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Sectional curvature of polygonal complexes with planar substructures

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A B S T R A C T

In this paper we introduce a class of polygonal complexes for which we consider a notion of sectional combinatorial curvature. These complexes can be viewed as generalizations of 2-dimensional Euclidean and hyperbolic buildings. We focus on the case of non-positive and negative combinatorial curvature. As geometric results we obtain a Hadamard–Cartan type theorem, thinness of bigons, Gromov hyperbolicity and estimates for the Cheeger constant. We employ the latter to get spectral estimates, show discreteness of the spectrum in the sense of a Donnelly–Li type theorem and present corresponding eigenvalue asymptotics. Moreover, we prove a unique continuation theorem for eigenfunctions and the solvability of the Dirichlet problem at infinity.

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1. Introduction

Since recent years there is an increasing interest in studying curvature notions on discrete spaces. First of all there are various approaches to Ricci curvature based on L^1-optimal transport on metric measure spaces starting with the work of Ollivier, [52,53]. These ideas were employed for graphs by various authors [7,38,45,46] to study geometric and spectral questions. A related and very effective definition using L^2-optimal transport was introduced in [24]. Secondly, in [38,46] there is the approach of defining curvature bounds via curvature-dimension-inequalities using a calculus of Bakry–Emery based on Bochner's formula for Riemannian manifolds. Similar ideas were used in [6] and very recently in [50] to prove a Li–Yau inequality for graphs. Finally let us mention the work on so-called Ricci-flat graphs [18] and [26] for another approach. All these approaches have in common that they model some kind of Ricci curvature and that they are very useful to study lower curvature bounds.

In contrast to these developments we are interested in sectional curvature and upper curvature bounds. The notion we develop has its origins in planar polygonal complexes or tessellations. For planar tessellations this notion is defined by an angular defect and these ideas go back as far as to works of Descartes [25] and often there is no obvious relation of this curvature to the recent notions of Ricci curvature above. For planar graphs this curvature notion has proven to be very effective to derive very strong spectral and geometric consequences of upper curvature bounds [8,9,33,42–44,61] which often relate to results to upper bounds on sectional curvature of Riemannian manifolds. (For consequences on lower bounds see, e.g., [19,36,35,51,58,63] as well.)

We introduce a notion of sectional curvature for more general non-planar polygonal complexes. A similar notion is found in the work of Wise [60] which uses in turn ideas of [30,55,57]. These works address primarily group theoretic questions, see also [48] for recent developments. In contrast, the aim of this work is to focus on geometric and spectral theoretic questions. So, we identify a class of polygonal complexes that is well suited for our purposes.

This class consists of *polygonal complexes with planar substructures*. They are 2-dimensional CW-complexes equipped with a family of subcomplexes homeomorphic to the Euclidean plane. We call these subcomplexes *apartments* since they have certain properties similar to the ones required for apartments in Euclidean and hyperbolic buildings. The 2-cells of a polygonal complex with planar substructures can be viewed as polygons and they are called faces and their closures are called chambers. The geometry is based on this set of faces and their neighboring structures. In particular, there is a combinatorial distance function on the set of faces.

Let us discuss the properties of apartments in more detail. First of all, we require that there are enough apartments, that is any two faces have to lie in at least one apartment (condition (PCPS1) in Definition 2.4 below). Sometimes, we require the stronger condition (PCPS1*) that every infinite geodesic ray of faces is included in an apartment. The second crucial property is that all apartments are *convex* (see condition (PCPS2)). These properties are also similar to the ones satisfied by flats in symmetric spaces. The definition of polygonal complexes with planar substructures comprises both planar tessellations and all 2-dimensional Euclidean and hyperbolic buildings.

We use the apartments of a polygonal complex with planar substructures to define combinatorial curvatures on them. Since these apartments could be seen in a vague sense as tangent planes of the polygonal complex with planar substructures, we call these curvatures *sectional curvatures*. We introduce sectional curvatures on the faces and on the corners of an apartment (see Definition 2.8), and they are invariants measuring the *local geometry* of the polygonal complex with planar substructures.

The definition of polygonal complexes with planar substructures and basic notions are introduced in Section 2. The results in this article are then given in Sections 3 and 4. While most of these results are known for planar tessellations, it seems to us that several of these results were not known for Euclidean and hyperbolic buildings. Next, we explain our results in more detail.

In Section 3 we discuss implications of negative and non-positive curvature to the *global and asymptotic geometry* of a polygonal complex with planar substructures. Many of the presented results have well-known counterparts in the smooth setting of Riemannian manifolds. Amongst our results, we present a combinatorial Cartan–Hadamard theorem for non-positively curved polygonal complexes with planar substructures (see Theorem 3.1) and we conclude Gromov hyperbolicity and positivity of the Cheeger isoperimetric constant for negatively curved polygonal complexes with planar substructures with certain bounds on the vertex and face degree (see Theorems 3.6 and 3.8). These results are based on negativity or non-positivity of the sectional corner curvature. We also state an analogue of Myers theorem in the case of strictly positive sectional face curvature (see Theorem 3.13).

Section 4 is devoted to spectral considerations of the Laplacian. We discuss combinatorial/geometric criteria to guarantee emptiness of the essential spectrum and to derive certain eigenvalue asymptotics on polygonal complexes with planar substructures (see Theorem 4.1). We also show that non-positive sectional corner curvature on polygonal complexes with planar substructures implies absence of finitely supported eigenfunctions (see Theorem 4.3). Finally, we derive solvability of the Dirichlet problem at infinity for polygonal complexes with planar substructures in the case of negative sectional corner curvature (see Theorem 4.6).

As mentioned before, 2-dimensional Euclidean and hyperbolic buildings provide large classes of examples of polygonal complexes with planar substructures. While all these spaces have non-positive sectional face curvature, their corner curvature is not always necessarily non-positively curved. The main purpose of the final Section 5 is to provide a self-contained short survey over these important classes.

In the appendix we discuss how Wise's definition of sectional curvature, which is in some sense an even more flexible notion, is related to our notion of curvature.

2. Basic definitions

In this section we introduce *polygonal complexes with planar substructures* and define a notion of sectional curvature on theses spaces. In order to do so we introduce polygonal complexes and planar tessellations first. In the second subsection we explore some basic consequences of the convexity assumption we impose. In the third subsection we introduce a combinatorial sectional curvature notions for these spaces.

2.1. Polygonal complexes with planar substructures

The following definition of polygonal complexes is found in [3].

Definition 2.1 *(Polygonal complex).* A *polygonal complex* is a 2-dimensional CW-complex X with the following properties:

(PC1) The attaching maps of X are homeomorphisms.
(PC2) The intersection of any two closed cells of X is either empty or exactly one closed cell.

For a polygonal complex X we denote the set of 0-cells by V and call them *vertices*, we denote the set of 1-cells by E and call them the *edges* and we denote the set of 2-cells by F and call them the *faces*. We write $X = (V, E, F)$. Note that the closures of all edges and faces in X are necessarily compact (since they are images of compact sets under the continuous characteristic maps, see [32, Appendix]). We call two vertices v and w *adjacent* or *neighbors* if they are connected by an edge in which case we write $v \sim w$. We call two different faces f and g *adjacent* or *neighbors* if their closures intersect in an edge and we write $f \sim g$. It is convenient to call the closure of a face a *chamber*.

The *degree* $|v| \in \mathbb{N}_0 \cup \{\infty\}$ of a vertex $v \in V$ is the number of vertices that are adjacent to v. The *degree* $|e| \in \mathbb{N}_0 \cup \{\infty\}$ of an edge $e \in E$ is the number of chambers containing e. The boundary ∂f of a face $f \in F$ is the set of all 1-cells $e \in E$ being contained in the closure $\overline{f}$. Since in CW-complexes every compact set can meet only finitely many cells (see [32, Prop. A.1]), we have $|\partial f| = \#\partial f < \infty$. The *degree* $|f|$ of a face $f \in F$ is the number of faces that are adjacent to f and, in contrast to $|\partial f|$, the face degree $|f|$ can be infinite.

We call a (finite, infinite or bi-infinite) sequence $\ldots, f_{i-1}, f_i, f_{i+1}, \ldots$ of pairwise distinct faces a *path* if successive faces are adjacent. The *length* of the path is one less than the number of components of the sequence. The *(combinatorial) distance* between two faces f and g is the length of the shortest path connecting f and g and the distance is denoted by $d(f, g)$. We call a (finite, infinite or bi-infinite) path (f_k) of faces a *geodesic* or a *gallery*, if we have for any two faces f_m and f_n in the path $d(f_m, f_n) = |m - n|$, i.e., the distance between f_m and f_n is realized by the path.

Definition 2.2 *(Convex).* A polygonal subcomplex Σ of a polygonal complex X is called *convex* if every geodesic in X connecting two faces in Σ is included in Σ.

We say a polygonal complex X is *planar* if X is homeomorphic to $\mathbb{R}^2$. We also say that a polygonal complex X is *spherical* if X is homeomorphic to the two-sphere $\mathbb{S}^2$.

Next we introduce the notion of a planar tessellation following [8,9].

Definition 2.3 *(Planar tessellation).* A polygonal complex $\Sigma = (V, E, F)$ is called a *(planar/spherical) tessellation* if Σ is planar/spherical and satisfies the following properties:

(T1) Any edge is contained in precisely two different chambers.
(T2) Any two different chambers are disjoint or have precisely either a vertex or a side in common.
(T3) For any chamber the edges contained in it form a closed path without repeated vertices.
(T4) Every vertex has finitely many neighbors.

Note that property (T3) is already implied by (PC1) and (PC2). The tessellations form the substructures which we will later need to define sectional curvature. Now, we are in a position to introduce polygonal complexes with planar substructures.

Definition 2.4. A *polygonal complex with planar substructures* is a polygonal complex $X = (V, E, F)$, together with a set $\mathcal{A}$ of subcomplexes whose elements $\Sigma = (V_\Sigma, E_\Sigma, F_\Sigma)$ are called *apartments*, with the following properties:

(PCPS1) For any two faces there is an apartment containing both of them.
(PCPS2) The apartments are convex.
(PCPS3) The apartments are planar tessellations.

Similarly, we introduce *polygonal complexes with spherical substructures* by replacing property (PCPS3) in Definition 2.4 by

(PCSS3) The apartments are spherical tessellations.

Prominent examples of polygonal complexes with planar substructures are 2-dimensional Euclidean and hyperbolic buildings (see Section 5 for the definition of a building as well as several examples). Moreover, every planar tessellation is trivially a polygonal complex with planar substructures. For reasons of illustration, we introduce the following example of a Euclidean building.

Example 1. Let X_0 be the finite simplicial complex constructed from the seven equilateral Euclidean triangles illustrated in Fig. 1 by identifying sides with the same labels x_i.

Then X_0 has a single vertex which we denote by p_0, seven edges and seven faces. Its fundamental group $\Gamma = \pi_1(\Pi_0, p_0)$ has the following presentation

$$\Gamma = \langle x_0, \dots, x_6 \mid x_i x_{i+1} x_{i+3} = \mathrm{id} \text{ for } i = 0, 1, \dots, 6 \rangle$$

(where i is taken modulo 7). Let $X = (V, E, F)$ be the universal covering of X_0 together with the lifted triangulation. Then it follows from [4, Theorem 6.5] that X is a *thick Euclidean building of type* $\widetilde{A}_2$ and every edge of X belongs to precisely 3 triangles. Therefore, X is a polygonal complex with planar substructures. The group of covering transformations is isomorphic to Γ and acts transitively on the vertices of this building (see [16]).

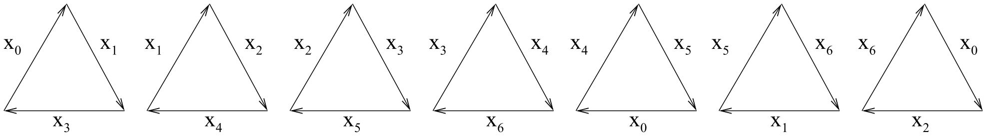

Fig. 1. Labeling scheme for the simplicial complex X_0.

