Bounding the Clique-Width of *H*-Free Chordal Graphs

Andreas Brandstädt,[1] Konrad K. Dabrowski,[2] Shenwei Huang,[3] and Daniël Paulusma[2]

[1] *INSTITUTE OF COMPUTER SCIENCE*
UNIVERSITÄT ROSTOCK
ALBERT-EINSTEIN-STRAßE 22, 18059 ROSTOCK, GERMANY
E-mail: ab@informatik.uni-rostock.de

[2] *SCHOOL OF ENGINEERING AND COMPUTING SCIENCES*
DURHAM UNIVERSITY, SCIENCE LABORATORIES
SOUTH ROAD, DURHAM DH1 3LE, UNITED KINGDOM
E-mail: konrad.dabrowski@durham.ac.uk; daniel.paulusma@durham.ac.uk

[3] *SCHOOL OF COMPUTING SCIENCE*
SIMON FRASER UNIVERSITY
8888 UNIVERSITY DRIVE, BURNABY B.C., V5A 1S6 CANADA
E-mail: shenweih@sfu.ca

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Abstract: A graph is *H*-free if it has no induced subgraph isomorphic to *H*. Brandstädt, Engelfriet, Le, and Lozin proved that the class of chordal graphs with independence number at most 3 has unbounded clique-width. Brandstädt, Le, and Mosca erroneously claimed that the gem and co-gem are the only two 1-vertex P_4-extensions *H* for which the class of *H*-free chordal graphs has bounded clique-width. In fact we prove that bull-free chordal and co-chair-free chordal graphs have clique-width at most 3 and 4, respectively. In particular, we find four new classes of *H*-free chordal graphs

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of bounded clique-width. Our main result, obtained by combining new and known results, provides a classification of all but two stubborn cases, that is, with two potential exceptions we determine *all* graphs H for which the class of H-free chordal graphs has bounded clique-width. We illustrate the usefulness of this classification for classifying other types of graph classes by proving that the class of $(2P_1 + P_3, K_4)$-free graphs has bounded clique-width via a reduction to K_4-free chordal graphs. Finally, we give a complete classification of the (un)boundedness of clique-width of H-free weakly chordal graphs. J. Graph Theory 00: 1–36, 2017

Keywords: *clique-width; chordal graph; forbidden induced subgraph; hereditary graph class*

1. INTRODUCTION

Clique-width is a well-studied graph parameter; see, for example, the surveys of Gurski [40] and Kamiński et al. [44]. In particular, there are numerous graph classes, such as those that can be characterized by one or more forbidden induced subgraphs,[1] for which it has been determined whether or not the class is of *bounded clique-width* (i.e. whether there is a constant c such that the clique-width of every graph in the class is at most c). Similar research has been done for variants of clique-width, such as linear clique-width [41] and power-bounded clique-width [5]. Clique-width is also closely related to other graph width parameters. For instance, it is known that every graph class of bounded treewidth has bounded clique-width but the converse is not true [21]. Moreover, for any graph class, having bounded clique-width is equivalent to having bounded rank-width [56] and also equivalent to having bounded NLC-width [43].

Clique-width is a very difficult graph parameter to deal with and our understanding of it is still very limited. We do know that computing clique-width is NP-hard [34], but we do not know if there exist polynomial-time algorithms for computing the clique-width of even very restricted graph classes, such as unit interval graphs. Also the problem of deciding whether a graph has clique-width at most c for some fixed constant c is only known to be polynomial-time solvable if $c \leq 3$ [19] and is a long-standing open problem for $c \geq 4$. Identifying more graph classes of bounded clique-width and determining what kinds of structural properties ensure that a graph class has bounded clique-width increases our understanding of this parameter. Another important reason for studying these types of questions is that certain classes of NP-complete problems become polynomial-time solvable on any graph class $\mathcal{G}$ of bounded clique-width.[2] Examples of such problems are those definable in Monadic second-order logic using quantifiers on vertices and vertex subsets, but not on edges or edge subsets.

[1] See also Information System on Graph Classes and their Inclusions [30], which keeps a record of graph classes for which (un)boundedness of clique-width is known.

[2] This follows from results [22, 33, 45, 57] that assume the existence of a so-called c-expression of the input graph $G \in \mathcal{G}$ combined with a result [55] that such a c-expression can be obtained in cubic time for some $c \leq 8^{\text{cw}(G)} - 1$, where $\text{cw}(G)$ is the clique-width of the graph G.

In this article, we primarily focus on chordal graphs. The class of chordal graphs has unbounded clique-width, as it contains the class of proper interval graphs and the class of split graphs, both of which have unbounded clique-width as shown by Golumbic and Rotics [38] and Makowsky and Rotics [51], respectively. We study the clique-width of subclasses of chordal graphs, but before going into more detail, we first give some necessary terminology and notation.

A. Notation

The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG. The *complement* of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G. For two graphs G and H, we write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. The graphs C_r, K_r, $K_{1,r-1}$, and P_r denote the cycle, complete graph, star, and path on r vertices, respectively. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h, i, and j from x, respectively. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph G is $(H_1, \ldots, H_p)$*-free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$. A graph G is *chordal* if it is $(C_4, C_5, \ldots)$-free and *weakly chordal* if both G and $\overline{G}$ are $(C_5, C_6, \ldots)$-free. Every chordal graph is weakly chordal.

B. Research Goal and Motivation

We want to determine all graphs H for which the class of H-free chordal graphs has *bounded* clique-width. Our motivation for this research is threefold.

(1) *Identify further graph classes for which a number of* NP*-complete problems can be solved in polynomial time.*
Although many such NP-complete problems, such as the colouring problem [37], are polynomial-time solvable on chordal graphs, many others continue to be NP-complete for graphs in this class. Examples of such problems are the well-known dominating set and Hamilton cycle problems. They are NP-complete even for split graphs [1, 20] and strongly chordal split graphs [52], respectively, but become polynomial-time solvable on any graph class of bounded clique-width [33, 36, 60]. Of course, in order to find new "islands of tractability," one may want to consider superclasses of H-free chordal graphs instead. However, already when one considers H-free weakly chordal graphs, one does not obtain any new tractable graph classes. Indeed, the clique-width of the class of H-free graphs is bounded if and only if H is an induced subgraph of P_4 [29], and as we prove later, the induced subgraphs of P_4 are also the only graphs H for which the class of H-free weakly chordal graphs has bounded clique-width. The same classification therefore also follows for superclasses, such as $(H, C_5, C_6, \ldots)$-free graphs (or H-free perfect graphs, to give another example). Since forests, or equivalently, $(C_3, C_4, \ldots)$-free graphs have bounded clique-width (see also Lemma 11), it follows that the class of $(H, C_3, C_4, \ldots)$-free graphs has bounded clique-width for every graph H. It is therefore a natural question to ask for which graphs H the class of $(H, C_4, C_5, \ldots)$-free (i.e. H-free chordal) graphs has bounded clique-width.

