor \frac{p}{2} \rfloor$ edges. Let the graph obtained be G' and note that it also has girth at least p . We will show that G has a connected vertex cover of size at most k if and only if G' has a connected odd cycle transversal of size at most k .

Let S be a connected vertex cover of G of at most k . As every cycle in G' contains two vertices that are adjacent in G , the set S is also an odd cycle transversal of G' , and it is also connected in G' as G' contains all the edges of G .

Let S be a connected odd cycle transversal of G' of at most k . We can assume that all vertices of S belong to G (if this is not the case, then we can move all vertices from $S \setminus V(G)$ to $G' - S$ without creating an odd cycle in $G' - S$ or disconnecting $G'[S]$). As each pair of adjacent vertices in G belong to the odd cycle of G' that contains them and the path on $2\lfloor \frac{p}{2} \rfloor$ edges added between, at least one of them must belong to S . Thus S is a connected vertex cover of G (of size at most k). \square

Theorems 2 and 4 imply the following corollary.

Corollary 1. *Let H be a graph. Then CONNECTED ODD CYCLE TRANSVERSAL is NP-complete for H -free graphs if H contains a cycle or a claw.*

2.2 Connected Feedback Vertex Set

We will now consider the CONNECTED FEEDBACK VERTEX SET problem. We first prove the following result.

Theorem 5. *CONNECTED FEEDBACK VERTEX SET is NP-complete for line graphs.*

Proof. Let G be an instance of HAMILTONIAN PATH with $n \geq 3$ vertices and m edges. We will describe a reduction to CONNECTED FEEDBACK VERTEX SET. We add two new vertices x and y to G and an edge between x and each vertex of G and an edge between y and each vertex of G . We also add two vertices x' and y' and edges $x'x$ and $y'y$. We call the resulting graph G' . We observe that G has a Hamiltonian path if and only if G' has a Hamiltonian path from x' to y' . Note that G' has $n + 4$ vertices and $m + 2n + 2$ edges. We claim that G' has a Hamiltonian path if and only if its line graph $L(G')$ has a connected feedback vertex set of size at most $m + n - 1$.

First suppose that G' has a Hamiltonian path P (on $n+4$ vertices). Let xa and yb be the two edges of P that join x and y to $V(G)$. Then every edge xu with $u \in V(G) \setminus \{a\}$ and every edge yv with $v \in V(G) \setminus \{b\}$ does not belong to P . Let $T \subseteq V(L(G'))$ be the set of size $n + 3$ in G' formed by the edges of P . Hence $S = V(L(G')) \setminus T$ is a feedback vertex set of $L(G')$ that has size $m + 2n + 2 - (n + 3) = m + n - 1$ and that contains all vertices of the form xu and yv except the vertices xa and yb . We need to show that S induces a connected subgraph of $L(G')$. This follows from the fact that every vertex $e = uv \in S \cap E(G)$ is adjacent to xu and yv , and at least one of them belongs to S . Moreover, the set of vertices $\{xu \mid u \in V(G) \setminus \{a\}\}$ is a clique in $L(G')$. Finally, every vertex $yv \in S$ with $v \in V(G) \setminus \{a, b\}$ is adjacent to xu in $L(G')$.

Now suppose that $L(G')$ has a connected feedback vertex set S of size at most $m + n - 1$. Then $L(G') - S$ has at least $n + 3$ vertices and must be a disjoint union of paths (as $L(G')$ is a line graph and thus claw-free). Every induced path in $L(G')$ corresponds to a path in G' with one more vertex. Moreover, G' has $n + 4$ vertices in total. Hence $L(G') - S$ must induce a single path on $n + 3$ vertices. The corresponding path in G' has $n + 4$ vertices and is thus a Hamiltonian path of G' . \square

We now show that the restriction of CONNECTED FEEDBACK VERTEX SET to graphs of arbitrarily large girth p is NP-hard. We note that it is known that FEEDBACK VERTEX SET is NP-complete for graphs of girth at least p for every constant $p \geq 3$; this follows immediately from the well-known observation (see for example [3,22]) that if a graph G' is obtained from a graph G by subdividing an edge, then G has a feedback vertex set of size at most k if and only if G' does. In order to prove the result for CONNECTED FEEDBACK VERTEX SET, we reduce from CONNECTED VERTEX COVER restricted to graphs of girth at least p using essentially the same construction and arguments as in the proof of Theorem 4. In this case, we can simply say that for an instance G, k of CONNECTED VERTEX COVER, we construct an instance G', k of CONNECTED FEEDBACK VERTEX SET by adding a path of length $p - 1$ between each pair of adjacent vertices in G (so that each edge of G corresponds to a cycle of length p in G').