For some of our results we need the following slightly stronger assumption than (PCPS1):

(PCPS1*) Every (one-sided) infinite geodesic is included in an apartment.

Condition (PCPS1*) is satisfied for all 2-dimensional Euclidean and hyperbolic buildings with a *maximal apartment system* (see Theorem 5.7 below).

Finally, let us mention the following important fact. To a polygonal complex $X = (V, E, F)$ we can naturally associate a graph G_X by letting F be the vertex set of G_X and by defining the edges of that graph via the adjacency relation of the corresponding faces. This "duality" becomes important when we use results for graphs in our context.

2.2. Consequences of convexity

The convexity assumption (PCPS2) is very important in our considerations. In this subsection we collect some of the immediate consequences.

Lemma 2.5. *Let X be a polygonal complex with planar substructures, Σ an apartment and let d_Σ be the combinatorial distance within the apartment. Then, for any two faces $f, g \in F_\Sigma$*

$$d(f, g) = d_\Sigma(f, g).$$

Proof. The inequality "$\le$" is clear. For the other direction "$\ge$" let $\gamma = (f_0, \dots, f_n)$ be a path connecting f and g minimizing $d(f, g)$. As γ is a geodesic with end-faces in Σ it is completely contained in Σ by (PCPS2). Hence, the statement follows. $\square$

We say a subset F_0 of F is *connected* if any two faces in F_0 can be joined by a path in F_0.

Lemma 2.6. *Let X be a polygonal complex with planar substructures. Let Σ_1 and Σ_2 be two apartments of X. Then the set $F_{\Sigma_1} \cap F_{\Sigma_2}$ is connected.*

Proof. Let f and g be two faces in $F_{\Sigma_1} \cap F_{\Sigma_2}$. Then, by (PCPS2), every geodesic connecting f and g is completely contained in Σ_1 and Σ_2. Thus, $F_{\Sigma_1} \cap F_{\Sigma_2}$ is convex and, therefore, connected. $\square$

For a fixed face $o \in F$ (called *center*), we define the (combinatorial) spheres and balls about o by

$$S_n = S_n(o) = \{f \in F \mid d(f, o) = n\},$$
$$B_n = B_n(o) = \bigcup_{k=0}^{n} S_k,$$

for $n \ge 0$. For $f \in F$, we let the *forward* and *backward degree* be given by

$$|f|_\pm = |\{g \in F \mid g \sim f,\ d(g, o) = d(f, o) \pm 1\}|,$$

and we call $g \in F$ with $g \sim f$ and $d(g, o) = d(f, o) + 1$ (respectively $d(g, o) = d(f, o) - 1$) a *forward* (respectively *backward*) *neighbor* of f. The next lemma shows that the convexity condition (PCPS2) imposes a lot of structure of the distance spheres.

Lemma 2.7. *Let X be a polygonal complex with planar substructures and $o \in F$ be a center. Let $f \in F$ with $f \in S_n$ for some $n \geq 0$ and $f_+ \in S_{n+1}$, $f_0 \in S_n$, $f_- \in S_{n-1}$ be neighbors of f. Then,*

(a) *Every face sharing the same edge with f and f_+ is in S_{n+1}.*
(b) *Every face sharing the same edge with f and f_0 is in $S_n \cup S_{n-1}$.*
(c) *Every face sharing the same edge with f and f_- is in S_n.*

Proof. (a) Let $g \in F$ be such that $\partial g \cap \partial f \cap \partial f^+ \neq \emptyset$. Since g is a neighbor of f_+, we have $d(o, g) \geq n$. Since g is a neighbor of f, we have $d(o, g) \leq n + 1$. Therefore, we have $g \in S_n \cup S_{n+1}$. If g was in S_n, then there are geodesics from the center o over g to f_+ and from o over f to f_+. By (PCPS2) both of these geodesics lie together in one apartment. Hence, g lies in one apartment together with f, f_+ and o. Then, there is an edge contained in three faces f, f_+ and g within one apartment Σ. This contradicts (T1) in the definition of a planar tessellation. But Σ is a planar tessellation, by (PCPS3). Thus, $g \in S_{n+1}$.

(b) Let $g \in F$ be such that $\partial g \cap \partial f \cap \partial f_0 \neq \emptyset$. If g was in S_{n+1} then there were two geodesics from o to g, one via f and the other one via f_0. By a similar argument as in (a), the faces g, f, f_0 and o lie in the same apartment. Again this is impossible by (T1) and (PCPS3).

(c) Let $g \in F$ be such that $\partial g \cap \partial f \cap \partial f_- \neq \emptyset$. Clearly, g is in $S_n \cup S_{n-1}$. If g was in S_{n-1} then, by similar arguments as in (a) and (b), the faces g, f, f_- and o lie in the same apartment which is again impossible by (T1) and (PCPS3). □

2.3. Sectional curvature

For an apartment $\Sigma = (V_\Sigma, E_\Sigma, F_\Sigma)$, let $|v|_\Sigma$ be the *degree of v in Σ* which is the number of neighboring vertices in V_Σ. We notice that the degree of an edge in Σ, i.e., the number of faces in F_Σ bounded by the edge, is always equal to 2 by (T1). Moreover, the *degree $|f|_\Sigma$ of a face f in Σ* is equal to $|\partial f|$. Therefore, $|f|_{\Sigma_1} = |f|_{\Sigma_2}$ for any two apartments Σ_1, Σ_2 that contain f. Furthermore, for a polygonal complex with planar substructures X and $\Sigma \in \mathcal{A}$ we let the set of corners of X and of Σ be given by

$$C = \{(v, f) \in V \times F \mid v \in \overline{f}\}, \quad C_\Sigma = \{(v, f) \in V_\Sigma \times F_\Sigma \mid v \in \overline{f}\}.$$

Definition 2.8 *(Sectional curvature).* Let Σ be an apartment of a polygonal complex with planar substructures X. The *sectional corner curvature* $\kappa_c^{(\Sigma)} : C_\Sigma \to \mathbb{R}$ with respect to Σ is given by

$$\kappa_c^{(\Sigma)}(v,f) = \frac{1}{|v|_\Sigma} - \frac{1}{2} + \frac{1}{|f|_\Sigma},$$

and the *sectional face curvature* $\kappa^{(\Sigma)} : F_\Sigma \to \mathbb{R}$ with respect to Σ is given as

$$\kappa^{(\Sigma)}(f) = \sum_{(v,f)\in C_\Sigma} \kappa_c^{(\Sigma)}(v,f) = 1 - \frac{|f|_\Sigma}{2} + \sum_{v\in V_\Sigma, v\in \overline{f}} \frac{1}{|v|_\Sigma}.$$

The above combinatorial curvature notions are motivated by a combinatorial version of the Gauß–Bonnet Theorem for closed surfaces. We have for polygonal tessellations $\Sigma = (V,E,F)$ of a closed surface S (see [8, Theorem 1.4])

$$\chi(S) = \sum_{f\in F_\Sigma} \kappa^{(\Sigma)}(f)$$

where $\chi(S)$ is the Euler characteristic of S. The sectional curvatures in Definition 2.8 are then the intrinsic curvatures in the apartments Σ, and the apartments Σ can be understood as discrete analogues of specific tangent planes. Note that curvature is a local concept and, for a given corner or face, only information of the nearest neighboring faces in the apartment are needed for its calculation.

Example 1 *(revisited).* The apartments in Example 1 are regular tessellations of a Euclidean plane by equilateral triangles. Thus, this example has vanishing sectional face and corner curvature. This is a special case covered by Proposition 5.5 in Section 5.2.1 which presents curvature properties in the general situation of Euclidean buildings.

Let us briefly comment on two other notions of curvature.

Remark 2.9. (a) Wise [60] introduces a sectional curvature which is closely related to the notion above, however, it is more flexible as it can be defined for general polygonal complexes. In contrast to our definition he considers a "sectional vertex curvature" rather than a sectional face curvature as above. However, both concepts are related. Precisely, we show in the appendix that non-negative sectional corner curvature in our sense implies non-negative sectional planar curvature in the sense of Wise (with a natural choice of angles). We refer to the appendix for a more detailed discussion.

(b) Metric spaces of non-positive Alexandrov curvature are characterized by a comparison of their geodesic triangles to the Euclidian case, see e.g. [37, Section 2.3] or [10, 11]. Specifically, a metric space (M,δ) is said to have non-positive Alexandrov curvature (or is NPC or CAT(0)) if, for all points $x,y,z \in M$ and every geodesic $\gamma : [0,1] \to M$ connecting x and z and all $0 \le t \le 1$, we have

$$\delta^2\big(y,\gamma(t)\big) \le (1-t)\delta^2\big(y,\gamma(0)\big) + t\delta^2\big(y,\gamma(1)\big) - t(1-t)\delta^2\big(\gamma(0),\gamma(1)\big).$$

This notion of curvature fits very well to the combinatorial curvature introduced above in the case of non-negative curvature, i.e., when the relation sign is flipped to "$\ge$", see e.g. [36,35]. In this case the inequality above can be translated into a statement about the angular defect about a vertex.

However, polygonal complexes equipped with the combinatorial metric that we consider above never have non-positive Alexandrov curvature: Namely, the inequality above implies by direct calculations that geodesics between two points must be unique, confer [37, Corollary 2.3.2]. But this is not the case for tessellations considered with the combinatorial metric. Indeed, two geodesics connecting the same points can have arbitrarily large "interior" in the case of vanishing combinatorial curvature, see e.g. [9, Fig. 1]. We show that in certain cases of negative curvature two geodesics connecting the same points can have at most distance one, see Theorem 3.3, but this bound can not be improved to be zero instead.

Nevertheless, we are optimistic that the following strategy is applicable instead. If we consider a geometric realization of our polygonal complexes, then these metric spaces should inherit the sign of the curvature of the corresponding combinatorial object. This is for example the case for Euclidean Bruhat–Tits buildings which have non-positive Alexandrov curvature, see e.g. [37, Example 3, p. 55]. Moreover, if there is a uniform bound on the vertex and the face degree, then one can compare the polygonal complex to its metric realization via rough isometries in the spirit of Kanai [39,40].

3. Geometry

In this section we discuss implications of the curvature sign to the global geometry of polygonal complexes with planar substructures like emptiness of cut-locus, Gromov hyperbolicity and positivity of the Cheeger constant. Before we enter into these topics, we first introduce some more useful combinatorial notions.

We say X is *locally finite* if for all $v \in V$ and $e \in E$

$$|v| < \infty \quad \text{and} \quad |e| < \infty.$$

Since $|f| = \sum_{e\in\partial f}(|e| - 1)$, we also have $|f| < \infty$ for locally finite polygonal complexes. For locally finite X, we define for a face $f \in F$

$$m_E(f) = \min_{e\in\partial f}(|e| - 1), \qquad M_E(f) = \max_{e\in\partial f}(|e| - 1)$$

the minimal and maximal number of neighbors over one edge of f. The *minimal* and *maximal thickness* of X is then given by

$$m_E = \min_{f\in F} m_E(f), \quad M_E = \sup_{f\in F} M_E(f).$$

The *maximal vertex* and *face degree* are defined by

$$M_V = \sup_{v\in V} |v|, \qquad M_F = \sup_{f\in F} |f|.$$

Note that we always have $M_E \leq M_F$ and both can be infinite.

3.1. Absence of cut-locus

We first present a theorem which is an analogue of the Hadamard–Cartan theorem from Riemannian manifolds. It is a rather immediate consequence of convexity and [9, Theorem 1] for plane tessellating graphs.

For a face $f \in F$ in a polygonal complex $X = (V, E, F)$, the *cut locus* of f is defined as

$$\mathrm{Cut}(f) = \{g \in F \mid d(f, \cdot) \text{ attains a local maximum in } g\}.$$

Absence of cut locus means that $\mathrm{Cut}(f) = \emptyset$ for all $f \in F$ which means that every finite geodesic starting in f can be continued to an infinite geodesic.

