

**Mock modular index of M2-M5 brane systems**Tadashi Okazaki<sup>\*</sup>*Department of Physics and Center for Theoretical Sciences,  
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We present Bogomolny-Prasad-Sommerfield (BPS) indices of the supergroup Wess-Zumino-Witten (WZW) models that live on intersecting M2-M5-brane systems. They can encode data of the stretched M2-branes between M5-branes and count the BPS states. They are generally expressed in terms of mock theta functions via the Kac-Wakimoto character formula of the affine Lie superalgebra. We give an explicit expression of the index for the  $PSL(2|2)_{k=1}$  WZW model in terms of the second-order multivariable Appell-Lerch sum. It indicates that wall crossing occurs in the BPS state counting due to the  $C$  field on the M5-branes.

DOI: [10.1103/PhysRevD.96.026017](https://doi.org/10.1103/PhysRevD.96.026017)**I. INTRODUCTION**

We recently proposed that a particular topologically twisted field theory arising from the intersection of M2-branes and M5-branes is described by a supergroup Wess-Zumino-Witten (WZW) model [1]. In this paper, we explore this description, and in particular, we present a Bogomolny-Prasad-Sommerfield (BPS) index for such a WZW model. It encodes data of the stretched M2-branes between the M5-branes. When all the M2-branes are suspended between the M5-branes, the index can be computed via the Kac-Wakimoto character formula [2,3]. This gives an explicit expression for the index in the case of supergroup  $PSL(2|2)$ . The result can be expressed in terms of Jacobi theta functions and second-order multivariable Appell-Lerch sums. However, while it is holomorphic, it is not modular. Based on Zweegers' method [4] and the results of Dabholkar *et al.* [5] and Ashok *et al.* [6], we demonstrate how to complete the expressions to give a modular index, which would contribute to the torus partition function. The Appell-Lerch sum of order 2 in the counting function is suggestive of the occurrence of the wall-crossing phenomenon due to the dependence of the Fourier expansion on the parameter region. The structure of the paper is as follows.

In Sec. II, we review some background material, including notation for supergroups and summarize our previous work [1]. Then, we review the main result of that paper, that, after topological twisting, a certain configuration of M2-branes stretched between M5-branes gives rise to a supergroup WZW model. We also comment on type-IIB brane configurations related to these M-branes configurations.

In Sec. III, we review properties of affine Lie superalgebras, which are relevant to the supergroup WZW index derived in

this paper. As well as defining notation, we discuss the important concept of atypical modules. In Sec. IV, the connection between these atypical modules and brane configurations is explained. This relation for M-branes is similar to the relation proposed by Mikhaylov and Witten [7] for branes in type IIB. We discuss the connection between these realizations of atypical modules from brane configurations.

Section V contains the main result of this paper, the derivation of an index for the supergroup WZW models. The details of the index are explained, including an explicit evaluation for the case of supergroup  $PSL(2|2)$ . The result is a holomorphic but not modular expression. In Sec. VI, we adapt results in Ref. [6] to find the modular completion of this index. We comment on the relation to wall crossing in counting of the BPS states of the M2-M5 system and black hole microstates. In Sec. VII, we summarize our results and discuss future directions.

**II. M2-M5 SYSTEM AND SUPERGROUP WZW MODEL**

We start with some preliminaries, reviewing some essential properties of supergroups before summarizing our previous results. In particular, we briefly review the M2-M5-branes construction and the resulting supergroup WZW model. We also give a description of type-IIB brane configurations related to these M-brane configurations through compactification and T duality.

**A. Preliminaries**

To formulate our result in detail, we first fix our notation and conventions. Let  $\mathfrak{sg} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be the Lie superalgebra where  $\mathfrak{g}_0$  and  $\mathfrak{g}_1$  are, respectively, the even and odd parts of the superalgebra  $\mathfrak{sg}$ . The bilinear form  $(\cdot, \cdot) : \mathfrak{sg} \otimes \mathfrak{sg} \rightarrow \mathbb{C}$  obeys the properties [8]

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$$(a, b) = 0 \quad \text{for } a \in \mathfrak{g}_0, \quad b \in \mathfrak{g}_1, \quad (\text{even}) \quad (2.1)$$

$$(a, b) = (-1)^{\deg a \cdot \deg b} (b, a), \quad (\text{supersymmetric}) \quad (2.2)$$

$$([a, b], c) = (a, [b, c]) \quad (\text{invariant}), \quad (2.3)$$

and the Lie superbracket  $[\cdot, \cdot]: \mathfrak{sg} \otimes \mathfrak{sg} \rightarrow \mathfrak{sg}$  satisfies the axioms [8]

$$[a, b] = ab + (-1)^{\deg a \cdot \deg b} ba, \quad (2.4)$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg a \cdot \deg b} [b, [a, c]] \quad (2.5)$$

where we have assigned the grade such that  $\deg a = 0$  for  $a \in \mathfrak{g}_0$  while  $\deg a = 1$  for  $a \in \mathfrak{g}_1$ . The relation (2.5) is the Jacobi identity.

Let  $\mathfrak{h}$  be the Cartan subalgebra of  $\mathfrak{sg} = \mathfrak{gl}(N|M)$ , which is a set of diagonal matrices with basis  $\{E_{1,1}, \dots, E_{N,N}; E_{N+1,N+1}, \dots, E_{N+M,N+M}\}$  where  $E_{ij}$  is the matrix of which the entries are all zero except for the  $ij$  entry, which is 1.

Let  $\{\epsilon_1, \dots, \epsilon_N; \delta_1, \dots, \delta_M\}$  be the basis for the dual space  $\mathfrak{h}^*$ . The bilinear form on  $\mathfrak{h}^*$  can be defined by  $(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$  and  $(\epsilon_i, \delta_j) = 0$ . We denote a set of roots by  $\Delta = \Delta_0 \cup \Delta_1$ , where

$$\Delta_0 = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq N\} \cup \{\delta_k - \delta_l | 1 \leq k \neq l \leq M\}, \quad (2.6)$$

$$\Delta_1 = \{\epsilon_i - \delta_k | 1 \leq i \leq N, 1 \leq k \leq M\}. \quad (2.7)$$

Simple roots are the elements  $\alpha_i \in \mathfrak{h}^*$  that obey  $\alpha_i(h_j) = a_{ij}$  where  $A = (a_{ij})$  is a symmetrized Cartan matrix. Let  $\Pi$  be a set of simple roots,  $Q := \mathbb{Z}\Pi$  be the root lattice, and  $Q^+ = \mathbb{Z}_{\geq 0}\Pi$ . We define a set of positive roots by  $\Delta^+ = \Delta \cap Q^+$ . A set  $\Pi$  of simple roots specifies the decomposition of  $\Delta$  into positive and negative roots  $\Delta = \Delta^+ \cup \Delta^-$  and the Borel decomposition  $\mathfrak{sg} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$  where  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$  is the Borel subalgebra and  $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{sg}_{\pm\alpha}$  with  $[\mathfrak{h}, \mathfrak{n}^+] \subset \mathfrak{n}^+$ ,  $[\mathfrak{h}, \mathfrak{n}^-] \subset \mathfrak{n}^-$ .

The Weyl vector is defined by  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha$ , and it depends on a choice of the set of positive roots. The Weyl group  $W \subset GL(\mathfrak{h}^*)$  of  $\mathfrak{sg}$  is the group generated by the reflections  $r_\alpha(\Lambda) = \Lambda - \frac{2(\alpha, \Lambda)\alpha}{(\alpha, \alpha)}$  with respect to nonisotropic roots  $\alpha \in \Delta_0$ . Let  $h^\vee$  be the dual Coxeter number, i.e., half of the eigenvalue of the Casimir operator associated to the bilinear form  $(\cdot, \cdot)$ . For  $h^\vee \neq 0$ , we define [2]

$$\begin{aligned} \Delta_0^\sharp &:= \{\alpha \in \Delta_0 | h^\vee(\alpha, \alpha) > 0\}, \\ W^\sharp &:= \{r_\alpha \in W | \alpha \in \Delta_0^\sharp\}. \end{aligned} \quad (2.8)$$

For  $h^\vee = 0$ , i.e.,  $\mathfrak{sg} = \mathfrak{gl}(N|N)$ ,  $\mathfrak{osp}(2N+2|2N)$ , and  $D(2, 1; \alpha)$ , the root system  $\Delta_0$  is a union of two orthogonal root subsystems. For  $\mathfrak{gl}(N|N)$ , we define  $\Delta_0^\sharp = \mathfrak{gl}(N)$  [9,10].

We set

$$\bar{\Delta}_0 = \left\{ \alpha \in \Delta_0 \mid \frac{1}{2} \alpha \notin \Delta \right\}, \quad \bar{\Delta}_1 = \{\alpha \in \Delta_1 | (\alpha, \alpha) = 0\} \quad (2.9)$$

and define

$$\text{sgn}^+(w) := (-1)^{l(w)}, \quad \text{sgn}^-(w) := (-1)^{m(w)}, \quad (2.10)$$

where  $l(w)$  is the length function on  $W$ , i.e., the number of reflections with respect to roots from  $\Delta_0^+$  appearing in the decompositions of  $w \in W$ , and  $m(w)$  is the number of reflections for the realization of  $w$  from  $\bar{\Delta}_0^+$ . In terms of the isotropic root  $\beta \in \Pi$ , we can define an odd reflection by [11]

$$r_\beta(\Delta^+) = (\Delta^+ \setminus \{\beta\}) \cup \{-\beta\}, \quad (2.11)$$

and it is also a set of simple roots for  $\mathfrak{sg}$  [12]. It turns out that any two sets of positive roots can be obtained from each other by applying a finite sequence of odd reflections.

A weight  $\lambda \in \mathfrak{h}^*$  is called dominant if  $\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \geq 0$  for all  $\alpha \in \Delta_0^+$ , strictly dominant if  $\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} > 0$  for all  $\alpha \in \Delta_0^+$ , and integral if  $\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$  for all  $\alpha \in \Delta_0^+$ . Let  $P$  be a set of integral weights and  $P^+$  be a set of dominant integral weight. We define

$$\mathbb{P}^+ = \{\lambda \in P^+ | (\lambda + \rho, \epsilon_i) \in \mathbb{Z}, (\lambda + \rho, \delta_k) \in \mathbb{Z}\}. \quad (2.12)$$

$\mathbb{P}^+$  does not depend on a choice of  $\Pi$ .

## B. M2-M5 system

The starting point for the brane construction is a set of  $N$  M2-branes suspended between two M5-branes. As is well known, the description of multiple M2-branes is given by supersymmetric Chern-Simons matter theories, either the Bagger-Lambert-Gustavsson (BLG) or Aharony-Bergman-Jafferis-Maldacena (ABJM) model. In fact, the configuration of a fuzzy funnel of M2-branes producing an M5-brane is described by the Basu-Harvey equation [13], a generalization of the Nahm equation. Requiring this to be a BPS equation of the M2-brane theory was a crucial ingredient used by Bagger and Lambert [14] in the derivation of the supersymmetry transformations, leading to the BLG model. Such BPS equations were already studied in the context of the a variety of M2-M5 systems [15,16], including a generalization of the Basu-Harvey equation by Berman and Copland [17], which was shown to apply to the BLG

model in Ref. [18]. However, as discussed in Ref. [19], it is not clear how the required geometry, a funnel with fuzzy 3-sphere cross section, can be realized for arbitrary numbers of M2-branes, although BPS configurations of the BLG or ABJM models describing the M2-M5 or D2-D4 systems have also been discussed in detail in Refs. [20,21].

When describing open M2-branes by the BLG or ABJM action, a crucial feature is that when we have a boundary a Chern-Simons term will give rise to a WZW model. On the other hand, the boundary of M2-branes on M5-branes corresponds to the self-dual strings in the M5-brane theory. The description of such systems has been considered in terms of boundary conditions for the Chern-Simons theories and through adding boundary degrees of freedom to restore the gauge symmetry of the Chern-Simons theory in the presence of a boundary [22–29].

In Ref. [1], with the aim of describing the internal dynamics of these strings, we choose a brane configuration in order to project out the transverse scalar degrees of freedom and to decouple the two-dimensional boundary theory from the “bulk” three-dimensional M2-brane world volume theory. Another motivation was the construction of Mikhaylov and Witten [7] building on results in Refs. [30–32] studying field theories in one higher dimension. There, four-dimensional twisted  $\mathcal{N} = 4$  Super-symmetric Yang-Mills (SYM) theory with a boundary was shown to give rise to a three-dimensional Chern-Simons theory with a supergroup.

In the type-IIB setting, this is realized for D3-branes ending on both sides of a single NS5-brane (see Fig. 1). When  $N$  D3-branes end on one side ( $x^6 < 0$ ) of a single NS5-brane at  $x^6 = 0$ , and  $M$  D3-branes on the other side ( $x^6 > 0$ ), the system supports four-dimensional  $\mathcal{N} = 4$   $U(N)$  SYM theory for  $x^6 < 0$  and  $U(M)$  SYM theory for  $x^6 > 0$ . With an appropriate choice of supercharges  $\mathcal{Q}$  via topological twist, the complete action of the effective theory is shown to be written as a sum of a  $\mathcal{Q}$ -exact term and a  $U(N|M)$  supergroup Chern-Simons theory at the common boundary at  $x^6 = 0$ ,

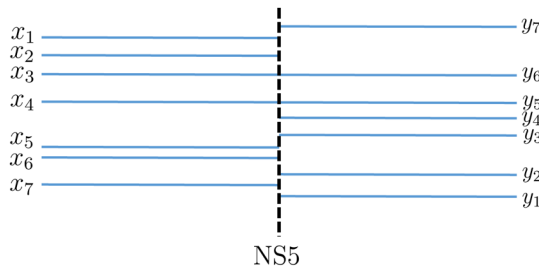


FIG. 1. A brane configuration with  $N$  D3-branes and  $M$  D3-branes terminating on a single NS5-brane at  $x^6 = 0$  from left and right, respectively. The horizontal lines are the D3-branes extending in the  $x^6$  direction. The sequences  $\{x_i\}$  and  $\{y_k\}$  of the heights of the D3-branes in the picture label the Ramond-Ramond (RR) charges of the D3-branes. In this example,  $N = M = 7$ .

$$S = \{\mathcal{Q}, \dots\} + \frac{i\mathcal{K}}{8\pi} \int \text{Str} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right), \tag{2.13}$$

where  $\mathcal{A}$  is a  $\mathfrak{u}(N|M)$ -valued field and  $\mathcal{K}$  is a complex parameter.

To meet the counterpart of the above construction in M theory, we considered the M2-branes to be suspended between two differently oriented M5-branes, labeled M5 and M5', which share a four-dimensional world volume. The details are summarized as

	0	1	2	3	4	5	6	7	8	9	10	
M5	◦	◦		◦	◦					◦	◦	
M5'	◦	◦				◦	◦			◦	◦	
M2	◦	◦	◦									

(2.14)

where ◦ are the directions spanned by the branes.

