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Flavon-induced connections between lepton flavour mixing and charged lepton flavour violation processes

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Abstract

In leptonic flavour models with discrete flavour symmetries, couplings between flavons and leptons can result in special flavour structures after they gain vacuum expectation values. At the same time, they can also contribute to the other lepton-flavour-violating processes. We study the flavon-induced LFV 3-body charged lepton decays and radiative decays and we take as example the A_4 discrete symmetry. In A_4 models, a Z_3 residual symmetry roughly holds in the charged lepton sector for the realisation of tri-bimaximal mixing at leading order. The only processes allowed by this symmetry are $\tau^- \to \mu^+ e^- e^-, e^+ \mu^- \mu^-$, and the other 3-body and all radiative decays are suppressed by small Z_3-breaking effects. These processes also depend on the representation the flavon is in, whether pseudo-real (case i) or complex (case ii). We calculate the decay rates for all processes for each case and derive their strong connection with lepton flavour mixing. In case i, sum rules for the branching ratios of these processes are obtained, with typical examples $\mathrm{Br}(\tau^- \to \mu^+ e^- e^-) \approx \mathrm{Br}(\tau^- \to e^+ \mu^- \mu^-)$ and $\mathrm{Br}(\tau^- \to e^- \gamma) \approx \mathrm{Br}(\tau^- \to \mu^- \gamma)$. In case ii, we observe that the mixing between two Z_3-covariant flavons plays an important role. All processes are suppressed by charged lepton masses and current experimental constraints allow the electroweak scale and the flavon masses to be around hundreds of GeV. Our discussion can be generalised in other flavour models with different flavour symmetries.

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1 Introduction

A series of solar [1], atmospheric [2], accelerator [3] and reactor [4] neutrino oscillation experiments have proven that neutrinos have masses and mix. The mixing is described by the so-called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [5], which is parametrised by [6]

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix} , \tag{1}$$

in which $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. The three mixing angles have been measured to a good accuracy. Their current best-fit and $\pm 1\sigma$ values from a global analysis of the available data [7] are given by

$$\sin^2\theta_{12} = 0.308^{+0.013}_{-0.012} , \quad \sin^2\theta_{23} = 0.574^{+0.026}_{-0.144} \, (0.579^{+0.022}_{-0.029}) , \quad \sin^2\theta_{13} = 0.0217^{+0.0013}_{-0.0010} \, (0.0221^{+0.0010}_{-0.0010}) \tag{2}$$

for the normal (inverted) ordering of neutrino masses, $m_1 < m_3$ ($m_1 > m_3$). We notice that the atmospheric angle θ_{23} and solar angle θ_{12} are rather large, with θ_{23} possibly being maximal, and the reactor angle θ_{13} takes a value around 0.1, $\theta_{13} \sim 9^\circ$.

The origin of this distinct mixing structure remains unexplained. Discrete flavour symmetries have been widely used to address these questions. It is assumed that at some high energy scale, there exists an underlying discrete flavour symmetry, $G_{\rm f}$, which unifies the three flavours together. The tetrahedral group A_4 [8], which is the smallest group containing 3-dimensional irreducible representations, is the most famous example of this type. There are other commonly-used groups, such as S_4 [9], A_5 [10], $\Delta(48)$ [11], and $\Delta(96)$ [12]. At a lower energy scale, the flavour symmetry is broken, leading to nontrivial flavour mixing. Most models are built in the framework of the so-called "direct" or "semi-direct" approaches [13]. In these cases, different residual symmetries, G_l and G_ν, subgroups of $G_{\rm f}$, are preserved in the charged lepton and neutrino sectors, respectively after the whole flavour symmetry $G_{\rm f}$ breaking. By choosing different G_l and G_ν, special flavour structures arise. In the direct approach, the mixing matrix is fully determined by G_l and G_ν up to Majorana phases and column or row permutations of the PMNS matrix. In the semi-direct approach, G_l and G_ν cannot fully determine flavour mixing and an accidental symmetry is necessary. For instance, in models based on A_4, the tri-bimaximal (TBM) mixing pattern [14], which predicts $s_{12} = 1/\sqrt{3}$, $s_{23} = 1/\sqrt{2}$ and $s_{13} = 0$, can be realised in the semi-direct approach [15, 16, 17]. The residual symmetries are chosen to be $G_l = Z_3$ and $G_\nu = Z_2$, while an additional Z_2' symmetry, not belonging to A_4, arises accidentally in the neutrino sector. In models of S_4, the residual Z_3, Z_2, Z_2' all belong to S_4, so TBM can be obtained in the direct approach [18, 19]. S_4 can predict another mixing pattern, the bimaximal mixing one ($s_{12} = s_{23} = 1/\sqrt{2}$, $s_{13} = 0$) [20] by choosing $G_l = Z_4$ [21]. Some other mixing patterns can be arranged using larger groups in the direct approach, for instance, the golden-ratio mixing ($s_{12} = \sqrt{2}/\sqrt{5+\sqrt{5}}$, $s_{23} = 1/\sqrt{2}$, $s_{13} = 0$) [22] predicted by A_5 [23, 24] and the Toorop-Feruglio-Hagedorn mixing ($s_{12} = s_{23} = \sqrt{2}/\sqrt{4+\sqrt{3}}$, $s_{13} = 1/(3+\sqrt{3})$) predicted by $\Delta(96)$ [12]. It should be noted that the predicted values of θ_{13} in all these mixing patterns is not in agreement with the data. This suggests that small corrections should be introduced and the residual symmetries should be slightly broken.

A common approach to realise the breaking of $G_{\rm f}$ is to introduce flavons, new scalars that couple to fermions and have non-trivial properties under the flavour symmetry. These scalars get vacuum expectation values (VEVs), leading to the spontaneous breaking of the flavour symmetry and leaving residual symmetries in the charged lepton and neutrino sectors, respectively. At least two flavon multiplets, one for charged leptons and the other for neutrinos, have to be introduced to guarantee different residual symmetries in the two sectors. The well-known and simplest case is the realisation of TBM in A_4 models [15, 16, 17, 25]. In models with larger symmetry groups, more flavon multiplets may be needed for model constructions [19, 21, 23, 11, 26].

The slight breaking of the residual symmetries can be provided by additional interactions of flavons. The latter may be directly from higher-dimensional operators in the couplings between flavons and leptons [13, 27]. In our recent paper [25], we observe that cross coupling between neutrino and charged lepton flavons can shift the VEVs from their original Z_3 and Z_2 symmetric values. In the models based on A_4, we studied in detail the modification to the TBM flavour mixing pattern, in particular conserving the origin of non-zero θ_{13} and Dirac-type CP violation.

The interactions of flavons and leptons, in addition to ensuring special Yukawa structures in the lepton sector, may also contribute to other processes and in particular lead to lepton-flavour-violating (LFV) processes. Most flavour models assume that the flavour symmetry is broken at a very high scale such that these processes are too suppressed to be observed. However, the scale of flavour symmetry is not known and could be much lower than commonly considered. An electroweak-scale flavour symmetry has recently been discussed [28], see also [29]. For instance, some flavons are formed by multi-Higgs ($SU(2)_L$ scalar doublets), their VEVs must be below the electroweak scale. If the scale and the flavon masses are sufficiently low, there would be some testable signatures, especially in charged LFV decay channels. Measuring these processes would provide important clues to identify the origins of leptonic flavour mixing.

The LFV decays of charged leptons induced by flavons can be divided into two classes: those preserving the residual symmetry in the charged lepton sector and those breaking it. In A_4 models, the only processes allowed by the Z_3 residual symmetry are $\tau^- \to \mu^+ e^- e^-$, $\tau^- \to e^+ \mu^- \mu^-$, and all other 3-body and radiative decays are forbidden [30, 31]. The latter can take place if the Z_3 symmetry is broken [32], but are typically suppressed due to the consistency with oscillation data, as shown later. Current experimental bounds of the branching ratios of LFV τ 3-body decays $\tau^- \to \mu^+ e^- e^-$, $e^+\mu^-\mu^-$, $\mu^+\mu^-\mu^-$, $e^+e^-\mu^-$, $\mu^+\mu^-e^-$, $e^+e^-e^-$, and ratiative decays $\tau^- \to \mu^-\gamma$, $e^-\gamma$ are in general around 10^{-8}, measured by Belle [33] and BaBar [34], respectively. The upper limit of the μ 3-body decay $\mu^- \to e^+e^-e^-$ decay is 1.0×10^{-12} at 90 % C.L., from the SINDRUM experiment [35]. The most stringent measurement is $\mu^- \to e^-\gamma$ in the MEG experiment, with branching ratio $\sim 4.2\times 10^{-13}$ at 90% C.L. [36]. A MEG upgrade (MEG II) is envisaged to reach the upper limit of the branching ratio to 4×10^{-14} in the near future [37]. One may expect these experiments provide important constraints to the scale of flavour symmetry.

In this paper, we develop a generic method to analyse charged LFV processes in models with discrete flavour symmetries. For definiteness, we choose to work on models based on A_4, which we review in section 2. In section 3, we give a model-independent discussion of charged LFV processes induced by flavons. We derive the expressions of leading flavon contributions to Z_3-preserving LFV processes and specify different Z_3-breaking effects contributions to Z_3-breaking processes. Since the latter are strongly dependent upon the model construction, we list two models and analyse them in detail in section 4. These models, which have been constructed in Ref. [25], are very economical and consistent with current oscillation data.

2 Flavour mixing in the A_4 symmetry

2.1 Residual symmetries and tri-bimaximal mixing

For definiteness, we assume the flavour symmetry is the tetrahedral group A_4, which is the group of even permutations of four objects. It is generated by S and T with the requirement $S^2 = T^3 = (ST)^3 = 1$, and contains 12 elements: 1, S, ST, TS, STS, T^2, ST^2, T^2S, TST, S, T^2ST, TST^2. It is the smallest discrete group which has a 3-dimensional irreducible representation $\mathbf{3}$. In addition, it has three 1-dimensional irreducible representations: $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$. The Kronecker product of two 3-dimensional irreducible representations can be reduced as $\mathbf{3}\times\mathbf{3} = \mathbf{1}+\mathbf{1}'+\mathbf{1}''+\mathbf{3}_S+\mathbf{3}_A$, where the subscripts $_S$ and $_A$ stands for the symmetric and anti-symmetric components, respectively.

We work in the Altarelli-Feruglio basis [16], where T is diagonal. T and S are respectively given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} , \qquad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} . \tag{3}$$

This basis is widely used in the literature since the charged lepton mass matrix invariant under T is diagonal in this basis. The products of two 3-dimensional irreducible representations $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ can be expressed as

$$\begin{aligned} (ab)_{\mathbf{1}} &= a_1b_1 + a_2b_3 + a_2b_3 \,, \\ (ab)_{\mathbf{1}'} &= a_3b_3 + a_1b_2 + a_2b_1 \,, \\ (ab)_{\mathbf{1}''} &= a_2b_2 + a_1b_3 + a_3b_1 \,, \end{aligned} \quad (ab)_{\mathbf{3}_S} = \frac{1}{2}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_3b_1 - a_1b_3 \end{pmatrix} , \quad (ab)_{\mathbf{3}_A} = \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix} . \tag{4}$$

We assume that A_4 is preserved at high energy scale and broken at some lower scale, which we refer to as the scale of flavour symmetry. In the charged lepton and neutrino sectors, residual symmetries Z_3 and Z_2, which are subgroups of A_4, are preserved, respectively. The generators of Z_3 and Z_2, T and S

respectively, generate the full symmetry A_4. The invariance of the charged lepton mass matrix under Z_3 and that of the neutrino mass matrix under Z_2 satisfy

$$T M_l M_l^\dagger T^\dagger = M_l M_l^\dagger \,, \quad S M_\nu S^T = M_\nu \,. \tag{5}$$

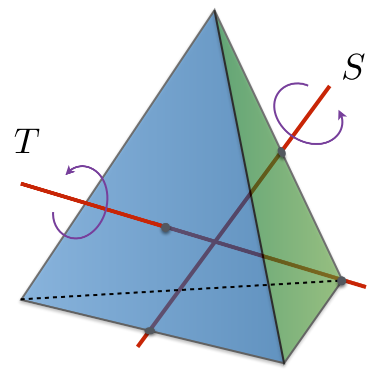

Figure 1: The tetrahedral group A_4 as the full flavour symmetry and its subgroups $Z_3 = <T>$ and $Z_2 = <S>$ as residual symmetries in the charged lepton and neutrino sectors, respectively.

In order to induce the flavour symmetry breaking, we introduce two flavon triplets φ, χ in the charged lepton and neutrino sectors, respectively. We consider two cases with different representation properties:

i. The flavons are pseudo-real triplets of A_4. In this case, the three components of φ satisfy $\varphi_1^* = \varphi_1$ and $\varphi_2^* = \varphi_3$ in the Altarelli-Feruglio basis. This is an economical case introducing as few degrees of freedom beyond the Standard Model as possible.

ii. The flavons are complex triplets of A_4. This case has been used more widely than case i due to the consistency with supersymmetric and multi-Higgs flavour models and can be regarded as a simplification of these models.

The representation properties of flavons will have important consequences for LFV processes, as will be discussed later.

In order to preserve Z_3 and Z_2, the flavon VEVs should be invariant under the transformations of T and S, respectively, i.e.,

$$T\langle\varphi\rangle = \langle\varphi\rangle \,, \qquad S\langle\chi\rangle = \langle\chi\rangle \,. \tag{6}$$

The non-vanishing solutions for the above equation are given by

$$\langle\varphi\rangle = (1,0,0)^T \frac{v_\varphi}{\sqrt{n}} \,, \qquad \langle\chi\rangle = (1,1,1)^T \frac{v_\chi}{\sqrt{3n}} \,. \tag{7}$$

Here, $n = 1, 2$ for cases i and ii, respectively, and v_φ and v_χ stand for the overall size of the VEVs and can be treated as A_4-breaking scale. Without loss of generality, we assume v_φ, $v_\chi > 0$.

The Lagrangian terms for generating lepton masses are represented by some higher-dimensional operators. The electroweak lepton doublet $\ell_L = (\ell_{eL}, \ell_{\mu L}, \ell_{\tau L})^T$ is often arranged to belong to a **3** of

A_4, and the right-handed charged leptons e_R, μ_R and τ_R belong to singlets $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$, respectively. The relevant Lagrangian terms are given by [1]

$$
\begin{aligned}
-\mathcal{L}_l &= \frac{y_e}{\Lambda}(\overline{\ell_L}\varphi)_{\mathbf{1}} e_R H + \frac{y_\mu}{\Lambda}(\overline{\ell_L}\varphi)_{\mathbf{1}''}\mu_R H + \frac{y_\tau}{\Lambda}(\overline{\ell_L}\varphi)_{\mathbf{1}'}\tau_R H + \text{h.c.}\,, \\
-\mathcal{L}_\nu &= \frac{y_1}{2\Lambda\Lambda_{\rm W}}\big((\overline{\ell_L}\tilde{H}\tilde{H}^T\ell_L^c)_{\mathbf{3}_S}\chi\big)_{\mathbf{1}} + \frac{y_2}{2\Lambda_{\rm W}}(\overline{\ell_L}\tilde{H}\tilde{H}^T\ell_L^c)_{\mathbf{1}} + \text{h.c.}\,.
\end{aligned}
\tag{8}
$$

Here, the Higgs H belongs to $\mathbf{1}$ of A_4. Λ is a new scale higher than v_φ, v_χ. It may be a consequence of the decoupling of some heavy A_4 multiplet particles. To generate tiny Majorana neutrino masses, we apply the traditional dimension-5 Weinberg operator $(\overline{\ell_L}\tilde{H}\tilde{H}^T\ell_L^c)$ and $\Lambda_{\rm W}$ is the related scale, which may be different from Λ. After the flavons get the VEVs in Eq. (7), we obtain the lepton mass matrices

$$
M_l = \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix} \frac{v v_\varphi}{\sqrt{2n}\Lambda}\,, \qquad M_\nu = \begin{pmatrix} 2a+b & -a & -a \\ -a & 2a & -a+b \\ -a & -a+b & 2a \end{pmatrix}\,,
\tag{9}
$$

where $v = 246$ GeV is the VEV of the Higgs H, $a \equiv y_1 v_\chi v^2/(4\sqrt{3n}\Lambda\Lambda_{\rm W})$ and $b \equiv y_2 v^2/2\Lambda_{\rm W}$. At leading order, M_l is diagonal and M_ν is diagonalised by the unitary matrix

$$
U_{\rm TBM} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}
\tag{10}
$$

and have eigenvalues $m_1 = |3a+b|$, $m_2 = |b|$ and $m_3 = |3a-b|$. This gives rise to $s_{13} = 0$, $s_{12} = 1/\sqrt{3}$ and $s_{23} = 1/\sqrt{2}$, i.e., the so-called TBM mixing pattern.