Theorem 3.1. *Let $X = (V, E, F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} \leq 0$ for all apartments $\Sigma \in \mathcal{A}$. Then,* $\mathrm{Cut}(f) = \emptyset$ *for all $f \in F$. Moreover, every geodesic within an apartment Σ can be continued to an infinite geodesic within Σ.*

We conclude from Theorem 3.1 that emptiness of cut-locus holds, e.g., for our Example 1 and Examples 6–9 (found in Section 5). Note also that the condition of non-positive sectional *corner curvature* in Theorem 3.1 cannot be weakened to non-positive sectional *face curvature* as Fig. 2 in [9] shows.

Proof. Let $f \in F$. Choose $g \in F$ and let Σ be an apartment which contains both f and g (which exists by (PCPS1)). By [9, Theorem 1] the cut locus of f within Σ is empty that is there is a face $h \in F_\Sigma$ with $g \sim h$ such that $d_\Sigma(f, h) = d_\Sigma(f, g) + 1$. (Note that [9, Theorem 1] is formulated in the dual setting which, however, can be carried over directly.) As $d = d_\Sigma$ on Σ, by Lemma 2.5, we conclude $g \notin \mathrm{Cut}(f)$. Since this holds for all $g \in F$, we have $\mathrm{Cut}(f) = \emptyset$. The second statement is an immediate consequence of [9, Theorem 1] and Lemma 2.5. $\square$

Corollary 3.2. *Let $X = (V, E, F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} \leq 0$ for all $\Sigma \in \mathcal{A}$. Then, every face has at least one forward neighbor.*

3.2. Thinness of bigons

In this subsection we show a useful hyperbolicity criterion.

Let $X = (V, E, F)$ be a polygonal complex. A *bigon* is a pair of geodesics $(f_0, \dots, f_n)$ and $(g_0, \dots, g_n)$ such that $f_0 = g_0$ and $f_n = g_n$. We say a bigon is *δ-thin* for $\delta \geq 0$, if $d(f_k, g_k) \leq \delta$ for all $k = 0, \dots, n$.

Theorem 3.3. *Let $X = (V, E, F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} < 0$ for all apartments $\Sigma \in \mathcal{A}$. Then, every bigon is 1-thin.*

Proof. Let $\gamma_1 = (f_0, \dots, f_n)$ and $\gamma_2 = (g_0, \dots, g_n)$ be a bigon and $\Sigma \in \mathcal{A}$ be an apartment that contains $f_0 = g_0$ and $f_n = g_n$. By the convexity assumption (PCPS2) the apartment Σ contains both geodesics γ_1 and γ_2 and, therefore, the pair (γ_1, γ_2) is a bigon within Σ. By [9, Theorem 2] it follows that $d_\Sigma(f_k, g_k) \leq 1$ for $k = 0, \dots, n$, and by Lemma 2.5 we conclude that $d(f_k, g_k) \leq 1$ for $k = 0, \dots, n$. □

We have an immediate consequence.

Corollary 3.4. *Let $X = (V, E, F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Let $f_1, f_2 \in F$ with $d(f_1, f_2) = n$. Then, we have for all $0 \leq k \leq n$*

$$|B_k(f_1) \cap B_{n-k}(f_2)| \leq 2.$$

In particular, if f_1 is considered as a center, f_2 has at most two backward neighbors.

Proof. By convexity we can restrict our considerations on any apartment $\Sigma \in \mathcal{A}$ containing f_1 and f_2. Every $f \in B_k(f_1) \cap B_{n-k}(f_2)$ must obviously satisfy $d(f_1, f) = k$ and $d(f, f_2) = n - k$. If there were three faces in the intersection $B_k(f_1) \cap B_{n-k}(f_2) \subset F_\Sigma$, then there would be three geodesics from f_1 to f_2 in Σ. Then, one of the three geodesics is enclosed by the other two in Σ and the other two geodesics form a bigon. Then this bigon in not 1-thin which contradicts the previous theorem. □

In fact, using the techniques of [9] the last statement of Corollary 3.4 holds even for non-positive sectional corner curvature.

Proposition 3.5. *Let $X = (V, E, F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} \leq 0$ for all $\Sigma \in \mathcal{A}$ and $o \in F$ be a center. Then every face has at most two backward neighbors.*

Proof. This is a consequence of the results in [9]. Let $f \in F$. Let $\Sigma \in \mathcal{A}$ be an apartment containing o and f. Then the ball $B_n \cap \Sigma$ is an admissible polygon in Σ in the sense of [9, Def. 2.2] and $\partial f \cap \partial B_n$ is a connected path of length ≤ 2, by [9, Prop. 2.5]. This shows that f can have at most two backward neighbors. □

3.3. Gromov hyperbolicity

Recall from the end of Subsection 2.1 that every polygonal complex $X = (V, E, F)$ can also be viewed as a metric space via the associated graph G_X and its natural combinatorial distance function. Geodesics (f_i) of faces in X correspond then to (vertex) geodesics in G_X. With this understanding, we call the polygonal complex (X, d) Gromov hyperbolic if there exists $\delta > 0$ such that any side of any geodesic triangle in G_X lies in the δ-neighborhood of the union of the two other sides of the triangle. We show

Gromov hyperbolicity of a polygonal complex with planar substructures (X, d) with negative sectional corner curvature as well as properties of the Gromov boundary $X(\infty)$ under the additional boundedness assumption of the vertex and face degree. For details on the Gromov boundary (and the Gromov product used to define it) we refer to [14, Chpt. III.H].

Theorem 3.6. *Let X be a polygonal complex with planar substructures with $M_V, M_F < \infty$ and $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Then, (X, d) and all its apartments are Gromov hyperbolic spaces. If additionally* (PCPS1*) *is satisfied then every connected component of the Gromov boundary $X(\infty)$ contains the Gromov boundary of an apartment which is homeomorphic to the unit circle S^1.*

By the theorem in the section above all bigons in (X, d) are 1-thin. For Cayley graphs, [54, Theorem 1.4] tells us that the statement of the theorem above is true. For general G_X, we need the following generalization given in the unpublished Master's dissertation of Pomroy (a proof of it can be found in [17, Appendix]). Here, a *ρ-bigon* in a geodesic metric space with metric d is a pair of $(1, \rho)$ quasi-geodesics γ_1, γ_2 with the same end points, i.e.,

$$|t - t'| - \rho \leq d\big(\gamma_i(t), \gamma_i(t')\big) \leq |t - t'| + \rho,$$

for all t, t'.

Theorem 3.7 *(Pomroy). If for a geodesic metric space there are $\varepsilon, \rho > 0$ such that ρ-bigons are uniformly ε-thin, then the space is Gromov hyperbolic.*

Proof of Theorem 3.6. By Theorem 3.3 all bigons in (X, d) are 1-thin. The same holds true within all apartments. For G_X to satisfy the requirement of a *geodesic* metric space, we view it as a *metric graph* with all its edge lengths equal to one. Choose $\rho < 1/2$ and $\varepsilon = 1$, we can then conclude from Theorems 3.3 and 3.7 that (X, d) and all its apartments are Gromov hyperbolic.

Next we prove the rest of the theorem assuming (PCPS1*). From $M_F < \infty$ we conclude that G_X is a *proper* (i.e., closed balls in G_X of finite radius are compact) hyperbolic geodesic space and, therefore, the geodesic boundary (defined via equivalence classes of geodesic rays, where rays are equivalent iff they stay in bounded distance to each other) and the Gromov boundary coincide (see, e.g., [14, Lm. III.H.3.1]) and we can think of any boundary point $\xi \in X(\infty)$ as being represented by a geodesic ray (f_i) of faces in F. Using (PCPS1*), there is an apartment $\Sigma \in \mathcal{A}$ such that all the faces f_i are in F_Σ and $\xi \in \Sigma(\infty) \subset X(\infty)$. We also know from [9, Cor. 5] that $\Sigma(\infty)$ is homeomorphic to S^1, finishing the proof. □

It is easy to see that the Euclidean buildings in Example 1 and 6 are not Gromov hyperbolic. Theorem 3.6 is not applicable since these examples have vanishing sectional corner curvature.

3.4. Cheeger isoperimetric constants

In this subsection we prove how negative curvature effects positivity of the Cheeger isoperimetric constant.

Let $X = (V, E, F)$ be a *locally finite* polygonal complex. We consider the following Cheeger constant which is very useful for spectral estimates. For $H \subseteq F$, we define

$$\alpha_H = \inf_{K \subseteq H \text{ finite}} \frac{|\partial K|}{\mathrm{vol}(K)}$$

with

$$\partial K = \{(f, g) \in K \times F \setminus K \mid f \sim g\}$$

and

$$\mathrm{vol}(K) = \sum_{f \in K} |f|.$$

Note that $\alpha_H \leq 1$. We set $\alpha = \alpha_F$.

Firstly, we present a result that shows positivity of the Cheeger isoperimetric constant for negative sectional corner curvature under the additional assumption of bounded geometry. This result is a consequence of a general result of Cao [15], which also holds in the smooth setting of Riemannian manifolds. Secondly, we give more explicit estimates for the Cheeger constant.

Theorem 3.8. *Let $X = (V, E, F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Assume that X additionally satisfies (PCPS1*) and $M_V, M_F < \infty$. Then, $\alpha > 0$.*

A straightforward consequence of Theorem 3.8 and Theorem 5.7 below is the following result.

Corollary 3.9. *Every 2-dimensional locally finite hyperbolic building with regular hyperbolic polygons as faces has a positive Cheeger constant $\alpha > 0$.*

Proof. Note that negative sectional curvature and the definition do not depend on the choice of the apartment systems. Hence, we switch to the corresponding building with maximal apartment system to obtain (*PCPS1**) by Theorem 5.7. We conclude the statement by Theorem 3.8. $\square$

In particular, all buildings in Examples 6–9 in Section 5 have positive Cheeger constant.

Proof of Theorem 3.8. Note that by the comment at the end of Subsection 2.1 we can associate to every polygonal complex with planar substructures $X = (V, E, F)$ a graph G_X by considering the faces of X as vertices in G_X and the edge relation given by the adjacency relation of the faces. In this light [15, Theorem 1] tells us that a polygonal complex (X, d) has positive Cheeger isoperimetric constant if the following four assumptions are satisfied

(1) (X, d) has bounded face degree $M_F < \infty$,
(2) (X, d) admits a quasi-pole,
(3) (X, d) is Gromov hyperbolic,
(4) every the Gromov boundary $X(\infty)$ has positive diameter (with respect to a fixed Gromov metric),

where (2) means that there is a finite set $\Omega \subset F$ of faces and a $\delta > 0$ such that every face $f \in F$ is found in a δ-neighborhood of a geodesic ray emanating from this finite set. Moreover, for (4) we follow [15] and define for two geodesic rays $(f_i), (f_i')$ of faces with the same initial face $f_0 = f_0'$ representing the points $\xi, \eta \in X(\infty)$:

$$d_{f_0,\varepsilon}(\xi, \eta) = \liminf_{n\to\infty} \exp\big(-\varepsilon\big(n - \tfrac{1}{2}d\big(f_n, f_n'\big)\big)\big).$$

Then, there is an $\varepsilon > 0$ such that $d_{f_0,\varepsilon}$ is a metric which is called a Gromov metric. Note that the Cheeger constant considered in [15] is defined as

$$h = \inf_{H\subseteq F} \frac{|\partial_F H|}{|H|},$$

where $\partial_F H = \{f \in F \mid d(f, H) = 1\}$. As every face in $\partial_F H$ is connected with H via at least one edge we have $|\partial H| \geq |\partial_F H|$. Also $\mathrm{vol}(H) \leq M_F|H|$ and, therefore,

$$\alpha \geq \frac{h}{M_F}.$$

Hence, by the assumption $M_F < \infty$ the constant α is positive whenever h is. Thus, it remains to check the conditions (1)–(4).