(2) *Classify the boundedness of the clique-width of* (H_1, H_2)*-free graphs.*
Classifying the boundedness of clique-width for H-free chordal graphs turns out to be useful for determining the (un)boundedness of the clique-width of graph classes characterized by two forbidden induced subgraphs H_1 and H_2, just as the full classification for H-free bipartite graphs [28] has proven to be [26, 27, 29]. To demonstrate this, we will prove that the class of $(2P_1 + P_3, K_4)$-free graphs has bounded clique-width via a reduction to K_4-free chordal graphs. We note that reducing from a target graph class to another class already known to have bounded clique-width is an important technique, which has also been used by others; for instance, by Brandstädt et al. [10] who proved that the class of $(C_4, K_{1,3}, 4P_1)$-free graphs has bounded clique-width by reducing these graphs to $(K_{1,3}, 4P_1)$-free chordal graphs. Moreover, in a previous paper [26], this technique was used for showing the boundedness of the clique-width of three other graph classes of (H_1, H_2)-free graphs [26]. In that paper, each of these classes was reduced to some known subclass of perfect graphs of bounded clique-width (perfect graphs form a superclass of chordal graphs). In particular, one of these three classes, namely the class of $(\overline{2P_1 + P_2}, 2P_1 + P_3)$-free graphs was reduced to the class of $\overline{2P_1 + P_2}$-free chordal graphs, also known as diamond-free chordal graphs (the diamond is the graph $\overline{2P_1 + P_2}$, see also Fig. 1), which has bounded clique-width [38].
Our new result for the class of $(2P_1 + P_3, K_4)$-free graphs and the three results of [26] belong to a line of research, trying to extend results [4, 10–14, 17, 25, 27, 51] on the clique-width of classes of (H_1, H_2)-free graphs in order to try to determine the boundedness or unboundedness of the clique-width of every such graph class [26, 29]. Including our new result for the case $(2P_1 + P_3, K_4)$ and five cases recently proved by Dabrowski et al. [24], this led to a classification of all but eight open cases (up to some equivalence relation, see [29]).

(3) *Complete a line of research on* H*-free chordal graphs.*
A classification of those graphs H for which the clique-width of H-free chordal graphs is bounded would complete a line of research in the literature, which we feel is an interesting goal on its own. As a start, using a result of Corneil and Rotics [21] on the relationship between treewidth and clique-width, it follows that the clique-width of the class of K_r-free chordal graphs is bounded for all $r \geq 1$. Brandstädt et al. [10] proved that the class of $4P_1$-free chordal graphs has unbounded clique-width. Brandstädt et al. [14] considered forbidding the graphs $\overline{P_1 + P_4}$ (gem) and $P_1 + P_4$ (co-gem) as induced subgraphs (see also Fig. 2). They showed that $(P_1 + P_4)$-free chordal graphs have clique-width at most 8 and also observed that $\overline{P_1 + P_4}$-free chordal graphs belong to the class of distance-hereditary graphs, which have clique-width at most 3 (as shown by Golumbic and Rotics [38]). Moreover, the same authors [14] erroneously claimed that the gem and co-gem are the only two 1-vertex P_4-extensions H for which the class of H-free chordal graphs has bounded clique-width. We prove that bull-free chordal graphs have clique-width at most 3, improving a known bound of 8, which was shown by Le [48]. We also prove that $\overline{S_{1,1,2}}$-free chordal graphs have clique-width at most 4, which Le posed as an open problem. Results [8, 38, 51] for split graphs and proper interval graphs lead to other classes of H-free chordal graphs of unbounded clique-width, as we will discuss in Section 2. However, in order to obtain our almost full dichotomy for H-free chordal graphs, new results are also need to be proved.

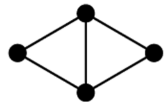

FIGURE 1. The graph $\overline{2P_1 + P_2}$, also known as the diamond.

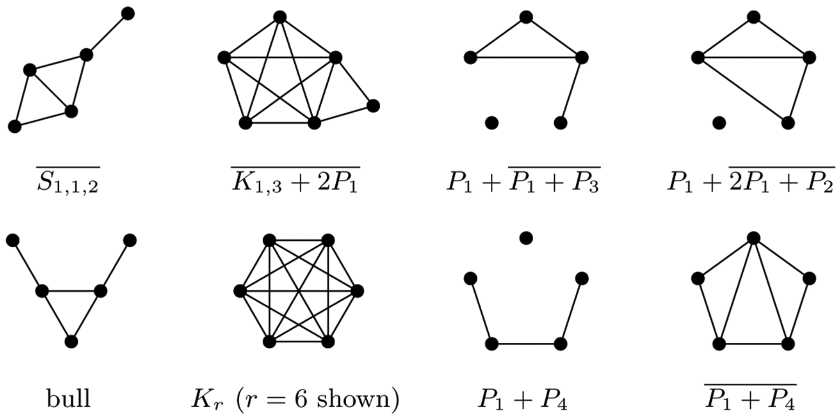

FIGURE 2. The graphs *H* listed in Theorem 1, for which the class of *H*-free chordal graphs has bounded clique-width; the four graphs at the top correspond to new cases proved in this article.

C. Our Results

In Section 2, we collect all previously known results for H-free chordal graphs and use a result of Olariu [54] to prove that bull-free chordal graphs have clique-width at most 3. In Section 3, we present four new classes of H-free chordal graphs of bounded clique-width,[3] namely when $H \in \{\overline{K_{1,3} + 2P_1}, P_1 + \overline{P_1 + P_3}, P_1 + \overline{2P_1 + P_2}, \overline{S_{1,1,2}}\}$ (see also Fig. 2). In particular, we show that $\overline{S_{1,1,2}}$-free graphs have clique-width at most 4. One of the algorithmic consequences of these results is that we have identified four new graph classes for which problems such as dominating set and Hamilton cycle are polynomial-time solvable. In Section 4, we combine all these results with previously known results [8, 10, 14, 38, 48] to obtain an almost complete classification for H-free chordal graphs (see also Fig. 2), leaving only two open cases (see also Fig. 3).

[3]In Theorems 25, 29, and 31, we do not specify our upper bounds as this would complicate our proofs for negligible gain. In our proofs, we repeatedly apply graph operations that exponentially increase the upper bound on the clique-width, which means that the bounds that could be obtained from our proofs would be very large and far from being tight. Furthermore, we make use of other results that do not give explicit bounds. We use different techniques to prove Lemma 17 and Theorem 34, and these allow us to give tight bounds for these cases.

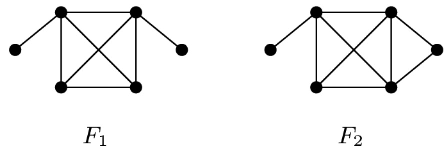

FIGURE 3. The two graphs H for which the boundedness of clique-width of the class of H-free chordal graphs is open.

Theorem 1. *Let H be a graph with $H \notin \{F_1, F_2\}$. The class of H-free chordal graphs has bounded clique-width if and only if*

(a) $H = K_r$ *for some* $r \geq 1$*;*
(b) $H \subseteq_i$ bull*;*
(c) $H \subseteq_i P_1 + P_4$*;*
(d) $H \subseteq_i \overline{P_1 + P_4}$*;*
(e) $H \subseteq_i \overline{K_{1,3} + 2P_1}$*;*
(f) $H \subseteq_i P_1 + \overline{P_1 + P_3}$*;*
(g) $H \subseteq_i P_1 + \overline{2P_1 + P_2}$ *or*
(h) $H \subseteq_i \overline{S_{1,1,2}}$.

In Section 4, we also show (using only previously known results) our aforementioned classification for H-free weakly chordal graphs.

Theorem 2. *Let H be a graph. The class of H-free weakly chordal graphs has bounded clique-width if and only if H is an induced subgraph of P_4.*

In Section 5, we illustrate the usefulness of having a classification for H-free chordal graphs by proving that the class of $(2P_1 + P_3, K_4)$-free graphs has bounded clique-width via a reduction to K_4-free chordal graphs. As such, up to an equivalence relation (see [29]), the number of pairs (H_1, H_2) for which we do not know whether the clique-width of the class of (H_1, H_2)-free graphs is bounded is eight. These remaining cases are given in Section 6 (see also [29]). In Section 6, we mention a number of future research directions.