Taking the limit where the separation between the M5- and M5'-branes vanishes would produce a two-dimensional theory but still with two transverse scalar degrees of freedom corresponding to the freedom of the M2-branes to move in the  $(x^9, x^{10})$  directions. To relate the standard Chern-Simons matter theory to a supergroup WZW model, we first implement a topological twist. We consider the theory in the Euclidean space with the  $(x^0, x^1, x^9, x^{10})$  directions a K3 manifold, and the M2-brane wrapping a Riemann surface  $\Sigma \subset K3$ . Then, we twist the theory by identifying the twisted two-dimensional Euclidean Lorentz group as

$$SO(2)_{E'} = \text{diag}(SO(2)_E \times SO(2)_R),$$

where  $SO(2)_E$  is the Euclidean Lorentz group on the two-dimensional Riemann surface and  $SO(2)_R$  is the rotation group in the  $(x^9, x^{10})$  directions (see Ref. [33] for details).

Now, it turns out that in the twisted theory the fields combine, with the result that the Chern-Simons matter theory becomes a Chern-Simons theory with complexified gauge fields. The boundary action then becomes a WZW model with the bosonic part described by the complexified gauge group, i.e.,  $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$  for the BLG theory. However, the fermionic fields couple in such a way that the complete description is in terms of a supergroup. In other words, in this construction, the two groups on the boundary are identified together as the even part of a supergroup. Specifically, for the BLG theory, we arrive at a boundary  $PSL(2|2)$  WZW model. This theory is summarized in the following section. Of course, this can also be viewed as a special case arising from the ABJM action. However, note that in detail, while  $SU(2) \times SU(2) \rightarrow SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow PSL(2|2)$ , with gauge group  $U(N) \times U(N)$ , we have  $U(N) \times U(N) \rightarrow GL(N, \mathbb{C}) \times GL(N, \mathbb{C}) \rightarrow GL(N|N)$ .

Before proceeding, we also note that the ABJM theory can be seen to arise from a type-IIB brane configuration. The basic connection is that D3-branes wrapped on a circle T-dualize to D2-branes in type IIA and then lift to M2-branes. However, to get a Chern-Simons theory rather than SYM theory, the D3-branes are taken to intersect two NS5-branes at points on this circle, and furthermore  $k$  D5-branes also intersect at the position of one of the NS5-branes. This is summarized as

	0	1	2	3	4	5	6	7	8	9	
NS5	○	○	○	○	○	○					
D5	○	○	○	○	○					○	(2.15)
D3 <sup>+</sup>	○	○	○				+				
D3 <sup>-</sup>	○	○	○				-				

The  $x^6$  direction is taken to have period  $2\pi R$ , and the two NS5-branes are located at  $x^6 = 0$  and  $x^6 = \pi R$ . The D3-branes split into two stacks of D3-branes suspended between the NS5-branes, each stack covering one-half of the circle and distinguished by  $\pm$  in the above table. The D5-branes are located at  $x^6 = 0$ . Note that here the D3-branes are free to move in the  $(x^3, x^4)$  directions (see Fig. 2).

Finally, the intersection of the NS5-brane with the  $k$  D5-branes is deformed to form a  $(p, q)$ -5-brane web in the  $(x^5, x^9)$  plane. Specifically, the parts of the NS5- and D5-branes with positive  $x^5$  and  $x^9$  are separated from the parts with negative  $x^5$  and  $x^9$ . The two ‘‘corners’’ are then linked by a  $(1, k)$ -5-brane with a suitable orientation in the  $(x^5, x^9)$  plane to preserve supersymmetry. As explained in Refs. [34–36], this gives rise to a SYM theory with massive fundamental chiral multiplets, and integrating those out produces a Chern-Simons theory.

T-dualizing the brane configuration along  $x^6$  and then lifting to 11 dimensions results in M2-branes with world volume directions  $(x^0, x^1, x^2)$ , while the 5-branes become KK monopoles and D6-branes in type IIA, both of which

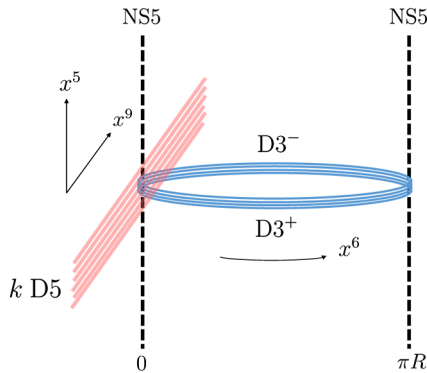


FIG. 2. The type-IIB configuration of the ABJM model. Here, the  $x^6$  direction is compact, and there are two stacks of D3-branes on the circle labeled by  $D3^+$  and  $D3^-$ .

arise from KK monopoles in 11 dimensions. The resulting low-energy background is given by a  $\mathbb{Z}_k$  orbifold in the  $\mathbb{C}^4$  transverse to the M2-branes.

It is interesting to observe that the type-IIB brane origin of ABJM theory contains D3-branes ending on an NS5-brane. As shown by Mikhaylov and Witten [7], after topological twisting, this intersection gives rise to a supergroup Chern-Simons theory at the intersection of the D3-branes with the NS5-brane. In the case, as here, with  $N$  D3-branes on either side of the NS5-brane, this can be interpreted as a codimension-1 defect in the four-dimensional  $\mathcal{N} = 4$  SYM theory, and at the defect, we have a  $U(N|N)$  supergroup Chern-Simons theory. It is tempting to speculate that the appearance of a supergroup in this way is related to the supergroup WZW model arising in Ref. [1].<sup>1</sup> However, the precise link is not clear as, in the case of M2-branes ending on an M5-brane, the supergroup theory arose due to the boundary for the M2-branes. In particular, the result did not require a supergroup Chern-Simons theory. However, it is certainly the case that the structure of the ABJM model is constrained, e.g., the conditions for such Chern-Simons matter theories to preserve large amounts of supersymmetry can be expressed in terms of supergroups [31,38,39].

Now, in order to relate to an M theory configuration with M5-branes, we need to introduce additional 5-branes in the type-IIB configuration. This has been discussed in the similar context of M2-branes between parallel M5-branes by Niarchos [28]. Of course, in the case of parallel 5-branes, the BPS index for the M strings has been calculated in Ref. [40] using various techniques including topological strings. However, the type-IIB construction as discussed by Niarchos can be used to provide an explicit Lagrangian description, albeit without all supersymmetry manifest.

In our case, the following additional D5-branes will give rise to the M5- and M5'-branes in 11 dimensions:

	0	1	2	3	4	5	6	7	8	9	
D5	○	○		○	○	○	○				(2.16)
D5'	○	○				○	○	○	○		

The complete brane configuration in type IIB now preserves two supercharges. However, this is not quite the right configuration, as in the 11-dimensional configuration, there is an obvious discrete symmetry relating the M5- and M5'-branes. In type IIB, we see that the D5-brane shares the world volume directions  $x^3$  and  $x^4$  with the NS5- and  $(1, k)$ -5-branes, while the D5'-brane has a lower-dimensional set of common directions. However, we can maintain this symmetry in the type-IIB configuration by taking the D5-brane to have embedding  $w_1 = w_2$  while

<sup>1</sup>Indeed, the appearance of supergroup WZW models in this way has subsequently been analyzed by Gaiotto and Rapčák [37] at junctions in webs of 5-branes which are filled by D3-branes.

the D5'-brane has  $w_1 = -w_2$ , where we define  $w_1 = x^3 + ix^4$  and  $w_2 = x^7 + ix^8$ . This preserves exactly the same supersymmetries in type IIB, while in 11 dimensions, this just corresponds to a change of coordinates. We can therefore schematically describe the D5- and D5'-branes embeddings as

$$\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 \text{D5} & \circ & \circ & & \nearrow & \nearrow & \circ & \circ & \nearrow & \nearrow & \\
 \text{D5}' & \circ & \circ & & \searrow & \searrow & \circ & \circ & \searrow & \searrow & 
 \end{array} \quad (2.17)$$

and these D5-branes would correspond to the following M5-branes:

$$\begin{array}{ccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 \text{M5} & \circ & \circ & & \nearrow & \nearrow & \circ & & \nearrow & \nearrow & \circ & \\
 \text{M5}' & \circ & \circ & & \searrow & \searrow & \circ & & \searrow & \searrow & \circ & 
 \end{array} \quad (2.18)$$

However, this is not the only way to introduce branes corresponding to the M5-branes in the type-IIB configuration. We can alternatively map the M5- and M5'-branes to NS5- and NS5'-branes in type IIB. Preserving the same supersymmetry, we can instead add the following NS5-branes (see Fig. 3):

$$\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 \text{NS5} & \circ & \circ & & \nearrow & \nearrow & \circ & & \nearrow & \nearrow & \circ \\
 \text{NS5}' & \circ & \circ & & \searrow & \searrow & \circ & & \searrow & \searrow & \circ
 \end{array} \quad (2.19)$$

Now, the map to 11 dimensions will result in the following M5-branes:

$$\begin{array}{ccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 \text{M5} & \circ & \circ & & \nearrow & \nearrow & \circ & & \nearrow & \nearrow & \circ & \\
 \text{M5}' & \circ & \circ & & \searrow & \searrow & \circ & & \searrow & \searrow & \circ & 
 \end{array} \quad (2.20)$$

In either of these cases (2.18) or (2.20), we end up with M5- and M5'-branes which share the  $(x^0, x^1)$  directions with the M2-branes; are at fixed  $x^2$  so they can provide a boundary for the M2-branes; and in the transverse space

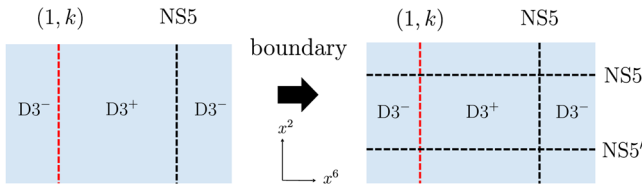


FIG. 3. The type-IIB configuration of the ABJM model with two boundaries of M5- and M5'-branes which correspond to NS5- and NS5'-branes. The D3-branes are in finite boxes in the  $(x^2, x^6)$  plane.

to the M2-branes, the M5- and M5'-branes share two directions and are orthogonal in the remaining space. Therefore, by simply changing coordinates in 11 dimensions, we can arrive at the brane configuration (2.14). Note also that in either type-IIB configuration, after introducing either D5- and D5'- or NS5- and NS5'-branes, the D3-branes can no longer move in the  $(x^3, x^4)$  directions.

In the type-IIB configuration, we will choose the case in which the M5- and M5'-branes are NS5- and NS5'-branes. The reason for this is that the boundary conditions for D3-branes ending on NS5-branes allow preservation of the full gauge symmetry, and in our M theory configuration, we took boundary conditions for the M2-branes so that the full gauge symmetry of the Chern-Simons theory could be preserved [41].

Now that we have a type-IIB configuration, we can consider generalizations of the M2-M5 system. In particular, we could have M2-branes ending on both sides of an M5-brane, and we could also consider more M5- or M5'-branes with M2-branes stretched between them. In type IIB, this would correspond to including D3-branes on both sides of the NS5- and NS5'-brane, and more generally including several such NS-branes. The advantage of the type-IIB configuration is that it is possible to describe the field theory on the D3-branes in terms of open strings. Mapping this back to M theory should indicate the effect of having two ABJM theories coupled through the brane configuration of M2-branes ending on both sides of an M5-brane. Some results in this direction have been derived by Niarchos [28], without M5'-branes or topological twisting. It would be interesting to understand the relation in detail.

We leave a full analysis of the type-IIB configurations to future work. However, we note that our expectation is that the configuration with  $N$  D3<sup>+</sup>- and  $N$  D3<sup>-</sup>-branes stretched between an NS5- and an NS5'-brane gives a  $GL(N|N)$  WZW model after taking the limit of coincident NS5- and NS5'-branes and dualizing to M theory. If we introduce a stack of  $M$  D3<sup>+</sup>- and  $M$  D3<sup>-</sup>-branes on the other side of the NS5'-brane and allow these to end on an additional NS5- or NS5'-brane, we will arrive at a  $GL(N|N) \times GL(M|M)$  WZW model with bifundamental matter from the open strings connecting the D3-branes across the NS5'-brane. In M theory, this would correspond to the configuration with (along increasing  $x^2$ ) M5- $N$  M2-M5'- $M$  M2-M5. While we hope to return to this type-IIB description in the future, for this paper, we now focus on the case with just the single stacks of  $N$  D3<sup>+</sup>- and D3<sup>-</sup>-branes.

### C. Supergroup WZW model

The action of the supergroup WZW model for maps  $s: \Sigma \rightarrow SG$  from a two-dimensional Euclidean Riemann surface  $\Sigma$  to the supergroup  $SG$  is given by

$$S[s] = -\frac{k}{8\pi} \int_{\Sigma} d^2x (s^{-1} \partial^\alpha s, s^{-1} \partial_\alpha s) - \frac{ik}{24\pi} \int_M d^3x \epsilon^{\mu\nu\lambda} (s^{-1} \partial_\mu s, [s^{-1} \partial_\nu \lambda, s^{-1} \partial_\lambda s]), \quad (2.21)$$

where  $k \in \mathbb{Z}$  is the level.<sup>2</sup> Here, the second term is the Wess-Zumino term integrated over a 3-manifold  $M$  of which the boundary is  $\Sigma$ .

The action (2.21) is invariant under the transformation

$$s(z, \bar{z}) \rightarrow \Omega(z) s(z, \bar{z}) \bar{\Omega}^{-1}(\bar{z}), \quad (2.22)$$

where  $\Omega(z)$  and  $\bar{\Omega}(\bar{z})$  are arbitrary  $SG$ -valued functions of the complex variables  $z = x^0 + ix^1$  and  $\bar{z} = x^0 - ix^1$ . This realizes the semilocal symmetry  $SG(z) \times SG(\bar{z})$ , the direct product symmetry group of left and right multiplications. Under the infinitesimal transformation  $\Omega(z) = \mathbb{1} + \omega(z)$ ,  $s$  transforms as  $\delta_\omega s = \omega s$ , and the action (2.21) is invariant. Hence, we find the conserved currents

$$J(z) = J^a(z) T_a = -k \partial_z s \cdot s^{-1}, \quad (2.23)$$

where  $T^a$  is a generator of  $\mathfrak{sg}$ . The conservation of the currents can be derived from the classical equations of motion  $\partial_{\bar{z}} J = 0$ , which ensure that  $J$  is holomorphic. Let us concentrate only on the holomorphic current  $J$ . Substituting the transformation  $\delta_\omega J$  into the Ward identity, we obtain the Operator Product Expansion

$$J^a(z) J^b(w) \sim \frac{k(T^a, T^b)}{(z-w)^2} + \frac{[T^a, T^b]_c J^c(w)}{z-w}. \quad (2.24)$$

Since the current is an analytic function of  $z$ , it can be expanded as

$$J^a(z) = \sum_{n=-\infty}^{\infty} \frac{J_n^a}{z^{n+1}}. \quad (2.25)$$

Then, the Operator Product Expansion (2.24) leads to the affine Lie superalgebra  $\widehat{\mathfrak{sg}}$ ,

$$[J_n^a, J_m^b] = [T^a, T^b]_c J_{n+m}^c + m(T^a, T^b) \delta_{n+m,0} k. \quad (2.26)$$

### III. AFFINE LIE SUPERALGEBRA

Because of the underlying symmetry algebra (2.26), we need to study the affine Lie superalgebra

$$\widehat{\mathfrak{sg}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{sg}) \oplus \mathbb{C}K \oplus \mathbb{C}d. \quad (3.1)$$

<sup>2</sup>As in Ref. [1], our description of the M2-M5 system is for the case  $k = 1$ , for which the BLG and ABJM models describe flat membranes.