Let $X = (V, E, F)$ be a polygonal complex with planar substructures which satisfies the assumptions of the theorem. Then, (1) is obviously satisfied. Secondly, by absence of cut-locus, Theorem 3.1, condition (2) is satisfied and by Theorem 3.6 condition (3) is satisfied. Finally, let us turn to (4). By Theorem 3.6 and the assumption (PCPS1*) we know that every connected component of the Gromov boundary of X includes the Gromov boundary of an apartment. Therefore, it suffices to show (4) for the Gromov boundary of an apartment. We observe that we find in every apartment a bi-infinite geodesic. This can be seen as follows: Let $(f_{-n}, \dots, f_n)$ be a geodesic in an apartment $\Sigma \in \mathcal{A}$. By [9, Theorem 1] the face f_n is not in $\mathrm{Cut}_\Sigma(f_{-n})$ and, therefore, there is $f_{n+1} \in \Sigma$ such that $(f_{-n}, \dots, f_{n+1})$ is a geodesic. Simultaneously, f_{-n} is not in $\mathrm{Cut}_\Sigma(f_{n+1})$ and, therefore, there is $f_{-(n+1)} \in \Sigma$ such that $(f_{-(n+1)}, \dots, f_{n+1})$ is a geodesic in Σ. In this way, we construct a bi-infinite geodesic $(f_n)_{n\in\mathbb{Z}}$. Let $\xi, \eta \in X(\infty)$ be the end points of the geodesics $(f_n)_{n\geq 0}, (f_{-n})_{n\geq 0} \subset F_\Sigma$. Since $(f_n)_{n\in\mathbb{Z}}$ is a bi-infinite geodesic, we have $d(f_n, f_{-n}))) = 2n$. So, we obtain for any $\varepsilon > 0$

$$d_{f_0,\varepsilon}(\xi,\eta) = \liminf_{n\to\infty} \exp\big(-\varepsilon\big(n - \tfrac{1}{2}d(f_n, f_{-n})\big)\big) = 1.$$

Hence, (4) is satisfied and we finished the proof. $\square$

Remark 3.10. The question whether a Gromov hyperbolic space has positive Cheeger constant is very subtle. Note that every infinite tree T is Gromov hyperbolic. But if we attach to one of its vertices the ray $[0,\infty)$ with integer vertices then the new tree $\widetilde{T}_1$ is still Gromov hyperbolic but it has vanishing Cheeger constant. This new ray adds an isolated point to the Gromov boundary of T and therefore assumption (4) is violated for $\widetilde{T}_1$. On the other hand, if we attach to a sequence of vertices $(v_n)_{n\in\mathbb{N}}$ in T the segments $[0,n]$ with integer vertices and denote the new tree by $\widetilde{T}_2$, then this new tree has again vanishing Cheeger constant. In this case both trees T and $\widetilde{T}_2$ even have the same boundaries, but $\widetilde{T}_2$ cannot have a quasi-pole since the newly added vertices do not lie in geodesic rays and, therefore, assumption (2) is violated (see end of Subsection 1.1 in [15]).

The next result provides explicit lower bounds for the Cheeger constant in terms of the face degrees and minimal and maximal thickness.

Theorem 3.11. *Let X be a locally finite polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} \leq 0$. Then,*

$$\alpha \geq \inf_{f\in F}\left(\frac{m_E(f)}{M_E(f)}\left(1 - \frac{6}{|\partial f|}\right)\right) \geq \frac{m_E}{M_E}\left(1 - \frac{6}{\min_{f\in F}|\partial f|}\right).$$

In particular, $\alpha > 0$ if $|\partial f| \geq 7$ and $M_E < \infty$. Secondly,

$$\alpha \geq \inf_{f\in F}\frac{m_E(f) - 2}{|f|} \geq \frac{m_E - 2}{M_F}.$$

In particular, $\alpha > 0$ if $m_E \geq 3$ and $M_F < \infty$.

The theorem implies in particular that all locally finite 2-dimensional Euclidean buildings with minimal thickness $m_E \geq 3$ (i.e., every edge is contained in at least 4 chambers) have positive Cheeger constant. Moreover, all locally finite hyperbolic buildings with generating polygon P at least a 7-gon have also positive Cheeger constant.

Proof. Translating [20, Lemma 1.15] into the "dual" language (as the comment at the end of Section 2.1 indicates) tells us that if there is a center $o \in V$ and $C \geq 0$ such that

$$|f|_+ - |f|_- \geq C|f|$$

for all $f \in F$, then $\alpha \geq C$. Thus, it suffices to estimate $\inf_{f \in F}(|f|_+ - |f|_-)/|f|$ to get a lower bound on α. For $f \in F$, let $n \geq 0$ be such that $f \in S_n$ and let Σ be an apartment that contains f. By Proposition 3.5 we immediately have $|f|_- \leq 2$. Moreover, by [8, Theorem 3.2] (combined with Theorem 3.1) there are at most two neighbors of f in $F_\Sigma \cap S_n$ and, therefore, $|f|_+ \geq m_E(f)|f|_{\Sigma,+} \geq m_E(f)(|\partial f| - 4)$. Here $|f|_{\Sigma,+}$ denotes the number of forward neighbors of f within Σ, which is $|\partial f|$ minus the number (≤ 2) of backward neighbors of f minus the number (≤ 2) of neighbors of f in $F_\Sigma \cap S_n$. Moreover, $|f| \leq M_E(f)|\partial f|$. Hence,

$$\frac{|f|_+ - |f|_-}{|f|} \geq \frac{m_E(f)}{M_E(f)}\left(1 - \frac{6}{|\partial f|}\right)$$

which yields the first inequality. On the other hand, we have by Theorem 3.1 and Lemma 2.7 (a) $|f|_+ \geq m_E(f)$. Hence, by $|f|_- \leq 2$

$$\frac{|f|_+ - |f|_-}{|f|} \geq \frac{m_E(f) - 2}{|f|}.$$

This finishes the proof. □

From the proof we may easily extract the following statement which turns out to be useful for studying the essential spectrum of the Laplacian. Define for a locally finite polygonal complex $X = (V, E, F)$ the *Cheeger constant at infinity* by

$$\alpha_\infty = \sup_{K \subseteq F \text{ finite}} \alpha_{F \setminus K}.$$

Corollary 3.12. *Let X be a locally finite polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} \leq 0$. Then,*

$$\alpha_\infty \geq \sup_{K \subseteq F \text{ finite}} \inf_{f \in F \setminus K} \frac{m_E(f)}{M_E(f)}\left(1 - \frac{6}{|\partial f|}\right).$$

3.5. Finiteness and infiniteness

In this subsection we show that positivity or non-positivity of *sectional face curvature* determines whether a locally finite polygonal complex with planar/spherical substructures is finite or infinite. The statement that positive curvature implies finiteness is an analogue of a theorem of Myers for Riemannian manifolds [49].

Theorem 3.13. *Let $X = (V, E, F)$ be a locally finite polygonal complex with planar or spherical substructures with apartment system $\mathcal{A}$.*

(a) *If we have $\kappa^{(\Sigma)}(f) > 0$ for all $\Sigma \in \mathcal{A}$ and all $f \in F_\Sigma$, then F is finite and X is a polygonal complex with spherical substructures.*
(b) *If we have $\kappa^{(\Sigma)}(f) \leq 0$ for all $\Sigma \in \mathcal{A}$ and all $f \in F_\Sigma$, then F is infinite and X is a polygonal complex with planar substructures.*

Proof. Note first that every planar tessellation has infinitely many faces (since the closure of every face is compact) while every spherical tessellation has finitely many faces. Therefore, F_Σ, $\Sigma \in \mathcal{A}$, is infinite if X is a polygonal complex with planar substructures and finite if X is a polygonal complex with spherical substructures.

We first prove (b) by contraposition. Assume that X is a polygonal complex with planar or spherical substructures with F a finite set. We will show that there is a face with positive sectional face curvature. Choose an apartment $\Sigma \in \mathcal{A}$. By the Gauß–Bonnet Theorem, we have

$$\sum_{f \in F_\Sigma} \kappa^{(\Sigma)}(f) = \chi\big(\mathbb{S}^2\big) = 2,$$

where χ denotes the Euler characteristic. Hence, $\kappa^{(\Sigma)}$ must be positive on some faces. This shows (b).

Turning to (a), we assume that $\kappa^{(\Sigma)}(f) > 0$ for all $\Sigma \in \mathcal{A}$ and all $f \in F_\Sigma$. By DeVos–Mohar's proof of Higuchi's conjecture [19, Theorem 1.7] (which is again stated in the dual formulation) every apartment must be finite. Moreover, the number of faces (in their case vertices) in an apartment is uniformly bounded by 3444 except for prisms and antiprisms.[1] A prism in our dual setting are two wheels of triangles glued together along their boundaries and an antiprism are two wheels of squares glued together along their boundaries (see Fig. 2). We can think of these two wheels as representing the lower and upper hemisphere of $\mathbb{S}^2$ and the boundaries as agreeing with the equator of the sphere $\mathbb{S}^2$.

If F is infinite, then there exists a face $f_0 \in F$ and a sequence of faces $f_n \in F$ with $d(f_0, f_n) \to \infty$ because of the local finiteness. Then, f_0 must lie in a sequence (Σ_n) of spherical apartments $\mathbb{S}^2$ tessellated by pairs of wheels with number of faces going to infinity, glued together along the equator. Assuming that f_0 lies always in the lower hemisphere of $\Sigma_n \cong \mathbb{S}^2$, then the south pole of all these apartments would be one and the same vertex $v_0 \in \overline{f}_0$. But this would imply that $|v_0| = \infty$, which contradicts the local finiteness. Therefore, F must be finite which implies that X is a polygonal complex with spherical substructures. □

[1] Note that in the meantime the bound has been improved by Zhang [63] to 580 vertices while the largest known graph with positive curvature has 208 vertices and was constructed by Nicholson and Sneddon [51].

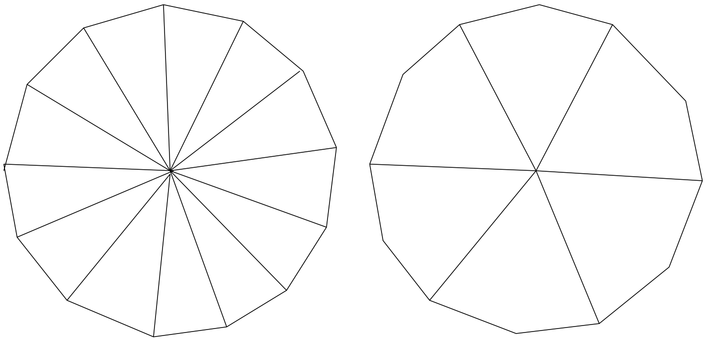

Fig. 2. A wheel of triangles and a wheel of squares.

4. Spectral theory

In this section we turn to the spectral theory of the Laplacian on polygonal complexes. As the geometric structure is determined by assumptions on the faces, it is only natural to consider the Laplacian for functions on the faces. The reader who prefers to think about the Laplacian as an operator on functions on the vertices is referred to the comment at the end of Section 2.1. That is, we can associate a graph G_X to each polygonal complex $X = (V, E, F)$ in a natural way.

Let $X = (V, E, F)$ be a locally finite polygonal complex and

$$\ell^2(F) = \left\{ \varphi : F \to \mathbb{C} \mid \sum_{f \in F} |\varphi(f)|^2 < \infty \right\}.$$

For functions $\varphi, \psi \in \ell^2(F)$ the standard scalar product is given by

$$\langle \varphi, \psi \rangle = \sum_{f \in F} \overline{\varphi(f)} \psi(f),$$

and the norm is given by $\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}$. Define the Laplacian Δ by

$$\Delta\varphi(f) = \sum_{g \in F, g \sim f} \big(\varphi(f) - \varphi(g)\big)$$

for functions in the domain

$$D(\Delta) = \big\{ \psi \in \ell^2(F) \mid \Delta\psi \in \ell^2(F) \big\}.$$

It can be checked directly that the operator is positive and, moreover, it is selfadjoint by [62, Theorem 1.3.1]. Note that the operator Δ can be seen to coincide with the graph Laplacian on $\ell^2(G_X)$.