2.2 The breaking of the residual symmetries

The TBM mixing has been excluded since it predicts a vanishing θ_{13}. To be consistent with neutrino oscillation data, corrections of order < 0.1

$$
s_{13} = \frac{r}{\sqrt{2}}\,, \qquad s_{12} = \frac{1+s}{\sqrt{3}}\,, \qquad s_{23} = \frac{1+a}{\sqrt{2}}\,,
\tag{11}
$$

must be included. In the Standard Model, TBM is modified by radiative corrections which break the residual symmetries. However, such corrections are too small to produce an $\mathcal{O}(0.1)$ θ_{13}. Couplings with different flavon multiplets provide another origin for flavour mixing corrections. In the charged lepton sector, a direct origin of Z_3-breaking corrections is the interrupt of the flavon triplet χ. And in the neutrino sector, the Z_2-breaking origin is from φ. They may contribute to the Yukawa couplings directly through higher-dimensional operators or indirectly through the shifts of the VEVs induced by cross couplings in the flavon potential. After these corrections are included, the PMNS mixing matrix is parametrised as

$$
U_{\rm PMNS} = U_l^\dagger U_{\rm TBM} U_\nu P_\nu\,,
\tag{12}
$$

where U_l and U_ν are unitary matrices representing corrections in the charged lepton sector and neutrino sector, respectively, and P_ν is a diagonal phase matrix to render positive neutrino masses. In this paper,

[1] Note that terms such as $(\overline{\ell_L}\chi)_{\mathbf{1}'}\tau_R H$ and $\big((\overline{\ell_L}\tilde{H}\tilde{H}^T\ell_L^c)_{\mathbf{3}_S}\varphi\big)_{\mathbf{1}}$ cannot be forbidden by A_4. These terms modify the mixing structures and should be forbidden at leading order. In concrete models, it can be required by introducing additional discrete Abelian symmetry, which will not be discussed here.

as we focus on the charged lepton sector, we assume corrections from the neutrino sector to be negligible, i.e., $U_\nu - \mathbf{1} \ll U_l - \mathbf{1}$. Then, the mixing matrix can be simplified to $U_{\rm PMNS} = U_l^\dagger U_{\rm TBM} P_\nu$.

We consider corrections from higher-dimensional operators in the charged lepton sector, which are written in the following form:

$$-\delta\mathcal{L}_l = \left(\overline{\ell_L}\chi_e\right)_{\mathbf{1}} e_R H + \left(\overline{\ell_L}\chi_\mu\right)_{\mathbf{1}''}\mu_R H + \left(\overline{\ell_L}\chi_\tau\right)_{\mathbf{1}'}\tau_R H + \text{h.c.}\,, \tag{13}$$

where

$$\chi_e \equiv \sum_{m,n}\frac{y_e^{m,n}}{\Lambda^{m+n}}(\varphi^m\chi^n)_{\mathbf{3}}\,, \quad \chi_\mu \equiv \sum_{m,n}\frac{y_\mu^{m,n}}{\Lambda^{m+n}}(\varphi^m\chi^n)_{\mathbf{3}}\,, \quad \chi_\tau \equiv \sum_{m,n}\frac{y_\tau^{m,n}}{\Lambda^{m+n}}(\varphi^m\chi^n)_{\mathbf{3}}\,, \tag{14}$$

with m, n sum for $m+n \geqslant 2$. $y_{e,\mu,\tau}^{m,n}$ are dimensionless complex coefficients. For $m+n=2$, we obtain the following combinations of VEVs

$$\langle(\varphi\chi)_{\mathbf{3}_S}\rangle \propto (2,-1,-1)^T\,, \quad \langle(\varphi\chi)_{\mathbf{3}_A}\rangle \propto (0,1,-1)^T\,, \tag{15}$$

and $\langle(\chi\chi)_{\mathbf{3}}\rangle$ vanishes at this order. For $m+n=3$, we get another direction of VEV combinations

$$\langle(\varphi\varphi)_{\mathbf{1}}\chi\rangle \propto \langle(\chi\chi)_{\mathbf{1}}\chi\rangle \propto (1,1,1)^T\,. \tag{16}$$

One can prove that any other Z_3-breaking combinations of the flavon VEVs must have the directions belonging to one of the aboves. After including these corrections, we obtain the most general form

$$\langle\chi_e\rangle \propto (1,\epsilon_{e2},\epsilon_{e3})\,, \qquad \langle\chi_\mu\rangle \propto (1,\epsilon_{\mu2},\epsilon_{\mu3})\,, \qquad \langle\chi_\tau\rangle \propto (1,\epsilon_{\tau2},\epsilon_{\tau3})\,. \tag{17}$$

Another type of correction comes from the vacuum shift of φ due to the Z_3-breaking couplings between φ and χ. The most general VEVs of φ takes the form

$$\varphi = (1,\epsilon_{\varphi2},\epsilon_{\varphi3})^T v_\varphi\,, \tag{18}$$

where ϵ_{φ_2} and ϵ_{φ_3} stand for the vacuum shift of φ. To calculate the exact expressions of the shifts, we expand the flavon potential around the Z_3-invariant VEV $\langle\varphi\rangle = (1,0,0)^T v_\varphi$ and separate it in the Z_3-preserving part $V_0(\varphi)$ and the Z_3-breaking part $V_1(\varphi)$. $V_0(\varphi)$ would result from the self couplings of φ, e.g., $\left((\varphi\varphi)_{\mathbf{3}}(\varphi\varphi)_{\mathbf{3}}\right)_{\mathbf{1}}$, and some trivial cross couplings with the other flavons, e.g., $(\varphi\varphi)_{\mathbf{1}}(\chi\chi)_{\mathbf{1}}$, after A_4 breaking. The Z_3-breaking $V_1(\varphi)$ include the cross couplings with other flavon multiplets whose VEVs do not respect the Z_3 symmetry, e.g., $(\varphi\varphi)_{\mathbf{1}''}(\chi\chi)_{\mathbf{1}'}$ with $\langle\chi\rangle$ only preserving Z_2. In the two cases for φ we are considering, we can obtain the expressions of the shifts.

- In case i, where φ is a pseudo-real triplet, the most general Z_3-preserving and Z_3-breaking terms that are relevant to the vacuum shift at first order are given by

$$\begin{aligned} V_0^{(2)}(\varphi) &= \frac{1}{2}m_{\varphi_1}^2\varphi_1^2 + m_{\varphi_2}^2\varphi_2^*\varphi_2\,, \\ V_1^{(1)}(\varphi) &= \varepsilon_1 v_\varphi^3\varphi_2^* + \text{h.c.}\,, \end{aligned} \tag{19}$$

  respectively, where the real parameters m_{φ_1}, m_{φ_2} are the masses of φ_1 and φ_2, respectively and ε_1 is a complex dimensionless parameter. The accidential Z_2' symmetry can be recovered if ε_1 is real. $V_0^{(2)}(\varphi)$ is invariant under the transformation $\varphi_2 \to \omega^2\varphi_2$, which is required by the Z_3 symmetry. The minimisation of $V_0^{(2)}(\varphi) + V_1^{(1)}(\varphi)$ leads to $\epsilon_{\varphi2} = \epsilon_{\varphi3}^* = \epsilon_\varphi$ with ϵ_φ defined by $\epsilon_\varphi \equiv -\varepsilon_1 v_\varphi^2/m_{\varphi_2}^2$.

- If φ is a complex scalar, i.e., case ii, the relevant terms of φ are modified to

$$V_0^{(2)}(\varphi) = \frac{1}{2}m_{\varphi_1}^2 h_1^2 + m_{\varphi_2}^2 \varphi_2^* \varphi_2 + m_{\varphi_3}^2 \varphi_3^* \varphi_3 + (m_{\varphi_2\varphi_3}^2 \varphi_2\varphi_3 + \text{h.c.}) ,$$
$$V_1^{(1)}(\varphi) = \varepsilon_1 v_\varphi^3 \varphi_2^* + \varepsilon_1' v_\varphi^3 \varphi_3^* + \text{h.c.} , \tag{20}$$

where m_{φ_1}, m_{φ_2}, m_{φ_3} are real and $m_{\varphi_2\varphi_3}$, ε_1 and ε_1' are in general complex. The phase of $m_{\varphi_2\varphi_3}$ is unphysical and can always be rotated away by an overall phase redefinition of the flavon φ. h_1 is the real component of φ_1, $\varphi_1 \equiv (v_\varphi + h_1 + ia_1)/\sqrt{2}$, and the pseudo-real scalar a_1 becomes an unphysical massless Goldstone particle after A_4 breaking to Z_3. From Eq. (20), we derive the vacuum shifts

$$\epsilon_{\varphi 2} = -\frac{\varepsilon_1 m_{\varphi_3}^2 - \varepsilon_1' m_{\varphi_2\varphi_3}^2}{m_{\varphi_2}^2 m_{\varphi_3}^2 - m_{\varphi_2\varphi_3}^4} v_\varphi^2 , \qquad \epsilon_{\varphi 3} = -\frac{\varepsilon_1' m_{\varphi_2}^2 - \varepsilon_1 m_{\varphi_2\varphi_3}^2}{m_{\varphi_2}^2 m_{\varphi_3}^2 - m_{\varphi_2\varphi_3}^4} v_\varphi^2 . \tag{21}$$

After considering the direct corrections to Yukawa couplings from higher-dimensional operators and the indirect corrections from the flavon vacuum shift, the charged lepton mass matrix becomes non-diagonal:

$$M_l = \begin{pmatrix} y_e & y_\mu(\epsilon_{\mu 3} + \epsilon_{\varphi 3}) & y_\tau(\epsilon_{\tau 2} + \epsilon_{\varphi 2}) \\ y_e(\epsilon_{e2} + \epsilon_{\varphi 2}) & y_\mu & y_\tau(\epsilon_{\tau 3} + \epsilon_{\varphi 3}) \\ y_e(\epsilon_{e3} + \epsilon_{\varphi 3}) & y_\mu(\epsilon_{\mu 2} + \epsilon_{\varphi 2}) & y_\tau \end{pmatrix} \frac{v v_\varphi}{\sqrt{2}\Lambda} , \tag{22}$$

leading to the mixing matrix

$$U_l^\dagger = \begin{pmatrix} 1 & -(\epsilon_{\mu 3} + \epsilon_{\varphi 3}) & -(\epsilon_{\tau 2} + \epsilon_{\varphi 2}) \\ \epsilon_{\mu 3}^* + \epsilon_{\varphi 3}^* & 1 & -(\epsilon_{\tau 3} + \epsilon_{\varphi 3}) \\ \epsilon_{\tau 2}^* + \epsilon_{\varphi 2}^* & \epsilon_{\tau 3}^* + \epsilon_{\varphi 3}^* & 1 \end{pmatrix} . \tag{23}$$

As e_L, μ_L and τ_L take different Z_3 charges, their mixing leads to the breaking of the Z_3 symmetry. Neglecting the corrections of U_ν, this can be recast in terms of the mixing angles given by

$$\sin\theta_{13} = \frac{1}{\sqrt{2}}|\epsilon_{\tau 2} - \epsilon_{\mu 3} + \epsilon_{\varphi 2} - \epsilon_{\varphi 3}| ,$$
$$\sin\theta_{12} = \frac{1}{\sqrt{3}}\big[1 - \text{Re}(\epsilon_{\tau 2} + \epsilon_{\mu 3} + \epsilon_{\varphi 2} + \epsilon_{\varphi 3})\big] ,$$
$$\sin\theta_{23} = \frac{1}{\sqrt{2}}\big[1 + \text{Re}(\epsilon_{\tau 3} + \epsilon_{\varphi 3})\big] . \tag{24}$$

The Dirac phase at leading order is given by

$$\delta = -\text{Arg}\big\{\epsilon_{\tau 2} - \epsilon_{\mu 3} + \epsilon_{\varphi 2} - \epsilon_{\varphi 3}\big\} , \tag{25}$$

and the Majorana phases cannot be determined. In a specific model, these corrections may not be independent with each other due to additional assumptions, and sum rules of mixing angles and the Dirac phase could appear.

3 Charged lepton flavour violation in flavour models

We now focus on the analysis of LFV decays of charged leptons mediated by flavons, including 3-body decays $l_1^- \to l_2^+ l_3^- l_4^-$ and radiative decays $l_1^- \to l_2^- \gamma$. All charged LFV processes mediated by flavons in

A_4 flavour models can be divided into two parts: those consistent with the Z_3 residual symmetry and those violating it. The only allowed Z_3-preserving LFV decays are $\tau^- \to \mu^+ e^- e^-$ and $\tau^- \to e^+ \mu^- \mu^-$. We only focus on charged LFV processes induced by flavons. Namely, these LFV processes originate from the couplings between flavons and charged leptons that generate charged lepton masses and give rise to special flavour structures [2].