Here,  $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{sg} \oplus \mathbb{C}K$  is a central extension of the loop algebra  $\widehat{\mathfrak{sg}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{sg}$  with  $\mathbb{C}[t, t^{-1}]$  being Laurent polynomial in variable  $t$ ,  $K$  being a central element called the level, and  $d = t \frac{d}{dt}$  being the derivation. The generators of  $\widehat{\mathfrak{sg}}$  obey the commutation relations

$$[at^n, bt^m] = [a, b] t^{n+m} + m \delta_{m+n,0} (a, b) K, \quad (3.2)$$

$$[d, at^n] = nat^n, \quad [K, \widehat{\mathfrak{sg}}] = 0, \quad (3.3)$$

and the nondegenerate supersymmetric invariant bilinear form is

$$(at^n, bt^m) = \delta_{m+n,0} (a, b), \quad (\widehat{\mathfrak{sg}}, \mathbb{C}K + \mathbb{C}d) = 0, \quad (3.4)$$

$$(K, K) = (d, d) = 0, \quad (K, d) = 1, \quad (3.5)$$

with  $a, b \in \mathfrak{sg}$ ,  $m, n \in \mathbb{Z}$ . Note that in the physical setup (2.26) the derivation  $d$  corresponds to the Virasoro generator  $-L_0$ , while the level  $K$  is the constant value for the  $SG$  WZW model (2.21).

The Cartan subalgebra  $\widehat{\mathfrak{h}}$  of  $\widehat{\mathfrak{sg}}$  can be defined in terms of a Cartan subalgebra  $\mathfrak{h}$  of the finite Lie superalgebra  $\mathfrak{sg}$ ,

$$\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}d + \mathbb{C}K. \quad (3.6)$$

We will introduce the coordinate on  $\widehat{\mathfrak{h}}$ ,

$$h := 2\pi i(-\tau d + z + tK), \quad (3.7)$$

with  $\tau, t \in \mathbb{C}$ ,  $z \in \mathfrak{h}$ .

#### A. Roots and weights

The nondegenerate bilinear form of  $\widehat{\mathfrak{sg}}$  is extended to  $\widehat{\mathfrak{h}}$  as  $(\mathfrak{h}, \mathbb{C}K + \mathbb{C}d) = 0$ , and one gets the dual  $\widehat{\mathfrak{h}}^*$  of  $\widehat{\mathfrak{h}}$ . The roots and weights belong to the dual  $\widehat{\mathfrak{h}}^*$  of  $\widehat{\mathfrak{h}}$ . The root space is

$$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0, \quad (3.8)$$

where the elements  $\delta$  and  $\Lambda_0$  of  $\widehat{\mathfrak{h}}^*$  are defined by

$$\delta|_{\mathfrak{h} + \mathbb{C}K} = 0, \quad \delta(d) = 1, \quad (3.9)$$

$$\Lambda_0|_{\mathfrak{h} + \mathbb{C}d} = 0, \quad \Lambda_0(K) = 1 \quad (3.10)$$

and they are represented by  $\delta = (0, 0, 1)$  and  $\Lambda_0 = (1, 0, 0)$ . As  $\widehat{\mathfrak{h}}$  is identified with  $\widehat{\mathfrak{h}}^*$  by the bilinear form (3.5), we have

$$\delta = K, \quad \Lambda_0 = d. \quad (3.11)$$

Let  $\hat{\Delta} \subset \hat{\mathfrak{h}}^*$ ,  $\hat{\Delta}_0$ , and  $\hat{\Delta}_1$  be the set of roots and the subsets of even and odd roots, respectively.  $\hat{\Delta}_0$  turns out to be a union of a finite number of root systems  $\hat{\Delta}^{\text{re}+} := \{\alpha + s\delta | \alpha \in \Delta, s > 0\} \cup \Delta^+$  of the affine Lie superalgebra with the same primitive imaginary roots  $\hat{\Delta}^{\text{im}+} := \{s\delta | s > 0\}$ . We define a coroot as  $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$  for nonisotropic root  $\alpha \in \hat{\Delta}$  and  $\alpha^\vee = \alpha$  for isotropic root  $\alpha \in \hat{\Delta}$ .

The set of simple roots of  $\widehat{\mathfrak{sg}}$  is given by  $\hat{\Pi} = \Pi \cup \alpha_0$ , where  $\Pi = (0, \alpha_i, 0)$  with  $\alpha_i$  being simple roots of  $\mathfrak{sg}$  and  $\alpha_0 := \delta - \theta = (0, -\theta, 1)$  with  $\theta$  being the highest root of  $\mathfrak{sg}$ , which is defined by  $\theta = \sum_{i=1}^{N+M-1} k_i \alpha_i \in \Delta_+$  so that  $\sum_{i=1}^{N+M-1} k_i$  is maximal for  $\mathfrak{gl}(N|M)$ . For example, the sets of simple roots of  $\widehat{\mathfrak{gl}}(N|N)$ , which consist of isotropic roots, are

$$\{\delta - \epsilon_1 - \delta_N, \epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \dots, \delta_{N-1} - \epsilon_N, \epsilon_N - \delta_N\}. \quad (3.12)$$

The Borel subalgebra  $\hat{\mathfrak{b}}$  of  $\widehat{\mathfrak{sg}}$  is given by

$$\hat{\mathfrak{b}} = \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+ = \hat{\mathfrak{h}} \oplus \mathfrak{n}^+ \oplus \left( \bigoplus_{n>0} t^n \otimes \mathfrak{sg} \right). \quad (3.13)$$

A weight  $\Lambda \in \hat{\mathfrak{h}}^*$  takes the form  $(k, \lambda, n)$  where  $\lambda$  is the weight of  $\mathfrak{sg}$ . The fundamental weight  $\Lambda_i \in \hat{\mathfrak{h}}^*$  is defined by

$$(\Lambda_i, \alpha_j^\vee) = \delta_{ij}, \quad (\Lambda_i, d) = 0, \quad (3.14)$$

and the label of the weight  $\lambda$  is defined by

$$m_i = (\Lambda, \alpha_i^\vee). \quad (3.15)$$

### B. Weyl group

The affine Weyl vector  $\hat{\rho}$  is defined by

$$\hat{\rho} = \rho + h^\vee \Lambda_0. \quad (3.16)$$

It obeys  $(\hat{\rho}, \alpha) = \frac{1}{2}(\alpha, \alpha)$  for  $\forall \alpha \in \hat{\Pi}$ ,  $(\hat{\rho}, d) = 0$  and  $(\hat{\rho}, K) = h^\vee$ . For  $\alpha \in \hat{\mathfrak{h}}^*$ , we define  $t_\alpha \in \text{Aut}(\hat{\mathfrak{h}}^*)$  by

$$t_\alpha(\Lambda) = \Lambda + \Lambda(K)\alpha - \left( (\Lambda, \alpha) + \frac{1}{2}(\alpha, \alpha)\Lambda(K) \right) \delta. \quad (3.17)$$

The affine Weyl group is

$$\hat{W} = W \ltimes \{t_\alpha | \alpha \in L\}, \quad (3.18)$$

where  $W$  is the Weyl group of  $\mathfrak{sg}$  and  $L \subset \mathfrak{h}$  is the coroot lattice.

### C. Representations

For each weight  $\Lambda \in \hat{\mathfrak{h}}^*$ , one can define the irreducible highest weight module  $L(\Lambda)$  over  $\widehat{\mathfrak{sg}}$  such that there exists a nonzero vector  $v_\Lambda$  satisfying

$$h v_\Lambda = \Lambda(h) v_\Lambda, \quad \text{for } h \in \hat{\mathfrak{h}}, \quad (3.19)$$

$$\mathfrak{n}^+ v_\Lambda = 0, \quad (3.20)$$

$$(t^n \otimes \mathfrak{sg}) v_\Lambda = 0, \quad \text{for } n > 0. \quad (3.21)$$

The central element  $K$  on  $L(\Lambda)$  is the scalar  $k = \Lambda(K)$  called the level in (3.8). The irreducible highest weight module  $L(\Lambda)$  is called integrable if (i)  $\dim L(\Lambda) < \infty$  and (ii)  $t^n \otimes \mathfrak{sg}_\alpha$  are locally nilpotent for all  $\alpha \in \Delta_0^\#$  and  $n \in \mathbb{Z}$ . It is known that  $L(\Lambda)$  is integrable if the numbers  $\frac{2(\Lambda, \alpha)}{(\alpha, \alpha)}$  and  $\frac{2(\Lambda, K - \theta)}{(\theta, \theta)}$  are non-negative integers for all simple roots  $\alpha \in \hat{\Pi}$  and the highest root  $\theta$ . The necessary condition of integrability of  $L(\Lambda)$  over  $\widehat{\mathfrak{gl}}(N|M)$  is [2]

$$m_i \in \mathbb{Z}_+, \quad m' = m_0 + m_N - \sum_{i=N+1}^{N+M-1} m_i \in \mathbb{Z}_+, \quad (3.22)$$

and the sufficient condition is [2]

$$m' \geq M \quad (3.23)$$

for  $N \geq 2$ .

Let  $S$  be a subset of a simple root system  $\Pi$ . We call it a  $(\lambda + \rho)$ -maximal isotropic subset if it consists of  $d$  pairwise orthogonal isotropic roots  $\{\beta_i\}$ ,  $i = 1, \dots, d$  that are also orthogonal to  $\lambda + \rho$ , i.e., [3,9],

$$(\lambda + \rho, \beta_i) = 0, \quad (\beta_i, \beta_j) = 0. \quad (3.24)$$

The number  $d$  of linearly independent pairwise orthogonal isotropic roots is called the atypicality of  $L(\lambda)$ . The atypicality of a simple finite-dimensional module does not depend on the choice of simple root system, and the maximal number  $d$  of the Lie superalgebra  $\mathfrak{sg}$  is called the defect and denoted by  $\text{def}(\mathfrak{sg})$ .

An irreducible highest weight module  $L(\lambda)$  over  $\mathfrak{sg}$  is called typical if  $S$  is empty and atypical or tame otherwise. Similarly, an irreducible highest weight module  $L(\Lambda)$  of level  $K$  over  $\widehat{\mathfrak{sg}}$  is called atypical or tame if the corresponding module  $L(\lambda)$  over the finite part  $\mathfrak{sg}$  of  $\widehat{\mathfrak{sg}}$  is atypical and if  $K + h^\vee \neq 0$  [3,9].

Note that the irreducible highest weight module  $L(\lambda)$  is characterized by the vectors annihilated by  $\mathfrak{n}^+$  acting as the raising operators. However, the choice of  $\mathfrak{n}^+$  is not unique but depends on the Weyl group  $W$  that permutes the different weights. To characterize  $L(\lambda)$  over  $\mathfrak{sg}$  so that the

choice of  $\mathfrak{n}^+$  does not depend on  $W$ , we need to take the shifted weight  $\lambda + \rho$  on which  $w \in W$  acts.

#### IV. BRANES AND WEIGHT DIAGRAM

##### A. Weight diagram

In terms of the basis  $\{\epsilon_1, \dots, \epsilon_N; \delta_1, \dots, \delta_M\}$  of  $\mathfrak{h}^*$ , one can write the dominant integrable weight  $\lambda$  of the irreducible highest weight modules  $L(\lambda)$  as

$$\lambda + \rho = \sum_{i=1}^N x_i \epsilon_i - \sum_{k=1}^M y_k \delta_k, \quad (4.1)$$

where the integral condition requires that the coefficients  $x_i$  and  $y_k$  are integers and the dominant condition is satisfied by the ordering  $x_1 \geq \dots \geq x_N$ ,  $y_1 \leq \dots \leq y_M$ . It can be represented diagrammatically in terms of the weight diagram, and the irreducible characters over the Lie superalgebras have been computed using a combinatorial algorithm [42–45]. Consider a horizontal number line with vertices labeled by a set of consecutive integers  $n$  in increasing order from left to right. Then, we label the vertex of  $n$  by

$$\begin{cases} \vee & \text{if } n \in \{x_i\} \cap \{y_k\} \\ > & \text{if } n \in \{x_i\} \setminus \{y_k\} \\ < & \text{if } n \in \{y_k\} \setminus \{x_i\} \\ \wedge & \text{if } n \notin \{x_i\} \cup \{y_k\} \end{cases} \quad (4.2)$$

Each  $\vee$  corresponds to an atypical root  $\beta$ , and the degree  $d$  of atypicality of  $\lambda$  is the number of  $\vee$ 's in the weight diagram. The dominant weight is uniquely determined by the weight diagram.

For example, the weight

$$\lambda + \rho = 9\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + 2\epsilon_4 - \delta_1 - 3\delta_2 - 7\delta_3 - 9\delta_4 \quad (4.3)$$

corresponds to the following weight diagram:

$$\begin{array}{cccccccccccc} \wedge & \wedge & < & > & \vee & \wedge & > & \wedge & < & \wedge & \vee & \wedge \\ -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \quad (4.4)$$

The  $\lambda + \rho$ -maximal isotropic subset is

$$S = \{\epsilon_1 - \delta_4, \epsilon_3 - \delta_2\}, \quad (4.5)$$

and the atypicality of the corresponding irreducible highest weight module  $L(\lambda)$  is  $d = 2$ .

One can consider certain combinatorial operations on the weight diagrams by moving  $\vee$ 's and  $\wedge$ 's at specific positions to other locations [42–44]. We define a right move  $R_{i \rightarrow j}(\lambda)$  on the weight diagram  $\lambda$  by exchanging (counting from the left) the  $i$ th  $\vee$  with a  $\wedge$  to its right. This

$\wedge$  is specified in such a way that there are exactly  $k \equiv j - i \vee$ 's and the same number of  $\wedge$ 's between the  $i$ th  $\vee$  and this  $\wedge$ . As a consequence, the  $i$ th  $\vee$  moves to become the  $j = (i + k)$ th  $\vee$ . For example, for the weight diagram (4.4),  $R_{1 \rightarrow 2} \circ R_{1 \rightarrow 1} \circ R_{1 \rightarrow 1} \circ R_{1 \rightarrow 1}(\lambda)$  is

$$\begin{array}{cccccccccccc} \wedge & \wedge & < & > & \wedge & \wedge & > & \wedge & < & \wedge & \vee & \wedge & \vee \\ -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \quad (4.6)$$

Note for the last step that all locations to the right (or left) of the weight diagram are filled by  $\wedge$ 's. The right move  $R_{i \rightarrow j}$  corresponds to a raising operator for the corresponding module [44].