2. PRELIMINARIES

All graphs considered in this article are finite, undirected, and have neither multiple edges nor self-loops. In this section, we first define some more standard graph terminology, some additional notation, and give some structural lemmas. We refer to the textbook of Diestel [32] for any undefined terminology. Afterwards, we give the definition of clique-width and present a number of known results on clique-width that we will use as lemmas for proving our results.

Let $G = (V, E)$ be a graph. For $S \subseteq V$, we let $G[S]$ denote the *induced* subgraph of G, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. For some set $T \subseteq V$, we may write $G - T = G[V \setminus T]$. Recall that for two graphs G and H, we write $H \subseteq_i G$ to indicate that H is an induced subgraph of G.

Let $G = (V, E)$ be a graph. The set $N(u) = \{v \in V \mid uv \in E\}$ is the *neighbourhood* of $u \in V$. The *degree* of a vertex $u \in V$ in G is the size $|N(u)|$ of its neighbourhood. The

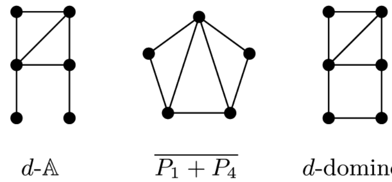

FIGURE 4. The minimal prime extensions of $\overline{2P_1 + P_2}$.

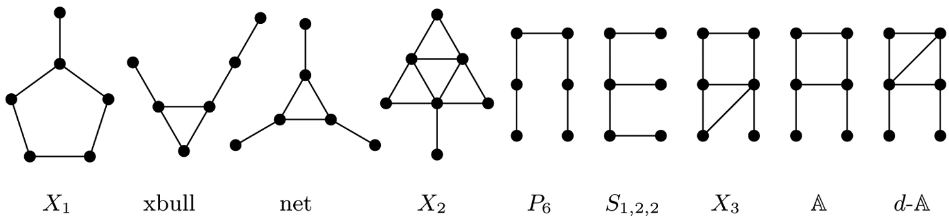

FIGURE 5. The minimal prime extensions of $P_1 + P_4$.

maximum degree of G is the maximum vertex degree. Let $S \subseteq V$. We define $N(S) = (\bigcup_{v \in S} N(v)) \setminus S$. For a vertex $u \in V$, we write $N_S(u) = N(u) \cap S$.

Let S and T be two vertex subsets of a graph $G = (V, E)$ with $S \cap T = \emptyset$. We say that S *dominates* T if every vertex of T is adjacent to at least one vertex of S. We say that S is a *dominating set* of G or that S *dominates* G if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. We say that S is *complete* to T if every vertex in S is adjacent to every vertex in T, and we say that S is *anticomplete* to T if every vertex in S is non-adjacent to every vertex in T. Similarly, a vertex $v \in V \setminus T$ is *complete* or *anticomplete* to T if it is adjacent or non-adjacent, respectively, to every vertex of T. A set of vertices M is a *module* if every vertex not in M is either complete or anticomplete to M. A module in a graph is *trivial* if it contains zero, one, or all vertices of the graph, otherwise it is *non-trivial*. We say that G is *prime* if every module in G is trivial. We say that a vertex v *distinguishes* two vertices x and y if v is adjacent to precisely one of x and y. Note that if a set $M \subseteq V$ is not a module then there must be vertices $x, y \in M$ and a vertex $v \in V \setminus M$ such that v distinguishes x and y.

The following two structural lemmas, both of which we need for the proofs of our results, are about prime graphs containing some specific induced subgraph H. They are examples of the well-developed technique of *prime extension*, that is, they show us that such prime graphs must also contain (as an induced subgraph) at least one of a list of possible extensions of H. The first prime extension lemma is due to Brandstädt, and the second one is due to Brandstädt, Le, and de Ridder.

Lemma 3 ([6]). *If a prime graph G contains an induced* $\overline{2P_1 + P_2}$*, then it contains an induced* $\overline{P_1 + P_4}$*, d-*$\mathbb{A}$ *or d-domino (see Fig. 4).*

Lemma 4 ([15]). *If a prime graph G contains an induced subgraph isomorphic to* $P_1 + P_4$*, then it contains one of the graphs in Figure 5 as an induced subgraph.*

We also use the following structural lemma due to Olariu.

Lemma 5 ([54]). *Every prime* (bull, house)*-free graph (see Fig. 6) is either* K_3*-free or the complement of a* $2P_2$*-free bipartite graph.*

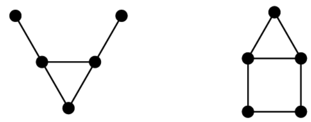

FIGURE 6. The graphs bull and house.

Let $G = (V, E)$ be a graph. An edge $e \in E$ is a *bridge* if deleting it would increase the number of components in G. A vertex $v \in V$ is a *cut-vertex* if $G[V \setminus \{v\}]$ has more connected components than G. If G is connected and has at least three vertices, but no cut-vertices then it is *2-connected*. For any two vertices u and v in a 2-connected graph, there are two paths from u to v that are internally vertex-disjoint (by Menger's Theorem, see, e.g. [32]). A *block* of G is a maximal 2-connected subgraph, a bridge or an isolated vertex. Note that two blocks of G have at most one common vertex, which must be a cut-vertex of G.

Recall that $K_{1,r}$ denotes the $(r + 1)$-vertex star. In this graph, the vertex of degree r is called the *central vertex*. A *double star* is the graph formed from two stars $K_{1,s}$ and $K_{1,r}$ by joining the central vertices of each star with an edge.

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is *independent* if $G[S]$ contains no edges. The *independence number* of G is the size of a largest independent set of G. If V can be partitioned into two (possibly empty) independent sets then G is *bipartite*. We say that G is *complete multipartite* if V can be partitioned into k independent sets $V_1, \ldots, V_k$ (called *partition classes*) for some integer k, such that two vertices are adjacent if and only if they belong to two different sets V_i and V_j.

The next result, which we will use later on, is due to Olariu [53] (note that the graph $\overline{P_1 + P_3}$ is also called the *paw*).

Lemma 6 ([53]). *Every connected* $(\overline{P_1 + P_3})$*-free graph is either complete multipartite or* K_3*-free.*

Let $G = (V, E)$ be a graph. A vertex $v \in V$ is *simplicial* if $G[N(v)]$ is complete. The following lemma is well known (see, e.g. [37]).

Lemma 7. *Every chordal graph has a simplicial vertex.*

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is said to be a *clique* if $G[S]$ is a complete graph. The *clique number* of G is the size of a largest clique of G. The *chromatic number* of G is the minimum number k for which G has a *k-colouring*, that is, for which there exists a mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever u and v are adjacent. We say than G is *perfect* if, for every induced subgraph $H \subseteq_i G$, the chromatic number of H equals its clique number. The graph G is a *split graph* if it has a *split partition*, that is, a partition of V into two (possibly empty) sets K and I, where K is a clique and I is an independent set; if K and I are complete to each other, then G is said to be a *complete* split graph.

It is well known that every split graph is chordal and that every chordal graph is perfect (see [37]). The first inclusion also follows from the next lemma, which is due to Földes and Hammer.

Lemma 8 ([35]). *A graph is split if and only if it is* $(C_4, C_5, 2P_2)$*-free.*

A graph G is a *thin spider* if its vertex set can be partitioned into a clique K, an independent set I and a set R such that $|K| = |I| \geq 2$, the set R is complete to K and anticomplete to I and the edges between K and I form an induced matching (i.e. every vertex of K has a unique neighbour in I and vice versa). Note that if a thin spider is prime, then $|R| \leq 1$. A *thick spider* is the complement of a thin spider. A graph is a *spider* if it is either a thin or a thick spider.