Flavon fields couple to charged leptons, as shown in Eq. (8) with subleading-order corrections shown in Eq. (13). These couplings are suppressed by charged lepton masses. After the flavour symmetry breaking, the flavon fields gain VEVs, masses and mixing, from the potential $V(\varphi)$. Generically, one can write out the effective operators of the 3-body LFV decay $l_1^- \to l_2^+ l_3^- l_4^-$ after the breakings of the flavour symmetry and the electroweak symmetry as

$$\mathcal{L}^{(6)} = \sum_{P_i} C^{l_4 l_2 l_3 l_1}_{P_4 P_2 P_3 P_1} (\overline{l_{4P_4}} l_{2P_2})(\overline{l_{3P_3}} l_{1P_1}) + C^{l_3 l_2 l_4 l_1}_{P_3 P_2 P_4 P_1} (\overline{l_{3P_3}} l_{2P_2})(\overline{l_{4P_4}} l_{1P_1}) \,, \tag{27}$$

where $P_i = L, R$ (for $i = 1,2,3,4$) and the coefficients $C^{l_4 l_2 l_3 l_1}_{P_4 P_2 P_3 P_1}$, $C^{l_3 l_2 l_4 l_1}_{P_3 P_2 P_4 P_1}$ are functions of charged lepton and flavon mass parameters. Later we will see that due to choices of representations $e_R \sim \mathbf{1}$, $\mu_R \sim \mathbf{1'}$, $\tau_R \sim \mathbf{1''}$ and the large hierarchy $m_e \ll m_\mu \ll m_\tau$, the contribution corresponding to $P_1 = L$ is subleading in the A_4 models and can be neglected in our discussion. Ignoring charged lepton masses in the final states, we derive the decay width of $l_1^- \to l_2^+ l_3^- l_4^-$ as

$$\begin{aligned}\Gamma(l_1^- \to l_2^+ l_3^- l_4^-) \approx \frac{\eta m_{l_1}^5}{3(16\pi)^3} \Big[&\big|C^{l_4 l_2 l_3 l_1}_{LRLR}\big|^2 + \big|C^{l_3 l_2 l_4 l_1}_{LRLR}\big|^2 - \mathrm{Re}\Big(C^{l_4 l_2 l_3 l_1}_{LRLR} (C^{l_3 l_2 l_4 l_1}_{LRLR})^*\Big) \\ &+ \big|C^{l_4 l_2 l_3 l_1}_{RLLR}\big|^2 + \big|C^{l_3 l_2 l_4 l_1}_{RLLR}\big|^2 - \mathrm{Re}\Big(C^{l_4 l_2 l_3 l_1}_{RLLR} (C^{l_3 l_2 l_4 l_1}_{RLLR})^*\Big)\Big] \,,\end{aligned} \tag{28}$$

where $\eta = 1, 2$ for $l_3 = l_4$, $l_3 \neq l_4$, respectively and m_{l_1} is the mass of l_1. As for the radiative decay $l_1^- \to l_2^- \gamma$, its amplitude is generically written as $\overline{u_{l_2}} \Gamma^{l_2 l_1}_\mu u_{l_1} \epsilon^{\mu *}$ [39] with

$$\Gamma^{l_2 l_1}_\mu = i\sigma_{\mu\nu} q^\nu (A^{l_2 l_1}_L P_L + A^{l_2 l_1}_R P_R) \,, \tag{29}$$

where $R_{L,R} = (1 \mp \gamma_5)/2$, and the coefficients $A^{l_2 l_1}_L$, $A^{l_2 l_1}_R$ are dependent upon charged lepton and flavon mass parameters. We mention that also due to choices of representations of e_R, μ_R, τ_R and the large hierarchy of charged lepton masses, $A^{l_2 l_1}_L \ll A^{l_2 l_1}_R$. Thus, the decay rate can be expressed as

$$\Gamma(l_1^- \to l_2^- \gamma) = \frac{m_{l_1}^3}{16\pi} |A^{l_2 l_1}_R|^2 \,. \tag{30}$$

3.1 Z_3-preserving LFV charged lepton decays

From the Lagrangian terms in Eq. (8), we can write the couplings between flavon and charged leptons explicitly. In the Altarelli-Feruglio basis, they are given by

$$\mathcal{L}_l^{\rm eff} = \frac{m_e}{v_\varphi} \left(\overline{e_L} e_R \varphi_1 + \overline{\mu_L} e_R \varphi_2 + \overline{\tau_L} e_R \varphi_3 \right) \sqrt{n}$$

[2]In the most general case, dimension-6 operators

$$\begin{aligned}&\frac{C_1}{\Lambda'^2} (\overline{\ell_L}\gamma_\mu \ell_L)_{\mathbf{1'}} (\overline{\ell_L}\gamma^\mu \ell_L)_{\mathbf{1''}} \,, && \frac{C_2}{\Lambda'^2} \big((\overline{\ell_L}\gamma_\mu \ell_L)_{\mathbf{3}} (\overline{\ell_L}\gamma^\mu \ell_L)_{\mathbf{3}}\big)_{\mathbf{1}} \,, && \frac{C_3}{\Lambda'^2} (\overline{\ell_L}\gamma_\mu \ell_L)_{\mathbf{1''}} (\overline{e_R}\gamma^\mu \mu_R) \,, \\ &\frac{C_4}{\Lambda'^2} (\overline{\ell_L}\gamma_\mu \ell_L)_{\mathbf{1'}} (\overline{e_R}\gamma^\mu \tau_R) \,, && \frac{C_5}{\Lambda'^2} (\overline{e_R}\gamma_\mu \mu_R)(\overline{e_R}\gamma^\mu \tau_R) \,, && \frac{C_6}{\Lambda'^2} (\overline{\mu_R}\gamma_\mu e_R)(\overline{\mu_R}\gamma^\mu \tau_R) \,.\end{aligned} \tag{26}$$

cannot be forbidden by A_4 and the electroweak symmetry, and allow the Z_3-preserving LFV decays. Assuming $C_i \sim \mathcal{O}(1)$, the experimental constraint to the scale Λ' is $\Lambda' > 15$ TeV [38]. However, they are not essential to generate lepton mass matrices with flavour structures, and thus will not be discussed in this paper.

$$
\begin{aligned}
&+ \frac{m_\mu}{v_\varphi} \left(\overline{\mu_L}\mu_R\varphi_1 + \overline{\tau_L}\mu_R\varphi_2 + \overline{e_L}\mu_R\varphi_3\right)\sqrt{n} \\
&+ \frac{m_\tau}{v_\varphi} \left(\overline{\tau_L}\tau_R\,\varphi_1 + \overline{e_L}\tau_R\varphi_2 + \overline{\mu_L}\tau_R\varphi_3\right)\sqrt{n} + \text{h.c.}\,. \qquad (31)
\end{aligned}
$$

The Z_3 symmetry corresponds to the invariance under the transformation

$$
(e_L,\, e_R,\, \varphi_1) \to (e_L,\, e_R,\, \varphi_1)\,, \quad (\mu_L,\, \mu_R,\, \varphi_2) \to \omega^2(\mu_L,\, \mu_R,\, \varphi_2)\,, \quad (\tau_L,\, \tau_R,\, \varphi_3) \to \omega(\tau_L,\, \tau_R,\, \varphi_3)\,. \quad (32)
$$

Namely, e_L, e_R, φ_1 are invariant under Z_3, τ_L, τ_R, φ_3 are covariant under Z_3 with a charge 1, and μ_L, μ_R, φ_2 are covariant under Z_3 with a charge 2 (or equivalently, contravariant under Z_3 with a charge 1). In the case of transfer momentum much lower than the scale of flavour symmetry and flavon masses, one can integrate out φ_1, φ_2, φ_3, and derive the effective 4-fermion interactions. While the Z_3-invariant flavon φ_1 induces flavour-conserving processes, the Z_3-covariant flavons φ_2 and φ_3 are the main sources for charged LFV processes. As e, μ and τ take different Z_3 charges in A_4 models, it is easy to prove that the only allowed processes are $\tau^- \to \mu^+e^-e^-$ and $\tau^- \to e^+\mu^-\mu^-$. The other 3-body decay and all radiative decay modes are forbidden at this level [30].

In case i, we recall that φ is a pseudo-real triplet, $\varphi_1^* = \varphi_1$, $\varphi_2^* = \varphi_3$. As shown in Eq. (19), φ_1 and φ_2 have different masses $m^2_{\varphi_1}$ and $m^2_{\varphi_2}$, and the Z_3 symmetry forbids the mixing between them at leading order. The 4-fermion interactions mediated by φ_1 and φ_2 are given by

$$
\begin{aligned}
\mathcal{L}_{\varphi_1} &= \frac{1}{m^2_{\varphi_1}}\Big[\frac{m_e}{v_\varphi}\overline{e}e + \frac{m_\mu}{v_\varphi}\overline{\mu}\mu + \frac{m_\tau}{v_\varphi}\overline{\tau}\tau\Big]^2\,, \\
\mathcal{L}_{\varphi_2} &= \frac{1}{m^2_{\varphi_2}}\Big[\frac{m_e}{v_\varphi}(\overline{\mu_L}e_R + \overline{e_R}\tau_L) + \frac{m_\mu}{v_\varphi}(\overline{\tau_L}\mu_R + \overline{\mu_R}e_L) + \frac{m_\tau}{v_\varphi}(\overline{e_L}\tau_R + \overline{\tau_R}\mu_L)\Big] \\
&\quad \times\Big[\frac{m_e}{v_\varphi}(\overline{e_R}\mu_L + \overline{\tau_L}e_R) + \frac{m_\mu}{v_\varphi}(\overline{\mu_R}\tau_L + \overline{e_L}\mu_R) + \frac{m_\tau}{v_\varphi}(\overline{\tau_R}e_L + \overline{\mu_L}\tau_R)\Big]\,. \qquad (33)
\end{aligned}
$$

The above Lagrangian terms are compatible with the flavour triality [30], which gives rise to the LFV decay modes $\tau^- \to \mu^+e^-e^-$ and $\tau^- \to e^+\mu^-\mu^-$ and their charged-conjugate processes. These are the only 3-body LFV charged lepton decays allowed by the Z_3 symmetry. In Model I, operators for $\tau^- \to \mu^+e^-e^-$ and $\tau^- \to e^+\mu^-\mu^-$ can be expressed as

$$
\frac{m_\mu m_\tau}{v_\varphi^2 m^2_{\varphi_2}}(\overline{e_L}\mu_R)(\overline{e_L}\tau_R)\,, \quad \frac{m_\mu m_\tau}{v_\varphi^2 m^2_{\varphi_2}}(\overline{\mu_R}e_L)(\overline{\mu_L}\tau_R)\,, \qquad (34)
$$

respectively, with both coefficients $C^{e\mu e\tau}_{LRLR}$ and $C^{\mu e\mu\tau}_{RLLR}$ suppressed by μ and τ masses. We just list the leading contribution here. Terms such as $(\overline{e_R}\mu_L)(\overline{e_L}\tau_R)$ and $(\overline{\mu_R}e_L)(\overline{\mu_R}\tau_L)$ are also allowed, but sub-leading, suppressed by $\frac{m_e m_\tau}{v_\varphi^2 m^2_{\varphi_2}}$ or $\frac{m_\mu^2}{v_\varphi^2 m^2_{\varphi_2}}$, and will not be considered in the following. Then, we get approximatively equal branching ratios of these two processes

$$
\text{Br}(\tau^- \to \mu^+e^-e^-) \approx \text{Br}(\tau^- \to e^+\mu^-\mu^-)\,, \qquad (35)
$$

both suppressed by $\big(\frac{m_\mu m_\tau v^2}{m^2_{\varphi_2} v_\varphi^2}\big)^2$. If we assume the scale of flavour symmetry v_φ and flavon masses to be around the electroweak scale $v = 246$ GeV, the branching ratios will be smaller than 10^{-11}. One highlighted feature is that both Z_3-preserving processes have the same branching ratios. This is because $\overline{e_L}\tau_R\varphi_2$ and $\overline{\mu_L}\tau_R\varphi_3$ have the same coefficient as shown in Eq. (31), and φ_2 is the complex conjugate of φ_3. Essentially, it is a consequence of the Z_3 symmetry remaining from the breaking of A_4 and the economical choice of the pseudo-real presentation of φ.

In case ii, as φ is a complex triplet of A_4, the mixing between the Z_3-covariant flavons φ_2 and φ_3^* should be considered in the Z_3-preserving processes. As shown in Eq. (20), the Z_3 symmetry cannot forbid the off-diagonal mass term $m^2_{\varphi_2\varphi_3}(\varphi_2\varphi_3 + \text{h.c.})$. We need to go into the mass basis φ_2' and φ_3' with mass eigenvalues $m_{\varphi_2'}$ and $m_{\varphi_3'}$, thanks to the rotation:

$$\begin{pmatrix} \varphi_2' \\ \varphi_3'^* \end{pmatrix} = \begin{pmatrix} c_\vartheta & -s_\vartheta \\ s_\vartheta & c_\vartheta \end{pmatrix} \begin{pmatrix} \varphi_2 \\ \varphi_3^* \end{pmatrix} , \tag{36}$$

with $s_\vartheta \equiv \sin\vartheta$, $c_\vartheta \equiv \cos\vartheta$ and $\tan 2\vartheta = 2m^2_{\varphi_2\varphi_3}/(m^2_{\varphi_3} - m^2_{\varphi_2})$, $-45^\circ < \vartheta \leqslant 45^\circ$. Then, integrating the massive scalars out, we obtain the coefficients of the 4-fermion processes $\tau^- \to \mu^+ e^- e^-$ and $\tau^- \to e^+ \mu^- \mu^-$ as

$$C^{e\mu e\tau}_{LRLR} = \frac{m_\mu m_\tau}{v_\varphi^2} \left(\frac{\sin 2\vartheta}{m^2_{\varphi_3'}} - \frac{\sin 2\vartheta}{m^2_{\varphi_2'}} \right) , \qquad C^{\mu e\mu\tau}_{RLLR} = \frac{m_\mu m_\tau}{v_\varphi^2} \left(\frac{2s^2_\vartheta}{m^2_{\varphi_2'}} + \frac{2c^2_\vartheta}{m^2_{\varphi_3'}} \right) , \tag{37}$$

respectively. Compared with case i, where the coefficients of $(\overline{e_L}\mu_R)(\overline{e_L}\tau_R)$ and $(\overline{\mu_R}e_L)(\overline{\mu_L}\tau_R)$ are the same, the coefficients in case ii are in general different. This feature could be used to establish if φ is in a pseudo-real or complex representation if future experiments observe the signatures. We emphasise that since the mixing between φ_2 and φ_3^* is in general large, even maximal, branching ratios for both channels should be at the same level, both proportional to $m_\mu m_\tau$. The above equation reduces to Eq. (34) in the limit $m^2_{\varphi_2'} \to \infty$ and $\vartheta \to 45^\circ$, or $m^2_{\varphi_3'} \to \infty$ and $\vartheta \to -45^\circ$. Only in the limit $\vartheta \to 0$, $\tau^- \to \mu^+ e^- e^-$ is suppressed, as discussed in [32].

3.2 Z_3-breaking LFV charged lepton decays

The Z_3-breaking LFV processes have three sources, depending on their connection with flavour mixing. One is the mixing of charged lepton mass eigenstates, characterised by $\epsilon_{\mu 3} + \epsilon_{\varphi 3}$, $\epsilon_{\tau 2} + \epsilon_{\varphi 2}$ and $\epsilon_{\tau 3} + \epsilon_{\varphi 3}$ in the last section. The left-handed charged lepton mass eigenstates are superpositions of e_L, μ_L and τ_L, which obviously break the Z_3 symmetry:

$$\begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} \to U_l \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} . \tag{38}$$

There is also mixing of the right-handed charged leptons e_R, μ_R and τ_R, but their mixing is suppressed by both ϵ_φ and the hierarchy of charged lepton masses [3], and thus can be safely neglected. Charged LFV processes induced by this effect is easy to be calculated.

The other two are related to the Z_3-breaking property of the flavon triplet φ. When the vacuum shift results in the mixing between charged leptons, it also results in the mixing and mass corrections of different components of φ. The mixing between the Z_3-invariant flavon φ_1 and the Z_3-covariant flavons φ_2 and φ_3, and the mass splitting of the two real degrees of freedom for each Z_3-covariant complex flavons contribute to LFV processes.

To calculate the mass corrections to and the mixing of φ, we expand the Z_3-preserving potential $V_0(\varphi)$ to the third order and the Z_3-breaking potential $V_1(\varphi)$ to the second order around the Z_3-invariant

[3] This is due to the typical feature in A_4 models that e_R, μ_R and τ_R are always arranged as singlets $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ of A_4.