A left move  $L_{i \leftarrow j}$  is similarly defined by swapping (still counting from the left) the  $j$ th  $\vee$  with a  $\wedge$  to its left, again separated by  $k \equiv j - i \vee$ 's and  $k \wedge$ 's. Then, the  $j$ th  $\vee$  is shifted to the  $i = (j - k)$ th  $\vee$ . For example, for the weight diagram (4.4),  $L_{1 \leftarrow 2} \circ L_{2 \leftarrow 2} \circ L_{2 \leftarrow 2} \circ L_{2 \leftarrow 2}(\lambda)$  gives

$$\begin{array}{cccccccccccc} \vee & \wedge & \wedge & < & > & \vee & \wedge & > & \wedge & < & \wedge & \wedge & \wedge \\ -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \quad (4.7)$$

This operation corresponds to a lowering operator in the corresponding module [44].

##### B. Brane construction

Now, we return to the  $GL(N|N)$  WZW model describing the M2-M5-brane system. We argue that the dominant integrable weight  $\lambda$  of the irreducible highest weight atypical module  $L(\lambda)$  over  $\hat{\mathfrak{gl}}(N|N)$  corresponds to the vacuum configuration of branes.

Let  $C$  (respectively,  $C'$ ) be the M theory 3-form “ $C$  field” on the M5-brane (respectively, M5'-brane), and let  $\Sigma_a$ ,  $a = 1, \dots, N$  be the 2-cycle wrapped by  $a$ th M2-brane. In the two-dimensional intersection with the M5-brane (respectively, M5'-brane)  $\Sigma_a$ , Abelian gauge fields  $\{A^i\}$ ,  $i = 1, \dots, N$  (respectively,  $\{A'^k\}$ ,  $k = 1, \dots, N$ ) arise from the Kaluza-Klein reduction of the M theory 3-form

$$C = \sum_{i=1}^N A^i \wedge \Sigma_i, \quad C' = \sum_k A'^k \wedge \Sigma_k. \quad (4.8)$$

The presence of the M5- and M5'-branes independently carrying  $N$  M2-brane charges of the  $C$  field implies that one can specify data of the M2-M5 system by a choice of two sets of vector bundles  $E \rightarrow \Sigma_i$ ,  $E' \rightarrow \Sigma_k$  and connections on  $E$ ,  $E'$ . From the M2-brane point of view, they are viewed as global charges. We denote the eigenvalue of the  $i$ th M2-brane charge in the M5-brane by  $x_i \in \mathbb{Z}$ ,  $i = 1, \dots, N$  and that of the  $k$ th M2-brane charge for the M5'-brane by  $y_k \in \mathbb{Z}$ ,  $k = 1, \dots, N$ . Then, we can obtain a unique weight



diagram from the brane configuration by considering an integer coordinate and putting a symbols  $\{\vee, >, <, \wedge\}$  on it in the same manner as (4.2).

Similarly, Mikhaylov and Witten [7] point out that a vacuum configuration of the brane system with  $N$  D3-branes ending on one side, and  $M$  D3-branes ending on the other side, of a single NS5-brane corresponds to the dominant integrable weight  $\lambda$  of  $\mathfrak{u}(N|M)$  and its weight diagram (see Fig. 1). In that case, the two sequences  $\{x_i\}$  and  $\{y_k\}$  would represent the charges of wrapped D3-branes under the RR fields.

This construction gives interesting physical implications of the weight diagram. The nonzero eigenvalues of M2-brane charge correspond to  $\vee$ 's that are shared by both M5-branes and to  $>$  or  $<$  that is taken by only one of the M5-branes. Since the limit in which the separation of the M5-branes is taken to zero requires the same eigenvalues for both M5-branes, the  $\vee$ 's are identified with the M2-branes, which are suspended between the M5- and M5'-brane. Thus the atypicality, that is, the number of  $\vee$ 's, is the number of M2-branes attached to both M5-branes. In particular, for  $GL(N|N)$  arising from  $N$  M2-branes all stretched between the two M5-branes, the modules of interest have maximal atypicality  $N$ .

For example, consider the brane configuration in Fig. 4 with  $d = 4$  M2-branes stretched between the M5- and M5'-brane and  $(N - d) = (7 - 4) = 3$  M2-branes attached to one of them. Set the eigenvalues of the  $i$ th M2-brane charge for the M5-brane as  $\{x_i\} = \{12, 10, 8, 7, 5, 4, 1\}$  and those of the  $k$ th M2-brane charge for the M5'-brane as  $\{y_k\} = \{3, 4, 5, 6, 8, 12, 13\}$ , which correspond to the heights of the M2-branes in Fig. 4. Then, the corresponding weight reads

$$\begin{array}{cccccccccccccccc} \wedge & \wedge & > & \wedge & < & \vee & \vee & < & > & \vee & \wedge & > & \wedge & \vee & < \\ -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}, \tag{4.9}$$

and the dominant weight is

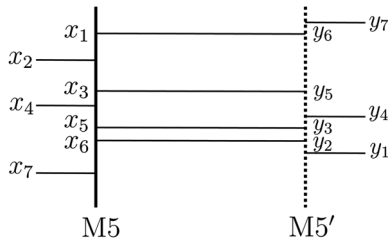


FIG. 4.  $(N - d)$  M2-branes attached to one of the M5-branes and  $d$  M2-branes stretched between the two M5-branes. The vertical bold (respectively, dotted) line represents the M5- (respectively, M5'-)brane, and the horizontal lines denotes M2-branes in the  $x^2$  direction. The M2-brane charges  $\{x_i\}$  and  $\{y_k\}$  are illustrated as the heights of the M2-branes. Here is the case with  $N = 7$  and  $d = 4$ .

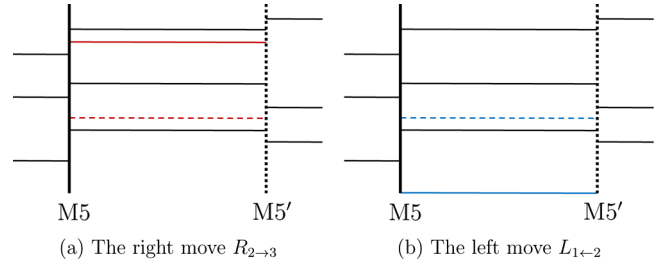


FIG. 5. The operation of the right move and the left move on the brane configuration in Fig. 4. The right move  $R_{2 \rightarrow 3}$  lifts the second mode at 5 (shown in dotted red) to the third at 11 (shown in red), while the left move  $L_{1 \leftarrow 2}$  reduces the second mode of the location 5 (shown in dotted blue) to 0 (shown in blue).

$$\begin{aligned} \lambda + \rho &= 12\epsilon_1 + 10\epsilon_2 + 8\epsilon_3 + 7\epsilon_4 + 5\epsilon_5 + 4\epsilon_6 + \epsilon_7 \\ &\quad - 3\delta_1 - 4\delta_2 - 5\delta_3 - 6\delta_4 - 8\delta_5 - 12\delta_6 - 13\delta_7. \end{aligned} \tag{4.10}$$

The  $\lambda + \rho$ -maximal isotropic subset  $S$  is

$$S = \{\epsilon_1 - \delta_6, \epsilon_3 - \delta_5, \epsilon_5 - \delta_3, \epsilon_6 - \delta_2\}, \tag{4.11}$$

and the atypicality of the module is  $d = 4$ , that is, the number of M2-branes stretched between the M5- and M5'-brane. The right move  $R_{2 \rightarrow 3}(\lambda)$  gives the weight diagram

$$\begin{array}{cccccccccccccccc} \wedge & \wedge & > & \wedge & < & \wedge & \vee & < & > & \vee & \wedge & > & \vee & \vee & < \\ -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}, \tag{4.12}$$

and the left move  $L_{1 \leftarrow 2}$  yields the weight diagram

$$\begin{array}{cccccccccccccccc} \wedge & \vee & > & \wedge & < & \vee & \wedge & < & > & \vee & \wedge & > & \wedge & \vee & < \\ -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}. \tag{4.13}$$

They correspond to new charge assignments of the brane configuration depicted in Fig. 5. The right move  $R_{i \rightarrow j}$  and the left move  $L_{i \leftarrow j}$  are, respectively, interpreted as a raising operator and a lowering operator of the suspended M2-brane charges.

Quantum mechanically, a transition amplitude is given by a weighted sum over all paths as the Feynman path integral. As shown in Fig. 5, it will be achieved by summing over all possible paths of excitation modes by acting with raising or lowering operators. However, it can be now rephrased as a sum over all possible paths of the sequence of left moves, or equivalently right moves with a weight characterized by multiplicity of the path.<sup>3</sup>

<sup>3</sup>Interestingly, the terminology *path* is also used for the collection of the left moves and right moves in the mathematical literature [43,44].

Therefore, the dominant weight of the irreducible highest weight atypical module  $L(\Lambda)$  over the underlying symmetry  $\widehat{\mathfrak{gl}}(N|N)$  can be determined by the vacuum configuration of the M2-M5 system.

In the absence of atypical roots, the dominant integral weight  $\lambda$  defines a typical highest weight module  $L(\lambda)$  [46]. In the M2-M5 system, there is no stretched M2-brane. It is known that most questions in the typical irreducible representations reduce to those in the ordinary affine Lie algebra  $\widehat{\mathfrak{g}}$ . For example, it was shown in Ref. [47] that the classical Weyl-Kac character formula holds for arbitrary typical finite-dimensional irreducible modules such that  $\dim \widehat{\mathfrak{sg}} < \infty$  and  $\dim L(\lambda) < \infty$ . In the context of the Alday-Gaiotto-Tachikawa (AGT) correspondence, the intersection of nonparallel M5-branes wrapping  $\Sigma$  leads to a relation between instanton partition functions in the four-dimensional  $\mathcal{N} = 2$  quiver gauge theories in the presence of certain surface operators and the conformal block of the affine Lie algebra  $\widehat{\mathfrak{g}}$  [48–52]. Since the typical modules of  $\widehat{\mathfrak{sg}}$  essentially contain the affine Lie algebra  $\widehat{\mathfrak{g}}$ , and likewise the M2-M5 system realizes two intersecting M5-branes without any suspended M2-branes as a special case (see Fig. 6), it may be possible to extend the AGT correspondence, in the presence of surface operators as a combination of M2-like and M5'-like surface operators, in terms of the affine Lie superalgebra  $\widehat{\mathfrak{sg}}$ .

A relation between brane configurations and atypical representations of a supergroup has also been described previously by Mikhaylov and Witten [7]. In that case, the supergroup arose from D3-branes ending on both sides on an NS5-brane. The labels  $\{x_i\}$  and  $\{y_k\}$  were associated with the D3-branes ending on the left and right, respectively, of the NS5-brane. In the type-IIB configuration dual to our M-brane construction, we have the stacks of D3<sup>+</sup>- and D3<sup>-</sup>-branes on each side of an NS5-brane. Since the M2-branes arise as a combination of these two stacks of branes, it is consistent that the two sets of labels are both associated with the same M2-branes. Also, as previously noted, the introduction of the M5- and M5'-branes corresponds to NS5- and NS5'-branes, which remove the freedom for the D3-branes to move in the 34 directions. Thus, it should not be surprising that in the limit we are considering

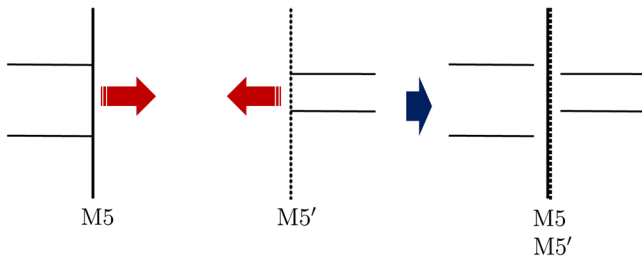


FIG. 6. The M2-M5 system for a typical module with  $N = 2$ ,  $d = 0$ , for which no M2-brane is stretched between the M5-branes.

the D3<sup>+</sup>- and D3<sup>-</sup>-branes should have the same vacuum configuration, and hence the  $\{x_i\}$  and  $\{y_k\}$  should be the same, giving maximum atypicality.

We could introduce further stacks of D3<sup>+</sup>- and D3<sup>-</sup>-branes on the other side of the NS5'-brane. We would expect the case in which some D3<sup>+</sup>-branes (and likewise for D3<sup>-</sup>-branes) on either side of the NS5'-brane carried the same charges to have special properties. This would give the M theory case in which M2-branes ended on both sides of the M5'-brane. However, further study of this is beyond the scope of this paper.

## V. MOCK MODULAR INDEX

### A. Definition

We have identified the highest weight atypical module  $L(\Lambda)$  over  $\widehat{\mathfrak{gl}}(N|N)$  for a given vacuum configuration of M2-M5 system. Now, we want to study these modules via the indices and partition functions. We define an index for the supergroup WZW models by

$$\mathcal{I}(\tau, z) := \text{Tr}_{\mathcal{H}}(-1)^F q^{H_L} \prod_{a=1}^d x_a^{F_a}. \quad (5.1)$$

Here,  $(-1)^F$  is the fermion number operator, and  $q := e^{2\pi i \tau}$  is a complex parameter associated with the left-moving Hamiltonian  $H_L = 2(H + iP) = L_0 - \frac{c}{24}$ . The vector  $F_a$  is the charge vector associated with the Cartan subalgebra for the atypical block of atypicality  $d$  in the bosonic subalgebra  $\mathfrak{g}_0$ , where  $a = 1, \dots, d$ . We have introduced the associated chemical potential  $x_a := e^{2\pi i z_a}$ . This index is an analog of the Witten index for the supersymmetric quantum mechanics in that the  $z_a \rightarrow 0$  limit gives the Witten index.

Now, we are ready to explain how the index (5.1) encodes the data of the M2-M5 system. We take the Hilbert space  $\mathcal{H}$  as the irreducible atypical highest weight modules with atypicality  $d$  being the number of the stretched M2-branes. The left-moving Hamiltonian  $H_L$  is an energy of the sandwiched M2-branes, i.e., a winding number of the stretched M2-branes along one of the cycles of  $\Sigma$ , viewed as the Euclidean time circle. The  $F_a$ ,  $a = 1, \dots, d$  are the  $U(1)$  charges for a holomorphic  $U(1)$  vector bundle over the Riemann surface wrapped by the stretched M2-branes, which originates from the 3-form  $C$  field (4.8). Therefore, the index (5.1) counts BPS states of the M2-M5 system.