Spiders play an important role in our result for $\overline{S_{1,1,2}}$-free chordal graphs and we will need the following lemmas. The first is due to Brandstädt and Mosca and the second is due to Brandstädt, Dragan, Le, and Mosca.

Lemma 9 ([18]). *If G is a prime $S_{1,1,2}$-free split graph then it is a spider.*

Lemma 10 ([9]). *Prime thick spiders have clique-width at most* 4.

In fact, using the software of Heule and Szeider [42], one can verify that the bound in the above lemma is tight.

A. Clique-Width

The *clique-width* of a graph G, denoted by $\text{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

(1) creating a new graph consisting of a single vertex v with label i (denoted by $i(v)$);
(2) taking the disjoint union of two labelled graphs G_1 and G_2 (denoted by $G_1 \oplus G_2$);
(3) joining each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$);
(4) renaming label i to j (denoted by $\rho_{i\rightarrow j}$).

An algebraic term that represents such a construction of G and uses at most k labels is said to be a *k-expression* of G (i.e. the clique-width of G is the minimum k for which G has a k-expression). For instance, an induced path on four consecutive vertices a, b, c, d has clique-width equal to 3, and the following 3-expression can be used to construct it:

$$\eta_{3,2}(3(d) \oplus \rho_{3\rightarrow 2}(\rho_{2\rightarrow 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))).$$

A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. We define the following operations. The *subdivision* of an edge uv replaces uv by a new vertex w with edges uw and vw. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a nonedge, and vice versa. Similarly, for two disjoint vertex subsets S and T in G, the *bipartite complementation* operation with respect to S and T acts on G by replacing every edge with one end-vertex in S and the other one in T by a nonedge and vice versa.

We now state some useful facts about how the above operations (and some other ones) influence the clique-width of a graph. We will use these facts throughout the article. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is *(k, γ)-obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$, there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

If we do not impose a finite upper bound k on the number of applications of γ then we write that $\mathcal{G}'$ is (∞, γ)-obtained from $\mathcal{G}$.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [49].
Fact 2. Subgraph complementation preserves boundedness of clique-width [44].
Fact 3. Bipartite complementation preserves boundedness of clique-width [44].
Fact 4. If $\mathcal{G}$ is a class of graphs, then $\mathcal{G}$ has bounded clique-width if and only if the class of 2-connected induced subgraphs of graphs in $\mathcal{G}$ has bounded clique-width [4, 49].
Fact 5. For a class of graphs $\mathcal{G}$ of *bounded* maximum degree, let $\mathcal{G}'$ be a class of graphs that is $(\infty, \texttt{es})$-obtained from $\mathcal{G}$, where $\texttt{es}$ is the edge subdivision operation. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}'$ has bounded clique-width [44].

We also use a number of other elementary results on the clique-width of graphs. The first two are well known and straightforward to check.

Lemma 11. *The clique-width of a forest is at most* 3.

Lemma 12. *The clique-width of a graph with maximum degree at most* 2 *is at most* 4.

The following lemma tells us that if $\mathcal{G}$ is a hereditary graph class (i.e. a graph class closed under vertex deletion), then in order to determine whether $\mathcal{G}$ has bounded clique-width we may restrict ourselves to the graphs in $\mathcal{G}$ that are prime.

Lemma 13 ([23])**.** *Let G be a graph and let $\mathcal{P}$ be the set of all induced subgraphs of G that are prime. Then* $\text{cw}(G) = \max_{H \in \mathcal{P}} \text{cw}(H)$.

B. Known Results on *H*-Free Chordal Graphs

To prove our results, we need to use a number of known results. We present these results as lemmas in this subsection. The first of these lemmas gives a classification for H-free graphs.

Lemma 14 ([29])**.** *Let H be a graph. The class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of P_4.*

We will use the following characterization of graphs H for which the class of H-free bipartite graphs has bounded clique-width (which is similar to a characterization of Lozin and Volz [50] for a different variant of the notion of H-freeness in bipartite graphs, see [28] for an explanation of the difference).

Lemma 15 ([28])**.** *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

(a) $H = sP_1$ *for some* $s \geq 1$;
(b) $H \subseteq_i K_{1,3} + 3P_1$;
(c) $H \subseteq_i K_{1,3} + P_2$;
(d) $H \subseteq_i P_1 + S_{1,1,3}$;
(e) $H \subseteq_i S_{1,2,3}$.

For a graph G, let $\text{tw}(G)$ denote the treewidth of G (see, e.g. Diestel [32] for a definition of this notion). Corneil and Rotics [21] showed that $\text{cw}(G) \leq 3 \times 2^{\text{tw}(G)-1}$ for every graph G. Because the treewidth of a chordal graph is equal to the size of a maximum clique minus 1 (see, e.g. [3]), this result leads to the following well-known lemma.

Lemma 16. *The class of K_r-free chordal graphs has bounded clique-width for all $r \geq 1$.*

The *bull* is the graph obtained from the cycle *abca* by adding two new vertices d and e with edges ad, be (see also Fig. 2). In [14], Brandstädt et al. erroneously claimed that the clique-width of $\overline{S_{1,1,2}}$-free chordal graphs and of bull-free chordal graphs is unbounded. Using a general result of De Simone [31], Le [48] proved that every bull-free chordal graph has clique-width at most 8. Using a result of Olariu [54], we can prove the following.

Lemma 17. *Every bull-free chordal graph has clique-width at most* 3.

Proof. Let G be a bull-free chordal graph. By Lemma 13, we may assume that G is prime. Note that the house contains an induced C_4, so G is house-free. Then, by Lemma 5, G is either K_3-free or the complement of a $2P_2$-free bipartite graph. Every K_3-free chordal graph is a forest, so by Lemma 11 it has clique-width at most 3. We may therefore assume that G is a prime graph that is the complement of a $2P_2$-free bipartite graph. Such graphs are known as k-webs in [54], where $k \geq 2$. A k-web consists of two cliques, $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$, such that for $i, j \in \{1, \ldots, k\}$ the vertex x_i is adjacent to y_j if and only if $i < j$. We will show how to use the operations of clique-width constructions to inductively build, using three labels, a copy of a k-web in which every vertex in the set X is labelled 1 and every vertex in the set Y is labelled 2. Consider a k-web labelled as described above for some $k \geq 0$ (if $k = 0$, this is the empty graph). Add a vertex labelled 3 to the graph, join it to every vertex of label 1 and to every vertex of label 2, then relabel it to have label 1. Next, add a vertex labelled 3 to the graph, join it to every vertex of label 2, then relabel it to have label 2. This is precisely the $(k+1)$-web, also labelled as described above. We conclude that every k-web can be constructed using at most 3 labels, so G has clique-width at most 3. ■

Since P_4 is a bull-free chordal graph and has clique-width 3, the bound in the above lemma is tight. Next, we recall the aforementioned results of Brandstädt et al.

Lemma 18 ([14])**.** *Every $P_1 + P_4$-free chordal graph has clique-width at most* 8.

Lemma 19 ([14])**.** *Every $\overline{P_1 + P_4}$-free chordal graph has clique-width at most* 3.

Lemma 20 ([10])**.** *The class of* $4P_1$*-free chordal graphs has unbounded clique-width (see also Fig. 7).*

Recall that Golumbic and Rotics [38] proved that the class of proper interval graphs has unbounded clique-width. Such graphs are well known to be $K_{1,3}$-free and chordal [58].