By standard Cheeger estimates [42] based on [21,27] we have

$$\lambda_0(\Delta) \geq m_F\big(1 - \sqrt{1-\alpha^2}\big),$$

where $\lambda_0(\Delta)$ denotes the bottom of the spectrum of Δ and

$$m_F = \min_{f \in F} |f|.$$

Applying Theorem 3.8 gives a criterion when the bottom of the spectrum is positive and Theorem 3.11 even gives explicit estimates.

4.1. Discreteness of spectrum and eigenvalue asymptotics

In this subsection we address the question under which circumstances the spectrum of Δ is purely discrete. We prove an analogue of a theorem of Donnelly–Li, [23], for Riemannian manifolds that curvature tending to $-\infty$ outside increasing compacta implies emptiness of the essential spectrum.

For a selfadjoint operator T we denote the eigenvalues below the essential spectrum in increasing order counted with multiplicity by $\lambda_n(T)$, $n \geq 0$. For two sequences (a_n), (b_n) we write $a_n \sim b_n$ if there is $c > 0$ such that $c^{-1}a_n \leq b_n \leq ca_n$. We denote the maximal operator of multiplication by the face degree by D_F. That is D_F is an operator from $\{\varphi \in \ell^2(F) \mid |\cdot|\varphi \in \ell^2(F)\}$ to $\ell^2(F)$ acting as

$$D_F\varphi(f) = |f|\varphi(f).$$

We call X *balanced* if there is $C > 0$ such that $Cm_E(f) \geq M_E(f)$ and *strongly balanced* if

$$\sup_{K \subseteq F \text{ finite}} \inf_{f \in F \setminus K} \frac{m_E(f)}{M_E(f)} = 1.$$

That means that C in the definition of balanced equals 1 asymptotically. An analogue of the Donnelly–Li result reads as follows. Let

$$\kappa_\infty := \inf_{K \subseteq F \text{ finite}} \sup_{\Sigma \in \mathcal{A}, f \in F_\Sigma \setminus K} \kappa^{(\Sigma)}(f).$$

Theorem 4.1. *Let $X = (V, E, F)$ be a locally finite polygonal complex with planar substructures that is balanced and $\kappa_c^{(\Sigma)} \leq 0$. If $\kappa_\infty = -\infty$, then the spectrum of Δ is purely discrete and*

$$\lambda_n(\Delta) \sim \lambda_n(D_F).$$

If, additionally, X is strongly balanced, then

$$\frac{\lambda_n(\Delta)}{\lambda_n(D_F)} \to 1 \text{ as } n \to \infty.$$

Finally, under the additional assumption $M_E < \infty$, purely discrete spectrum of Δ implies $\kappa_\infty = -\infty$.

We like to mention that the result here holds for the *generally unbounded* discrete Laplacian. The first result on the essential spectrum of graphs analogous to Donnelly–Li was proved by Fujiwara [27] and he considered the *normalized* Laplacian. The very different spectral behavior of these two operators is discussed in [42].

The proof of Theorem 4.1 is based on the following proposition.

Proposition 4.2. *Let $X = (V, E, F)$ be a locally finite polygonal complex with planar substructures. If*

$$a := \sup_{K\subseteq F \text{ finite}} \inf_{f\in F\setminus K} \frac{m_E(f)}{M_E(f)}\left(1 - \frac{6}{|\partial f|}\right) > 0,$$

then the spectrum of Δ is discrete if and only if

$$\sup_{K\subseteq F \text{ finite}} \inf_{f\in F\setminus K} |f| = \infty.$$

In this case,

$$\left(1 - \sqrt{1-a^2}\right) \le \liminf_{n\to\infty} \frac{\lambda_n(\Delta)}{\lambda_n(D_F)} \le \limsup_{n\to\infty} \frac{\lambda_n(\Delta)}{\lambda_n(D_F)} \le \left(1 + \sqrt{1-a^2}\right).$$

Proof. The characterization of discreteness of spectrum follows from Corollary 3.12 and [42, Theorem 2]. The asymptotics of eigenvalues follow combining Corollary 3.12 and [12, Thms. 2.2. and 5.3.]. $\square$

Proof of Theorem 4.1. We observe that for all $\Sigma \in \mathcal{A}$ and $f \in F_\Sigma$

$$-\frac{|f|_\Sigma}{2} \le \kappa^{(\Sigma)}(f).$$

Hence, $\kappa_\infty = -\infty$ implies $\sup_{K\subseteq F \text{ finite}} \inf_{f\in F\setminus K} |f| = \infty$. Combining this with the assumption that X is balanced with constant C implies that $a \ge 1/C$, where a is taken from Proposition 4.2. In the case of X being strongly balanced we have $a = 1$. Thus, the first part of the theorem follows from Proposition 4.2. Conversely, if there is $c > 0$ such that $\kappa_\infty \ge -c > -\infty$, then there is a sequence of faces f_n with $d(f, f_n) \to \infty$ for any fixed face $f \in F$ and apartments Σ_n, $n \ge 0$, such that

$$-c < \kappa^{(\Sigma_n)}(f_n) \le 1 - \frac{|f_n|_\Sigma}{6} \le 1 - \frac{|f_n|}{6M_E},$$

where we used $|v|_\Sigma \ge 3$ for all $v \in V_\Sigma$ which holds as Σ is a tessellation. We conclude that $|f_n|$ is uniformly bounded by some constant $c' > 0$. Thus, the essential spectrum of Δ starts below c' (confer [42, Theorem 1]) and Δ does not have purely discrete spectrum. $\square$

Example 2. The simplest example of a polygonal complex with planar substructures satisfying the conditions of Theorem 4.1 is a planar tessellation $X = (V, E, F)$ with one apartment $\Sigma = X$ and center $o \in F$ such that $\lim_{n\to\infty} \inf_{f\in S_n} |f| = \infty$. In this case we have

$$\kappa^{(\Sigma)}(f) \leq 1 - \frac{|f|}{6},$$

and we see that $\kappa_\infty = -\infty$. Moreover, X is strongly balanced since we have $m_E(f) = M_E(f) = 1$. Therefore, the spectrum of Δ is purely discrete and $\lambda_n(\Delta)/\lambda_n(D_F) \to 1$.

Note that purely discrete spectrum can also be established by increasing $m_E(f)$ instead of $|\partial f|$ for all faces outside compact sets (by keeping the polygonal complex balanced) and applying Proposition 4.2 directly. The condition $\sup_{K\subseteq F \text{ finite}} \inf_{f\in F\setminus K} |f| = \infty$ follows then directly from $|f| \geq m_E(f)$.

4.2. Unique continuation of eigenfunctions

While unique continuation results hold in great generality for continuum models with very mild assumptions, there are very natural examples for graphs with finitely supported eigenfunctions, see e.g. [22] and various other references. In this subsection we prove that for non-positive curvature there are no finitely supported eigenfunctions.

Theorem 4.3. *Let $X = (V, E, F)$ be a locally finite polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} \leq 0$ for all $\Sigma \in \mathcal{A}$. Then, Δ does not admit finitely supported eigenfunctions.*

Cases where we do not have finite supported eigenfunctions are therefore Example 1 and Examples 6–9.

In [44,43] results like Theorem 4.3 are found for the planar case and more general operators. Indeed, we consider here also nearest neighbor operators, where we even do not have to assume local finiteness.

Definition 4.4. Let $X = (V, E, F)$ be a polygonal complex. We call A a *nearest neighbor operator* on X if there is $a : F \times F \to \mathbb{C}$

(NNO1) $a(f, g) \neq 0$ if $f \sim g$,
(NNO2) $a(f, g) = 0$ if $f \nsim g$,
(NNO3) $\sum_{g\in F} |a(f, g)| < \infty$ for all $f \in F$,

and A acts as

$$A\varphi(f)=\sum_{g\in F}a(f,g)\varphi(g),$$

on functions φ in

$$\widetilde{D}(A)=\Big\{\varphi:F\to\mathbb{C}\mid\sum_{g\in F}|a(f,g)\varphi(g)|<\infty\text{ for all }\ f\in F\Big\}.$$

The summability assumption (NNO3) guarantees that the functions of finite support are included in $\widetilde{D}(A)$. Clearly, the Laplacian introduced at the beginning of this section is a nearest neighbor operator, where we can also add an arbitrary potential to be in the general setting of Schrödinger operators. Theorem 4.3 is an immediate consequence of the following theorem.

Theorem 4.5. *Let $X=(V,E,F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)}\leq 0$ for all $\Sigma\in\mathcal{A}$ and A be a nearest neighbor operator on X. Then A does not admit eigenfunctions supported within a distance ball.*

Proof. Let $\varphi\in\widetilde{D}(A)$ be an eigenfunction of A to the eigenvalue λ. Let k be such that φ vanishes completely on all distance spheres at levels larger or equal than k from a center $o\in F$. Let $f_0\in F$ be a face at distance $k-1$. We want to show that $\varphi(f_0)=0$. Let Σ be an apartment containing o and f_0. Since we do not have cut-locus in any of the apartments due to non-positive sectional corner curvature, cf. Theorem 3.1, there exists a face $g_0\in F_\Sigma$ adjacent to f_0 with $d(o,g_0)=k$. By assumption, we have $\varphi(g_0)=0$. Now, by convexity, all faces $f\in F$ with $d(f,o)=k-1$ adjacent to g_0 lie within Σ. By Proposition 3.5 there can be at most two such faces, one of them equal to f_0. If there is only one such face, namely f_0, we conclude from the eigenfunction identity evaluated at g_0 that we have $\varphi(f_0)=0$. If there are two such faces, say f_0,f_1, then we conclude from the eigenfunction identity evaluated at g_0 that $a(g_0,f_0)\varphi(f_0)=-a(g_0,f_1)\varphi(f_1)$. With the notation of [9, Section 2.2] the vertex v_0 in the intersection of $\overline{f_0},\overline{f_1}$ and $\overline{g_0}$ has label b with respect to the tessellation Σ (label b means that there is more than one adjacent face to v_0 within B_{k-1} or if one of the faces adjacent to v_0 is a triangle then there are even more than three adjacent faces in B_{k-1}; however, the case that v_0 has a neighboring triangle can be excluded by $\kappa_c^{(\Sigma)}\leq 0$). The vertex v_0 has two neighbors in the boundary of B_{k-1} in Σ. One of these neighbors is in the intersection of $\overline{f_0}\cap\overline{g_0}$ and the other one which we denote by v_1 is in the intersection of $\overline{f_1}\cap\overline{g_0}$. By [9, Cor. 2.7.] the vertex v_1 has label a^+ (which means that v_1 has only one adjacent face within B_{k-1}). This implies that the face f_1 has another neighbor g_1 in S_k. By assumption $\varphi(g_1)=0$ and applying the same arguments to g_1 we find $f_2\in S_{k-1}\cap F_\Sigma$, $f_2\sim g_1$ such that $a(g_1,f_1)\varphi(f_1)=-a(g_1,f_2)\varphi(f_2)$. Proceeding inductively we find the sequences $(f_0,\ldots,f_n)$, $f_0=f_n$, and $(g_0,\ldots,g_n)$, $g_0=g_n$ of faces in Σ that form a closed boundary walk and boundary vertices $(v_0,\ldots,v_{2n})$, $v_0=v_{2n}$, with labels $b,a^+,b,a^+,b,\ldots$. However, this is geometrically impossible [44, Prop. 13]. Hence, we conclude $\varphi(f_0)=0$. As this argument applies for all faces in S_{k-1} we deduce

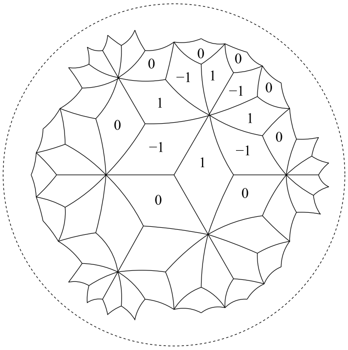

Fig. 3. Part of the tessellation Σ_4 with a finitely supported eigenfunction which is zero at the faces with no entry.

that φ vanishes on S_{k-1}. Repeating this argument for S_{k-j}, $j = 2, \ldots, k$, yields that φ vanishes on B_k and thus by assumption on F. We finished the proof. $\square$

We conclude this subsection by giving examples of tessellations with negative sectional face curvature that admit finitely supported eigenfunctions. This shows the assumption in the theorem cannot be modified to negative sectional face curvature instead of non-positive sectional corner curvature.