In addition, we consider a partition function,

$$\mathcal{Z}(\tau, \bar{\tau}, z) := \text{Tr}_{\mathcal{H}}(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \prod_{a=1}^d x_a^{F_a}. \quad (5.2)$$

Here,  $\bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$  insert the right-moving Hamiltonian  $H_R = 2(H - iP) = \bar{L}_0 - \frac{\bar{c}}{24}$  into the index (5.1). The partition function has the same form as the equivariant elliptic genus. It can be formulated by a path integral on a torus with a

coordinate  $w = \sigma_1 + \tau\sigma_2$  where  $\sigma_1$  and  $\sigma_2$  are periodic with periodicity  $2\pi$  and  $\tau$ . Here,  $\tau = \tau_1 + i\tau_2$  characterizes the complex structure of a torus  $w \simeq w + 2\pi \simeq w + 2\pi\tau$ , on which the WZW model is defined. From the point of view of the M2-M5 system, the right-moving Hamiltonian  $H_R$  is a momentum of the stretched M2-branes along the other cycle of  $\Sigma$ , viewed as the Euclidean spatial circle.

A torus partition function should be the same for equivalent tori. A holomorphic function  $\varphi$  on the upper half-plane  $\mathbb{H}$  transforming under the modular group  $SL(2, \mathbb{Z})$  of reparametrizations of the torus as

$$\varphi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \varphi(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (5.3)$$

is called a modular form of weight  $k$ . The effect of a chemical potential  $x_a$  is equivalent to the coupling of external gauge fields  $A^a$  on the torus to the current so that

$$x_a = e^{2\pi i z_a} = e^{2\pi i \left[ \oint_t A^a - \tau \oint_s A^a \right]}, \quad (5.4)$$

where  $\oint_t$  (respectively,  $\oint_s$ ) is the temporal (respectively, spatial) cycle of the torus. Such coupling is translated into the twisted boundary conditions of the fields  $\phi(w, \bar{w})$  along the two cycles,

$$\phi(w + 2\pi, \bar{w} + 2\pi) = \prod_a e^{2\pi i F_a} \oint_t A^a \phi_t(w, \bar{w}), \quad (5.5)$$

$$\phi(w + 2\pi\tau, \bar{w} + 2\pi\bar{\tau}) = \prod_a e^{2\pi i \tau F_a} \oint_s A^a \phi_s(w, \bar{w}), \quad (5.6)$$

where  $\phi_t$  (respectively  $\phi_s$ ) is the untwisted boundary condition along the temporal (respectively spatial) cycle. A function  $\varphi(\tau, z)$  is called elliptic with index  $m$  in  $z$  if it has a transformation law

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z), \quad \lambda, \mu \in \mathbb{Z} \quad (5.7)$$

under the translation of  $z$ . A holomorphic function  $\varphi(\tau, z)$  on  $\mathbb{H} \times \mathbb{C}$  with the ellipticity (5.7) which transforms under the modular group  $SL(2, \mathbb{Z})$  as

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \varphi(\tau, z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (5.8)$$

is called a Jacobi form of weight  $k$  and index  $m$ .

## B. Kac-Wakimoto formula

To compute the indices, we recall the definition of the character  $\text{ch}_{L(\Lambda)}$  and the supercharacter  $\text{sch}_{L(\Lambda)}$  of the module  $L(\Lambda)$ :

$$\text{ch}_{L(\Lambda)} := \sum_{h \in \hat{\mathfrak{h}}} \dim L(\Lambda) e^h, \quad \text{sch}_{L(\Lambda)} := \sum_{h \in \hat{\mathfrak{h}}} \text{sdim} L(\Lambda) e^h. \quad (5.9)$$

The module  $L(\Lambda)$  is integrable if and only if the character is invariant under  $\hat{W}^\sharp = W^\sharp \ltimes t_{L^\sharp}$ , which is the subgroup of the affine Weyl group  $\hat{W}$  where  $L^\sharp$  is the sublattice of the coroot lattice  $L$  corresponding to the root system (2.8).

Using the coordinate (3.7) for  $h \in \hat{\mathfrak{h}}$ , the supercharacter can be written explicitly as

$$\text{sch}_{L(\Lambda)}(\tau, z, t) = \text{Str}_{L(\Lambda)} e^{2\pi i(-\tau d + z + tK)}. \quad (5.10)$$

It is demonstrated in Ref. [53] that for an integrable  $L(\Lambda)$  the supercharacter absolutely converges in the convex domain  $D = \{h \in \hat{\mathfrak{h}} | \text{Re} \alpha_i(h) > 0, i = 1, \dots, l\}$  to a holomorphic function. Also, for all known examples, it converges in the upper half-plane  $\mathbb{H} = \{(\tau, z, t) | \text{Im} \tau > 0\}$  to a meromorphic function.

Since the replacement of  $\Lambda$  with  $\Lambda + a\delta$  for  $a \in \mathbb{C}$  keeps  $L(\Lambda)$  irreducible, we further consider the supercharacter multiplied by  $q^a$ . The normalized supercharacter  $\text{sch}_\Lambda$  is defined by multiplying the supercharacter  $\text{sch}_{L(\Lambda)}$  by  $q^{m_\Lambda}$  [53],

$$\text{sch}_\Lambda = q^{m_\Lambda} \text{sch}_{L(\Lambda)}(\tau, z, t), \quad (5.11)$$

where  $m_\Lambda = \frac{(\Lambda + \hat{\rho}, \Lambda + \hat{\rho})}{2(k+h^\vee)} - \frac{\text{sdim} \mathfrak{g}}{24} = h_\Delta - \frac{c}{24}$  is called the modular anomaly. The normalized factor  $q^{m_\Lambda}$  is necessary to realize the contributions from the zero mode of the Virasoro generator  $L_0$ . It is associated to the modular invariance for the bosonic WZW models. However, for the supergroup WZW models, it is needed to acquire the intriguing mock modular property, as we will see later.

The supercharacter formula for the atypical integrable module  $L(\Lambda)$  given by the Kac-Wakimoto formula [9,54]

$$e^{\hat{\rho}} \hat{R}^- \text{sch}_{L(\Lambda)} = \sum_{w \in \hat{W}^\sharp} \text{sgn}^-(w) \frac{e^{w(\Lambda + \hat{\rho})}}{\prod_{\beta \in S} (1 - e^{-w(\beta)})}, \quad (5.12)$$

where

$$\hat{R}^- = \frac{\prod_{\alpha \in \hat{\Delta}_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \hat{\Delta}_1^+} (1 - e^{-\alpha})} \quad (5.13)$$

is the affine superdenominator,  $\hat{W}^\sharp = W^\sharp \ltimes t_{L^\sharp}$ , is the subgroup of the affine Weyl group  $\hat{W}$ ,  $L^\sharp$  is the corresponding

sublattice of the coroot lattice  $L$ , and  $\text{sgn}^-(w)$  is the sign factor defined by (2.10).

Furthermore, from Eqs. (5.11) and (5.12), the normalized supercharacter is expressed as [2,3]

$$\text{sch}_\Lambda = \frac{\sum_{w \in W^\#} \text{sgn}^-(w) \Theta_{\Lambda+\hat{\rho}, S}^{L^\pm, -}}{q^{\frac{\text{dim} \mathfrak{g}}{24}} \hat{R}^-}. \quad (5.14)$$

It turns out that the denominator in the formula (5.14) consists of the theta functions  $\vartheta_{11}$  and  $\vartheta_{10}$  and powers of the eta function  $\eta(\tau)$  (see Appendix), which are members of a modular invariant family. On the other hand, the function  $\Theta_{\Lambda, S}^{Q, \pm}$  in the numerator is a Ramanujan mock theta function [55–57] defined as the series [3,58,59]

$$\Theta_{\Lambda, S}^{Q, \pm} = q^{-\frac{2(\Lambda, \Lambda)}{2k}} \sum_{\gamma \in Q} \text{sgn}^\pm(t_\gamma) \frac{e^{t_\gamma(\Lambda)}}{\prod_{\beta \in S} (1 - e^{-t_\gamma(\beta)})}, \quad (5.15)$$

where  $t_\gamma$  is the element of the affine Weyl group  $\hat{W}$  defined in (3.17). The mock theta function  $\Theta_{\Lambda, S}^{Q, \pm}$  is determined by four data: (i) the weight  $\Lambda \in \hat{\mathfrak{h}}^*$  with  $\Lambda(K) > 0$ , (ii) the positive definite integral root lattice  $Q$  of  $\hat{\mathfrak{h}}_{\mathbb{R}}^*$ , (iii) the finite subset  $S \subset \hat{\mathfrak{h}}_{\mathbb{R}}^*$  composed of pairwise orthogonal isotropic vectors orthogonal to  $\Lambda$ , and (iv) the homomorphism  $\text{sgn}^\pm(\gamma): Q \rightarrow \{\pm\}$ , with  $\gamma \in Q$ . The degree of the mock theta function (5.15) is  $\Lambda(K) = k$ , and the  $\Theta_{\Lambda+\hat{\rho}, S}^{L^\pm, -}$  in the Kac-Wakimoto formula (5.14) is a mock theta function of degree  $k + h^\vee$ .

### C. Computation

Comparing (5.1) with (5.10), we find that the index (5.1) is the specialization of the supercharacter

$$\mathcal{I}(\tau, z) = \text{sch}_\Lambda(\tau, z, 0) \quad (5.16)$$

for  $k = 1$ . From now on, we restrict our attention to the atypical module  $L(\Lambda)$  and take it as the Hilbert space  $\mathcal{H}$  in the definition of the indices. Applying the Kac-Wakimoto formula (5.14), we see that the index  $\mathcal{I}(\tau, z)$  can be expressed in terms of the mock theta function. We thus call this index, which is analogous to the Witten index, a mock modular index.

Next, consider the torus partition function  $\mathcal{Z}(\tau, \bar{\tau}, z)$ . For the equivariant elliptic genus in compact superconformal field theories, the Hilbert spaces only contain discrete sets of primary fields. The additional factor  $\bar{q}^{\bar{L}_0 - \frac{c}{24}}$  requires the combined left- and right-moving sectors. However, there is a cancellation between bosonic and fermionic fluctuations from supersymmetry. Then, due to the discreteness of the spectrum in the Ramond sector, there is just an algebraic sum of the spectrum in the Ramond sector, and the contribution only arises from the ground states of the

Ramond sector. This ensures the holomorphicity of the elliptic genus.

However, the emergence of the mock theta function does not allow us to extend  $\mathcal{I}(\tau, z)$  to  $\mathcal{Z}(\tau, \bar{\tau}, z)$  by naively inserting the factor  $\bar{q}^{\bar{L}_0 - \frac{c}{24}}$  without any modification of the result. This is because the index  $\mathcal{Z}(\tau, \bar{\tau}, z)$  should be modular invariant due to the path integral formalism, while the index  $\mathcal{I}(\tau, z)$  is not. This indicates that some pieces in  $\mathcal{Z}(\tau, \bar{\tau}, z)$  are missing in  $\mathcal{I}(\tau, z)$ , and a proper completion must be added to restore the modular invariant  $\mathcal{Z}(\tau, \bar{\tau}, z)$ .

Such a property of the spectrum stems from the structure of the Hilbert space  $\mathcal{H}$  of the theory under consideration. The holomorphic elliptic genus relies on the fact that  $\mathcal{H}$  has a holomorphically factorized form

$$\mathcal{H} = \bigoplus_{\mu} \mathcal{H}_{\mu} \otimes \bar{\mathcal{H}}_{\mu}, \quad (5.17)$$

where  $\mathcal{H}_{\mu}$  (respectively,  $\bar{\mathcal{H}}_{\mu}$ ) is the holomorphic (respectively, antiholomorphic) sector. However, for the supergroup WZW models, the space of the states has been argued to have the form [60–63]

$$\mathcal{H} = \left( \bigoplus_{\mu \in \text{typical}} \mathcal{H}_{\mu} \otimes \bar{\mathcal{H}}_{\mu} \right) \oplus \left( \bigoplus_{\nu \in \text{atypical}} \hat{\mathcal{H}}_{\nu} \right). \quad (5.18)$$

Although there is the holomorphic factorization  $\mathcal{H}_{\mu} \otimes \bar{\mathcal{H}}_{\mu}$  in the typical sector, in the atypical sector  $\hat{\mathcal{H}}_{\nu}$ , the holomorphic and antiholomorphic parts are entangled with each other in a complicated way. This observation is consistent with our conclusion as we are now dealing with  $\hat{\mathcal{H}}_{\nu}$ , the Hilbert space of an atypical module.

The appearance of the mock theta function  $\Theta_{\Lambda, S}^{Q, \pm}$  in the normalized supercharacter is remarkable in that, although the mock modular functions are not exactly modular invariant, they can be made modular invariant by adding suitable nonholomorphic completions developed by Zwegers [4]. The basic idea is that a new nonholomorphic function

$$\hat{h}(\tau, \bar{\tau}) = h(\tau) + g^*(\tau, \bar{\tau}), \quad (5.19)$$

created by the addition of the nonholomorphic Eichler integral

$$g^* = \left( \frac{i}{2\pi} \right)^{k-1} \int_{-\bar{\tau}}^{\infty} dz (z + \tau)^{-k} \overline{g(-\bar{z})} \quad (5.20)$$

constructed from a holomorphic modular form  $g(\tau)$  of weight  $2 - k$ , called a shadow of  $h(\tau)$ , turns out to be modular invariant at the cost of holomorphicity. This naturally leads to a prescription for the evaluation of the nonholomorphic part of the modular invariant partition function  $\mathcal{Z}(\tau, \tau, z)$  defined by (5.2) on an elliptic curve as

$$\mathcal{Z}(\tau, \bar{\tau}, z) = \hat{\mathcal{I}}(\tau, \bar{\tau}, z) + (\text{holomorphic modular function}). \quad (5.21)$$

The first term  $\hat{\mathcal{I}}(\tau, \bar{\tau}, z)$  is the modular completion of  $\mathcal{I}(\tau, z)$  via Zweegers's method (5.19), which is the contribution from the atypical sector  $\hat{\mathcal{H}}_\nu$ , while the remnant is the holomorphic modular function arising from the typical sector  $\mathcal{H}_\mu \otimes \bar{\mathcal{H}}_\mu$ . Note that the index  $\mathcal{Z}(\tau, \bar{\tau}, z)$  is no longer holomorphic due to  $\hat{\mathcal{I}}(\tau, \bar{\tau}, z)$ , but it is modular invariant.