Theorem 6. *For every constant $p \geq 3$, CONNECTED FEEDBACK VERTEX SET is NP-complete for graphs of girth at least p .*

Theorems 5 and 6 imply the following corollary.

Corollary 2. *Let H be a graph. Then CONNECTED FEEDBACK VERTEX SET is NP-complete for H -free graphs if H contains a cycle or a claw.*

3 Polynomial-Time Results

In this section we prove our polynomial-time results (see also Table 1). The following lemma provides a useful technique that we will be able to apply several times. We say that a set of vertices in a graph is *connected* if it induces a connected subgraph.

Lemma 4. *Let P be a property of a set of vertices in a graph that if it holds for a set also holds for all supersets. Given a graph G , let $f(G)$ be the minimum size of a set of vertices in G with property P . Let \mathcal{G} be a graph class and let c be a constant. Suppose that for every graph $G \in \mathcal{G}$, all minimal sets with property P can be enumerated in polynomial time, and the minimum size of a connected set of vertices with property P is at most $f(G) + c$. Then a minimum size connected set of vertices of G with property P can be found in polynomial time.*

Proof. The algorithm for finding a minimum size connected set with property P is simple. Enumerate the minimal sets of G with property P , and, for each such set consider all the sets that can be obtained by adding c or fewer additional vertices and check whether each is connected. Return the smallest connected set found (if any). It is clear that the algorithm runs in polynomial time. We must prove correctness.

Let S be a minimum size connected set of vertices in G with property P . We will show that the algorithm finds S and so returns it (or another connected set with property P of the same size). We know that $|S| \leq f(G) + c$. From S remove as many vertices as possible until a minimal set S' with property P is obtained. Then S' is one of the sets that will be enumerated and as $|S'| \geq f(G)$, we have that $|S| - |S'| \leq c$ and so S will be found. \square

For an algorithm that enumerates all solutions to a problem, the *delay* is the maximum of the time taken before finding the first solution and that required between any pair of consecutive outputs. We need two well-known results. The first one is due to Balas and Yu (who gave a sharper bound, which we do not need for our purposes) and the second one is due to Tsukiyama, Ide, Ariyoshi, and Shirakawa.

Theorem 7 ([1]). *For every constant $s \geq 1$, the number of maximal independent sets of an sP_2 -free graph on n vertices is at most $n^{2s} + 1$.*

Theorem 8 ([30]). *For every constant $s \geq 1$, it is possible to enumerate all maximal independent sets of a graph G on n vertices and m edges with a delay of $O(nm)$.*

It is well-known that VERTEX COVER is polynomial-time solvable for sP_2 -free graphs; in fact this follows immediately from Theorems 7 and 8. Our proof of the same result for CONNECTED VERTEX COVER can be found in Section 3.1, for FEEDBACK VERTEX SET and CONNECTED FEEDBACK VERTEX SET in Section 3.2, and for ODD CYCLE TRANSVERSAL and CONNECTED ODD CYCLE TRANSVERSAL in Section 3.3.

3.1 Connected Vertex Cover

We consider the CONNECTED VERTEX COVER problem. We have the following result on the price of connectivity for vertex cover restricted to sP_3 -free graphs by Lemma 10 of a previous paper of four of the current authors. The constant $4s^2 + 2s - 10$ is determined in the proof of Lemma 10 of [2] (the statement of the lemma only mentions the existence of a constant without further specification).

Theorem 9 ([18]). *Let $s \geq 1$ and let G be a connected sP_3 -free graph. Let f be the size of a minimum vertex cover of G . Then the size of a minimum connected vertex cover of G is at most $f + 4s^2 + 2s - 10$.*

We can use Theorem 9 to determine the complexity of CONNECTED VERTEX COVER for sP_2 -free graphs.