Example 3. Let Σ_n, $n \geq 3$, be a bipartite tessellation of the plane $\mathbb{R}^2$ with squares as follows. There are two infinite sets of vertices V_1 and V_2, where the vertices in V_1 have degree $2n$ and the vertices in V_2 have degree 3. The tessellation Σ_n is now given such that vertices in V_1 are only connected to vertices in V_2 and vice versa. Hence, each face contains two vertices of V_1 and two of V_2. See Fig. 3 for the tessellation Σ_4, realized in the hyperbolic Poincaré unit disk.

The face curvature is then given by

$$\kappa(f) = 1 - \frac{|f|}{2} + \sum_{v \in f} \frac{1}{|v|} = 1 - 2 + \frac{2}{3} + \frac{2}{2n} = -\frac{n-3}{3n}.$$

For $n > 3$ the face curvature is negative and in the interval $(-1/3, -1/12)$. On the other hand, we have for the corner curvatures

$$\kappa_c(v_1, f) = -\frac{n-2}{4n}, \quad \kappa_c(v_2, f) = \frac{1}{12} > 0,$$

with $v_1 \in V_1$ and $v_2 \in V_2$ and $v_1, v_2 \in \overline{f}$. Moreover, for a vertex with degree $2n$ let $F_0 = \{f_1, \ldots, f_{2n}\}$ be the faces around it in cyclic order. Let a function φ with support in F_0 be given such that $\varphi(f_{2j}) = 1$ and $\varphi(f_{2j-1}) = -1$ for $j = 1, \ldots, n$. Then, φ is a finitely supported eigenfunction of Δ to the eigenvalue 6. Looking at the dual regular graph ${\Sigma_n}^*$ with constant vertex degree 4, we see that the Δ-eigenfunction φ of Σ_n corresponds to an eigenvector of the adjacency matrix of ${\Sigma_n}^*$ to the eigenvalue -2.

4.3. The Dirichlet problem at infinity

We assume that $X = (V, E, F)$ is a polygonal complex with planar substructures with strictly negative sectional corner curvature and that (PCPS1*) holds. Moreover, we assume $M_V, M_F < \infty$. Then we know from Theorem 3.6 that (X, d) is Gromov hyperbolic and that the boundary $X(\infty)$ carries a natural topological structure. Moreover, $\overline{X} = X \cup X(\infty)$ is compact (see [14, Prop. III.H.3.7(4)]). Given a function $U \in C(X(\infty))$, the *Dirichlet problem at infinity* asks whether there is a unique continuous function $u \in C(\overline{X})$ which agrees with U on $X(\infty)$ and such that the restriction $u_0 = u|_X$ is harmonic (i.e., $\Delta u = 0$). The existence of such a function u is the main problem since uniqueness of the solution is typically obtained from a maximum principle type argument. Applying the general theory of [2] to Theorem 3.8 answers this question positively.

Theorem 4.6. *Let $X = (V, E, F)$ be a polygonal complex with planar substructures such that $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Assume that X additionally satisfies (PCPS1*) and $M_V, M_F < \infty$. Then (X, d) is Gromov hyperbolic and the Dirichlet problem at infinity is solvable on X.*

Examples of spaces, where the theorem is applicable and the Dirichlet problem at infinity can be solved, are all locally finite 2-dimensional hyperbolic buildings with regular hyperbolic polygons as faces.

Proof. Let $P = \frac{1}{M_F}(M_F - \Delta)$ be the operator

$$P\varphi(f) = \sum_{g \in F} p(f, g)\varphi(g),$$

where $p(f, f) = (M_F - |f|)/M_F$ and $p(f, g) = 1/M_F$ if $f \sim g$, and $p(f, g) = 0$ in all other cases. Then P is symmetric with respect to $(\varphi_1, \varphi_2) = \sum_{f \in F} \varphi_1(f)\varphi_2(f)$, i.e., $p(f, g) = p(g, f)$, and Markovian, i.e., $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denotes the constant one-function on the set of faces. Moreover, it is easy to see that P satisfies the properties of [2, Assumptions 1.1], i.e., P is admissible. Note further that a function φ on F satisfies $\Delta\varphi = 0$ if and only if $P\varphi = \varphi$.

Following the theory in [2], we first use the fact that X has positive Cheeger constant and, therefore, the Dirichlet problem at infinity is solvable *with respect to the P-Martin boundary.*

Note that the P-Martin boundary of X is based on the associated Green function $G : F \times F \to [0, \infty)$, which is defined as $G(f, g) = \sum_{n \geq 0} p^n(f, g)$. We know from Theorem 3.8 that the Cheeger constant α of X is positive. We conclude from [2, Prop. 4.4] that $\|P\|_2 < 1$ and, therefore, that there exists $\varepsilon > 0$ such that $G^\varepsilon(f, g) = \sum_{n \geq 0}(1-\varepsilon)^{-n-1}p^n(f, g)$ is finite (this is the crucial condition (*) in [2], establishing a Harnack inequality at infinity). For a given reference point $f_0 \in F$, note that the P-Martin boundary consists of (equivalence classes) of sequences f_j with $d(f_0, f_j) \to \infty$

and $K(f) = \lim_{j\to\infty} G(f, f_j)/G(f_0, f_j)$ exists for all $f \in F$. (Two sequences are equivalent if they lead to the same limit function K.) Then [2, Cor. 5.4] guarantees that the Dirichlet problem at infinity is solvable with respect to the P-Martin boundary.

Moreover, we know from Theorem 3.6 that (X, d) is Gromov hyperbolic. This allows us to apply [2, Cor. 6.10] and to conclude that the Gromov compactification satisfies the assumptions (G.A) in [2, Theorem 5.2] which, in turn agrees with the P-Martin compactification. This shows that the P-Martin boundary, the Gromov boundary and the geodesic boundary coincide and, therefore, that the Dirichlet problem at infinity is solvable for each one of these boundaries. □

5. Examples

In this section, we will mainly focus on non-positively curved polygonal complexes with planar substructures. Rich classes of examples are provided by 2-dimensional Euclidean and hyperbolic buildings. Before we consider these classes more closely, let us start with particularly simple examples of non-buildings.

5.1. Simple examples and basic notions

As mentioned earlier, every planar tessellation $\Sigma = (V, E, F)$ is trivially a polygonal complex with planar substructures with just one apartment, i.e., $\mathcal{A} = \{\Sigma\}$.

Next, let us introduce *morphisms* between two complexes X_1 and X_2: These are continuous maps from X_1 to X_2 mapping k-cells of X_1 homeomorphically to k-cells of X_2, for all k. A morphism $f : X_1 \to X_2$ is an *isomorphism* if both f and f^{-1} are morphisms. In this case we call X_1 and X_2 isomorphic complexes.

Example 4 *("Book").* Let $\mathcal{H} = (V, E, F)$ be the tessellation of the upper half space $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ where

$$V = \big\{(x, y) \in \mathbb{Z}^2 \mid y \geq 0\big\},$$

E is the set of horizontal and vertical straight Euclidean line segments of length 1 connecting two vertices of V, and F is the set of all Euclidean unit squares with vertices in V. Let $k \geq 2$ be an integer and X_k be the polygonal complex obtained by taking k copies of $\mathcal{H}$ and identifying them along their boundaries $\mathbb{R} \times \{0\} \subset \mathcal{H}$. We can think of X_k as a book with the copies of $\mathcal{H}$ as its pages. Note that the union of any two pages can be understood as a tessellation of the plane by squares. Every such choice represents an apartment of the polygonal complex with planar substructures X_k. It is straightforward to see that X_k has non-positive sectional corner curvature. Books can also be obtained by combining pages with more general and different polygonal structures by using isomorphisms between their boundaries (considered as 1-dimensional cell complexes). Moreover, it is also possible to consider books with *infinitely many* pages. They are obviously non-locally finite polygonal complexes with planar substructures.

Example 5. Let us present an example of polygonal complexes that have no planar substructures satisfying (PCPS1) and (PCPS2). Let $X = (V, E, F)$ be given by $V = \mathbb{Z}^3$, E be the set of straight Euclidean line segments of length 1 connecting two vertices of V, and F be the set of all unit squares with vertices in V. The triple X is obviously a polygonal complex, but there does not exist a choice of apartments (planes tessellated by squares) satisfying both conditions (PCPS1) and (PCPS2). The set of all planes parallel to the coordinate planes does not satisfy (PCPS1). Thus, we also need to declare certain topological planes which are bent to be apartments. But it is easy to see that the convexity property (PCPS2) is violated for any such bent plane.

Next, we come to two important notions in the local combinatorial description of polygonal complexes. Our purpose is to use these notions later to define certain buildings in the next sections.

Definition 5.1 *(Link).* Let $X = (V, E, F)$ be a polygonal complex. The *link* $L(v)$ of a vertex $v \in V$ is a graph defined as follows: Every edge adjacent to v is represented by a vertex in $L(v)$, and two vertices w_1, w_2 in $L(v)$ are connected by an edge in $L(v)$ if the edges in X corresponding to w_1, w_2 are edges of a face f in F.

As an easy example one finds that the link of a vertex of degree d in a planar tessellation is a d-gon. Similarly, one finds that the link of a vertex in $\mathbb{Z}^3$ is an octahedron.

Furthermore, polygonal complexes are often described via the type of their faces and the graphs appearing as links. It is proven in [3, Theorem 1] that for given $p \geq 6$, $n \geq 3$ there is a continuum of non-isomorphic simply connected polygonal complexes such that the faces are p-gons and the links of all vertices are the 1-skeletons of an n-simplex.

Next, we give the definition of generalized m-gons that appear as links of Euclidean and hyperbolic buildings which are introduced in the next section.

Definition 5.2 *(Generalized m-gon).* Let $m \geq 2$ be an integer. A *generalized m-gon* is a connected bipartite graph of diameter m and of girth $2m$ such that each vertex has degree ≥ 2.

Next to ordinary $2m$-gons, important examples of generalized m-gons are the Heawood graph ($m = 3$) and complete bipartite graphs ($m = 2$). As it shall be discussed in the next sections, they appear as examples of links of vertices of buildings.

Let us make another remark to stress the relevance of these notions. The adjacency matrices of regular generalized m-gons have interesting spectral properties. In particular, they are Ramanujan graphs (see [47, Section 8.3]). Spectral properties of the links of vertices of 2-dimensional simplicial complexes were also very useful to obtain Kazhdan property (T) for groups acting cocompactly in these complexes (see [5]).

5.2. Euclidean and hyperbolic buildings

Let us give a quick introduction into 2-dimensional Euclidean and hyperbolic buildings, following essentially [28]. In contrast to our Definition 2.1, the cells in the polygonal complexes used for Coxeter complexes and buildings have an additional metric structure, namely, the 1-cells are open Euclidean or hyperbolic geodesic segments and the 2-cells are Euclidean or hyperbolic polygons (we restrict our considerations to compact ones), and the attaching maps are isometries (see also [14, Sct. I.7.37]). We call an *isometric isomorphism* between two polygonal complexes an *isometry*, for simplicity. The closures of the 2-cells are called *chambers* of the polygonal complex.