#### D. $PSL(2|2)_{k=1}$ WZW model

In this subsection, we will provide a simple example of the index computation for the  $PSL(2|2)_{k=1}$  WZW model. The corresponding brane configuration is illustrated in Fig. 7 where  $N = d = 2$  M2-branes are stretched between the M5- and M5'-brane. For example, given M2-brane charges  $\{x_i\} = \{4, 2\}$  and  $\{y_k\} = \{2, 4\}$ , the weight of the irreducible highest weight module with maximal atypicality  $d = 2$  is given by

$$\lambda + \rho = 4\epsilon_1 + 2\epsilon_2 - 2\delta_1 - 4\delta_2, \quad (5.22)$$

and the weight diagram has only  $\vee$ 's and  $\wedge$ 's as follows:

$$\begin{array}{cccccccc} \wedge & \wedge & \vee & \wedge & \vee & \wedge & \wedge & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & \end{array}. \quad (5.23)$$

The Cartan subalgebra  $\hat{\mathfrak{h}}$  of  $\widehat{\mathfrak{psl}}(2|2)$  takes the form of (3.6), where  $\mathfrak{h}$  is the quotient of diagonal matrices of  $\mathfrak{sl}(2|2)$  by  $\mathbb{C}I_4$ . We choose a simple root system of  $\widehat{\mathfrak{psl}}(2|2)$  as

$$\Pi = \{\alpha_1, \alpha_2, \alpha_3\} = \{\epsilon_1 - \delta_1, \delta_1 - \delta_2, \delta_2 - \epsilon_2\}, \quad (5.24)$$

where  $\alpha_1 = \alpha_3$ . The corresponding Cartan matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (5.25)$$

and the Dynkin diagram is shown in Fig. 8. We then have inner products

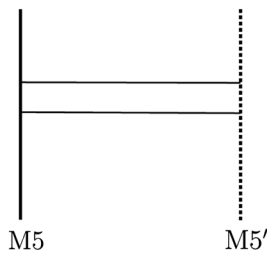


FIG. 7.  $N = d = 2$  M2-branes stretched between the M5- and M5'-branes.

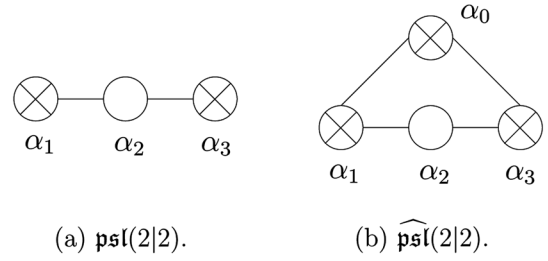


FIG. 8. The Dynkin diagrams corresponding to (5.24). The white dot  $\circ$  represents a simple even root, and the gray dot  $\otimes$  represents a simple odd root of zero length.

$$\begin{aligned} (\alpha_1, \alpha_1) = (\alpha_3, \alpha_3) = (\alpha_1, \alpha_3) &= 0, & (\alpha_2, \alpha_2) &= -2, \\ (\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) &= 1, & (\theta, \theta) &= 2, \end{aligned} \quad (5.26)$$

where  $\theta = \alpha_1 + \alpha_2 + \alpha_3 = \epsilon_1 - \epsilon_2$  is a highest root. The positive root systems and the Weyl vectors of  $\widehat{\mathfrak{psl}}(2|2)$  are

$$\Delta_0^+ = \{\alpha_2, \theta\} = \{\delta_1 - \delta_2, \epsilon_1 - \epsilon_2\}, \quad (5.27)$$

$$\begin{aligned} \Delta_1^+ &= \{\alpha_1, \alpha_3, \alpha_{12}, \alpha_{23}\} \\ &= \{\epsilon_1 - \delta_1, \delta_2 - \epsilon_2, \epsilon_1 - \delta_2, \delta_1 - \epsilon_2\}, \end{aligned} \quad (5.28)$$

$$\rho_0 = \frac{1}{2}(\alpha_{12} + \alpha_{23}) = \frac{1}{2}(\epsilon_1 - \epsilon_2 + \delta_1 - \delta_2), \quad (5.29)$$

$$\rho_1 = \theta = \alpha_{123} = \epsilon_1 - \epsilon_2, \quad (5.30)$$

$$\rho = \rho_0 - \rho_1 = -\frac{1}{2}\alpha_{13} = -\frac{1}{2}(\epsilon_1 - \epsilon_2 - \delta_1 + \delta_2), \quad (5.31)$$

where  $\alpha_{ij} := \alpha_i + \alpha_j$  and  $\alpha_{ijk} := \alpha_i + \alpha_j + \alpha_k$ . Let us choose a coordinate (3.7) on  $\hat{\mathfrak{h}}$  as

$$h := 2\pi i(-\tau d - (z_1 + z_2)\alpha_1 - z_1\alpha_2 + tK), \quad (5.32)$$

where  $\tau, z_1, z_2, t \in \mathbb{C}$  and  $z := -(z_1 + z_2)\alpha_1 - z_1\alpha_2$  is a coordinate on  $\mathfrak{h}$  with an inner product  $(z, z) = 2z_1z_2$ .

For  $\widehat{\mathfrak{psl}}(2|2)$ , the normalized affine superdenominator (5.13) is expressed as [3]

$$\hat{R}^-(\tau, z_1, z_2) = \eta(\tau)^4 \frac{\vartheta_{11}(\tau, z_1 - z_2)\vartheta_{11}(\tau, z_1 + z_2)}{\vartheta_{11}(\tau, z_1)^2\vartheta_{11}(\tau, z_2)^2}, \quad (5.33)$$

where  $\eta(\tau)$  is the Dedekind eta function (A1) and  $\vartheta_{11}(\tau, z)$  is the Jacobi theta function (A6). Since  $\widehat{\mathfrak{psl}}(2|2)$  has zero dual Coxeter number,  $\hat{R}^-(\tau, z_1, z_2)$  has no dependence on parameter  $t \in \mathbb{C}$  in (5.32). From Eqs. (A2), (A8), and (A10), the modular transformations of  $\hat{R}^-(\tau, z_1, z_2)$  read

$$\hat{R}^-\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}\right) = i\tau e^{\frac{\pi i z_1 z_2}{\tau}} \hat{R}^-(\tau, z_1, z_2), \quad (5.34)$$

$$\hat{R}^-(\tau + 1, z_1, z_2) = e^{-\frac{\pi i}{\tau}} \hat{R}^-(\tau, z_1, z_2). \quad (5.35)$$

Following Ref. [3], we consider here the normalized supercharacter of the atypical module  $L(\Lambda)$  for  $\Lambda$  admissible [64,65].<sup>4</sup> The admissible weight  $\Lambda$  is classified by the so-called simple subset  $S = \varphi^{*-1}(\hat{\Pi}) \in \hat{\Delta}_+$  [65] for the compatible homomorphism  $\varphi: \widehat{\mathfrak{sl}} \rightarrow \widehat{\mathfrak{sl}}$ . Let  $\varphi(K) = MK$  where  $M$  is a positive integer called the degree of  $\varphi$ . For  $\widehat{\mathfrak{sl}}(2|2)$ , the conditions for the admissible weights are given by [3]

$$K = \frac{m}{M}, \quad \gcd(m, M) = 1, \quad (5.36)$$

where  $m$  is a nonzero integer. There exist four simple subsets  $S$  [3],

$$\begin{aligned} S_1 &= \left\{ k_i \delta + \alpha_i | i = 0, \dots, 3, \sum_{i=0}^3 k_i = M - 1 \right\}, \\ S_2 &= \left\{ k_i \delta - \alpha_i | i = 0, \dots, 3, \sum_{i=0}^3 k_i = M + 1, k_i > 0 \right\}, \\ S_3 &= \left\{ k_0 \delta + \alpha_0, k_1 \delta + \alpha_{12}, k_2 \delta - \alpha_2, k_3 \delta \right. \\ &\quad \left. + \alpha_{23} | \sum_{i=0}^3 k_i = M - 1, k_2 > 0 \right\}, \\ S_4 &= \left\{ k_0 \delta - \alpha_0, k_1 \delta - \alpha_{12}, k_2 \delta + \alpha_2, k_3 \delta \right. \\ &\quad \left. - \alpha_{23} | \sum_{i=0}^3 k_i = M - 1, k_2 > 0 \right\}, \end{aligned} \quad (5.37)$$

where we have introduced the integers  $k_i \in \mathbb{Z}_{\geq 0}$ ,  $i = 0, 1, 2, 3$  with  $k_1 = k_3$ . Setting  $(j, k) := (k_1, k_1 + k_2)$ ,  $j, k \in \mathbb{Z}_{\geq 0}$ , we obtain all the possible admissible highest weights  $\Lambda_{jk}$  labeled by  $(j, k)$  as

$$\Lambda_{jk} = \begin{cases} k \geq j \geq 0, & j + k \leq M - 1 & \text{for } s = 1 \\ M - 1 \geq j \geq k \geq 1, & j + k \geq M & \text{for } s = 2 \\ 0 \leq k < j, & j + k \leq M - 1 & \text{for } s = 3 \\ 1 \leq j \leq k \leq M - 1, & j + k \geq M & \text{for } s = 4, \end{cases} \quad (5.38)$$

<sup>4</sup>It has been conjectured in Refs. [64,65] that if the highest weight module  $L(\Lambda)$  is modular invariant  $\Lambda$  is realized as an admissible weight.

with  $s = 1, 2, 3, 4$  labeling the four simple subsets (5.37). Collecting all the results, the Kac-Wakimoto supercharacter formula (5.14) for the admissible representations  $\Lambda_{jk}$  of  $\widehat{\mathfrak{psl}}(2|2)$  reads [3]

$$\text{sch}_{\Lambda_{jk}} = \frac{(-1)^{\frac{(s-1)(s-2)}{2}} q^{\frac{mjk}{M}} e^{\frac{2\pi i m}{M}} \Phi^{[m]}(M\tau, z_1 + j\tau, z_2 + k\tau, \frac{\tau}{M})}{\hat{R}^-}, \quad (5.39)$$

where

$$\Phi^{[m]}(\tau, z_1, z_2, t) = e^{2\pi n m t} \sum_{n \in \mathbb{Z}} \left[ \frac{e^{2\pi i n m (z_1 + z_2)} e^{2\pi i z_1} q^{m n^2 + n}}{(1 - e^{2\pi i z_1} q^n)^2} - \frac{e^{-2\pi i n m (z_1 + z_2)} e^{-2\pi i z_2} q^{m n^2 + n}}{(1 - e^{-2\pi i z_2} q^n)^2} \right]. \quad (5.40)$$

To proceed with the index computation of the  $PSL(2|2)_{k=1}$  WZW models, we first observe that the fixed level  $k = 1$  requires that the degree  $M$  is equal to 1. Furthermore, the conditions (5.36), (5.37), and (5.38) are realized only when  $K = M = m = 1$ ,  $(j, k) = (0, 0)$  for  $s = 1$ . Making use of the formulas (5.16) and (5.39), we obtain the mock modular index  $\mathcal{I}(\tau, z)$  for the  $PSL(2|2)_{k=1}$  WZW model

$$\begin{aligned} \mathcal{I}(\tau, z_1, z_2) &= \frac{1}{\eta(\tau)^4} \frac{\vartheta_{11}(\tau, z_1)^2 \vartheta_{11}(\tau, z_2)^2}{\vartheta_{11}(\tau, z_1 - z_2) \vartheta_{11}(\tau, z_1 + z_2)} \\ &\quad \times \sum_{n \in \mathbb{Z}} \left[ \frac{e^{2\pi i n (z_1 + z_2)} e^{2\pi i z_1} q^{n^2 + n}}{(1 - e^{2\pi i z_1} q^n)^2} - \frac{e^{-2\pi i n (z_1 + z_2)} e^{-2\pi i z_2} q^{n^2 + n}}{(1 - e^{-2\pi i z_2} q^n)^2} \right]. \end{aligned} \quad (5.41)$$

## VI. APPELL-LERCH SUMS

The holomorphic index (5.41) takes the form

$$\begin{aligned} \mathcal{I} &= \frac{1}{\eta^4(\tau)} \frac{\vartheta_{11}^2(z_1; \tau) \vartheta_{11}^2(z_2; \tau)}{\vartheta_{11}(z_1 - z_2; \tau) \vartheta_{11}(z_1 + z_2; \tau)} \\ &\quad \times (\mathcal{A}_{2,1}(\tau, z_1, z_1 + z_2) - \mathcal{A}_{2,1}(\tau, -z_2, -z_1 - z_2)), \end{aligned} \quad (6.1)$$

where the second-order Appell-Lerch sum is given by

$$\mathcal{A}_{2,1}(\tau, u, v) = U \sum_{n \in \mathbb{Z}} \frac{q^{n(n+1)} V^n}{(1 - U q^n)^2} \quad (6.2)$$

and we have denoted  $U = \exp(2\pi i u)$  and  $V = \exp(2\pi i v)$ . As previously noted for the atypical modules, the issue, which we will now address, is that the Appell-Lerch sums are not modular.

Following closely the method in Ref. [6], based on Refs. [4,5], we can complete the second-order Appell-Lerch sums. The idea is to express the second-order sum as a derivative of a first-order sum. It is already known how to complete the first-order sum, so replacing it by its modular completion gives the modular completion of the second-order sum, once we have taken into account the modular transformation properties coming from the derivative operator.

Explicitly, the modular completions of the first-order Appell-Lerch sums

$$\mathcal{A}_{1,k}(\tau, u, v) = U^k \sum_{n \in \mathbb{Z}} \frac{q^{kn(n+1)} V^n}{1 - Uq^n} \quad (6.3)$$

are the weight-1 Jacobi forms

$$\hat{\mathcal{A}}_{1,k}(\tau, u, v) = \mathcal{A}_{1,k}(\tau, u, v) + \mathcal{R}_{1,k}(\tau, u, v), \quad (6.4)$$

where

$$\begin{aligned} \mathcal{R}_{1,k}(\tau, u, v) &= \frac{i}{4k} U^{k-1/2} \sum_{m=0}^{2k-1} \vartheta_{11} \left( \frac{v+m}{2k} + \frac{(2k-1)\tau}{4k}; \frac{\tau}{2k} \right) \\ &\times R \left( u - \frac{v+m}{2k} - \frac{(2k-1)\tau}{4k}; \frac{\tau}{2k} \right) \end{aligned} \quad (6.5)$$

$$\begin{aligned} R(w; \tau) &= \sum_{\nu \in \mathbb{Z} + 1/2} \left( \text{sgn}(\nu) - \text{Erf} \left( \sqrt{2\pi\tau_2} \left( \nu + \frac{\Im(w)}{\tau_2} \right) \right) \right) \\ &\times (-1)^{\nu-1/2} W^{-\nu} q^{-\nu^2/2} \end{aligned} \quad (6.6)$$

and  $\tau_2 = \Im(\tau)$ .