Lemma 21. *The class of $K_{1,3}$-free chordal graphs has unbounded clique-width (see also Fig. 7).*

The next lemma is obtained by combining Lemma 8 with the aforementioned result of Makowsky and Rotics, who showed that the class of split graphs has unbounded clique-width.

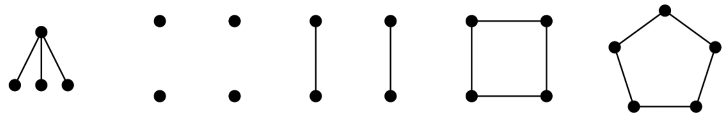

FIGURE 7. Graphs H for which the class of H-free chordal graphs was previously known to have unbounded clique-width.

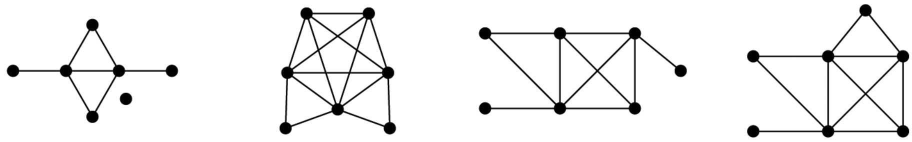

FIGURE 8. The four graphs for which it is not known whether or not the class of H-free split graphs has bounded clique-width. (Recall that the cases F_4 and $\overline{F_4}$ are equivalent to each other and the cases F_5 and $\overline{F_5}$ are also equivalent to each other.)

Lemma 22 ([51]). *The class of* $(C_4, C_5, 2P_2)$*-free graphs (or equivalently split graphs) has unbounded clique-width.*

We note that Lemma 22 also follows from a result of Korpelainen et al. [46], who proved that the class of split permutation graphs has unbounded clique-width. Moreover, Lemma 22 implies that the class of H-free chordal graphs has unbounded clique-width for $H \in \{C_4, C_5, 2P_2\}$ (see also Fig. 7).

Recall that by Lemma 8, every split graph is a chordal graph. Therefore, if the class of H-free chordal graphs has bounded clique-width then the class of H-free split graphs must also have bounded clique-width. To prove Theorem 1, we will make heavy use of the following lemma. This lemma can be seen as a refinement of Lemma 22, as it classifies all but two graphs H (up to complementation) for which the class of H-free split graphs has bounded clique-width. (Note that a graph is a split graph if and only if its complement is a split graph, so by Fact 2, the class of H-free split graphs has bounded clique-width if and only if the class of $\overline{H}$-free split graphs has bounded clique-width.)

Lemma 23 ([8]). *Let H be a graph such that neither H nor $\overline{H}$ is in $\{F_4, F_5\}$ (see Fig. 8). The class of H-free split graphs has bounded clique-width if and only if*

(a) H or $\overline{H}$ is isomorphic to rP_1 for some $r \geq 1$;
(b) H or $\overline{H} \subseteq_i F_4$; or
(c) H or $\overline{H} \subseteq_i F_5$.

A graph $G = (V, E)$ is a *permutation graph* if there exists a set of straight line segments between two parallel lines, where each vertex $v \in V$ corresponds to a straight line segment l_v such that there is an edge between two vertices u and v if and only if l_u and l_v intersect. A graph is *bipartite permutation* if it is both bipartite and permutation. We need the following result due to Brandstädt and Lozin, which we will use in the proof of Theorem 2.

Lemma 24 ([16]). *The class of bipartite permutation graphs has unbounded clique-width.*

3. NEW CLASSES OF BOUNDED CLIQUE-WIDTH

We present four new classes of H-free chordal graphs that have bounded clique-width, namely when $H \in \{\overline{K_{1,3} + 2P_1}, P_1 + \overline{P_1 + P_3}, P_1 + \overline{2P_1 + P_2}, \overline{S_{1,1,2}}\}$. We prove that these classes have bounded clique-width in the subsections below, making use of known results from Section 2. In particular, we will often use Facts 1–5. Note that Facts 1 and 4 can be used safely, since every class of H-free chordal graphs is closed under vertex deletion (when applying the other three facts we need to be more careful).

A. The Case $H = \overline{K_{1,3} + 2P_1}$

Here is our first result. To prove it, we use the celebrated Menger's Theorem (see, e.g. [32]) and the facts from Section 2. In particular Fact 4, which states that a graph G has bounded clique-width if and only if every block of G has bounded clique-width, will play an important role in our proof.

Theorem 25. *The class of $\overline{K_{1,3} + 2P_1}$-free chordal graphs has bounded clique-width.*

Proof. Let G be a $\overline{K_{1,3} + 2P_1}$-free chordal graph. By Fact 4, we may assume that G is 2-connected. Let K be a maximum clique in G on k vertices. We may assume that $k \geq 7$, otherwise G is K_7-free, in which case G has bounded clique-width by Lemma 16. We let S be the set of vertices outside K with at least two neighbours in K. Because K is maximum, $k \geq 5$ and G is $\overline{K_{1,3} + 2P_1}$-free, every vertex in S has either exactly one or exactly two non-neighbours in K.

We will prove that $V(G) = K \cup S$. To this end, we first prove that $G - S$ is connected. Suppose, for contradiction, that there exists a vertex x that is in a connected component D of $G - S$ other than the component containing K. Let $u \in K$. Because G is 2-connected, it contains two paths P_1 and P_2 from x to u that are internally vertex-disjoint (by Menger's Theorem). Note that we may assume that each P_i is induced. For $i = 1, 2$, let $s_i \in P_i$ be the first vertex that is not in D and let x_i be the predecessor of s_i on P_i. Note that $s_1, s_2 \in S$. Since $k \geq 5$ and every vertex in S has at most two non-neighbours in K, there must be a vertex $u' \in K$ adjacent to both s_1 and s_2. For $i = 1, 2$, let P_i' be the path from x to u' formed by taking the part of the path P_i from x to s_i and adding u'. Note that P_1' and P_2' are both induced paths in G and each contains exactly one vertex from K and one from S. Since G is chordal, s_1 and s_2 must be adjacent and at least one of x_1 and x_2 must be adjacent to both s_1 and s_2. Without loss of generality, we assume that x_1 is adjacent to both s_1 and s_2. On the other hand, since $k \geq 7$ and every vertex in S has at most two non-neighbours in K, the vertices s_1 and s_2 have at least three common neighbours in K. Let $k_1, k_2, k_3 \in K$ be three common neighbours of s_1 and s_2. Then $G[x_1, k_1, k_2, k_3, s_1, s_2]$ is a $\overline{K_{1,3} + 2P_1}$, a contradiction. Thus $G - S$ is indeed connected.

Suppose, for contradiction, that $V(G) \neq K \cup S$. Since $G - S$ is connected, there must be a vertex $y \in V(G) \setminus (K \cup S)$ adjacent to a vertex $v \in K$. As $y \notin S$, y is anticomplete to $K \setminus \{v\}$. Let $u \in K \setminus \{v\}$. Since G is 2-connected, there must exist an induced path P from y to u with $v \notin V(P)$ (by Menger's Theorem). Then v is complete to $V(P)$ since G is chordal. Let y' be the last vertex (from y to u) on P that is not in $K \cup S$ (note that y' is not necessarily distinct from y). Let s be the successor of y' on P. Since $y' \notin S$ and y' is adjacent to v, we find that y' is anticomplete to $K \setminus \{v\}$. Hence, $s \notin K$, so $s \in S$. Moreover, s and v have at least four common neighbours in $K \setminus \{v\}$, since $k \geq 7$ and every vertex in S has at most two non-neighbours in K. Let $k_1, k_2, k_3 \in K \setminus \{v\}$ be three common neighbours of s and v. Then $G[y', k_1, k_2, k_3, s, v]$ is a $\overline{K_{1,3} + 2P_1}$, a contradiction.