Important *planar* polygonal complexes are Coxeter complexes, which we introduce first (for more details see, e.g., [34]). Let $\mathbb{X}$ stand for either the Euclidean plane $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}^2$. Let $P \subset \mathbb{X}$ be a compact polygon with $k \geq 3$ vertices such that the interior angle at vertex i is of the form π/m_i with $m_i \geq 2$. We call such a polygon P a *Coxeter polygon*. Let $S = \{s_1, \dots, s_k\}$ be the set of reflections along the sides of P and W be the group generated by the elements of S. Then it is a well known fact due to Poincaré that W is a discrete subgroup of the isometry group $\mathrm{Iso}(\mathbb{X})$ with P as its fundamental domain, i.e., the translates $\{gP \mid g \in W\}$ form a tessellation of $\mathbb{X}$, which is a planar polygonal complex in the above sense. We refer to it as the *Coxeter complex* $C(W, S)$ and call the polygon P the *generating polygon* of the *Coxeter group* (W, S).

Definition 5.3 *(Building).* Let $\mathbb{X} \in \{\mathbb{R}^2, \mathbb{H}^2\}$, $P \subset \mathbb{X}$ be a Coxeter polygon and (W, S) be the associated Coxeter group. A (2-dimensional) *building of type* (W, S) is a polygonal complex $X = (V, E, F)$, together with a set $\mathcal{A}$ of subcomplexes whose elements $\Sigma = (V_\Sigma, E_\Sigma, F_\Sigma)$ are called *apartments*, with the following properties:

- (B1) For any two cells of X there is an apartment containing both of them.
- (B2) If Σ_1 and Σ_2 are two apartments containing two cells c_1, c_2 of X, then there exists an isometry $f : \Sigma_1 \to \Sigma_2$ which fixes c_1 and c_2 pointwise.
- (B3) Each apartment Σ is isometric to the planar tessellation $C(W, S)$.

The building X is called *Euclidean* if $\mathbb{X} = \mathbb{R}^2$ and *hyperbolic* if $\mathbb{X} = \mathbb{H}^2$. A building is called *thick* if every edge is contained in at least three chambers. A building which is not thick is called a *thin* building.

Proposition 5.4. *Every 2-dimensional Euclidean or hyperbolic building is a polygonal complex with planar substructures, i.e., it satisfies the axioms (PCPS1), (PCPS2), (PCPS3).*

Proof. Disregarding the additional Euclidean or hyperbolic structure of the cells of a building, we can view it and its apartments as polygonal complexes in the sense of Definitions 2.1 and 2.3. Since the apartments of buildings are always convex (see

[28, p. 164, l. −5] and also [1, Corollary 5.54] or [29, Proposition on p. 59] for simplicial buildings), we see that every building is a polygonal complex with planar substructures. □

5.2.1. Euclidean buildings

In this subsection we discuss how our theory applies to Euclidean buildings and give two specific examples.

As discussed above the Coxeter polygon P has to be a k-gon whose interior angles are given by $\pi/m_1, \ldots, \pi/m_k$ with integers $m_1, \ldots, m_k \geq 2$ which have to satisfy

$$(k-2)\pi = \frac{\pi}{m_1} + \ldots + \frac{\pi}{m_k}$$

due to the Euclidean structure. This implies $k \leq 4$. As for P being a triangle, $k = 3$, one has either of the interior angles $\{\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\}$, $\{\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\}$ or $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\}$. Each of these choices leads to a unique Coxeter group and to a class of Euclidean buildings which are said to be of type $\widetilde{A}_2$, $\widetilde{C}_2$ and $\widetilde{G}_2$, respectively, see [1, Example 10.14]. For $k = 4$ the only possibility for P is to be the regular equilateral, the square.

By this discussion the following proposition can be checked immediately. We highlight it as it clarifies the applicability of the results of the previous sections to Euclidean buildings.

Proposition 5.5. *For every 2-dimensional Euclidian building, we have $\kappa^{(\Sigma)} = 0$, for every apartment Σ. Moreover, the sectional corner curvature $\kappa_c^{(\Sigma)}$ is constantly zero on every apartment Σ if and only if the Coxeter polygon is an equilateral triangle (type $\widetilde{A}_2$) or a square. Otherwise, some of the sectional corner curvatures are strictly positive.*

Proof. For the Coxeter polygon P with interior angles given by $\pi/m_1, \ldots, \pi/m_k$, the vertex degrees in the apartments of corresponding buildings have to be $2m_1, \ldots, 2m_k$ in order to sum up to 2π about each vertex. This gives the result by direct calculation. □

Let us stress that even though there are only three types of Euclidean triangles as Coxeter polygons, a classification of all buildings of one of these types is impossible because of their abundance (see [56, p. 157]).

Next we focus on two examples in more detail. First we revisit Example 1 in Subsection 2.1 in more detail.

Example 1 *(revisited).* This example is a thick Euclidean building based on an equilateral Euclidean triangle. Thus, it is of type $\widetilde{A}_2$ and has, therefore, zero sectional corner curvature.

To get a better understanding of this building, it is worth looking at the links of its vertices. It can be checked, that these links are all isomorphic to the Heawood graph which is a generalized 3-gon.

Next, we consider a natural class of Euclidean buildings based on a square.

Example 6 *(Product of trees).* Let $r, s \geq 2$ and T_r and T_s be infinite regular metric trees of vertex degrees r and s, respectively. All edge lengths are chosen to be 1. We can think of one of the trees, say T_r, to be *horizontal* and the other one to be *vertical*. Then the product $T_r \times T_s$ carries a natural structure of a thick Euclidean building $X = (V, E, F)$ with $P = [0,1]^2 \subset \mathbb{R}^2$. The set V consists of all pairs (x, y) where x and y are vertices in T_r and T_s respectively. Two vertices $(x_1, y_1), (x_2, y_2) \in V$ are connected by an edge in E, if either ($x_1 = x_2$ and $y_1 \sim_{T_s} y_2$) or ($y_1 = y_2$ and $x_1 \sim_{T_r} x_2$). In the first case we call the edge in E horizontal and in the second case we call the edge in E vertical. The chambers are the unit squares with boundary vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$ for any choice $x_1 \sim_{T_r} x_2$ and $y_1 \sim_{T_s} y_2$. All vertices in $T_r \times T_s$ have degree $r + s$ (with r emanating horizontal and s emanating vertices edges). Moreover, a vertical edge is contained in precisely r chambers while a horizontal edge is contained in precisely s chambers.

Two bi-infinite combinatorial geodesics $g_1 \subset T_r$ and $g_2 \subset T_s$ can be viewed as infinite regular trees of vertex degrees 2. The corresponding subcomplex $\Sigma = \Sigma_{g_1,g_2} = g_1 \times g_2$ is isomorphic to a regular tessellation of $\mathbb{R}^2$ by unit squares. We choose $\mathcal{A}$ to be the set of all those subcomplexes.

From the proposition above we learn that the sectional corner curvatures are constantly zero, i.e., $\kappa_c^{(\Sigma)} = 0$ for every apartment.

Another interesting fact about these buildings is that the link of every vertex in $T_r \times T_2$ is the complete bipartite graph $K_{r,s}$.

5.2.2. Hyperbolic buildings

Finally, let us consider some examples of hyperbolic buildings.

In the hyperbolic case, the Coxeter polygon P has to be a k-gon whose interior angles $\pi/m_1, \ldots, \pi/m_k$ with integers $m_1, \ldots, m_k \geq 2$ have to satisfy

$$(k-2)\pi > \frac{\pi}{m_1} + \ldots + \frac{\pi}{m_k}$$

due to the hyperbolic structure.

This gives the following immediate consequence.

Proposition 5.6. *For every 2-dimensional hyperbolic building, we have $\kappa^{(\Sigma)} < 0$, for every apartment Σ. Moreover, the sectional corner curvature satisfies $\kappa_c^{(\Sigma)} < 0$ if the Coxeter polygon is a regular hyperbolic polygon.*

Proof. Again, the vertex degrees in the apartments of corresponding buildings have to be $2m_1, \ldots, 2m_k$ in order to sum up to 2π about each vertex. This gives the result by direct calculation using the discussion above. $\square$

Note that while all hyperbolic buildings have negative sectional face curvature they do not always have also non-positive sectional corner curvature: consider a tessellation of the hyperbolic plane by triangles with interior angles $\frac{\pi}{r}, \frac{\pi}{s}, \frac{\pi}{t}$ with $r, s, t \geq 2$ and $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} < 1$ (which has to be satisfied as the sum over the angles of a hyperbolic triangle has to be less than π). This tessellation is a thin hyperbolic building and it has non-positive corner curvature if and only if $r, s, t \geq 3$.

Henceforth, we only consider hyperbolic buildings with *regular polygons* as faces. These hyperbolic buildings have always negative sectional corner curvature by the above proposition.

Below, we briefly outline three examples of hyperbolic buildings and refer the interested readers to the corresponding references.

We start with hyperbolic buildings whose faces are right-angled polygons.

Example 7 *("Bourdon buildings").* Let $p \geq 5$ and $q \geq 3$. Then there is a *unique* hyperbolic building $X_{p,q}$ with the following properties (see [13]): All chambers are regular right-angled hyperbolic p-gons and the link $L(v)$ of every vertex is the complete bipartite graph $K_{q,q}$. Since every edge of $X_{p,q}$ lies in q chambers, $X_{p,q}$ is a thick building. Moreover, $X_{p,q}$ has constant negative sectional corner curvature $\kappa_c^{(\Sigma)} = 1/p - 1/4 < 0$.

Next, we mention a general method to obtain hyperbolic buildings admitting a cocompact group action. First, we choose finitely many hyperbolic polygons, label their oriented edges and identify edges with the same labels (these edges must obviously have the same length). We call such a compact polygonal complex a *polyhedron*. Then its universal covering is again a polygonal complex (admitting a cocompact group action with this polyhedron as its quotient) and the links of its vertices provide useful information in the decision whether it is a building (see, e.g., [28]).

Next, we give an example which uses this construction.

Example 8 *(see [59,41]).* Let K be a *polygonal presentation* associated to the disjoint connected bipartite graphs $G_1, \ldots, G_n$ in the sense of [41, Definition 1.2]. Assume that all G_i are copies of the same generalized m-gon. Every cyclic p-tuple in K provides a clockwise labeling of the oriented edges of a regular hyperbolic p-gon with angles $\frac{\pi}{m}$. If $mp > 2m + p$ then the universal covering of the polyhedron corresponding to K is a hyperbolic building, see [59, p. 472]. It has constant sectional corner curvature $\kappa_c^{(\Sigma)} = (2m + p - mp)/(2mp) < 0$. This approach provides examples of hyperbolic buildings with p-sided chambers for arbitrary $p \geq 3$ *with a cocompact group action.*

In particular, the triangle presentations given in [41] lead to explicit hyperbolic buildings with regular triangles as faces.

Finally, techniques of Haglund [31] provide us with the following result.

Example 9 *(see [28, Thme. 3.6]).* Let $P \subset \mathbb{H}^2$ be a regular hyperbolic polygon with angles $\frac{\pi}{m}$, $m \geq 3$ and an even number of sides. Let (W, S) be the associated Coxeter group.

Let L be an algebraic generalized m-gon over a field with large enough cardinality. (The term "algebraic" refers to the fact that the m-gon is based on a Chevalley quadruple, see [28, Definition 3.3].) Then there are *uncountably many* hyperbolic buildings of type (W, S) with faces isometric to P such that all links are isomorphic to L.

5.3. Maximal apartment systems in buildings

Since any union of apartment systems of a building $X = (V, E, F)$ forms again an apartment system (see [1, Thm. 4.54] for a proof in the case of simplicial buildings), there exists a unique maximal system of apartments by Zorn's lemma. In the proof of positive Cheeger constant as a consequence of negative curvature (Theorem 3.8), we used the stronger axiom (PCPS1*) instead of (PCPS1). Below we show that, for a building with maximal apartment system, (PCPS1*) is satisfied. We give the full reference for the simplicial case and we believe that the result remains true in the polygonal case as well. The proof was indicated to us by Shahar Mozes.