Now, it is simple to check that

$$\mathcal{D}\mathcal{A}_{1,k}(\tau, u, v) = (k-1)\mathcal{A}_{1,k}(\tau, u, v) + U^k \sum_{n \in \mathbb{Z}} \frac{q^{kn(n+1)} V^n}{(1 - Uq^n)^2}, \quad (6.7)$$

where we define

$$\mathcal{D} = \frac{1}{2\pi i} \frac{\partial}{\partial u}. \quad (6.8)$$

So, we have for  $k = 1$  the simple relation

$$\mathcal{A}_{2,1}(\tau, u, v) = \mathcal{D}\mathcal{A}_{1,1}(\tau, u, v). \quad (6.9)$$

Since the modular transform of  $\hat{\mathcal{A}}_{1,k}$  is

$$\begin{aligned} \hat{\mathcal{A}}_{1,k} \left( \frac{a\tau + b}{c\tau + d}, \frac{u}{c\tau + d}, \frac{v}{c\tau + d} \right) \\ = (c\tau + d) \exp \left( \frac{2\pi i c}{c\tau + d} u(v - ku) \right) \hat{\mathcal{A}}_{1,k}(\tau, u, v), \end{aligned} \quad (6.10)$$

we can easily see that there is an extra term in the transformation of the derivative. Specifically,

$$\begin{aligned} \mathcal{D}\hat{\mathcal{A}}_{1,k} \left( \frac{a\tau + b}{c\tau + d}, \frac{u}{c\tau + d}, \frac{v}{c\tau + d} \right) \\ = (c\tau + d) \exp \left( \frac{2\pi i c}{c\tau + d} u(v - ku) \right) \mathcal{D}\hat{\mathcal{A}}_{1,k}(\tau, u, v) \\ + c(v - 2ku) \exp \left( \frac{2\pi i c}{c\tau + d} u(v - ku) \right) \hat{\mathcal{A}}_{1,k}(\tau, u, v), \end{aligned} \quad (6.11)$$

but then it is easy to see that by shifting the derivative operator we get the following expression, which transforms as a weight-2 Jacobi form:

$$\left( \mathcal{D} + \frac{\Im(v)}{\tau_2} - 2k \frac{\Im(u)}{\tau_2} \right) \hat{\mathcal{A}}_{1,k}(\tau, u, v). \quad (6.12)$$

Combining the above results, we see that the modular completion of  $\mathcal{A}_{2,1}(\tau, u, v)$  is

$$\hat{\mathcal{A}}_{2,1}(\tau, u, v) = \left( \mathcal{D} + \frac{\Im(v)}{\tau_2} - 2 \frac{\Im(u)}{\tau_2} \right) \hat{\mathcal{A}}_{1,1}(\tau, u, v). \quad (6.13)$$

Note that this works for the index since for both cases  $u = z_1, v = z_1 + z_2$  and  $u = -z_2, v = -z_1 - z_2$  we see that  $u(v - u) = z_1 z_2$ . So, the combination

$$\hat{\mathcal{A}}_{2,1}(\tau, z_1, z_1 + z_2) - \hat{\mathcal{A}}_{2,1}(\tau, -z_2, -z_1 - z_2)$$

also transforms as a Jacobi form of weight 2 (with index 1), i.e., with a factor

$$(c\tau + d)^2 \exp \left( \frac{2\pi i c}{c\tau + d} z_1 z_2 \right)$$

under a modular transformation.

If we include the  $\vartheta$  and  $\eta$  factors, the whole completed index

$$\hat{\mathcal{I}} = \frac{1}{\eta^4(\tau)} \frac{\vartheta_{11}^2(z_1; \tau) \vartheta_{11}^2(z_2; \tau)}{\vartheta_{11}(z_1 - z_2; \tau) \vartheta_{11}(z_1 + z_2; \tau)} \times (\hat{\mathcal{A}}_{2,1}(\tau, z_1, z_1 + z_2) - \hat{\mathcal{A}}_{2,1}(\tau, -z_2, -z_1 - z_2)) \quad (6.14)$$

transforms as a Jacobi form of weight 1 and index 1.

Now, to analyze the result, we define

$$\begin{aligned} \mathcal{R}_{2,1}(\tau, u, v) &= \hat{\mathcal{A}}_{2,1}(\tau, u, v) - \mathcal{A}_{2,1}(\tau, u, v) \\ &= \mathcal{DR}_{1,1}(\tau, u, v) + \left( \frac{\Im(v)}{\tau_2} - 2 \frac{\Im(u)}{\tau_2} \right) \\ &\quad \times \hat{\mathcal{A}}_{1,1}(\tau, u, v). \end{aligned} \quad (6.15)$$

$$\begin{aligned} \frac{\partial}{\partial \bar{\tau}} \hat{\mathcal{A}}_{2,1}(\tau, u, v) &= \frac{\partial}{\partial \bar{\tau}} \mathcal{R}_{2,1}(\tau, u, v) \\ &= \frac{\partial}{\partial \bar{\tau}} (\mathcal{DR}_{1,1}) + (\Im(v) - 2\Im(u)) \\ &\quad \times \frac{\partial}{\partial \bar{\tau}} \left( \frac{1}{\tau_2} \hat{\mathcal{A}}_{1,1}(\tau, u, v) \right) \\ &= \frac{-i}{2\tau_2^2} (\Im(v) - 2\Im(u)) \hat{\mathcal{A}}_{1,1}(\tau, u, v) \\ &\quad + \left( \mathcal{D} + \frac{\Im(v) - 2\Im(u)}{\tau_2} \right) \frac{\partial}{\partial \bar{\tau}} \mathcal{R}_{1,1}. \end{aligned} \quad (6.16)$$

From the definition of  $\mathcal{R}_{1,1}$  and noting that

$$\frac{d}{dz} \text{Erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}, \quad (6.17)$$

### A. Holomorphic anomaly

The completed index is not holomorphic, and we can calculate a holomorphic anomaly equation by taking its  $\bar{\tau}$  derivative. Specifically, we can calculate

we find

$$\begin{aligned} \frac{\partial}{\partial \bar{\tau}} \mathcal{R}_{1,1}(\tau, u, v) &= \frac{1}{8\sqrt{2\tau_2}} e^{\pi i u} \sum_{m=0}^1 \vartheta_{11} \left( \frac{v+m}{2} + \frac{\tau}{4}; \frac{\tau}{2} \right) \sum_{\mu \in \mathbb{Z}} \exp \left( -\frac{\pi \tau_2}{2} \left( \mu + \frac{2\Im(u) - \Im(v)}{\tau_2} \right)^2 \right) \left( \mu - \frac{2\Im(u) - \Im(v)}{\tau_2} \right) (-1)^\mu \\ &\quad \times \exp \left( -2\pi i \left( \mu + \frac{1}{2} \right) \left( u - \frac{v+m}{2} - \frac{\tau}{4} \right) \right) \exp \left( -\frac{\pi i \tau}{2} \left( \mu + \frac{1}{2} \right)^2 \right). \end{aligned} \quad (6.18)$$

Now, note that the factor of  $\mu$  in the sum can arise from differentiating, with respect to  $u$ , the exponential with an exponent linear in  $\mu$ . The structure of the sum is also of the form of a theta function. After some manipulation, we find

$$\begin{aligned} \frac{\partial}{\partial \bar{\tau}} \mathcal{R}_{1,1}(\tau, u, v) &= \frac{\partial}{\partial u} \frac{1}{8\sqrt{2\tau_2}} e^{\pi i u} \exp \left( -\frac{\pi}{2\tau_2} \left( 2\Im(u) - \Im(v) - \frac{\tau_2}{2} \right)^2 \right) \\ &\quad \times \sum_{m=0}^1 \vartheta_{11} \left( \frac{v+m}{2} + \frac{\tau}{4}; \frac{\tau}{2} \right) \vartheta_{11} \left( -\Re(u) + \frac{\Re(v)}{2} + \frac{\Re(\tau)}{4} + \frac{m}{2}; -\frac{\Re(\tau)}{2} \right). \end{aligned} \quad (6.19)$$

Using some theta function identities, we can write the sum over  $m$  of the product of  $\vartheta_{11}$  functions as products of  $\vartheta_{00}$  and  $\vartheta_{01}$ . The result is

$$\frac{\partial}{\partial \bar{\tau}} \mathcal{R}_{1,1}(\tau, u, v) = \frac{\partial}{\partial u} \frac{1}{8\sqrt{2\tau_2}} \exp \left( -\frac{\pi}{2\tau_2} (2\Im(u) - \Im(v))^2 \right) \sum_{m=0}^1 \vartheta_{0m} \left( \frac{v}{2}; \frac{\tau}{2} \right) \vartheta_{0m} \left( -\Re(u) + \frac{\Re(v)}{2}; -\frac{\Re(\tau)}{2} \right). \quad (6.20)$$

Now, we can simplify the notation a little by defining  $z \equiv v - 2u$ , and using variables  $z$  and  $v$ , we just replace  $\frac{\partial}{\partial u}$  with  $-2 \frac{\partial}{\partial z}$ . The result is

$$\frac{\partial}{\partial \bar{\tau}} \mathcal{R}_{1,1}(\tau, u, v) = \frac{\partial}{\partial z} \frac{-1}{4\sqrt{2\tau_2}} \exp \left( -\frac{\pi}{2\tau_2} (\Im(z))^2 \right) \sum_{m=0}^1 \vartheta_{0m} \left( \frac{v}{2}; \frac{\tau}{2} \right) \vartheta_{0m} \left( \frac{\Re(z)}{2}; -\frac{\Re(\tau)}{2} \right). \quad (6.21)$$

The most useful aspect of this notation is when we note that for  $u = z_1$  and  $v = z_1 + z_2 \equiv w$ , and for  $u = -z_2$  and  $v = -z_1 - z_2 = -w$ , we have  $z = z_2 - z_1$ . So, in both cases, we find (differing only in  $v = w$  or  $v = -w$ )



$$\begin{aligned}
 & \left( \mathcal{D} + \frac{\Im(v) - 2\Im(u)}{\tau_2} \right) \frac{\partial}{\partial \bar{\tau}} \mathcal{R}_{1,1} \\
 &= \left( \frac{i}{\pi} \frac{\partial}{\partial z} + \frac{\Im(z)}{\tau_2} \right) \frac{\partial}{\partial z} \frac{-1}{4\sqrt{2}\tau_2} \exp \left( -\frac{\pi}{2\tau_2} (\Im(z))^2 \right) \\
 & \times \sum_{m=0}^1 \vartheta_{0m} \left( \frac{\pm w}{2}; \frac{\tau}{2} \right) \vartheta_{0m} \left( \frac{\Re(z)}{2}; -\frac{\Re(\tau)}{2} \right), \quad (6.22)
 \end{aligned}$$

but since  $\vartheta_{0m}(-z; \tau) = \vartheta_{0m}(z; \tau)$ , we get exactly the same expression in both cases. This means that when we calculate the  $\bar{\tau}$  derivative of the completed index (6.14) the terms arising from the  $\bar{\tau}$  derivative of  $\mathcal{R}_{1,1}$  in (6.16) cancel. So, we finally get the result, which is indicative of a recursion relation for the holomorphic anomaly:

$$\begin{aligned}
 & \frac{\partial}{\partial \bar{\tau}} \hat{\mathcal{I}}(\tau, z_1, z_2) \\
 &= \frac{-i(z_2 - z_1)}{2\tau_2^2} \frac{1}{\eta^4(\tau)} \frac{\vartheta_{11}^2(z_1; \tau) \vartheta_{11}^2(z_2; \tau)}{\vartheta_{11}(z_1 - z_2; \tau) \vartheta_{11}(z_1 + z_2; \tau)} \\
 & \times (\hat{\mathcal{A}}_{1,1}(\tau, z_1, z_1 + z_2) - \hat{\mathcal{A}}_{1,1}(\tau, -z_2, -z_1 - z_2)). \quad (6.23)
 \end{aligned}$$

### B. Modular and elliptic transformations

If we define

$$\Phi(\tau, z_1, z_2) = \hat{\mathcal{A}}_{2,1}(\tau, z_1, z_1 + z_2) - \hat{\mathcal{A}}_{2,1}(\tau, -z_2, -z_1 - z_2), \quad (6.24)$$

then we find the following transformation properties, noting that both  $\hat{\mathcal{A}}$  terms transform in the same way under these transformations:

$$\begin{aligned}
 & \Phi(\tau + 1, z_1, z_2) = \Phi(\tau, z_1, z_2), \\
 & \Phi \left( -\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau} \right) = \tau^2 e^{\frac{2\pi i}{\tau} z_1 z_2} \Phi(\tau, z_1, z_2), \quad (6.25)
 \end{aligned}$$

$$\begin{aligned}
 & \Phi(\tau, z_1 + 1, z_2) = \Phi(\tau, z_1, z_2), \\
 & \Phi(\tau, z_1 + \tau, z_2) = e^{-2\pi i z_2} \Phi(\tau, z_1, z_2), \quad (6.26)
 \end{aligned}$$

$$\begin{aligned}
 & \Phi(\tau, z_1, z_2 + 1) = \Phi(\tau, z_1, z_2), \\
 & \Phi(\tau, z_1, z_2 + \tau) = e^{-2\pi i z_1} \Phi(\tau, z_1, z_2). \quad (6.27)
 \end{aligned}$$

If we also include the theta and eta functions, the index transforms as

$$\begin{aligned}
 & \hat{\mathcal{I}}(\tau + 1, z_1, z_2) = e^{\frac{\pi i}{6}} \hat{\mathcal{I}}(\tau, z_1, z_2), \\
 & \hat{\mathcal{I}} \left( -\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau} \right) = -i\tau e^{\frac{2\pi i}{\tau} z_1 z_2} \hat{\mathcal{I}}(\tau, z_1, z_2), \quad (6.28)
 \end{aligned}$$

$$\begin{aligned}
 & \hat{\mathcal{I}}(\tau, z_1 + 1, z_2) = \hat{\mathcal{I}}(\tau, z_1, z_2), \\
 & \hat{\mathcal{I}}(\tau, z_1 + \tau, z_2) = e^{-2\pi i z_2} \hat{\mathcal{I}}(\tau, z_1, z_2), \quad (6.29)
 \end{aligned}$$

$$\begin{aligned}
 & \hat{\mathcal{I}}(\tau, z_1, z_2 + 1) = \hat{\mathcal{I}}(\tau, z_1, z_2), \\
 & \hat{\mathcal{I}}(\tau, z_1, z_2 + \tau) = e^{-2\pi i z_1} \hat{\mathcal{I}}(\tau, z_1, z_2). \quad (6.30)
 \end{aligned}$$

### C. Wall-crossing

The Appell-Lerch sum of order 2 is associated to meromorphic Jacobi forms of weight 2. It is shown in Ref. [5] that any meromorphic Jacobi form  $\varphi_m(\tau, z)$  with double poles at  $z = z_s = \alpha\tau + \beta$ ,  $\alpha, \beta \in S \subset \mathbb{Q}^2$  has a decomposition

$$\varphi_m(\tau, z) = \varphi_m^F(\tau, z) + \varphi_m^P(\tau, z). \quad (6.31)$$

Here,

$$\varphi_m^F(\tau, z) = \sum_{l \in \mathbb{Z}/2m\mathbb{Z}} h_l(\tau) \vartheta_{m,l}(\tau, z) \quad (6.32)$$

is a finite part, and

$$\varphi_m^P(\tau, z) = \sum_{s \in S/\mathbb{Z}^2} (D_s(\tau) A_{1,m}^s(\tau, z) + E_s(\tau) A_{2,m}^s(\tau, z)) \quad (6.33)$$

is a polar part. Here,  $D_s(\tau)$  and  $E_s(\tau)$  are residue functions defined by

$$e^{2\pi i m \alpha z_s} \varphi(\tau, z_s + \epsilon) = \frac{E_s(\tau)}{(2\pi i \epsilon)^2} + \frac{D_s(\tau) - 2m\alpha E_s(\tau)}{2\pi i \epsilon} + \mathcal{O}(1), \quad (6.34)$$

while  $A_{1,m}^s(\tau, z)$  and  $A_{2,m}^s(\tau, z)$  are universal Appell-Lerch sums [5] of order 1 and 2. In our analysis, we saw multivariable order 1 and 2 Appell-Lerch sums. In the single variable case, these corresponding to taking  $s = 0$  above and are defined by

$$A_{1,m}(\tau, z) = -\frac{1}{2} \sum_{n \in \mathbb{Z}} q^{mn^2} x^{2mn} \frac{1 + xq^n}{1 - xq^n}, \quad (6.35)$$

$$A_{2,m}(\tau, z) = \sum_{n \in \mathbb{Z}} \frac{q^{mn^2} x^{2mn+1}}{(1 - xq^n)^2}. \quad (6.36)$$