VEV $\langle\varphi\rangle = (1,0,0)^T v_\varphi$. Taking a pseudo-real flavon triplet as an example, the Z_3-preserving terms are expressed as

$$V_0^{(3)}(\varphi) = \frac{1}{3}k_1 v_\varphi \varphi_1^3 + k_2 v_\varphi \varphi_1 \varphi_2^* \varphi_2 + \frac{1}{3} v_\varphi (k_3 \varphi_2^3 + k_3^* \varphi_2^{*3}) \,. \tag{39}$$

Here, k_1, k_2 are real, required by the Hermiticity of the potential. If the potential is renormalisable, k_3 is also real, since all the renormalisable A_4-invariant combinations of φ, including $(\varphi\varphi)_{\mathbf{1}}$, $\big((\varphi\varphi)_{\mathbf{3}_S}\varphi\big)_{\mathbf{1}}$, $(\varphi\varphi)_{\mathbf{1}'}(\varphi\varphi)_{\mathbf{1}''}$, $\big((\varphi\varphi)_{\mathbf{3}_S}(\varphi\varphi)_{\mathbf{3}_S}\big)_{\mathbf{1}}$, are real. Once higher-dimensional operators are included in the potential, e.g., $\lambda\big((\varphi\varphi)_{\mathbf{1}'}\big)^3 + \lambda^*\big((\varphi\varphi)_{\mathbf{1}''}\big)^3$, k_3 can be complex. Since the vacuum is shifted, a small mass term which splits the masses of two components of the complex φ_2 and a mixing term between φ_1 and φ_2 can be generated by the cubic couplings proportional to k_2 and k_3 in Eq. (39), respectively. After the flavon VEV shifts from $\langle\varphi\rangle = (1,0,0)^T v_\varphi$ to $\langle\varphi\rangle = (1,\epsilon_\varphi,\epsilon_\varphi^*)^T v_\varphi$, the cubic term $V_0^{(3)}(\varphi)$ will contribute a small mass term for the flavons

$$\delta V_0^{(2)} = \epsilon_\varphi v_\varphi^2 (k_2 \varphi_1 \varphi_2^* + k_3 \varphi_2^2) + \text{h.c.} \,. \tag{40}$$

The quadratic terms of $V_1(\varphi)$ can also generate such mass terms. In general, they are given by

$$V_1^{(2)}(\varphi) = \varepsilon_2 v_\varphi^2 \varphi_1 \varphi_2^* + \varepsilon_3 v_\varphi^2 \varphi_2^2 + \text{h.c.} \,, \tag{41}$$

where ε_2 and ε_3 are in general complex parameters. $V_1^{(2)}(\varphi)$ obviously can originate from cross couplings between φ and χ, including the renormalisable terms, i.e., $(\varphi\chi)_{\mathbf{1}}$, $\big((\varphi\varphi)_{\mathbf{3}_S}\chi\big)_{\mathbf{1}}$, $(\varphi\varphi)_{\mathbf{1}''}(\chi\chi)_{\mathbf{1}'}$, and the higher-dimensional terms, i.e., $\Big(\big((\varphi\varphi)_{\mathbf{3}_S}(\varphi\varphi)_{\mathbf{3}_S}\big)_{\mathbf{3_S}}\varphi\Big)_{\mathbf{1}}$, $[(\varphi\varphi)_{\mathbf{1}'}]^2(\chi\chi)_{\mathbf{1}'}$. The ε_2 and ε_3 terms can give another mass term, splitting masses of the two components of φ_2 and mixing of φ_1 and φ_2, respectively.

We include the contributions from $\delta V_0^{(2)}(\varphi)$ and $V_1^{(2)}(\varphi)$, as well as the leading order result $V_0^{(2)}(\varphi)$. The two Z_3-breaking effects of φ mentioned above can be discussed analytically. The first one is the mixing between the Z_3-invariant (φ_1) and Z_3-covariant flavons (φ_2). To rotate it to the mass eigenstates of flavons, we need do the following transformation:

$$\begin{aligned} \varphi_1 &\to \varphi_1 + \left(\epsilon_{\varphi_1\varphi_2}^* \varphi_2 + \epsilon_{\varphi_1\varphi_2} \varphi_2^*\right) \,, \\ \varphi_2 &\to \varphi_2 - \epsilon_{\varphi_1\varphi_2} \varphi_1 \,, \end{aligned} \tag{42}$$

where

$$\epsilon_{\varphi_1\varphi_2} = \frac{(k_2\epsilon_\varphi + \varepsilon_2) v_\varphi^2}{m_{\varphi_2}^2 - m_{\varphi_1}^2} \,. \tag{43}$$

The second one is the mass splitting of the two components of the complex scalar φ_2:

$$m_{h_2,a_2}^2 = m_{\varphi_2}^2 (1 \pm 2|\epsilon_{h_2a_2}|) \,. \tag{44}$$

with the two components h_2 and a_2 defined by

$$\begin{aligned} h_2 &= \frac{1}{\sqrt{2}}\Big(\varphi_2 \exp(i\frac{\theta_{h_2a_2}}{2}) + \varphi_2^* \exp(-i\frac{\theta_{h_2a_2}}{2})\Big) \,, \\ a_2 &= \frac{-i}{\sqrt{2}}\Big(\varphi_2 \exp(i\frac{\theta_{h_2a_2}}{2}) - \varphi_2^* \exp(-i\frac{\theta_{h_2a_2}}{2})\Big) \,, \end{aligned} \tag{45}$$

$\epsilon_{h_2a_2} = k_3\epsilon_\varphi + \varepsilon_3$ and $\theta_{h_2a_2}$ being the phase of $\epsilon_{h_2a_2}$.

Branching ratios for the Z_3-breaking charged LFV processes can be derived after including the three Z_3-breaking sources: the correction to the mixing of charged leptons, the correction to the mixing of flavons and flavon mass splitting. For the pseudo-real triplet flavon in case i, these three effects are analytically listed in Eqs. (38), (42) and (44). For the complex triplet flavon in case ii, the flavon potential is more complicated and it is hard to extract the general analytical expressions for the later two Z_3-breaking sources. By imposing additional assumptions, e.g., some Abelian symmetries, can simplify the flavon potential and also the Z_3-breaking sources. This scenario will be discussed in the second model in the next section.

4 Z_3-breaking LFV charged lepton decays in concrete models

In this section, we will study Z_3-breaking LFV charged lepton decays in two models. These models have been proposed and the phenomenology of their flavour mixing has been studied in detail in Ref. [25]. After a brief introduction of these models, we will calculate contributions of flavons to the Z_3-breaking processes and discuss the experimental constraints. For more details about the model constructions and properties of flavour mixings in both models, we refer to [25].

4.1 Model constructions

The flavour symmetry is assumed to be $A_4 \times Z_2^\varphi \times Z_4^\chi$ in both models, with field contents listed in Table 1. One right-handed neutrino triplet N and two flavon triplets φ, χ and one flavon singlet η of A_4 are introduced. The singlet η is used to obtain the correct neutrino mass spectrum. The only difference between these two models is that the flavon multiplets in Model I are pseudo-real representations of A_4 (case i), while all flavon multiplets in Model II are in complex representations (case ii).

Fields	ℓ_L	e_R, μ_R, τ_R	N	H	φ	χ	η
A_4	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \mathbf{1}''$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$
Z_2^φ	1	-1	1	1	-1	1	1
Z_4^χ	i	i	i	1	1	-1	-1

Table 1: Transformation properties of fields in the flavour symmetry $A_4 \times Z_2^\varphi \times Z_4^\chi$.

Model I

The most general renormalisable flavon potential of φ invariant under the symmetry is written as

$$V(\varphi) = \frac{1}{2}\mu_\varphi^2(\varphi\varphi)_{\mathbf{1}} + \frac{1}{4}\left[f_1\big((\varphi\varphi)_{\mathbf{1}}\big)^2 + f_2(\varphi\varphi)_{\mathbf{1}'}(\varphi\varphi)_{\mathbf{1}''} + f_3\big((\varphi\varphi)_{\mathbf{3}_S}(\varphi\varphi)_{\mathbf{3}_S}\big)_{\mathbf{1}}\right], \tag{46}$$

in which all the coefficients μ_φ^2 and $f_{1,2,3}$ are real. Once the relations $\mu_\varphi^2 < 0$ and $f_2 > f_3 > -f_1$ are required, we can derive the VEV $\langle\varphi\rangle$ in Eq. (7). The masses of φ_1 and φ_2 are given by

$$m_{\varphi_1}^2 = 2(f_1 + f_3)v_\varphi^2\,, \quad m_{\varphi_2}^2 = (f_2 - f_3)v_\varphi^2\,. \tag{47}$$

With replacements $\varphi \to \chi$ and $f_i \to g_i$ in Eq. (46), we get the potential of χ. By assuming a different relation $g_3 > g_2 > -g_1$, we obtain the VEV $\langle\chi\rangle$ in Eq. (7). All the cross couplings between φ and χ are expressed as

$$V(\varphi,\chi) = \frac{1}{2}\epsilon_1(\varphi\varphi)_{\mathbf{1}}(\chi\chi)_{\mathbf{1}} + \frac{1}{4}\big[\epsilon_2(\varphi\varphi)_{\mathbf{1}''}(\chi\chi)_{\mathbf{1}'} + \text{h.c.}\big] + \frac{1}{2}\epsilon_3\big((\varphi\varphi)_{\mathbf{3}_S}(\chi\chi)_{\mathbf{3}_S}\big)_{\mathbf{1}}\,, \tag{48}$$

where ϵ_1 and ϵ_3 are real and ϵ_2 is complex, and the ϵ_2 term is the only term that will break the Z_3 residual symmetry at first order. The relation of ϵ_2 and ε_1 in Eq. (19) is given by $\varepsilon_1 = \frac{1}{2}\epsilon_2 v_\chi^2/v_\varphi^2$. The ϵ_3 term will contribute to the breaking of Z_2 in $\langle\chi\rangle$. By assuming the VEV v_χ to be significantly higher than v_φ, e.g., $|v_\chi| \gtrsim 2|v_\varphi|$, this contribution can be neglected.

The Lagrangian generating lepton masses is the same as in Eq. (8). Here, we assume the contribution from higher-dimensional operators to be negligible and all the correction to TBM come from one single complex parameter ϵ_φ. In this case, the mixing parameters are simplified to [25]

$$\begin{aligned}
\sin\theta_{13} &= \sqrt{2}|\epsilon_\varphi \sin\theta_\varphi|\,,\\
\sin\theta_{12} &= \frac{1}{\sqrt{3}}\big(1-2|\epsilon_\varphi|\cos\theta_\varphi\big)\,,\\
\sin\theta_{23} &= \frac{1}{\sqrt{2}}\big(1+|\epsilon_\varphi|\cos\theta_\varphi\big)\,,\\
\delta &= \begin{cases} 270^\circ - 2|\epsilon_\varphi|\sin\theta_\varphi\,, & \theta_\varphi > 0\,,\\ 90^\circ - 2|\epsilon_\varphi|\sin\theta_\varphi\,, & \theta_\varphi < 0\,. \end{cases}
\end{aligned} \tag{49}$$

From the above expression, we see that both θ_{13} and δ originate from the same source, the imaginary part of ϵ_2, and almost-maximal CP violation is predicted, with $\delta + \sqrt{2}\theta_{13} \approx 90^\circ, 270^\circ$. In addition, there are sum rules of mixing angles

$$r^2 + s^2 = 4|\epsilon_\varphi|^2\,, \quad s + 2a = 0\,. \tag{50}$$

As shown in Ref. [25], this scenario is compatible with current neutrino oscillation data in the case $r \gg s$, which results in $|\epsilon_\varphi| \approx r/2 \approx 0.1$, or equivalently, $|\epsilon_\varphi| \approx \theta_{13}/\sqrt{2}$.

Model II

In this model, the flavon multiplets are in complex representations. The potential for φ is altered to

$$\begin{aligned}
V(\varphi) = {}& \mu_\varphi^2(\tilde\varphi\varphi)_{\mathbf{1}} + f_1\big((\tilde\varphi\varphi)_{\mathbf{1}}\big)^2 + f_2(\tilde\varphi\varphi)_{\mathbf{1}'}(\tilde\varphi\varphi)_{\mathbf{1}''} + f_3\big((\tilde\varphi\varphi)_{\mathbf{3}_S}(\tilde\varphi\varphi)_{\mathbf{3}_S}\big)_{\mathbf{1}}\\
&+ f_4\big((\tilde\varphi\varphi)_{\mathbf{3}_A}(\tilde\varphi\varphi)_{\mathbf{3}_A}\big)_{\mathbf{1}} + f_5\big((\tilde\varphi\varphi)_{\mathbf{3}_S}(\tilde\varphi\varphi)_{\mathbf{3}_A}\big)_{\mathbf{1}}\,,
\end{aligned} \tag{51}$$

where f_i are real and $\tilde\varphi = (\varphi_1^*, \varphi_3^*, \varphi_2^*)$ also transforms as a $\mathbf{3}$ of A_4. Terms related to the antisymmetric combination $(\tilde\varphi\varphi)_{\mathbf{3}_A}$ are included due to the complex property of φ. After φ gets the VEV $\langle\varphi\rangle = (1,0,0)^T v_\varphi/\sqrt{2}$ and A_4 is broken to Z_3, φ_1, φ_2 and φ_3 get masses with mass eigenvalues

$$m_{\varphi_1}^2 = 2(f_1+f_3)v_\varphi^2\,, \quad m_{\varphi_2'}^2, m_{\varphi_3'}^2 = \left(2f_2 - 5f_3 + f_4 \pm \sqrt{(2f_2+f_3-f_4)^2 + 4f_5^2}\right)\frac{v_\varphi^2}{4}\,. \tag{52}$$

The mixing angle ϑ is given by $\cot 2\vartheta = 2f_5/(2f_2+f_3-f_4)$ [4]. The potential of χ can be obtained with the replacements $\varphi \to \chi$, and $f_i \to g_i$ in Eq. (51). In order to achieve the successful breaking of

[4] As mentioned in the last section, the convention $-45^\circ \leqslant \vartheta \leqslant 45^\circ$ is used. This is equivalent to that in [25], in which both $-90^\circ \leqslant \vartheta \leqslant 90^\circ$ and the mass ordering $m_{\varphi_2'} \leqslant m_{\varphi_3'}$ are required.