For $i = 1, 2$, let S_i consist of those vertices with exactly i non-neighbours in K. Because every vertex in S has either one or two non-neighbours in K, we find that $S = S_1 \cup S_2$.

We will now prove, via Claims 1–5, that $G[S]$ is a forest.

Claim 1. *Any two adjacent vertices in S_2 have the same pair of non-neighbours in K.*

This follows directly from the fact that G is chordal.

Claim 2. *Any two non-adjacent vertices in S_2 have a common non-neighbour.*

Suppose that this is not the case. Then there exist two non-adjacent vertices $t, t' \in S_2$ and four distinct vertices $a, b, c, d \in K$ with t non-adjacent to a and b and with t' non-adjacent to c and d. As t and t' belong to S_2, it follows that t is adjacent to c and d, and that t' is adjacent to a and b. Since $k \geq 7$, we find that t and t' have two common neighbours in K. These two common neighbours, together with c, d, t, t' form an induced $\overline{K_{1,3} + 2P_1}$, a contradiction.

Claim 3. *If a vertex $s \in S_1$ is adjacent to a vertex $t \in S_2$ then s and t must have a common non-neighbour in K.*

Indeed, let v be the unique non-neighbour of s in K. Then v must be a non-neighbour of t otherwise a non-neighbour of t in K, together with s, t and v would induce a C_4 in G. This contradicts the fact that G is chordal.

Claim 4. *S_1 is an independent set.*

This holds as no two vertices in S_1 with a common non-neighbour in K are adjacent since K is maximum, while no two vertices in S_1 with different non-neighbours in K are adjacent since G is chordal.

Claim 5. *$G[S]$ is a forest.*

Suppose, for contradiction, that $G[S]$ is not a forest. Then, since G is chordal, $G[S]$ must contain a C_3, on vertices c_1, c_2, c_3, say. By Claim 4, we may assume without loss of generality that $c_2, c_3 \notin S_1$ and thus $c_2, c_3 \in S_2$. Then c_2 and c_3 must have the same pair of non-neighbours $a, b \in K$ by Claim 1. If $c_1 \in S_2$ then by Claim 1, the non-neighbours of c_1 in K are also a and b. If $c_1 \in S_1$ then by Claim 3, the non-neighbour of c_1 in K is either a or b. Hence, in both these cases, $(K \setminus \{a, b\}) \cup \{c_1, c_2, c_3\}$ is a clique of size more that $|K|$, contradicting the maximality of K.

We will consider two cases depending on whether or not $G[S]$ is $2P_2$-free. To do so, we first need to prove two more claims.

Claim 6. *If two vertices $s_1, s_2 \in S$, together with a vertex $w \in K$ form a triangle then w is complete to $S \setminus (N(s_1) \cup N(s_2))$.*

Indeed, suppose, for contradiction, that $t \in S \setminus (N(s_1) \cup N(s_2))$ is not adjacent to w. Since $|K| \geq 7$, there must be vertices $x, y \in K$ that are complete to $\{s_1, s_2, t\}$. Since t is non-adjacent to s_1 and s_2, we find that $\{t, s_1, s_2, w, x, y\}$ induces a $\overline{K_{1,3} + 2P_1}$, a contradiction.

Claim 7. *For any connected component D in $G[S]$ that contains at least one edge, there exist two vertices a and b in K such that $K \setminus \{a, b\}$ is complete to $S \setminus V(D)$.*

To see this, let D be a connected component with an edge st. Since S_1 is independent by Claim 4, we may assume that $t \in S_2$. Let a and b in K be the two non-neighbours of t. It follows from Claims 1 and 3 that a and b are the only possible non-neighbours of s or t in K. In other words, $K \setminus \{a, b\}$ is complete to $\{s, t\}$, and hence to $S \setminus V(D)$ by Claim 6.

We are now ready to consider the two cases.

Case 1: $G[S]$ *contains an induced* $2P_2$.
First, suppose that $G[S]$ has only one connected component that contains an edge. Then, since $G[S]$ is a forest and $G[S]$ contains an induced $2P_2$, deleting one vertex from S, which we may do by Fact 1, yields two connected components D_1 and D_2 that contain edges s_1t_1 and s_2t_2, respectively. It follows from Claim 7 that there exist vertices a and b in K such that $S \setminus D_1$ is complete to $K \setminus \{a, b\}$. In particular, s_2 and t_2 are complete to $K \setminus \{a, b\}$. Hence, $K \setminus \{a, b\}$ is also complete to D_1, by Claim 6. Thus, $K \setminus \{a, b\}$ is complete to S. Deleting a and b (which we may do by Fact 1) and applying a bipartite complementation between $K \setminus \{a, b\}$ and S (which we may do by Fact 3) splits the graph into two disjoint parts: a complete graph $G[K \setminus \{a, b\}]$, which has clique-width 2, and a forest $G[S]$, which has clique-width at most 3 by Lemma 11. We conclude that G has bounded clique-width.[4]

Case 2: $G[S]$ *is* $2P_2$*-free.*
In this case, S contains at most one connected component with an edge. If such a connected component exists, then it is a $2P_2$-free tree, and hence it must be a P_2, $K_{1,r}$, or a double star. In all three cases, deleting at most two vertices from S, which we may do by Fact 1, yields a split graph. If $S_2 \neq \emptyset$ then let s be a vertex in S_2 and let k_1 and k_2 be its two (only) non-neighbours in K. By Claim 2, any other vertex of S_2 is non-adjacent to at least one of k_1, k_2. Hence, after removing k_1 and k_2 (which we may do by Fact 1), every vertex of S is adjacent to all but at most one vertex of K. (In the case where $S_2 = \emptyset$, we do not need to remove any vertices of K.) Next, we perform a bipartite complementation between K and S, which we may do by Fact 3. This results in a new split graph in which each vertex of S is adjacent to at most one vertex of K. Hence, this graph, and consequently G, has bounded clique-width by Fact 4. ■

B. The Case $H = P_1 + \overline{P_1 + P_3}$

We first prove three useful lemmas.

Lemma 26. *The class of* $(P_1 + \overline{P_1 + P_3})$*-free split graphs has bounded clique-width.*

Proof. Let G be an arbitrary $(P_1 + \overline{P_1 + P_3})$-free split graph with split partition (C, I). By Fact 2, we may apply a subgraph complementation on the clique C. The resulting graph G' is bipartite. Because G is $(P_1 + \overline{P_1 + P_3})$-free, G' is $(P_2 + P_4)$-free and thus

[4]We mean to say that the clique-width of G is bounded by a constant that does not depend on the size of G but only on the class of graphs under consideration. We allow this minor abuse of notation throughout the article.

$S_{1,2,3}$-free. Then the result follows from the fact that $S_{1,2,3}$-free bipartite graphs have bounded clique-width by Lemma 15. ■

Lemma 27. *Every connected $\overline{P_1 + P_3}$-free chordal graph is a tree or a complete split graph.*

Proof. Let G be a connected $\overline{P_1 + P_3}$-free chordal graph. By Lemma 6, we find that G is C_3-free or complete multipartite. If G is C_3-free, then it must be a tree, since G is chordal. If G is complete multipartite, then at most one partition class of G can contain more than one vertex, otherwise G would contain an induced C_4. This means that G is a complete split graph. ■

Note that every induced $\overline{P_1 + P_3}$ in a $(P_1 + \overline{P_1 + P_3})$-free graph G is a dominating set of G. The proof of the next lemma, in which disconnected graphs are considered, heavily relies on this fact. We will also heavily exploit this property in the proof for the general case.