Theorem 5.7. *Every locally finite 2-dimensional Euclidean or hyperbolic building with a maximal apartment system satisfies the axioms (PCPS1*), (PCPS2), (PCPS3).*

Proof. By Proposition 5.4 we only have to show (PCPS1*), that is, every one-sided infinite geodesic is included in an apartment. Consider a one-sided infinite geodesic $(f_j)_{j\geq 0}$ of faces. Define $\mathcal{A}_0$ to be the set of all apartments that contain f_0. Define a metric δ on $\mathcal{A}_0$ viz

$$\delta(\Sigma_1, \Sigma_2) = 1/\max\{r \in \mathbb{N} \mid \Sigma_1 \cap B_r(f_0) = \Sigma_2 \cap B_r(f_0)\}$$

for $\Sigma_1 \neq \Sigma_2$ and 0, otherwise. We show that the metric space $(\mathcal{A}_0, \delta)$ is compact by showing that it is totally bounded and complete. Note that total boundedness of $(\mathcal{A}_0, \delta)$, (i.e., the metric space can be covered by finitely many ε balls for every $\varepsilon > 0$) follows from local finiteness, as local finiteness implies the set $\{\Sigma \cap B_r(f_0) \mid \Sigma \in \mathcal{A}_0\}$ is finite for all r. In order to see completeness, we let (Σ_n) be a Cauchy sequence in $\mathcal{A}_0$ and observe that, for a given r, there is N such that $b_r = \Sigma_n \cap B_r(f_0)$ are constant for $n \geq N$. One can check that $\Sigma = \bigcup_{r\geq 1} b_r$ is isometric to the Coxeter complex $C(W, S)$ and, thus, Σ is contained in the system of maximal apartments by [1, Proposition 4.59]. Hence, $\Sigma \in \mathcal{A}_0$ and, thus, Σ is a limit of (Σ_n) in $\mathcal{A}_0$. Hence, $(\mathcal{A}_0, \delta)$ is totally bounded and complete and, thus, compact. Now, let $\Sigma_n \in \mathcal{A}_0$ be an apartment that contains f_n and, by convexity of the apartments, $f_0, \ldots, f_n \in \Sigma_n$. By compactness, there is a convergent subsequence with limit $\Sigma \in \mathcal{A}_0$ which therefore contains the faces of the geodesic $(f_j)_{j\geq 0}$. $\square$

Let us close this section by a one-dimensional example that shows that the choice of the apartment system is not unique. Analogues in higher dimensions are easy to find.

Example 10. Let $T_r = (V, E)$ be a regular metric tree of edge length 1 and vertex degree $r \geq 3$, and let $\phi : E \to \{1, 2, \ldots, r\}$ be a labeling of the edges such that the r edges emanating from every vertex carry pairwise different labels. Let $\mathcal{A}$ be the set of bi-infinite paths (f_k) such that the bi-infinite sequence $x_k = \phi(f_k)$ has no doublings (i.e., $x_k \neq x_{k+1}$ for all $k \in \mathbb{Z}$) and is periodic (i.e., there exists $t \geq 1$ such that $x_{k+t} = x_k$ for all $k \in \mathbb{Z}$). Then it is easy to see that T_r together with $\mathcal{A}$ as its system of apartments forms a one-dimensional Euclidean building. Another choice $\mathcal{A}'$ of an apartment system is the set of all bi-infinite paths without doublings in the above sense, which is the maximal apartment system. It is obvious that $\mathcal{A}'$ is a strictly bigger apartment system than $\mathcal{A}$.

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Appendix A. Comparison to Wise's curvature

In this appendix we compare our notion of curvature for polygonal complexes with planar substructures to the definition of sectional curvature by Wise.

We start by briefly introducing the sectional curvature notion of Wise. For more details we refer to [60] and references therein.

Let X be a polygonal complex. We restrict ourselves to the case where all polygons are regular, that is, all angles in an n-gon f have degree $\pi(n-2)/n$ (which implies that Wise's curvature of the face f vanishes).

For a vertex v in X, a *section* is a based immersion, i.e. locally injective map, $\sigma : (S, s) \to (X, v)$ from a polygonal complex $S = (V_S, E_S, F_S)$ to X such that $\sigma(s) = v$. A section is called *planar* if the link $L(s)$ of s is a circle. In this case the curvature κ_σ at v with respect to σ is defined by

$$\kappa_\sigma(v) = 1 - \frac{|s|_S}{2} + \sum_{f \in F_S, s \in \overline{f}} \frac{1}{|f|_S}.$$

We say that a polygonal complex has *non-positive planar sectional curvature in the sense of Wise* at v if $\kappa_\sigma(v) \leq 0$ for all vertices v and all planar sections σ.

In comparison to our definition it is obvious that Wise's definition is "much more local", i.e., it does not need any planar substructures but only planar sections.

Nevertheless, a natural question is how non-positive curvature in the sense of our paper is related to non-positive curvature in the sense of Wise.

To get the obvious out of the way let us mention that for the sectional face curvature as we defined it, there is no relation to the "vertex" curvature of Wise. In particular, already in the case of planar tessellations vertex and face curvature are not related, namely, there are graphs with somewhere positive vertex curvature but non-positive face curvature everywhere and vice versa.

Let us turn to a more subtle question. Many of our results use the assumption of non-positive sectional corner curvature. This raises the question whether this already implies non-positive planar sectional curvature in the sense of Wise for polygonal complexes with planar substructures.

The non-obvious part here stems from the fact that for a polygonal complex with planar substructures there might be a based section $\sigma : (S, s) \to (X, v)$ such that S is locally not isomorphic to an apartment of X. On the other hand, convexity of the apartments is a rather strong assumption. So, we will show that non-positive sectional corner curvature implies non-positive planar sectional curvature in the sense of Wise. This is the main result of this appendix.

Theorem A.1. *If a polygonal complex with planar substructures has non-positive sectional corner curvature, then it has non-positive planar sectional in the sense of Wise.*

A key ingredient for the proof of this theorem is the following lemma.

Lemma A.2. *Let Σ be a non-positively corner curved apartment. For two faces f, f' sharing only a common vertex v, let $f_0, f_1, \ldots, f_n$ be the faces around v with cyclic enumeration such that $f_0 = f$. Let $1 \leq k \leq n$ such that $f_k = f'$. Then, at least one of the two paths $\gamma_1 = (f = f_0, f_1, f_2, \ldots, f_k = f')$ and $\gamma_2 = (f_0 = f, f_n, f_{n-1}, \ldots, f_k = f')$ is a geodesic.*

Proof. We give an indirect proof. Every geodesic $\gamma = (g_0 = f, \ldots, g_m = f')$ encloses a (possibly empty) interior domain of faces (since f and f' touch in the vertex v) and this interior together with the geodesic γ encloses one of the two paths γ_1 or γ_2. Since we assume that none of these two paths is a geodesic itself, every geodesic γ connecting f and f' has a non-empty interior of faces. Let $\gamma_0 = (g_0 = f, g_1, \ldots, g_m = f')$ be a geodesic connecting f and f' with minimal number of interior faces and let $\hat{f}_0$ be a face in the strictly interior of γ_0. Then we can find a geodesic $\hat{\gamma} = (h_0 = f, h_1, \ldots, h_n = \hat{f}_0)$ consisting only of faces of γ_0 itself and its interior (subpaths of $\hat{\gamma}$ lying outside γ_0 and with end faces in γ_0 can be replaced by the corresponding subpaths along γ_0). Since Σ has non-positive corner curvature, the cut locus of f is empty [9, Theorem 1] and we can extend the geodesic $\hat{\gamma}$ consecutively by interior faces $\hat{f}_1, \hat{f}_2, \ldots$ with $d(f, \hat{f}_i) = n + i$ until $\hat{f}_r$ coincides again with a face g_j of γ_0. Then, the geodesic $\gamma' = (h_0, \ldots, h_n = \hat{f}_0, \ldots, \hat{f}_r = g_j, \ldots, g_m)$ connects f and f' with less inner faces than γ_0, which is a contradiction. $\square$

Proof of Theorem A.1. Assume there is a vertex v of the polygonal complex with planar substructures X with a planar section $\sigma : (S, s) \to (X, v)$ such that $\kappa_\sigma(v) > 0$. We show that there must be a corner in an apartment of X with positive corner curvature.

We prove this statement by contradiction. So, we assume, in particular, there is no apartment Σ such that there is an immersion $(S, s) \to (\Sigma, v)$. Let n be the number of neighbors of s in S. We distinguish three cases $n = 3, 4, 5$ as for the case $n \geq 6$ we always have $\kappa_\sigma(v) \leq 0$. In each case, we enumerate the faces around v with respect to σ by $f_0, \dots, f_{n-1}$ in cyclic order and denote the common edge of f_j and f_{j+1} by e_j.

Case $n = 3$: Let Σ be an apartment that contains f_0 and f_1 (but not f_2). Then, there is a face $g \neq f_2$ in Σ adjacent to f_1 and v in e_1. Then, (f_0, f_1, g) is a geodesic: Otherwise, f_0 and g were adjacent and intersect in one of the edges e_0 or e_2. In the first case, g intersects f_1 in two edges which is a contradiction to the axioms of a tessellation for the apartment that contains g and f_1, cf. (PCPS3). In the second case, g intersects f_2 in two edges and we obtain again a contradiction by the same argument. Hence, (f_0, f_1, g) is a geodesic. But, then also (f_0, f_2, g) is a geodesic which implies that f_0, f_1, f_2 are all contained in one apartment, by (PCPS2), which is a contradiction.

Case $n = 4$: Since f_0 and f_2 are not adjacent, both paths (f_0, f_1, f_2) and (f_0, f_3, f_2) are geodesics and $f_0, \dots, f_3$ are in the same apartment, which is a contradiction.

Case $n = 5$: Note that positive planar sectional curvature in the sense of Wise implies that at least four of the faces $f_0, \dots, f_4$ must be triangles and the sum of the angles of any two corners (v, f_i), (v, f_j) is $< \pi$. Since $f_0, \dots, f_4$ are not in contained in an apartment, there must be a face g adjacent to one of the edges $e_0, \dots, e_4$, say w.l.o.g. e_2. Consider the path $\gamma_0 = (f_0, f_1, f_2, g)$. If γ_0 was a geodesic so is (f_0, f_4, f_3, g). Since both geodesics must be contained in the same apartment, by (PCPS2), this implies that $f_0, \dots, f_4$ are contained in one apartment contrary to the assumption.

So, γ_0 is not a geodesic and $d(f_0, g) = 1$ or $d(f_0, g) = 2$. In the first case, g and f_0 must be adjacent along one of the edges e_0 or e_4. If they are adjacent along e_0, then f_1, f_2, g are three faces around v with positive planar sectional curvature in the sense of Wise (since the sum of the angles in the corners (v, f_1) and (v, f_2) is $< \pi$) and we can return to Case $n = 3$ to obtain a contradiction. An analogous argument applies if g and f_0 are adjacent along e_4. So, we have $d(f_0, g) = 2$. Let Σ be an apartment containing f_0 and g. Since both faces touch in v, we can apply Lemma A.2 and find a face $h \in \Sigma$ adjacent to f_0 and g and containing v, such that (f_0, h, g) is a geodesic. Moreover, h and f_0 share one of the two edges e_0 or e_4. We can assume, w.l.o.g., that f_0 is a triangle. (Namely, if f_0 is not a triangle, then f_3 is a triangle and we rename the faces f_0, h, g by f_3, g, h, respectively. Note that (f_3, g, h) is also a geodesic.) Since we assume that the corner curvature of (v, f_0) is non-positive in Σ, we have $|v|_\Sigma \geq 6$ and, by Lemma A.2, $d(f_0, g') = 3$ for the second neighbor $g' \neq h$ of g along v in Σ. Note that g' meets g in the edge e_2 and that $g' \neq f_2, f_3$ since $d(f_2, f_0) = d(f_3, f_0) = 2$. Now, we consider the apartment Σ containing f_0 and g', which must contain both geodesics (f_0, f_1, f_2, g') and (f_0, f_4, f_3, g'), in contradiction to the assumption that $f_0, \dots, f_4$ do not lie in a common apartment. $\square$

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