$A_4 \to Z_3$ and Z_2 in charged lepton and neutrino sectors, respectively, the following conditions must be satisfied:

$$f_1 + f_3 > 0\,, \quad 2f_2 - 5f_3 + f_4 > 0\,, \quad 2(f_2 - f_3)(f_4 - 3f_3) - f_5^2 > 0\,,$$
$$g_1 + g_2 > 0\,, \quad 3g_3 - 6g_2 + g_4 > 0\,, \quad 4(g_2 - g_3)(3g_2 - g_4) - g_5^2 > 0\,. \tag{53}$$

The cross couplings between φ and χ are given by

$$\begin{aligned} V(\varphi,\chi) =\ & 2\epsilon_1(\tilde{\varphi}\varphi)_{\mathbf{1}}(\tilde{\chi}\chi)_{\mathbf{1}} + \left[\epsilon_2(\tilde{\varphi}\varphi)_{\mathbf{1}''}(\tilde{\chi}\chi)_{\mathbf{1}'} + \text{h.c.}\right] + 2\epsilon_3\big((\tilde{\varphi}\varphi)_{\mathbf{3}_S}(\tilde{\chi}\chi)_{\mathbf{3}_S}\big)_{\mathbf{1}} \\ & +2\epsilon_4\big((\tilde{\varphi}\varphi)_{\mathbf{3}_A}(\tilde{\chi}\chi)_{\mathbf{3}_A}\big)_{\mathbf{1}} + 2\epsilon_5\big((\tilde{\varphi}\varphi)_{\mathbf{3}_S}(\tilde{\chi}\chi)_{\mathbf{3}_A}\big)_{\mathbf{1}} + 2\epsilon_6\big((\tilde{\varphi}\varphi)_{\mathbf{3}_A}(\tilde{\chi}\chi)_{\mathbf{3}_S}\big)_{\mathbf{1}}\,, \end{aligned} \tag{54}$$

in which ϵ_2 is complex and ϵ_1, ϵ_3, ϵ_4, ϵ_5 and ϵ_6 are real parameters, which we assume to be small. Here, the ϵ_2 term is the only one that modifies the VEV of φ at first order. Taking into account its contribution of this term, we finally obtain the corrections to the VEVs, see Eq. (18),

$$\epsilon_{\varphi 2} = (1-\kappa)\epsilon_\varphi\,, \qquad \epsilon_{\varphi 3} = (1+\kappa)\epsilon_\varphi^* \tag{55}$$

with

$$\epsilon_\varphi = -\frac{\epsilon_2 v_\chi^2\big[(m^2_{\varphi'_3} + m^2_{\varphi'_2}) - (m^2_{\varphi'_3} - m^2_{\varphi'_2})\sin 2\vartheta\big]}{4m^2_{\varphi'_3}m^2_{\varphi'_2}}\,, \quad \kappa = \frac{(m^2_{\varphi'_3} - m^2_{\varphi'_2})\cos 2\vartheta}{(m^2_{\varphi'_3} + m^2_{\varphi'_2}) - (m^2_{\varphi'_3} - m^2_{\varphi'_2})\sin 2\vartheta}\,. \tag{56}$$

When the mixing between φ_2 and φ_3 is maximal, $\sin 2\vartheta = \pm 1$, κ vanishes, and we get the same structure of the VEV shift as in Model I. Furthermore, ϵ_φ takes the value $-\epsilon_2 v_\chi^2/(2m^2_{\varphi'_3})$ and $-\epsilon_2 v_\chi^2/(2m^2_{\varphi'_2})$ for $\vartheta = 45^\circ$ and -45°, respectively.

The Lagrangian terms led to lepton masses is also Eq. (8), the same as Model I. Since the κ-related asymmetric correction is included in the VEV $\langle\varphi\rangle$, the expressions for the mixing parameters are modified, approximating to [25]

$$\begin{aligned} \sin\theta_{13} &= |\epsilon_\varphi|\sqrt{2\kappa^2\cos^2\theta_\varphi + 2\sin^2\theta_\varphi}\,, \\ \sin\theta_{12} &= \frac{1}{\sqrt{3}}\big[1 - 2|\epsilon_\varphi|\cos\theta_\varphi\big]\,, \\ \sin\theta_{23} &= \frac{1}{\sqrt{2}}\big[1 + (1+\kappa)|\epsilon_\varphi|\cos\theta_\varphi\big]\,, \\ \delta &= \text{Arg}\Big\{\Big[-i\sin\theta_\varphi - \kappa\cos\theta_\varphi\Big]\Big[1 - i|\epsilon_\varphi|(2+\kappa)\sin\theta_\varphi\Big]\Big\}\,. \end{aligned} \tag{57}$$

The combination of κ and the real part of ϵ_2 provides new sources for θ_{13} and CP violation. It can induce sizable θ_{13} while not affecting CP conservation in some specific region of the parameter space [25]. In the case of maximal mixing between φ_2 and φ_3, $\cos 2\vartheta = 0$, leading to $\kappa = 0$, we recover $\theta_{13} = \sqrt{2}|\epsilon_\varphi \sin\theta_\varphi|$ and nearly maximal Dirac-type CP violation, the same result as in Model I.

4.2 Z_3-breaking LFV charged lepton decays in Model I

We now discuss the Z_3-breaking charged LFV processes induced by the cross couplings between φ and χ in Model I. The mixing and mass splitting of different components of φ induced by ϵ_2 can be expressed in terms of $\epsilon_{\varphi_1\varphi_2} = c\epsilon_\varphi$ and $\epsilon_{h_2a_2} = \epsilon_\varphi$ with $c = (m^2_{\varphi_2} + m^2_{\varphi_1})/(m^2_{\varphi_2} - m^2_{\varphi_1})$. As all the Z_3-breaking effects are essentially dependent upon ϵ_φ, we expect all the LFV charged lepton decay modes to be suppressed by $|\epsilon_\varphi|^2$, or equivalently, suppressed by $r^2 + s^2 = 2s_{13}^2 + (1-\sqrt{3}s_{12})^2$.

The effective 4-fermion operators for $\tau^- \to \mu^+\mu^- e^-$ after integrating out φ_1 and φ_2 are given by

$$-2\epsilon_\varphi \frac{m_\mu m_\tau}{v_\varphi^2}\left[\frac{1}{m_{\varphi_1}^2}(\overline{\mu}\mu)(\overline{e_L}\tau_R) + \frac{1}{m_{\varphi_2}^2}(\overline{\mu_R}\mu_L)(\overline{e_L}\tau_R) + \frac{1}{m_{\varphi_2}^2}(\overline{e_L}\mu_R)(\overline{\mu_L}\tau_R)\right]. \tag{58}$$

Here, we have considered all contributions of order $\epsilon_\varphi m_\mu m_\tau / v_\varphi^4$. They include those due to the mixing of left-handed charged leptons e_L, μ_L, τ_L and to the mixing between flavons φ_1 and φ_2. The effective 4-fermion interaction relevant for $\tau^- \to \mu^+\mu^-\mu^-$ and $\tau^- \to e^+e^-\mu^-$ are given by

$$\begin{aligned}
&-2\epsilon_\varphi^* \frac{m_\mu m_\tau}{v_\varphi^2}\left[\frac{1}{m_{\varphi_1}^2}(\overline{\mu}\mu)(\overline{\mu_L}\tau_R) + \frac{1}{m_{\varphi_2}^2}(\overline{\mu_R}\mu_L)(\overline{\mu_L}\tau_R)\right], \\
&-2\epsilon_\varphi^* \frac{m_\mu m_\tau}{v_\varphi^2}\frac{1}{m_{\varphi_2}^2}(\overline{\mu_R}e_L)(\overline{e_L}\tau_R),
\end{aligned} \tag{59}$$

respectively. From Eqs. (58) and (59), we obtain simple relations of the non-zero effective coefficients of the 4-fermion interactions of these processes

$$\begin{aligned}
(C_{LRLR}^{\mu\mu\mu\tau})^* &= C_{LRLR}^{\mu\mu e\tau} = -2\epsilon_\varphi \frac{m_\mu m_\tau}{v_\varphi^2 m_{\varphi_1}^2}, \\
(C_{RLLR}^{\mu ee\tau})^* &= C_{LRLR}^{e\mu\mu\tau} = -2\epsilon_\varphi \frac{m_\mu m_\tau}{v_\varphi^2 m_{\varphi_2}^2}, \\
(C_{RLLR}^{\mu\mu\mu\tau})^* &= C_{RLLR}^{\mu\mu e\tau} = (C_{LRLR}^{\mu\mu\mu\tau})^* + (C_{RLLR}^{\mu ee\tau})^*,
\end{aligned} \tag{60}$$

and the other coefficients will not contribute to the decays at leading order. These simple relations result in sum rules for the branching ratios of the τ LFV decays

$$\begin{aligned}
2(B_{\mu^+\mu^-e^-} - 2B_{\mu^+\mu^-\mu^-})^2 + (5B_{e^+e^-\mu^-} + 10B_{\mu^+\mu^-\mu^-} - 6B_{\mu^+\mu^-e^-})B_{e^+e^-\mu^-} &= 0, \\
B_{e^+e^-\mu^-} &\approx 2(r^2+s^2)\mathrm{Br}(\tau^- \to \mu^+e^-e^-),
\end{aligned} \tag{61}$$

where $B_{\mu^+\mu^-e^-}$, $B_{\mu^+\mu^-\mu^-}$, $B_{e^+e^-\mu^-}$ are branching ratios of $\tau^- \to \mu^+\mu^-e^-$, $\tau^- \to \mu^+\mu^-\mu^-$ and $\tau^- \to e^+e^-\mu^-$, respectively. In the limit $m_{\varphi_1} \ll m_{\varphi_2}$, we get $B_{\mu^+\mu^-e^-} \approx 2B_{\mu^+\mu^-\mu^-} \gg B_{e^+e^-\mu^-}$, and on the contrary, we have $B_{\mu^+\mu^-e^-} \approx 4B_{\mu^+\mu^-\mu^-} \approx 2B_{e^+e^-\mu^-}$.

For the processes $\tau^- \to e^+e^-e^-$ and $\mu^- \to e^+e^-e^-$, coefficients for the related operators are

$$\begin{aligned}
C_{LRLR}^{eee\tau} = C_{RLLR}^{eee\tau} &= -2\epsilon_\varphi \frac{m_e m_\tau}{v_\varphi^2}\left(\frac{1}{m_{\varphi_1}^2} + \frac{1}{m_{\varphi_2}^2}\right), \\
C_{LRLR}^{eee\mu} = C_{RLLR}^{eee\mu} &= -2\epsilon_\varphi^* \frac{m_e m_\mu}{v_\varphi^2}\left(\frac{1}{m_{\varphi_1}^2} + \frac{1}{m_{\varphi_2}^2}\right),
\end{aligned} \tag{62}$$

both suppressed by the electron mass. Taking $v_\varphi \sim m_{\varphi_1} \sim m_{\varphi_2}$ around the electroweak scale, we obtain branching ratios smaller than 10^{-16} and 10^{-18}, respectively, far below the current experimental upper limit. The Z_3-breaking terms will also contribute to the channels $\tau^- \to \mu^+e^-e^-$ and $\tau^- \to e^+\mu^-\mu^-$. They are subleading, generically suppressed by ϵ_φ, and will not be considered here.

The radiative decay $l_1^- \to l_2^-\gamma$ is allowed after the flavon cross couplings are included. We calculate them in detail in the Appendix and we show the results here. For $\tau^- \to e^-\gamma$, there are two main contributions: one is due to the mixing between e_L and τ_L and the other to the mixing between Z_3-invariant and Z_3-covariant flavons φ_1 and φ_2. After the transformation in Eqs. (38) and (42), we obtain the corresponding operators [5]:

$$\frac{m_\tau}{v_\varphi}\left[\overline{e_L}\tau_R\left(\varphi_2 - \tilde{\epsilon}_\varphi\varphi_1\right) + \overline{\tau}\tau\left(\varphi_1 + \tilde{\epsilon}_\varphi\varphi_2^*\right)\right], \tag{63}$$

[5] We just keep the terms directly to the process. Some Hermitian terms not relevant to $l_1^- \to l_2^-\gamma$ are not given here.

where $\tilde{\epsilon}_\varphi = (1+c)\epsilon_\varphi = 2\epsilon_\varphi m_{\varphi_2}^2/(m_{\varphi_2}^2 - m_{\varphi_1}^2)$. For $\tau^- \to \mu^-\gamma$, the main sources are the mixing between μ_L and τ_L and the one between φ_1 and φ_2. Operators related to these contributions are

$$\frac{m_\tau}{v_\varphi}\left[\overline{\mu_L}\tau_R\left(\varphi_2^* - \tilde{\epsilon}_\varphi^*\varphi_1\right) + \overline{\tau}\tau\left(\varphi_1 + \tilde{\epsilon}_\varphi^*\varphi_2\right)\right] \tag{64}$$

For $\mu^- \to e^-\gamma$, the main contributions come from the mixing between μ_L and τ_L, the mixing between e_L and τ_L, and the mass splitting between h_2 and a_2. φ_1 is not involved in this channel. After going into the charged lepton mass basis and replacing φ_2 by $e^{-i\theta_\varphi/2}(h_2 + ia_2)/\sqrt{2}$ in Eq. (72), the relevant operators are

$$\begin{aligned} &\frac{e^{-i\frac{\theta_\varphi}{2}}}{\sqrt{2}}\frac{m_\tau}{v_\varphi}\overline{e_L}\tau_R\Big[\left(1-|\epsilon_\varphi|\right)h_2 + i\left(1+|\epsilon_\varphi|\right)a_2\Big] \\ +\; &\frac{e^{-i\frac{\theta_\varphi}{2}}}{\sqrt{2}}\left[\frac{m_\tau}{v_\varphi}\overline{\tau_R}\mu_L + \frac{m_\mu}{v_\varphi}\overline{\tau_L}\mu_R\right]\Big[\left(1+|\epsilon_\varphi|\right)h_2 + i\left(1-|\epsilon_\varphi|\right)a_2\Big]. \end{aligned} \tag{65}$$

In the above equation, we keep the term $\overline{\tau_R}\mu_L$ at the m_τ/v_φ level but keep $\overline{\tau_L}\mu_R$ at the m_μ/v_φ one. The reason is that the former contribution is proportional to m_μ and the latter contribution is proportional to m_τ, as shown in Eq. (79) of the Appendix.

Starting from the effective couplings in Eqs. (63), (64) and (65), and following the calculation in the Appendix, we finally arrive at

$$\begin{aligned} A_R^{e\tau} &= \frac{ie\tilde{\epsilon}_\varphi m_\tau}{(4\pi)^2 v_\varphi^2}\left[F(\varphi_2) - F(\varphi_1)\right], \\ A_R^{\mu\tau} &= \frac{ie\tilde{\epsilon}_\varphi^* m_\tau}{(4\pi)^2 v_\varphi^2}\left[F(\varphi_2) - F(\varphi_1)\right], \\ A_R^{e\mu} &= \frac{-2ie\epsilon_\varphi^* m_\mu}{(4\pi)^2 v_\varphi^2}\frac{m_\tau^2}{m_{\varphi_2}^2}\Big(\log\frac{m_\tau^2}{m_{\varphi_2}^2} + \frac{7}{3}\Big), \end{aligned} \tag{66}$$

where

$$F(\varphi_i) = \frac{m_\tau^2}{m_{\varphi_i}^2}\Big(\log\frac{m_\tau^2}{m_{\varphi_i}^2} + \frac{4}{3}\Big). \tag{67}$$

We emphasise that we have taken account of all three contributions from the mixing of charged leptons, the mixing between Z_3-invariant and Z_3-covariant flavons φ_1 and φ_2, and the mass splitting between two components of the Z_3-covariant flavon φ_2, as shown in Eqs. (38), (42), and (44), respectively. As mentioned in the last subsection, we prove in the Appendix that $A_L^{l_2 l_1} \ll A_R^{l_2 l_1}$ is satisfied in all three channels, as imposed by the structure of charged lepton mass matrix. Here, one more sum rule for branching ratios is satisfied:

$$\mathrm{Br}(\tau^- \to e^-\gamma) \approx \mathrm{Br}(\tau^- \to \mu^-\gamma). \tag{68}$$

Taken the scale of the flavour symmetry to be at the electroweak scale, the branching ratio of $\tau^- \to e^-\gamma$ and $\tau^- \to \mu^-\gamma$ are in general less than 10^{-11}, at least 3 orders of magnitude below the current experimental upper limits. The $\mu^- \to e^-\gamma$ channel has been measured most precisely and gives the strongest constraint on the model. In Fig. 2, we show regions of v_φ and m_{φ_2} allowed by current experiments and testable at the near future experiments. The current upper limit of the branching ratio is 4.2×10^{-13}, measured by the MEG experiment [36]. By fixing the flavon VEV $v_\varphi = v/\sqrt{2}$, v, $2v =$

175, 246, 492 GeV, we obtain the lower limit of the φ_2 mass: $m_{\varphi_2} > 700, 500, 200$ GeV, respectively. The MEG II experiment will reach the upper limit of the branching ratio to 4×10^{-14} in the near future [37]. This experiment has the potential to prove the scale of flavour symmetry around electroweak scale or push it to a higher scale.