Lemma 28. *The class of disconnected $(P_1 + \overline{P_1 + P_3})$-free chordal graphs has clique-width at most 3.*

Proof. Let G be a disconnected $(P_1 + \overline{P_1 + P_3})$-free chordal graph. Since G has at least two connected components and each connected component contains a P_1, every connected component of G must therefore be $\overline{P_1 + P_3}$-free. By Lemma 27, every connected component of G must be a complete split graph or a tree. In the first case, the clique-width of the connected component is readily seen to be at most 2. In the second case, the clique-width of that connected component is at most 3 by Lemma 11. ■

We are now ready to prove our second result.

Theorem 29. *The class of $(P_1 + \overline{P_1 + P_3})$-free chordal graphs has bounded clique-width.*

Proof. Let G be a $(P_1 + \overline{P_1 + P_3})$-free chordal graph. Let x be a simplicial vertex in G, which exists by Lemma 7. Let $X = N(x)$ and $Y = V(G) \setminus (X \cup \{x\})$. Note that no vertex of Y is adjacent to x, so $G[Y]$ must be $\overline{P_1 + P_3}$-free. By Lemma 27, every connected component of $G[Y]$ is either a tree or complete split graph. We say that a connected component of $G[Y]$ is *trivial* if it consists of a single vertex. Otherwise it is *non-trivial*.

We will distinguish between two cases depending on whether or not $G[Y]$ is $2P_2$-free. In the first case, we will need the following claim.

Claim 1. *Suppose that $G[Y]$ contains at least two non-trivial components and $y \in Y$ is in such a component. If y is adjacent to $z \in X$ then y is complete to X or z is complete to Y.*

In order to prove this claim, suppose that y is not complete to X. We will show that z is complete to Y. Let D be the connected component of $G[Y]$ containing y. Since y is not complete to X, there must be a vertex $z' \in X$ that is not adjacent to y. Now $G[z, x, y, z']$ is a $\overline{P_1 + P_3}$. Since G is $(P_1 + \overline{P_1 + P_3})$-free, we find that $\{x, y, z, z'\}$ must dominate G. No vertex of $Y \setminus V(D)$ is adjacent to x or y. Therefore $Y \setminus V(D)$ is dominated by $\{z, z'\}$.

Let y_1y_2 be an edge in some non-trivial component D' of Y other than D (recall that such a component exists by our assumption). If y_1 and y_2 are both adjacent to z', then $G[y, z', y_1, x, y_2]$ would be a $P_1 + \overline{P_1 + P_3}$. Therefore, we may assume without loss of generality that y_1 is not adjacent to z'. Since $\{z, z'\}$ dominates y_1, we find that y_1 must be adjacent to z. If y_2 is not adjacent to z then, since $\{z, z'\}$ dominates y_2, we find that y_2 must be adjacent to z'. In this case $G[z, z', y_2, y_1]$ would be a C_4, contradicting the fact that G is chordal. Hence, both y_1 and y_2 are adjacent to z. Now $G[z, y_1, x, y_2]$ induces a $\overline{P_1 + P_3}$. Therefore, z is complete to $Y \setminus D'$, since G is $(P_1 + \overline{P_1 + P_3})$-free. Recall that y_1 is adjacent to z and non-adjacent to z'. By the same argument, with y_1 taking the role of y, since D is a non-trivial component of $G[Y]$, we find that z is complete to $Y \setminus V(D)$. Hence, z is complete to Y. This completes the proof of Claim 1.

We are now ready to consider the two possible cases.

Case 1: $G[Y]$ *contains an induced* $2P_2$.

First, suppose that all vertices of this $2P_2$ are in the same connected component D of $G[Y]$. Since split graphs are $2P_2$-free by Lemma 8, we find that D is a tree by Lemma 27. In this case, by Fact 1, we may delete one vertex in D so that the two edges of the $2P_2$ are in two different connected components of $G[Y]$. We may therefore assume without loss of generality that $G[Y]$ contains two non-trivial components.

Let Y' be the set of vertices in Y that are in non-trivial components of $G[Y]$. Let Y'' be the set of vertices in Y' that are complete to X. Let X' be the set of vertices in X that are complete to Y. It follows from Claim 1 that $X \setminus X'$ is anticomplete to $Y' \setminus Y''$. We can apply two bipartite complementation operations, one between X' and $Y' \cup \{x\}$ and the other between $Y'' \cup \{x\}$ and $X \setminus X'$. This will separate $G[Y' \cup \{x\}]$ from the rest of the graph. By Lemma 28, we find that $G[Y' \cup \{x\}]$ has bounded clique-width. Because $G[V \setminus (Y' \cup \{x\})]$ is a $(P_1 + \overline{P_1 + P_3})$-free split graph, it has bounded clique-width by Lemma 26. By Fact 3, we find that G has bounded clique-width. This completes the proof of Case 1.

Case 2: $G[Y]$ *is* $2P_2$*-free*.

If $G[Y]$ contains only trivial components then G is a $(P_1 + \overline{P_1 + P_3})$-free split graph, so it has bounded clique-width by Lemma 26. Since $G[Y]$ is $2P_2$-free, it can contain at most one non-trivial component. We may therefore assume that $G[Y]$ contains exactly one non-trivial component D.

First, suppose that D is a tree. In this case, $G[D]$ must be a P_2, $K_{1,r}$, or a double star. In all three cases, deleting at most two vertices in D (which we may do by Fact 1) makes Y an independent set, in which case we argue as before. By Lemma 27, we may therefore assume that $G[Y]$ is a complete split graph. We can partition $V(D)$ into two sets, D_B and D_W, such that D_B is a clique, D_W is an independent set and D_B is complete to D_W in G. We may assume that $|D_B| \geq 3$. Indeed, if $|D_B| \leq 2$ then by Fact 1 we may delete at most two vertices to obtain a graph in which $G[Y]$ has only trivial components, in which case we may argue as before.

Let X' be the set of vertices in X that have neighbours in D. We claim that X' is complete to $Y \setminus V(D)$. Suppose, for contradiction, that $x' \in X'$ is not adjacent to some vertex $y \in Y \setminus V(D)$. Then x' cannot have two neighbours $y_1, y_2 \in D_B$ otherwise $G[y, x', y_1, x, y_2]$ would be a $P_1 + \overline{P_1 + P_3}$. Let $y_1 \in V(D)$ be a neighbour of x'. Since $|D_B| \geq 3$, x' must have two non-neighbours $y_2, y_3 \in D_B$. However, now $G[y, y_1, y_2, x', y_3]$ is a $P_1 + \overline{P_1 + P_3}$. This contradiction means that X' is indeed complete to $Y \setminus V(D)$.

As X is a clique and X' is complete to $Y \setminus V(D)$, we find that $(Y \setminus V(D)) \cup (X \setminus X')$ is complete to X'. By Fact 3, we may apply a bipartite complementation between $(Y \setminus V(D)) \cup (X \setminus X')$ and X' and another between $X \setminus X'$ and $\{x\}$. This separates $G[(Y \setminus V(D)) \cup (X \setminus X')]$ from the rest of the graph, which is $G[\{x\} \cup X' \cup V(D)]$. The first graph is a $(P_1 + \overline{P_1 + P_3})$-free split graph, so it has bounded clique-width by Lemma 26. It remains to show that $G[\{x\} \cup X' \cup V(D)]$ has bounded clique-width.