Theorem 10. *For every constant $s \geq 1$, CONNECTED VERTEX COVER can be solved in polynomial time for sP_2 -free graphs.*

Proof. We know by Theorems 7 and 8 that all maximal independent sets, and thus all minimal vertex covers, of sP_2 -free graphs can be enumerated in polynomial time. Thus, as the property of being a vertex cover holds for supersets, the theorem follows from Theorem 9 and Lemma 4 (note that we may restrict ourselves to connected input graphs). \square

3.2 Feedback Vertex Set and Connected Feedback Vertex Set

To prove our results for FEEDBACK VERTEX SET and CONNECTED FEEDBACK VERTEX SET we need the following result of Schwikowski and Speckenmeyer [28].

Theorem 11 ([28]). *It is possible to enumerate all minimal feedback vertex sets of a graph G on n vertices and m edges with a delay of $O(n^3 + n^2m)$.*

We also need the following lemma.

Lemma 5. *For every constant $s \geq 1$ there is a constant c_s such that the number of minimal feedback vertex sets of an sP_2 -free graph G on n vertices is $O(n^{c_s})$.*

Proof. Let S be a minimal feedback vertex set of G . Let $F_S = G - S$, which, by definition, is a forest. Let $F'_S \subset F_S$ contain each vertex of degree at least 2 in F_S plus one vertex (arbitrarily chosen) from each component of F_S isomorphic to P_2 (if there are any). We say that F'_S is a *skeleton* of F_S . Let $\ell(F'_S) = V(F_S) \setminus V(F'_S)$ and notice that $\ell(F'_S)$ is an independent set and each of its vertices has at most one neighbour in F'_S . We say that $\ell(F'_S)$ is the *leave* of F'_S .

Claim. $|V(F'_S)| \leq 3s^2 - 5s + 2$.

To prove the claim, first consider the following subset A of $V(F'_S)$: all isolated vertices of F'_S , one arbitrarily chosen vertex from each P_2 , and every vertex of degree 1 from each other tree. Pair each vertex in A with one of its neighbours in $\ell(F'_S)$ (it must have at least one by the definition of F'_S). These pairs induce a collection of pairwise non-adjacent P_2 's, and, as G is sP_2 -free, $|A| \leq s - 1$. Notice that from a vertex in A , it is not possible to find a path containing more than $3s - 2$ vertices within F'_S since a longer path contains sP_2 as an induced subgraph. We choose a path of length at most $3s - 2$ from each vertex $v \in A$ to a fixed vertex of the component of F'_S containing v . In this way the union of all these paths cover all the vertices of F'_S , and the claim follows.

Let $J(F'_S) \subseteq S$ be the set of vertices in $G - V(F'_S)$ that have at most one neighbour in F'_S . Notice that $\ell(F'_S) \subseteq J(F'_S)$.

Claim. $\ell(F'_S)$ is a maximal independent set in $G[J(F'_S)]$.

To prove the claim, suppose instead that there is a vertex $v \in J(F'_S) \setminus \ell(F'_S)$ that has no neighbour in $\ell(F'_S)$. Then $G[V(F_S) \cup \{v\}]$ is a forest as v has at most one neighbour in F_S , and so $S \setminus \{v\}$ is a feedback vertex set of G , contradicting the minimality of S and proving the claim.

We can find minimal feedback vertex sets of G as follows. First we choose a skeleton on at most $3s^2 - 5s + 2$ vertices, and then, from amongst those vertices that have at most one neighbour in the chosen skeleton, we choose a maximal independent set. Then the set of vertices in neither the forest nor the independent set is tested to see whether it forms a minimal feedback vertex set. If we consider all possible choices of forest and independent set, then each minimal feedback set will be found as at some point a corresponding skeleton and its leave will be chosen. By our first claim, there are $O(n^{3s^2})$ choices for the forest, and, by Theorem 7, there are $O(n^{2s})$ choices for the independent set. The theorem is proved. \square

We also need the following result on the price of connectivity for feedback vertex set due to Belmonte, van 't Hof, Kamiński, and Paulusma. Notice that this result holds even for sP_3 -free graphs. The constant $12s^2 - 2s - 2$ is determined in the proof of Lemma 9 of [2].

Theorem 12 ([2]). *Let $s \geq 1$ and let G be a connected sP_3 -free graph. Let f be the size of a minimum feedback vertex set of G . Then the size of a minimum connected feedback vertex set of G is at most $f + 12s^2 - 2s - 2$.*

We are now ready to prove our result on the complexities of FEEDBACK VERTEX SET and CONNECTED FEEDBACK VERTEX SET restricted to sP_2 -free graphs.