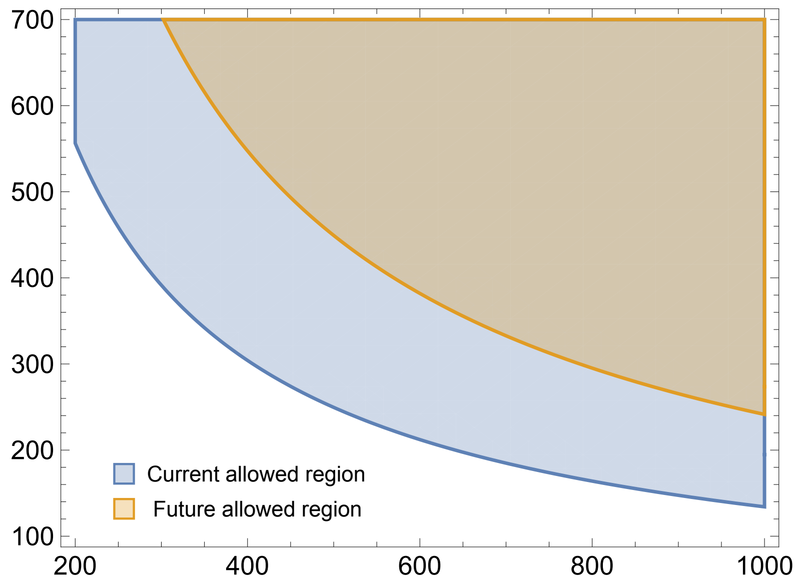

Figure 2: The current and near future constraints on Model I from the $\mu^- \to e^-\gamma$ experiments. $|\epsilon_\varphi|$ is fixed at 0.1 for generating the reactor angle θ_{13}. The current constraint of the MEG experiment is set to be $\text{Br}(\mu^- \to e^-\gamma) < 4.2 \times 10^{-13}$ [36], and the future constraint of MEG II is set to be $\text{Br}(\mu^- \to e^-\gamma) < 4 \times 10^{-14}$ [37].

4.3 Z_3-breaking LFV charged lepton decays in Model II

Flavon cross couplings shift the Z_3-preserving VEVs and open the Z_3-breaking LFV channels. Similar to Model I, the shift of flavon VEVs in Model II results in three contributions to the Z_3-breaking processes. They are

- the mixing between left-handed charged leptons. To get the Lagrangian in the charged lepton mass eigenstates, we do the transformation

$$\begin{aligned} e_L &\to e_L + (1+\kappa)\epsilon_\varphi^* \mu_L + (1-\kappa)\epsilon_\varphi \tau_L \,, \\ \mu_L &\to \mu_L + (1+\kappa)\epsilon_\varphi^* \tau_L - (1-\kappa)\epsilon_\varphi e_L \,, \\ \tau_L &\to \tau_L - (1+\kappa)\epsilon_\varphi^* e_L - (1-\kappa)\epsilon_\varphi \mu_L \,. \end{aligned} \tag{69}$$

- the mixing between Z_3-invariant flavon h_1 and Z_3-covariant flavons φ_2', φ_3'. It leads to the following transformation to the flavon mass eigenstates:

$$\begin{aligned} h_1 &\to h_1 + c_{12}\left(\epsilon_\varphi^* \varphi_2' + \epsilon_\varphi \varphi_2'^*\right) + c_{13}\left(\epsilon_\varphi \varphi_3' + \epsilon_\varphi^* \varphi_3'^*\right) \,, \\ \varphi_2' &\to \varphi_2' - c_{12}\epsilon_\varphi h_1 + c_{23}|\epsilon_\varphi| \varphi_3'^* \,, \\ \varphi_3' &\to \varphi_3' - c_{13}\epsilon_\varphi^* h_1 - c_{23}|\epsilon_\varphi| \varphi_2'^* \,, \end{aligned} \tag{70}$$

in which

$$c_{12} = \frac{m^2_{\varphi'_2} + m^2_{\varphi_1}}{m^2_{\varphi'_2} - m^2_{\varphi_1}} \frac{m^2_{\varphi'_3}\sqrt{2 - 2\sin 2\vartheta}}{(m^2_{\varphi'_3} + m^2_{\varphi'_2}) - (m^2_{\varphi'_3} - m^2_{\varphi'_2})\sin 2\vartheta}\,,$$
$$c_{13} = \frac{m^2_{\varphi'_3} + m^2_{\varphi_1}}{m^2_{\varphi'_3} - m^2_{\varphi_1}} \frac{m^2_{\varphi'_2}\sqrt{2 + 2\sin 2\vartheta}}{(m^2_{\varphi'_3} + m^2_{\varphi'_2}) - (m^2_{\varphi'_3} - m^2_{\varphi'_2})\sin 2\vartheta}\,,$$
$$c_{23} = \frac{2\big[(m^2_{\varphi'_3} + m^2_{\varphi'_2})^2 - 3(m^2_{\varphi'_3} - m^2_{\varphi'_2})^2\cos 4\vartheta\big]\cos 2\vartheta}{(m^2_{\varphi'_3} - m^2_{\varphi'_2})\big[(m^2_{\varphi'_3} + m^2_{\varphi'_2}) - (m^2_{\varphi'_3} - m^2_{\varphi'_2})\sin 2\vartheta\big]}\,. \tag{71}$$

- the mass splitting between h'_2 and a'_2, and that between h'_3 and a'_3, in which h'_2, a'_2, h'_3 and a'_3 are mass eigenstates of φ'_2 and φ'_3 given by

$$h_2 = \frac{1}{\sqrt{2}}\Big(\varphi_2\exp(i\frac{\theta_\varphi}{2}) + \varphi_2^*\exp(-i\frac{\theta_\varphi}{2})\Big)\,, \qquad h_3 = \frac{1}{\sqrt{2}}\Big(\varphi_3\exp(i\frac{\theta_\varphi}{2}) + \varphi_3^*\exp(-i\frac{\theta_\varphi}{2})\Big)\,,$$
$$a_2 = \frac{-i}{\sqrt{2}}\Big(\varphi_2\exp(i\frac{\theta_\varphi}{2}) - \varphi_2^*\exp(-i\frac{\theta_\varphi}{2})\Big)\,, \qquad a_3 = \frac{i}{\sqrt{2}}\Big(\varphi_3\exp(i\frac{\theta_\varphi}{2}) - \varphi_3^*\exp(-i\frac{\theta_\varphi}{3})\Big) \tag{72}$$

The corrected masses of them are respectively given by

$$m^2_{h'_2,a'_2} = m^2_{\varphi'_2}(1 \pm c_{22}|\epsilon_\varphi|)\,, \quad m^2_{h'_3,a'_3} = m^2_{\varphi'_3}(1 \pm c_{33}|\epsilon_\varphi|)\,, \tag{73}$$

with

$$c_{22} = \frac{(1-\sin 2\vartheta)\big[(m^2_{\varphi'_3} + m^2_{\varphi'_2})^2 - (m^2_{\varphi'_3} - m^2_{\varphi'_2})^2(1 + 6(1+\sin 2\vartheta)\sin 2\vartheta)\big]}{2m^2_{\varphi'_2}\big[(m^2_{\varphi'_3} + m^2_{\varphi'_2}) - (m^2_{\varphi'_3} - m^2_{\varphi'_2})\sin 2\vartheta\big]}\,,$$
$$c_{33} = \frac{(1+\sin 2\vartheta)\big[(m^2_{\varphi'_3} + m^2_{\varphi'_2})^2 - (m^2_{\varphi'_3} - m^2_{\varphi'_2})^2(1 - 6(1-\sin 2\vartheta)\sin 2\vartheta)\big]}{2m^2_{\varphi'_3}\big[(m^2_{\varphi'_3} + m^2_{\varphi'_2}) - (m^2_{\varphi'_3} - m^2_{\varphi'_2})\sin 2\vartheta\big]}\,. \tag{74}$$

The terms $1 \pm c_{22}|\epsilon_\varphi|$ and $1 \pm c_{33}|\epsilon_\varphi|$ must be positive to stabilise the vacuum.

Here, we would like to mention two special cases, $\vartheta = \pm 45°$:

- In the case $\vartheta = 45°$, we have c_{12}, c_{23} and c_{22} vanish, c_{13} takes the same value as c in Model I, and $c_{33} = 2$. In this case, h_1 and φ'^*_3 (as well as h'_3 and a'_3) are identical with φ_1, φ_2 (as well as h_2 and a_2) in Model I, respectively. There is no ϵ_φ-induced mixing between φ'_2 and h_1, nor that between φ'_2 and φ'^*_3.

- In the case $\vartheta = -45°$, we have c_{13}, c_{23} and c_{33} vanish, c_{12} takes the same value as c in Model I, and $c_{22} = 2$. Therefore, h_1 and φ'_2 (as well as h'_2 and a'_2) are identical with φ_1, φ_2 (as well as h_2 and a_2) in Model I, respectively, and there is no ϵ_φ-induced mixing between φ'^*_3 and h_1, nor that between φ'^*_3 and φ'_2.

After considering the three effects of Z_3-breaking, we can repeat the procedure as in Model I to calculate the LFV charged lepton decays and derive the 4-fermion interactions for $\tau^- \to \mu^+\mu^-e^-$, $\tau^- \to \mu^+\mu^-\mu^-$, $\tau^- \to e^+e^-\mu^-$, $\tau^- \to e^+e^-e^-$ and $\mu^- \to e^+e^-e^-$. Here, we list the coefficients that will contribute to the decays at leading order:

$$C^{\mu\mu e\tau}_{LRLR} = -\epsilon_\varphi \frac{m_\mu m_\tau}{v_\varphi^2}\left[\frac{1+\kappa}{m^2_{\varphi_1}} + \sqrt{2}c_{12}c_\vartheta \frac{\Delta m^2_{\varphi'_2\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi'_2}} + \sqrt{2}c_{13}s_\vartheta \frac{\Delta m^2_{\varphi'_3\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi'_3}} + (1+\kappa)\sin 2\vartheta\Big(\frac{1}{m^2_{\varphi'_2}} - \frac{1}{m^2_{\varphi'_3}}\Big)\right]\,,$$

$$C^{\mu\mu e\tau}_{RLLR} = -\epsilon_\varphi \frac{m_\mu m_\tau}{v_\varphi^2}\left[\frac{1+\kappa}{m^2_{\varphi_1}} + \sqrt{2}c_{12}c_\vartheta \frac{\Delta m^2_{\varphi_2'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_2'}} + \sqrt{2}c_{13}s_\vartheta \frac{\Delta m^2_{\varphi_3'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_3'}} + 2(1-\kappa)\Big(\frac{c^2_\vartheta}{m^2_{\varphi_2'}} + \frac{s^2_\vartheta}{m^2_{\varphi_3'}}\Big)\right] ,$$

$$C^{e\mu\mu\tau}_{LRLR} = -\epsilon_\varphi \frac{m_\mu m_\tau}{v_\varphi^2}\left[2\Big(\frac{c_{22}s^2_\vartheta}{m^2_{\varphi_2'}} + \frac{c_{33}c^2_\vartheta}{m^2_{\varphi_3'}}\Big)\right] ;$$

$$C^{\mu\mu\mu\tau}_{LRLR} = -\epsilon^*_\varphi \frac{m_\mu m_\tau}{v_\varphi^2}\left[\frac{1+\kappa}{m^2_{\varphi_1}} - \sqrt{2}c_{12}s_\vartheta \frac{\Delta m^2_{\varphi_2'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_2'}} + \sqrt{2}c_{13}c_\vartheta \frac{\Delta m^2_{\varphi_3'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_3'}} + (1-\kappa)\sin 2\vartheta\Big(\frac{1}{m^2_{\varphi_3'}} - \frac{1}{m^2_{\varphi_2'}}\Big)\right] ,$$

$$C^{\mu\mu\mu\tau}_{LRLR} = -\epsilon^*_\varphi \frac{m_\mu m_\tau}{v_\varphi^2}\left[\frac{1+\kappa}{m^2_{\varphi_1}} - \sqrt{2}c_{12}s_\vartheta \frac{\Delta m^2_{\varphi_2'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_2'}} + \sqrt{2}c_{13}c_\vartheta \frac{\Delta m^2_{\varphi_3'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_3'}} - 2(1+\kappa)\Big(\frac{s^2_\vartheta}{m^2_{\varphi_2'}} + \frac{c^2_\vartheta}{m^2_{\varphi_3'}}\Big)\right] ;$$

$$C^{\mu ee\tau}_{RLLR} = -\epsilon^*_\varphi \frac{m_\mu m_\tau}{v_\varphi^2}\left[\frac{2+2\kappa\cos\vartheta - c_{22}\sin 2\vartheta}{m^2_{\varphi_2'}} + \frac{2-2\kappa\cos\vartheta + c_{33}\sin 2\vartheta}{m^2_{\varphi_3'}}\right](\overline{\mu_R}e_L)(\overline{e_L}\tau_R) ;$$

$$C^{eee\tau}_{LRLR} = -\epsilon_\varphi \frac{m_e m_\tau}{v_\varphi^2}\left[\frac{1+\kappa}{m^2_{\varphi_1}} - \sqrt{2}c_{12}c_\vartheta \frac{\Delta m^2_{\varphi_2'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_2'}} - \sqrt{2}c_{13}s_\vartheta \frac{\Delta m^2_{\varphi_3'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_3'}} - (1+\kappa)\sin 2\vartheta\Big(\frac{1}{m^2_{\varphi_2'}} - \frac{1}{m^2_{\varphi_3'}}\Big)\right] ,$$

$$C^{eee\tau}_{RLLR} = -\epsilon_\varphi \frac{m_e m_\tau}{v_\varphi^2}\left[\frac{1+\kappa}{m^2_{\varphi_1}} - \sqrt{2}c_{12}c_\vartheta \frac{\Delta m^2_{\varphi_2'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_2'}} - \sqrt{2}c_{13}s_\vartheta \frac{\Delta m^2_{\varphi_3'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_3'}} + 2(1-\kappa)\Big(\frac{s^2_\vartheta}{m^2_{\varphi_2'}} + \frac{c^2_\vartheta}{m^2_{\varphi_3'}}\Big)\right] ;$$

$$C^{eee\mu}_{LRLR} = -\epsilon^*_\varphi \frac{m_e m_\mu}{v_\varphi^2}\left[\frac{1-\kappa}{m^2_{\varphi_1}} - \sqrt{2}c_{12}s_\vartheta \frac{\Delta m^2_{\varphi_2'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_2'}} + \sqrt{2}c_{13}c_\vartheta \frac{\Delta m^2_{\varphi_3'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_3'}} - (1-\kappa)\sin 2\vartheta\Big(\frac{1}{m^2_{\varphi_2'}} - \frac{1}{m^2_{\varphi_3'}}\Big)\right] ,$$

$$C^{eee\mu}_{RLLR} = -\epsilon^*_\varphi \frac{m_e m_\mu}{v_\varphi^2}\left[\frac{1-\kappa}{m^2_{\varphi_1}} - \sqrt{2}c_{12}s_\vartheta \frac{\Delta m^2_{\varphi_2'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_2'}} + \sqrt{2}c_{13}c_\vartheta \frac{\Delta m^2_{\varphi_3'\varphi_1}}{m^2_{\varphi_1}m^2_{\varphi_3'}} + 2(1+\kappa)\Big(\frac{s^2_\vartheta}{m^2_{\varphi_2'}} + \frac{c^2_\vartheta}{m^2_{\varphi_3'}}\Big)\right] , \quad (75)$$

respectively. Here, $\Delta m^2_{\varphi_2'\varphi_1} \equiv m^2_{\varphi_2'} - m^2_{\varphi_1}$ and $\Delta m^2_{\varphi_3'\varphi_1} \equiv m^2_{\varphi_3'} - m^2_{\varphi_1}$. All these channels are suppressed by both ϵ_φ and charged lepton masses strongly dependent upon the mixing between the two Z_3-covariant flavons φ_2 and φ_3. In addition, the last two channels $\tau^- \to e^+e^-e^-$ and $\mu^- \to e^+e^-e^-$ are highly suppressed by the electron mass. From the above effective interactions, one can directly obtain branching ratios for these channels, which are of the same orders of magnitude as those in Model I.