In the context of black hole microstate counting, the degeneracy of four-dimensional  $\mathcal{N} = 4$  quarter-BPS dyonic black holes with a set of three fixed charges  $(m, n, l)$  is given by Fourier coefficients of the partition function, that is a meromorphic Siegel modular form of weight  $-10$  [66–68],

$$\mathcal{Z}_{\text{dyon}} = \frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{m=-1}^{\infty} \varphi_m(\tau, z) y^m. \quad (6.37)$$

Here,  $\Phi_{10}(\tau, z, \sigma)$  is the Igusa cusp form of weight 10, and  $\varphi_m(\tau, z)$  is a meromorphic Jacobi-form of weight 2 and index  $m$ . According to the above decomposition theorem (6.31) of meromorphic Jacobi forms,  $\varphi_m(\tau, z)$  in (6.37) can be decomposed as

$$\varphi_m(\tau, z) = \varphi_m^F(\tau, z) + \frac{p_{24}(m+1)}{\Delta(\tau)} A_{2,m}(\tau, z). \quad (6.38)$$

Here, the first term  $\varphi_m^F(\tau, z)$  is a finite part without a pole and counts the single-centered black holes, while the second is a polar part with double poles and counts the multicentered black holes that decay into its single-centered constituents upon wall-crossing phenomena [5]. In fact, the Appell-Lerch sum of order 2 is intimately related to an occurrence of wall crossing due to its polar structure. To see this, it is useful to introduce an operation of averaging the residues at poles  $z = z_s = \alpha + \beta\tau$ ,

$$\text{Av}^{(m)}[f(x)] := \sum_{\lambda \in \mathbb{Z}} q^{m\lambda^2} x^{2m\lambda} f(q^\lambda x). \quad (6.39)$$

This averaging operator constructs a Jacobi form of index  $m$  out of an arbitrary function  $f(x)$ . Making use of the averaging operator, one can express the Appell-Lerch sum of order 2 as

$$\text{Av}^{(m)} \left[ \frac{x}{(1-x)^2} \right] = A_{2,m}. \quad (6.40)$$

The function  $f(x)$  has an expansion,

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots, \quad (6.41)$$

in the range  $|x| < 1$ , but it does not for  $|x| > 1$ . This implies wall crossing because different expansions of the meromorphic Jacobi form for  $|x| < 1$  and  $|x| > 1$  give different degeneracies as its coefficients. Correspondingly, we have

$$A_{2,m} = \left( \sum_{n \geq 0} \sum_{l \geq 0}^* - \sum_{n < 0} \sum_{l \leq 0}^* \right) l q^{mn^2 + ln} x^{2mn+l} \quad (6.42)$$

for  $|q| < |x| < 1$ . Here,  $\sum_l^*$  is the sum for the term  $l = 0$  with multiplicity  $\frac{1}{2}$ .

These quarter-BPS black holes can be realized as a configuration of M2-M5 bound states in M theory on  $K3 \times T^2$  [5]. Let  $T^2$  be a product of two circles  $S_\alpha^1 \times S_\beta^1$ . Let  $C^1$  be a homology 2-cycle of  $T^2$  and  $C^2$  and  $C^3$  be two 2-cycles in  $K3$  which have intersection number

$$\int_{K3 \times T^2} C^1 \wedge C^2 \wedge C^3 = 1. \quad (6.43)$$

Let  $\{D_a\}$  be 4-cycles dual to  $\{C^a\}$ , i.e.,  $D_a \cap C^b = \delta_a^b$ . We consider the M2-M5 bound states with  $w$  units of momentum along M circle  $S_0^1$  where  $\tilde{K}$  units of M5-brane charge wrap  $D_1 \times S_0^1$ ,  $Q_1$  units of M5-brane charge wrap  $D_2 \times S_0^1$ ,  $Q_5$  units of M5-brane charge wrap  $D_3 \times S_0^1$ , and  $\tilde{n}$  units of M2-brane charge wrap  $T^2$ . Then,

$$m = Q_1 Q_5, \quad n = w \tilde{K}, \quad l = \tilde{n} \tilde{K} \quad (6.44)$$

can be identified with the charges of the quarter-BPS dyonic black hole states, and the number of BPS bound states of the brane configuration can be viewed as the degeneracy of the black holes. When the M5-brane charges  $\tilde{K}$ ,  $Q_1$ , and  $Q_5$  are fixed, the charges  $(m, n, l)$  of the black hole are determined by the momentum  $w$  and the M2-brane charge  $\tilde{n}$ , which would be specified by the quantum numbers of the derivation  $d = -L_0$  and those of Cartan elements of the Lie superalgebra  $\mathfrak{sg}$ , respectively. Hence, under certain circumstances, our index would have an interpretation in terms of black hole microstate counting. An appearance of the second-order multivariable Appell-Lerch sum  $A_{2,m}$  would suggest that such multicentered black holes may decay into single-centered black holes [69–72].

From the perspective of the M2-M5 system, it is expected that wall crossing occurs due to the configuration of stretched M2-branes so that the moduli space of the M2-M5 system may develop a new branch at a particular critical value of the  $C$  field on the M5-branes.

## VII. DISCUSSION

We have described BPS indices for supergroup WZW models which we have argued count the degeneracies of BPS states of the intersecting M2-M5 system considered in Ref. [1]. The BPS states are specified by the highest weight modules of the affine Lie superalgebra in such a way that the number of stretched M2-branes is equal to the degree of atypicality. In addition, the momenta along a wrapped circle are given by the Virasoro modes that amount to the derivation, and the M2-brane charges under the  $C$  fields are given by the Cartan elements of the finite Lie superalgebra. When all these M2-branes are sandwiched between the M5-branes, in which case the BPS states are the modules with maximal atypicality, the indices can be evaluated using the Kac-Wakimoto character formula [2,3]. Quite remarkably, they are written in terms of the  $q$ -series known as Ramanujan's mock theta functions [55–57]. Our result is an encounter of the mock Jacobi forms in the BPS indices of the M strings, which are defined in the supergroup WZW models in the same manner as the equivariant elliptic genus studied in Refs. [40,73–76]. The indices have a structure which suggests there is wall crossing in the BPS state

counting of the M2-M5 system, related to universal features of the Appell-Lerch sums. We have argued that the mock modularity of the supercharacters of affine Lie superalgebras reflects the nonholomorphic atypical sector of the Hilbert space of the supergroup WZW models. To obtain the nonholomorphic modular parts of the torus partition function of supergroup WZW models, we have invoked Zweegers' method [4], closely following the discussion in Ref. [5] and particularly Ref. [6].

There are many future directions to consider. Clearly, it is desirable to extend our explicit evaluation of the indices for  $PSL(2|2)_{k=1}$  to other cases. The indices reduce to a specialization of the supercharacters of integrable highest weight modules over affine Lie superalgebras. However, at present, explicit calculation of supercharacters is only available for  $\widehat{\mathfrak{gl}}(N|1)$  and  $\widehat{\mathfrak{sl}}(N|1)$  in Ref. [2], for  $\widehat{\mathfrak{psl}}(2|2)$  in Ref. [3], for  $\widehat{\mathfrak{osp}}(3|2)$  in Ref. [58], and for some general basic Lie algebras  $\widehat{\mathfrak{sg}}$  in Ref. [59]. The case of most relevance for our application is  $\widehat{\mathfrak{gl}}(N|N)$ , which arises in the case of  $N$  M2-branes between the M5-branes. Understanding the dependence of the spectrum on  $N$  is an obvious issue, and perhaps some aspects can be studied even without the complete explicit expression for the supercharacter.

Another interesting question is whether our result can be described using a quantum mechanics description. Here, there are two possibilities, we could reduce the 2D WZW model on a circle, or instead the Quantum Mechanics (QM) description could arise using the interval between the M5-branes as the Euclidean time. In the latter case, the QM description would arise in the limit opposite of that we consider, in which the volume of the Riemann surface  $\Sigma$  is taken to be very small compared to the length of the interval separating the M5-brane from the M5'-brane. We note that recently an intriguing connection between Chern-Simons with boundary and WZW models, and a specific QM model in a large- $N$  limit, has been given by Dorey *et al.* [77,78]. In particular, it was shown that the partition function of the  $SU(p)$  WZW model could be reproduced exactly from a QM action with an  $SU(p)$  symmetry coupled to an  $SU(N)$  gauge field at large  $N$ . It would be interesting to explore generalizations of this QM system to include supersymmetry or supergroup symmetry, with the hope of describing supersymmetric or supergroup WZW models. Of course, if such generalizations could include models related to M-branes, we would hope that such QM systems could be independently derived from the brane configurations or some dual configurations. We are currently exploring such possibilities.

Going beyond the M-brane configurations considered in Ref. [1], we could consider configurations with M2-branes on both sides of an M5-brane and more than one M5- and M5'-brane. In the case of parallel M5-branes, the index has been calculated [40] using various techniques including topological strings. The type-IIB description of

such systems in flat space has been considered by Niarchos [28] by the addition of D5-branes to the ABJM configuration. We have commented on the description with both M5- and M5'-branes, including either D5- and D5'-branes or NS5- and NS5'-branes in type IIB. We expect this will lead to further understanding of the M2-M5 system, with or without the topological twisting. Certainly, as we discussed, we expect this to lead to an understanding of the detailed coupling between ABJM models describing M2-branes on either side of an M5-brane. In the type-IIB configuration, this can be studied in terms of open strings connecting the D3-branes, and recent works [41,79] on the supersymmetric boundary conditions in three-dimensional  $\mathcal{N} = 4$  gauge theories will play a key role in giving the description of these brane tiling models as two-dimensional gauge theories. In the case of supergroup WZW models, we expect that this would give a specific model based on  $GL(N|N) \times GL(M|M)$ . It may also be possible to extend this analysis in type IIB to include generalizations of the ABJM model, such as those based on the ABJ theory or with orthogonal and symplectic gauge groups [38,39,80]. Although it is not clear how to relate all these cases to M-brane configurations, we would expect some (but not all) to correspond to supergroup WZW models.

The Appell-Lerch sums, which we have found in the indices, are known to play an important role in mathematics and physics. In particular, they appear as the Fourier coefficients of the generating functions in various counting problems. We expect that the appearance of these sums from M-brane constructions will lead to a more unified formalism, relating different aspects of the Appell-Lerch sums. To seek gauge theoretical descriptions, we could start from the world volume theory of M5-branes wrapping a 4-manifold to obtain four-dimensional twisted  $\mathcal{N} = 4$  gauge theories [81,82]. In Ref. [83], the generating function of topological invariants of the moduli space of vector bundles over 4-manifolds was evaluated as the partition function of four-dimensional twisted  $\mathcal{N} = 4$  gauge theories, which is expressed in terms of multivariable Appell-Lerch sums. Also, in the weak string coupling region, one could calculate indices in the world volume theory of branes as the generating functions of certain topological invariants. In Ref. [84], the generating functions of Gromov-Witten invariants of elliptic orbifolds are given by multivariable Appell-Lerch sums. In the strong string coupling region, the brane system would involve the gravitational interaction, and the indices would count the microstates of the black holes. As we have seen, the partition functions of the multicentered black holes are expressed in terms of the Appell-Lerch sums [5]. We hope to report on progress from these viewpoints in subsequent works.

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### APPENDIX: MODULAR FORMS

The Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad (\text{A1})$$

satisfies the modular transformation properties

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} \eta(\tau), \quad \eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau). \quad (\text{A2})$$

The four Jacobi theta functions are defined by [85]

$$\begin{aligned} \vartheta_{00}(\tau, z) &= \vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n z} q^{\frac{n^2}{2}} \\ &= \prod_{n=1}^{\infty} (1 - q^n) (1 + e^{2\pi i z} q^{n-\frac{1}{2}}) (1 + e^{-2\pi i z} q^{n-\frac{1}{2}}), \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \vartheta_{01}(\tau, z) &= \vartheta_{00}\left(\tau, z + \frac{1}{2}\right) \\ &= \prod_{n=1}^{\infty} (1 - q^n) (1 - e^{2\pi i z} q^{n-\frac{1}{2}}) (1 - e^{-2\pi i z} q^{n-\frac{1}{2}}), \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \vartheta_{10}(\tau, z) &= e^{\frac{\pi i z}{4}} e^{\pi i z} \vartheta_{00}\left(\tau, z + \frac{\tau}{2}\right) \\ &= e^{\frac{\pi i z}{4}} e^{-\pi i z} \prod_{n=1}^{\infty} (1 - q^n) (1 + e^{2\pi i z} q^{n-1}) \\ &\quad \times (1 + e^{-2\pi i z} q^n), \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \vartheta_{11}(\tau, z) &= i e^{\frac{\pi i z}{4}} e^{\pi i z} \vartheta_{00}\left(\tau, z + \frac{\tau}{2} + \frac{1}{2}\right) \\ &= e^{\frac{\pi i z}{4}} e^{-\pi i(z+\frac{1}{2})} \prod_{n=1}^{\infty} (1 - q^n) (1 - e^{2\pi i z} q^{n-1}) \\ &\quad \times (1 - e^{-2\pi i z} q^n). \end{aligned} \quad (\text{A6})$$

We have the transformation laws

$$\begin{aligned} \vartheta_{00}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} e^{\frac{\pi i z^2}{\tau}} \vartheta_{00}(\tau, z), \\ \vartheta_{01}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} e^{\frac{\pi i z^2}{\tau}} \vartheta_{10}(\tau, z), \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} \vartheta_{10}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} e^{\frac{\pi i z^2}{\tau}} \vartheta_{01}(\tau, z), \\ \vartheta_{11}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= (i\tau)^{\frac{1}{2}} e^{\frac{\pi i z^2}{\tau}} \vartheta_{11}(\tau, z), \end{aligned} \quad (\text{A8})$$

and

$$\vartheta_{00}(\tau + 1, z) = \vartheta_{01}(\tau, z), \quad \vartheta_{01}(\tau + 1, z) = \vartheta_{00}(\tau, z), \quad (\text{A9})$$

$$\vartheta_{10}(\tau + 1, z) = e^{\frac{\pi i}{4}} \vartheta_{10}(\tau, z), \quad \vartheta_{11}(\tau + 1, z) = e^{\frac{\pi i}{4}} \vartheta_{11}(\tau, z). \quad (\text{A10})$$

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