Theorem 13. *For every constant $s \geq 1$, FEEDBACK VERTEX SET and CONNECTED FEEDBACK VERTEX SET can be solved in polynomial time for sP_2 -free graphs.*

Proof. We will show that there are polynomial-time algorithms that find the minimum size sets. The theorem, which concerns decision problems, follows.

Let G be an sP_2 -free graph. We may assume without loss of generality that G is connected. By the combination of Lemma 5 with Theorem 11, we can in fact enumerate all minimal feedback vertex sets of G in polynomial time. Thus the result for FEEDBACK VERTEX SET follows.

For CONNECTED FEEDBACK VERTEX SET, we can apply Lemma 4. We need only note that the property of being a feedback vertex set is inherited by supersets and that, by Theorem 12, within the class of sP_2 graphs, the minimum size of a connected feedback vertex set differs from the size of a minimum size set by at most a constant. \square

3.3 Odd Cycle Transversal and Connected Odd Cycle Transversal

We now consider the ODD CYCLE TRANSVERSAL problem and its connected variant. Again we can use a result on the price of connectivity, which is also covered by Lemma 10 of [18].

Theorem 14 ([18]). *Let $s \geq 1$ and let G be a connected sP_3 -free graph. Let f be the size of a minimum odd cycle transversal of G . Then the size of a minimum connected odd cycle transversal of G is at most $f + 4s^2 + 2s - 10$.*

We can now prove our final result.

Theorem 15. *For every constant $s \geq 1$, ODD CYCLE TRANSVERSAL and CONNECTED ODD CYCLE TRANSVERSAL can be solved in polynomial time for sP_2 -free graphs.*

Proof. Let S be a minimal odd cycle transversal of an sP_2 -free graph G . Let $F_S = G - S$. Then F_S is a bipartite graph. We choose a bipartition (X, Y) of F_S such that X has maximum size (so every vertex in Y has at least one neighbour in X). Then X is a maximal independent set of G , as otherwise there exists a vertex $u \in S$ not adjacent to any vertex of X , and thus $S \setminus \{u\}$ is an odd cycle transversal of G , contradicting the minimality of S . Moreover, by a similar argument, Y is a maximal independent set of $G - X$.

We describe a procedure to find all minimal odd cycle transversals of G , and, as the minimum size transversals of G will be amongst them this provides an algorithm for ODD CYCLE TRANSVERSAL. We enumerate all maximal independent sets of G , and for each maximal independent set X , we enumerate all maximal independent sets

of $G - X$. For each such set Y , we note that $V(G) \setminus (X \cup Y)$ is an odd cycle transversal of G . By the arguments above, we will find every minimal odd cycle transversal in this way, and, by Theorems 7 and 8, this takes polynomial time.

For **CONNECTED ODD CYCLE TRANSVERSAL**, we can now apply Theorem 14 and Lemma 4 as the property of being an odd cycle transversal holds for supersets. \square

Remark 1. Just as for minimal vertex covers, we can compute all minimal feedback vertex sets and all minimal odd cycle transversals of an sP_2 -free graph in polynomial time for every constant $s \geq 1$.

4 Conclusions

We first showed that just like the **CONNECTED VERTEX COVER** problem, the **CONNECTED FEEDBACK VERTEX SET** and **CONNECTED ODD CYCLE TRANSVERSAL** problems are NP-complete for H -free graphs if H contains a cycle or a claw. Hence, we could restrict ourselves to graphs H that are linear forests. We then proved that all three problems can be solved in polynomial time for sP_2 -free graphs.

Our main goal was to show an application of the price of connectivity for cycle transversals (Theorems 9, 12, and 14). In fact these three structural results on the price of connectivity hold even for sP_3 -free graphs. Hence, a natural future research direction would be to try to extend our algorithmic results to sP_3 -free graphs. However, then we can no longer bound the number of minimal transversals as before, so the situation is not clear.

It would also be interesting to research whether it is possible to compute, just as for the three original problems (see Remark 1), all minimal connected transversals in polynomial time for the three variants studied, even when we restrict the input to $2P_2$ -free graphs. With respect to this question, we recall an open problem of Golovach, Heggernes, and Kratsch [16]. They asked if there exists a graph class with a polynomial number of minimal vertex covers but an exponential number of minimal connected vertex covers.

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