Finally, we discuss the modification to the branching ratio of $l_1^- \to l_2^-\gamma$. The coefficients $A_R^{l_2 l_1}$ corresponding to $\tau^- \to e^-\gamma$, $\tau^- \to \mu^-\gamma$ and $\mu^- \to e^-\gamma$ are respectively given by

$$\begin{aligned} A_R^{e\tau} = \ & \frac{ie\epsilon_\varphi m_\tau}{(4\pi)^2 v_\varphi^2}\Big[-(1+\kappa+\sqrt{2}c_{12}c_\vartheta + \sqrt{2}c_{13}s_\vartheta)F(h_1) \\ & + \sqrt{2}c_\vartheta(c_{12} - \sqrt{2}s_\vartheta(1+\kappa))F(\varphi_2') + \sqrt{2}s_\vartheta(c_{13} + \sqrt{2}c_\vartheta(1+\kappa))F(\varphi_3') \\ & - \frac{(2c_\vartheta s_\vartheta(1+\kappa) + 2c^2_\vartheta(1-\kappa))m^2_\tau}{6(4\pi)^2 m^2_{\varphi_2'}} + \frac{(2c_\vartheta s_\vartheta(1+\kappa) - 2s^2_\vartheta(1-\kappa))m^2_\tau}{6(4\pi)^2 m^2_{\varphi_3'}}\Big] , \\ A_R^{\mu\tau} = \ & \frac{ie\epsilon^*_\varphi m_\tau}{(4\pi)^2 v_\varphi^2}\Big[-(1-\kappa-\sqrt{2}c_{12}s_\vartheta + \sqrt{2}c_{13}c_\vartheta)F(h_1) \\ & - \sqrt{2}s_\vartheta(c_{12} + \sqrt{2}c_\vartheta(1-\kappa))F(\varphi_2') + \sqrt{2}c_\vartheta(c_{13} + \sqrt{2}s_\vartheta(1-\kappa))F(\varphi_3') \\ & - \frac{(2c_\vartheta s_\vartheta(1-\kappa) + 2s^2_\vartheta(1+\kappa))m^2_\tau}{6(4\pi)^2 m^2_{\varphi_2'}} + \frac{(2c_\vartheta s_\vartheta(1-\kappa) - 2c^2_\vartheta(1+\kappa))m^2_\tau}{6(4\pi)^2 m^2_{\varphi_3'}}\Big] , \\ A_R^{e\mu} = \ & \frac{-2ie\epsilon^*_\varphi m_\mu}{(4\pi)^2 v_\varphi^2}\Bigg\{\frac{m^2_\tau}{m^2_{\varphi_2'}}\Big[c_{22}c^2_\vartheta\Big(\log\frac{m^2_\tau}{m^2_{\varphi_2'}} + \frac{5}{2}\Big) + \frac{1}{12}(c_{22}\sin 2\vartheta + 2\cos 2\vartheta + 2\kappa)\Big] \end{aligned}$$

$$+\frac{m_\tau^2}{m_{\varphi_3'}^2}\left[c_{33}c_\vartheta^2\left(\log\frac{m_\tau^2}{m_{\varphi_3'}^2}+\frac{5}{2}\right)-\frac{1}{12}(c_{33}\sin 2\vartheta+2\cos 2\vartheta+2\kappa)\right]\Bigg\}. \quad (76)$$

Here, the analytical result of $\mu^- \to e^-\gamma$ is only valid in the case of small mass difference between h_2' and a_2' and that between h_3' and a_3', i.e., $|c_{22}|, |c_{33}| \lesssim 2$. Again, φ_1 does not affect $\mu^- \to e^-\gamma$. We recover the results of Model I in the limit $\vartheta \to 45°$ and $m_{\varphi_2'}^2 \to \infty$, or in the limit $\vartheta \to -45°$ and $m_{\varphi_3'}^2 \to \infty$.

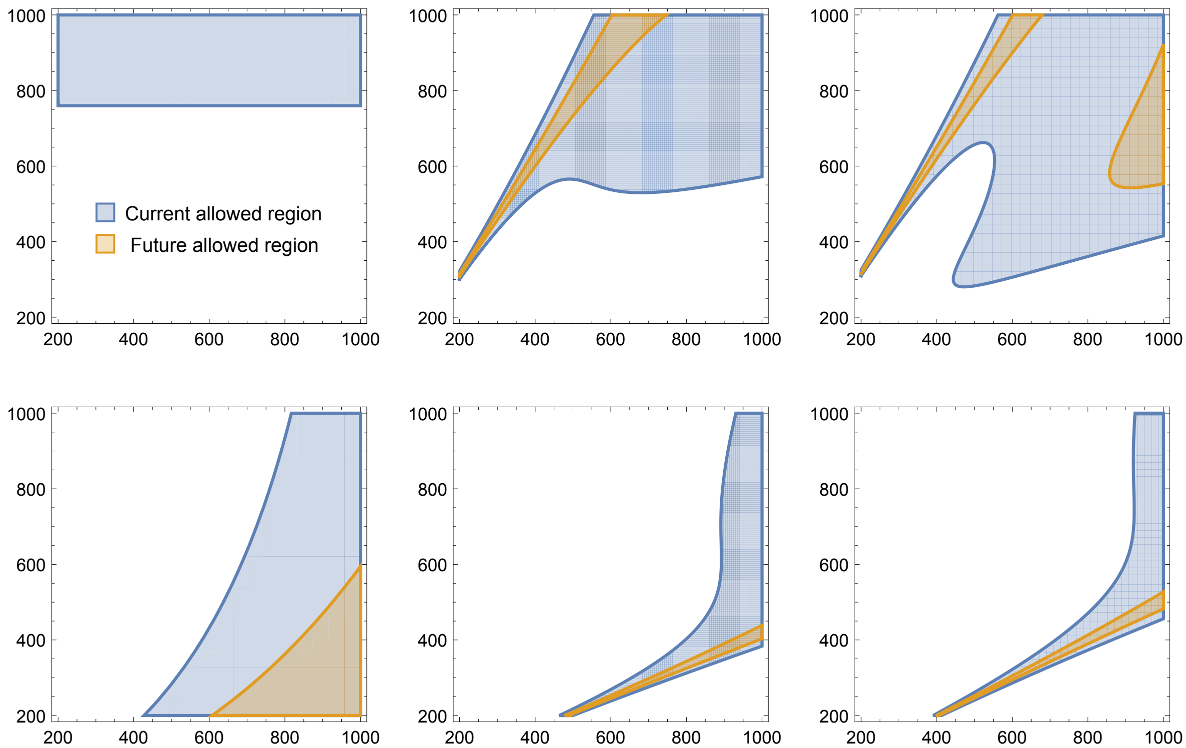

Figure 3: The current and near future constraints on the mass parameters $m_{\varphi_2'}$ and $m_{\varphi_3'}$ from the $\mu \to e\gamma$ experiments. $v_\varphi = 175$ GeV is assumed.

We perform the numerical analysis to show how Model II is constrained by current experiments. $\tau^- \to e^-\gamma$ and $\tau^- \to \mu^-\gamma$ are safe for current experimental limit by assuming the flavour symmetry scale around the electroweak one. Also for Model II, the strongest constraint is also from the $\mu^- \to e^-\gamma$ searches. In order to avoid the situation where the perturbation theory is not valid, we directly apply Eqs. (82) and (83) in the Appendix into our numerical calculation. We fix $v_\varphi = v/\sqrt{2} = 175$ GeV, vary the mixing angle ϑ and show the allowed regions of the mass parameters $m_{\varphi_2'}$ and $m_{\varphi_3'}$ by current and the expected future experiments in Fig. 3. Some comments follow:

- Masses $m_{\varphi_2'}$ and $m_{\varphi_3'}$ as low as 200 to 400 GeV are still allowed for some values of ϑ, much lower than the mass m_{φ_2} in Model I. This corresponds to the cancellation of contributions of φ_2 and φ_3.
- The mass eigenvalues of h_2', a_2' and h_3', a_3' in general deviate from $m_{\varphi_2'}$ and $m_{\varphi_3'}$, and the relative deviations are characterisd by $\pm c_{22}|\epsilon_\varphi|$ and $\pm c_{33}|\epsilon_\varphi|$, respectively. Numerically, we have checked that $|c_{22}|, |c_{33}| \lesssim 2$ hold in most of the allowed parameter space in Fig. 3. Thus, the deviations of $m_{h_2'}$, $m_{a_2'}$ from $m_{\varphi_2'}$, as well as those of $m_{h_3'}$, $m_{a_3'}$ from $m_{\varphi_3'}$, are in general small, and we can treat $m_{\varphi_2'}$ and $m_{\varphi_3'}$ as their masses at leading order.

- We have also checked that in all the allowed parameter space, $|c_{22}\epsilon_\varphi|$, $|c_{33}\epsilon_\varphi| < 1$. In other words, positive masses $m^2_{h'_2}$, $m^2_{a'_2}$, $m^2_{h'_3}$, $m^2_{a'_3} > 0$ are guaranteed.
- For $\vartheta = 45°$, c_{22} vanishes, no mass deviation between h'_2 and a'_2 arises, and the contributions of h'_2 and a'_2 cancel with each other. Thus, there is no constraint on $m_{\varphi'_2}$. The current experimental upper limit allows the mass of φ'_3 to be larger than 700 GeV. In the future, a mass smaller than 1 TeV would be ruled out. This is consistent with the results of φ_2 in Model I in Fig. 2. We have also checked that for $\vartheta = -45°$, there will be no constraint on $m_{\varphi'_3}$ and the upper limit of the φ'_2 mass is also around 700 GeV, which is not shown in Fig. 3.

5 Conclusion

Varies of flavour models with discrete flavour symmetries have been proposed to understand the mystery of lepton flavour mixing. Essential ingredients are flavon fields which couple among themselves and with leptons. The flavon potential generates special vacuum expectation values for flavons and trigger flavour symmetry breaking. And the couplings with leptons are responsible for Yukawa couplings with special flavour structures after the flavons get VEVs. These couplings will unavoidably contribute to other lepton-flavour-violating processes beyond neutrino oscillations. In this paper, we discuss the LFV decays of charged leptons induced by these couplings.

All charged LFV processes have the same origin as leptonic flavour mixing, as they originate from the effective couplings of flavons and couplings between flavons and leptons. For definiteness, we assume that the flavour symmetry is A_4 and lepton flavour mixing is tri-bimaximal at leading order after A_4 breaking. The flavon coupling to the charged leptons, φ, is a triplet of A_4 and its VEV should roughly preserve a Z_3 residual symmetry after A_4 breaking. Z_3 is phenomenologically necessary for realising TBM at leading order and has theoretically been realised in a lot of models. Depending on the different representation properties of φ, we consider two cases: i. φ is a pseudo-real triplet of A_4, an economical case which introduces as few degrees of freedom as possible to the model, and ii. φ is a complex triplet, a generalised case which can be regarded as a simplification of supersymmetric and multi-Higgs models. The breaking of $A_4 \to Z_3$ results in three physical parameters in case i: the scale of flavour symmetry breaking v_ϕ and flavon masses m_{φ_1} and m_{φ_2}. In case ii, the mixing between two Z_3-covariant flavons introduces a mixing angle ϑ, as well as one more flavon mass, which cannot be neglected. It is natural to assume the flavon masses to be of the same order of magnitude as v_φ.

The only Z_3-preserving LFV processes are $\tau^- \to \mu^+ e^- e^-$, $\tau^- \to e^+ \mu^- \mu^-$. They are triggered by the exchange of Z_3-covariant flavons and their branching ratios are dependent upon the flavour symmetry scale v_φ and the flavon masses. In case i, both channels are mediated by the same flavon φ_2 and their branching ratios are approximately equal. In case ii, the two branching ratios are in general different from each other due to the presence of the mixing between the two Z_3-covariant flavons. If their mixing is maximal, the branching ratios are equal again. To be compatible with charged lepton masses, these processes in both cases are suppressed by ratios of charged lepton masses to the scale of flavour symmetry, to be exact, suppressed by $m_\mu m_\tau / v_\varphi^2$. Once we assume v_φ around the electroweak scale, their branching ratios are much lower than current experimental limits.

Z_3 is not an exact symmetry due to the interrupt with other fields. The breaking of Z_3 is also supported by the phenomenological requirement that TBM must gain corrections to match current oscillation data. In presence of Z_3 breaking, other 3-body LFV decays and all radiative decays forbidden by Z_3 can take place, but are suppressed by the small Z_3-breaking effects. We identify three Z_3-breaking effects: the mixing of charged lepton flavour eigenstates, the mixing between the Z_3-invariant and Z_3-covariant flavons, and the mass splitting of the two real degrees of freedom of each Z_3-covariant complex flavons. Under the assumption that the corrections to TBM come mainly from the Z_3-breaking effects, strong relations between mixing angles, especially θ_{13}, with charged LFV processes arise.

Z_3-breaking charged LFV processes depend on the explicit structure of a concrete model. We consider two models in cases i and ii, respectively. These models have been proposed in our former work [25] and fit well with current oscillation data. We derive analytical expressions for branching ratios of all processes and numerically check the constraints on the A_4-breaking scale and flavon masses for each model. The results in Model I are quite simple. In special, a sum rule of $\tau^- \to \mu^+\mu^-e^-$, $\tau^- \to \mu^+\mu^-\mu^-$ and $\tau^- \to e^+e^-\mu^-$ is obtained and all these processes are suppressed by $2s_{13}^2 + (1-\sqrt{3}s_{12})^2$ compared with the Z_3-preserving ones. Another relation is $\mathrm{Br}(\tau^- \to \mu^-\gamma) \approx \mathrm{Br}(\tau^- \to e^-\gamma)$. The most stringent constraint is from the $\mu^- \to e^-\gamma$ measurement. The main contributions are the mixing of charged leptons and the mass splitting of the Z_3-covariant flavons induced by the breaking of the Z_3 symmetry. In Model I, the only unknown parameters contribute to this process are the scale v_φ and the Z_3-covariant flavon mass m_{φ_2}. Their allowed parameter space can be derived from the current upper limit for this process. Setting v_φ around the electroweak scale, we arrive at $m_{\varphi_2} > 500$ GeV. The results in Model II are strongly dependent upon the mixing between the Z_3-covariant flavons. They match with those in Model I in the limit $\vartheta = \pm 45^\circ$ and one of the Z_3-covariant flavons decouple from the processes. In both models, branching ratios of $\tau^- \to e^+e^-e^-$ and $\mu^- \to e^+e^-e^-$ are much weaker than the other processes due to the suppression of electron mass. The flavon masses and their mixing angle ϑ will influence the branching ratio. Tuning these parameters, tiny masses, 300-400 GeV, for the Z_3-breaking flavons are allowed by experiment constraints.

In conclusion, we have studied flavon-induced charged LFV processes. These flavons give explanation to lepton flavour mixing in flavour models and contribute to the other LFV processes beyond neutrino oscillations. Different from most models assuming the flavour symmetry at very high energy scale to avoid the strong constraints from charged LFV processes, we have checked that a relatively low-scale flavour symmetry, not far from the electroweak scale, is consistent with current experiment constraints from charged LFV processes. The main reason is that since these models give explanation to the flavour mixing, the couplings between flavons and charged leptons should be reasonably suppressed by the charged lepton masses. Furthermore, the approximative residual symmetry in the charged lepton sector, strongly suggested in most flavour models and supported by current oscillation data, can give an additional suppression factor $\mathcal{O}(0.01)$ for most LFV processes of 3-body charged lepton decays and all radiative decays. The method we developed here for calculating flavon-induced charged LFV processes can be extended into models with different flavour symmetries, such as S_4 and A_5.

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A Radiative decays mediated by complex scalars

Some of the scalars in our models (e.g., φ_2 in Model I, and φ_2', φ_3' in Model II) are complex scalars. Their couplings to fermions are strongly dependent upon the chirality of fermions. Thus, the chirality of fermions should be taken into account carefully when we calculate $l_1^- \to l_2^- \gamma$. It is useful for us to re-express $A_L^{l_2 l_1} P_L + A_R^{l_2 l_1} P_R$ in Eq. (29) as

$$A_L^{l_2 l_1} P_L + A_R^{l_2 l_1} P_R = g_{LL} O_{LL}^{l_2 l_1}(\varphi) + g_{RR} O_{RR}^{l_2 l_1}(\varphi) + g_{RL} O_{RL}^{l_2 l_1}(\varphi) + g_{LR} O_{LR}^{l_2 l_1}(\varphi) \,, \tag{77}$$

where $O_{P_2 P_1}^{l_2 l_1}(\varphi)$ (for $P_1, P_2 = L, R$) stand for the following loop integral excluding the $\sigma_{\mu\nu} q^\nu$ part:

$$\int \frac{d^4 p}{(2\pi)^4} P_{P_2} \frac{i(\not{p}_2 - \not{p}) + m_\tau}{(p_2 - p)^2 - m_\tau} \gamma_\mu \frac{i(\not{p}_1 - \not{p}) + m_\tau}{(p_1 - p)^2 - m_\tau} P_{P_1} \frac{i}{p^2 - m_\varphi^2} \,, \tag{78}$$

and $g_{P_2 P_1}$ are the relevant coefficients. Neglecting subleading terms, we derive the following expression for $O_{P_1 P_2}^{l_2 l_1}(\varphi)$ as

$$\begin{aligned}
O_{LL}^{l_2 l_1}(\varphi) &= \frac{1}{(4\pi)^2} \frac{m_\tau}{m_\varphi^2} \big(\log \frac{m_\tau^2}{m_\varphi^2} + \frac{3}{2} \big) P_L \,, \\
O_{RR}^{l_2 l_1}(\varphi) &= \frac{1}{(4\pi)^2} \frac{m_\tau}{m_\varphi^2} \big(\log \frac{m_\tau^2}{m_\varphi^2} + \frac{3}{2} \big) P_R \,, \\
O_{RL}^{l_2 l_1}(\varphi) &= \frac{1}{(4\pi)^2} \frac{m_{l_1}}{m_\varphi^2} \times \frac{-1}{6} P_L \,, \\
O_{LR}^{l_2 l_1}(\varphi) &= \frac{1}{(4\pi)^2} \frac{m_{l_1}}{m_\varphi^2} \times \frac{-1}{6} P_R \,,
\end{aligned} \tag{79}$$

For $\tau^- \to e^- \gamma, \mu^- \gamma$, $m_{l_1} = m_\tau$. For $\mu^- \to e^- \gamma$, $m_{l_1} = m_\mu$, and thus, $O_{LR}^{e\mu}, O_{RL}^{e\mu} \ll O_{LL}^{e\mu}, O_{RR}^{e\mu}$. These results are compatible with [39].

In Model I, we use the Lagrangian in Eqs. (63), (64) and (65) to calculate the radiative decays. With the help of Eq. (79), we derive

$$\begin{aligned}
A_L^{e\tau} P_L + A_R^{e\tau} P_R &= ie\tilde{\epsilon}_\varphi \frac{m_\tau^2}{v_\varphi^2} \big[O_{RR}^{e\tau}(\varphi_2) + O_{LR}^{e\tau}(\varphi_2) - O_{RR}^{e\tau}(\varphi_1) - O_{LR}^{e\tau}(\varphi_1) \big] \,, \\
A_L^{\mu\tau} P_L + A_R^{\mu\tau} P_R &= ie\tilde{\epsilon}_\varphi^* \frac{m_\tau^2}{v_\varphi^2} \big[O_{RR}^{\mu\tau}(\varphi_2) + O_{LR}^{\mu\tau}(\varphi_2) - O_{RR}^{\mu\tau}(\varphi_1) - O_{LR}^{\mu\tau}(\varphi_1) \big] \,, \\
A_L^{e\mu} P_L + A_R^{e\mu} P_R &= iee^{-i\theta_\varphi} \frac{m_\tau}{v_\varphi^2} \big\{ m_\mu \big[O_{RR}^{e\mu}(h_2) - O_{RR}^{e\mu}(a_2) \big] + m_\tau \big[O_{LR}^{e\mu}(h_2) - O_{LR}^{e\mu}(a_2) \big] \big\}
\end{aligned} \tag{80}$$

for $\tau^- \to e^- \gamma$, $\tau^- \to \mu^- \gamma$ and $\mu^- \to e^- \gamma$, respectively. A direct calculation shows that $O_{RR}^{l_2 \tau}(\varphi) + O_{LR}^{l_2 \tau}(\varphi) = F(\varphi)/m_\tau P_R$ (for $l_2 = e, \mu$) and $m_\mu O_{RR}^{e\mu}(\varphi) + m_\tau O_{LR}^{e\mu}(\varphi) = m_\mu F(\varphi)/m_\tau P_R$. And using the approximation

$$F(h_2) - F(a_2) = -4|\epsilon_\varphi| \frac{m_\tau^2}{m_{\varphi_2}^2} \big(\log \frac{m_\tau^2}{m_{\varphi_2}^2} + \frac{7}{3} \big) \,, \tag{81}$$

we finally obtain the results in Eq. (66).

In Model II, The related Lagrangian terms for $\tau^- \to e^- \gamma$, $\tau^- \to \mu^- \gamma$ and $\mu^- \to e^- \gamma$ are respectively given by

$$\frac{m_\tau}{v_\varphi} \overline{e_L} \tau_R \Big[-(1 + \sqrt{2} c_{12} c_\vartheta + \sqrt{2} c_{13} s_\vartheta) \epsilon_\varphi h_1 + \sqrt{2} c_\vartheta \varphi_2' + \sqrt{2} s_\vartheta \varphi_3'^* \Big]$$

$$
\begin{aligned}
&+ \frac{m_\tau}{v_\varphi}\overline{\tau_L}\tau_R\Big[h_1 + (c_{12} - \sqrt{2}s_\vartheta)\epsilon_\varphi\varphi_2'^* + (c_{13} + \sqrt{2}c_\vartheta)\epsilon_\varphi\varphi_3'\Big] \\
&+ \frac{m_\tau}{v_\varphi}\overline{\tau_R}\tau_L\Big[h_1 + (c_{12} + \sqrt{2}c_\vartheta)\epsilon_\varphi\varphi_2'^* + (c_{13} + \sqrt{2}s_\vartheta)\epsilon_\varphi\varphi_3'\Big]\,, \\
&\quad \frac{m_\tau}{v_\varphi}\overline{\mu_L}\tau_R\Big[-(1 - \sqrt{2}c_{12}s_\vartheta + \sqrt{2}c_{13}c_\vartheta)\epsilon_\varphi^* h_1 - \sqrt{2}s_\vartheta\varphi_2'^* + \sqrt{2}c_\vartheta\varphi_3'\Big] \\
&+ \frac{m_\tau}{v_\varphi}\overline{\tau_L}\tau_R\Big[h_1 + (c_{12} + \sqrt{2}c_\vartheta)\epsilon_\varphi^*\varphi_2' + (c_{13} + \sqrt{2}s_\vartheta)\epsilon_\varphi^*\varphi_3'^*\Big] \\
&+ \frac{m_\tau}{v_\varphi}\overline{\tau_R}\tau_L\Big[h_1 + (c_{12} - \sqrt{2}s_\vartheta)\epsilon_\varphi^*\varphi_2' + (c_{13} + \sqrt{2}c_\vartheta)\epsilon_\varphi^*\varphi_3'^*\Big]\,, \\
&\quad e^{-i\frac{\theta_\varphi}{2}}\frac{m_\tau}{v_\varphi}\overline{e_L}\tau_R\Big[\,(c_\vartheta + (1-\kappa)|\epsilon_\varphi|s_\vartheta)\,h_2' + i\,(c_\vartheta - (1-\kappa)|\epsilon_\varphi|s_\vartheta)\,a_2' \\
&\qquad\qquad + (s_\vartheta - (1-\kappa)|\epsilon_\varphi|c_\vartheta)\,h_3' + i\,(s_\vartheta + (1-\kappa)|\epsilon_\varphi|c_\vartheta)\,a_3'\Big] \\
&+ e^{-i\frac{\theta_\varphi}{2}}\frac{m_\tau}{v_\varphi}\overline{\tau_R}\tau_L\Big[\,(-s_\vartheta + (1+\kappa)|\epsilon_\varphi|c_\vartheta)\,h_2' - i\,(s_\vartheta + (1+\kappa)|\epsilon_\varphi|c_\vartheta)\,a_2' \\
&\qquad\qquad + (c_\vartheta + (1+\kappa)|\epsilon_\varphi|s_\vartheta)\,h_3' + i\,(c_\vartheta - (1+\kappa)|\epsilon_\varphi|s_\vartheta)\,a_3'\Big] \\
&+ e^{-i\frac{\theta_\varphi}{2}}\frac{m_\mu}{v_\varphi}\overline{\tau_L}\tau_R\Big[\,(c_\vartheta - (1-\kappa)|\epsilon_\varphi|s_\vartheta)\,h_2' + i\,(c_\vartheta + (1-\kappa)|\epsilon_\varphi|s_\vartheta)\,a_2' \\
&\qquad\qquad + (s_\vartheta + (1-\kappa)|\epsilon_\varphi|c_\vartheta)\,h_3' + i\,(s_\vartheta - (1-\kappa)|\epsilon_\varphi|c_\vartheta)\,a_3'\Big]\,. \qquad (82)
\end{aligned}
$$

They results in expressions of $A_L^{l_2 l_1}P_L + A_R^{l_2 l_1}P_R$ given by

$$
\begin{aligned}
& ie\epsilon_\varphi\frac{m_\tau^2}{v_\varphi^2}\ \Big\{\ -(1+\kappa+\sqrt{2}c_\vartheta c_{12} + \sqrt{2}s_\vartheta c_{13})\big[O_{RR}^{e\tau}(h_1) + O_{LR}^{e\tau}(h_1)\big] \\
&\quad +\sqrt{2}c_\vartheta c_{12}\big[O_{RR}^{e\tau}(\varphi_2) + O_{LR}^{e\tau}(\varphi_2)\big] - 2s_\vartheta c_\vartheta(1+\kappa)O_{RR}^{e\tau}(\varphi_2) + 2c_\vartheta^2(1-\kappa)O_{LR}^{e\tau}(\varphi_2)\big] \\
&\quad +\sqrt{2}s_\vartheta c_{13}\big[O_{RR}^{e\tau}(\varphi_3) + O_{LR}^{e\tau}(\varphi_3)\big] + 2s_\vartheta c_\vartheta(1+\kappa)O_{RR}^{e\tau}(\tau,\varphi_3) + 2s_\vartheta^2(1-\kappa)O_{LR}^{e\tau}(\varphi_3)\big]\Big\}\,, \\
& ie\epsilon_\varphi^*\frac{m_\tau^2}{v_\varphi^2}\ \Big\{\ -(1-\kappa-\sqrt{2}s_\vartheta c_{12} + \sqrt{2}c_\vartheta c_{13})\big[O_{RR}^{\mu\tau}(h_1) + O_{LR}^{\mu\tau}(h_1)\big] \\
&\quad -\sqrt{2}s_\vartheta c_{12}\big[O_{RR}^{\mu\tau}(\varphi_2) + O_{LR}^{\mu\tau}(\varphi_2)\big] - 2s_\vartheta c_\vartheta(1-\kappa)O_{RR}^{\mu\tau}(\varphi_2) + 2s_\vartheta^2(1+\kappa)O_{LR}^{\mu\tau}(\varphi_2)\big] \\
&\quad +\sqrt{2}c_\vartheta c_{13}\big[O_{RR}^{\mu\tau}(\varphi_3) + O_{LR}^{\mu\tau}(\varphi_3)\big] + 2s_\vartheta c_\vartheta(1-\kappa)O_{RR}^{\mu\tau}(\varphi_3) + 2c_\vartheta^2(1+\kappa)O_{LR}^{\mu\tau}(\varphi_3)\big]\Big\}\,, \\
& ie\frac{m_\tau}{v_\varphi^2}\ \Big\{\ e^{-i\theta_\varphi}m_\mu c_\vartheta^2\big[O_{RR}^{e\mu}(h_2) - O_{RR}^{e\mu}(a_2)\big] - e^{-i\theta_\varphi}m_\tau c_\vartheta s_\vartheta\big[O_{LR}^{e\mu}(h_2) - O_{LR}^{e\mu}(a_2)\big] \\
&\quad +e^{-i\theta_\varphi}m_\mu s_\vartheta^2\big[O_{RR}^{e\mu}(h_3) - O_{RR}^{e\mu}(a_3)\big] + e^{-i\theta_\varphi}m_\tau c_\vartheta s_\vartheta\big[O_{LR}^{e\mu}(h_3) + O_{LR}^{e\mu}(a_3)\big] \\
&\quad +\epsilon_\varphi^* m_\tau(c_\vartheta^2 - s_\vartheta^2 + \kappa)\big[O_{LR}^{e\mu}(h_2) + O_{LR}^{e\mu}(a_2) - O_{LR}^{e\mu}(h_3) - O_{LR}^{e\mu}(a_3)\big]\Big\} \qquad (83)
\end{aligned}
$$

for $\tau^- \to e^-\gamma$, $\tau^- \to \mu^-\gamma$ and $\mu^- \to e^-\gamma$, respectively. We express $O_{LR}^{l_2 l_1}(\varphi)$ and $O_{RR}^{l_2 l_1}(\varphi)$ in $F(\varphi)$, expand $m_{h_2'}$, $m_{a_2'}$ and $m_{h_3'}$, $m_{a_3'}$ around $m_{\varphi_2'}$ and $m_{\varphi_3'}$, respectively, and finally, we obtain Eq. (76).

Compared with $A_R^{l_2 l_1}$, $A_L^{l_2 l_1}$ is negligibly small in all three channels. The reason is that e_R, μ_R and τ_R are singlets of A_4. Each column of the charged lepton mass matrix in Eq. (22) is proportional to one charged lepton mass. This kind of flavour structure leads to very small mixing of e_R, μ_R and τ_R (suppressed by both ϵ_φ and charged lepton mass ratio), and thus terms such as $\overline{e}_L\tau_R\varphi_i$ are highly suppressed. The feature $A_L^{l_2 l_1} \ll A_R^{l_2 l_1}$ is a feature in models where right-handed charged leptons belong to singlet representations of the flavour symmetry. Considerable mixing among right-handed charged leptons is possible in some other models. In that case, the contribution of $A_L^{l_2 l_1}$ should be included in the decay $l_1^- \to l_2^-\gamma$.

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