We partition the vertices of X' as follows: Let Z be the set of vertices in X' that are complete to D_B, let Z' be the set of vertices in $X' \setminus Z$ that are complete to D_W and let $Z'' = X' \setminus (Z \cup Z')$. Let D'_W be the set of vertices in D_W that are complete to $Z' \cup Z''$ and let $D''_W = D_W \setminus D'_W$.

We claim that D''_W is anticomplete to Z''. Suppose, for contradiction, that $w \in D''_W$ is adjacent to $z \in Z''$. By definition, w must be non-adjacent to some vertex $z' \in Z' \cup Z''$ and z must be non-adjacent to some vertex $w' \in D_W$. Furthermore, z must be non-adjacent to some vertex $b \in D_B$. Note that w is not adjacent to w' since D_W is independent. Moreover, z and z' are adjacent because X' is a clique, and b is adjacent to both w and w' as D is a complete split graph. Then b and z' must be non-adjacent, otherwise $G[b, w, z, z']$ would be a C_4. Then w' must be adjacent to z', otherwise $G[w', z, x, w, z']$ would be a $P_1 + \overline{P_1 + P_3}$. However, this means that $G[z', z, w, b, w']$ induces a C_5, contradicting the fact that G is chordal. Therefore, D''_W is indeed anticomplete to Z''.

By Fact 1, we may delete the vertex x from G. Now $D_B \cup Z'$ is complete to $D'_W \cup D''_W \cup Z$, while Z'' is complete to $D'_W \cup Z$ and anticomplete to D''_W. By Fact 3, we may apply two bipartite complementations: one between $Z' \cup D_B$ and $D'_W \cup D''_W \cup Z$ and the other between Z'' and $D'_W \cup Z$. The resulting graph will be the disjoint union of two graphs: $G[D_W \cup Z]$ and $G[D_B \cup Z' \cup Z'']$. The first of these is a $(P_1 + \overline{P_1 + P_3})$-free split graph, so it has bounded clique-width by Lemma 26. Taking the complement of $G[D_B \cup Z' \cup Z'']$ (which we may do by Fact 2) yields the bipartite graph $\overline{G}[D_B \cup Z' \cup Z'']$, which is $2P_2$-free since G is chordal and therefore has bounded clique-width by Lemma 15. We conclude that G has bounded clique-width. This completes the proof of Theorem 29. ■

C. The Case $H = P_1 + \overline{2P_1 + P_2}$

A graph $G = (V, E)$ is *quasi-diamond-free* if its vertex set V can be partitioned into a clique V_1 and some other (possibly empty) set $V_2 = V \setminus V_1$ so that $G[V_2]$ is a $\overline{2P_1 + P_2}$-free chordal graph, every connected component of which has at most one neighbour in V_1.

We prove the following lemma, which will play an important role in our proof.

Lemma 30. *The class of quasi-diamond-free graphs has bounded clique-width.*

Proof. Let G be a quasi-diamond-free graph with corresponding clique V_1. Let B be a block of G. Then B is either equal to V_1 or contains at most one vertex of V_1 with all its other vertices belonging to V_2. In the first case, the clique-width of B is at most 2. In the second case, we may delete the vertex of $B \cap V_1$ from B (if such a vertex exists) by Fact 1. This yields a $\overline{2P_1 + P_2}$-free chordal graph G'. By Theorem 25, we find that G' has bounded clique-width. Therefore, G has bounded clique-width by Fact 4. ■

We are now ready to prove the following result.

Theorem 31. *The class of* $(P_1 + \overline{2P_1 + P_2})$*-free chordal graphs has bounded clique-width.*

Proof. Let $G = (V, E)$ be a $(P_1 + \overline{2P_1 + P_2})$-free chordal graph. We may assume without loss of generality that G is connected. Let v be a simplicial vertex in G, which exists by Lemma 7. Let $L_1 = N(v)$, $L_2 = N(L_1) \setminus (L_1 \cup \{v\})$ and $L_3 = N(L_2) \setminus (L_2 \cup L_1 \cup \{v\})$. Note that L_1 is a clique, because v is simplicial.

Claim 1. *If* $s, t \in L_2 \cup L_3$ *are non-adjacent then* s *is adjacent to all but at most one vertex of* $N_{L_1}(t)$.

Indeed, suppose, for contradiction, that s is non-adjacent to distinct vertices $a, b \in N_{L_1}(t)$. Then $G[s, a, b, t, v]$ is a $P_1 + \overline{2P_1 + P_2}$, a contradiction.

Let x be a vertex of L_2 such that $\Delta = |N_{L_1}(x)|$ is maximized. Note that $G[V \setminus (\{v\} \cup L_1)]$ is $\overline{2P_1 + P_2}$-free and $\{v\} \cup L_1$ is a clique. Hence, if $\Delta = 1$, then we can apply Lemma 30 to G with $V_1 = \{v\} \cup L_1$. Thus, from now on we may assume that $\Delta \geq 2$. This means that x and v have at least two common neighbours in L_1. Hence, as G is $(P_1 + \overline{2P_1 + P_2})$-free, we find that

$$V = \{v\} \cup L_1 \cup L_2 \cup L_3.$$

Claim 2. *We may assume that every vertex in* L_1 *has a neighbour in* L_2.

In order to show this, let $L_1' \subseteq L_1$ be the set of vertices with no neighbour in L_2. We apply a bipartite complementation between $(L_1 \setminus L_1') \cup \{v\}$ and L_1'. We may do so due to Fact 3. As $G[L_1']$ is a complete graph, it has clique-width at most 2, and we are left to consider $G[V \setminus L_1']$.

As $\Delta = |N_{L_1}(x)|$, we find that $\Delta \leq |L_1|$. We now consider two cases, depending on the difference between $|L_1|$ and Δ.

Case 1: $\Delta \leq |L_1| - 2$.
For $z \in L_1 \setminus N_{L_1}(x)$, let A_z be the set of neighbours of z in L_2. By Claim 2, we find that $A_z \neq \emptyset$ for all $z \in L_1 \setminus N_{L_1}(x)$.
Suppose that $z \in L_1 \setminus N_{L_1}(x)$ and $u \in A_z$. By our choice of x, we have that $|N_{L_1}(u)| \leq |N_{L_1}(x)|$, and so u must have a non-neighbour $y_u \in N_{L_1}(x)$. Then u is non-adjacent to x otherwise $G[u, x, y_u, z]$ would be a C_4, contradicting the fact that G is chordal. Now by Claim 1, we find that

$$N_{L_1}(u) = (N_{L_1}(x) \setminus \{y_u\}) \cup \{z\}.$$

The above implies that $A_z \cap A_{z'} = \emptyset$ for all $z, z' \in L_1 \setminus N_{L_1}(x)$ with $z \neq z'$.
We now show that $y_u = y_{u'}$ for any two vertices $u \in A_z$ and $u' \in A_{z'}$ and for any two (not necessarily distinct) vertices $z, z' \in L_1 \setminus N_{L_1}(x)$. First, suppose that $z, z' \in L_1 \setminus N_{L_1}(x)$ are distinct. Let $u \in A_z$ and $u' \in A_{z'}$. We may assume that such vertices exist since A_z and $A_{z'}$ are not empty by Claim 2. If $y_u \neq y_{u'}$ then, since $y_u, y_{u'} \in N_{L_1}(x)$ and $z, z' \in L_1 \setminus N_{L_1}(x)$, we find that $y_u, y_{u'}, z$ and z' are distinct vertices in L_1. Since $N_{L_1}(u) = (N_{L_1}(x) \setminus \{y_u\}) \cup \{z\}$ and $N_{L_1}(u') = (N_{L_1}(x) \setminus \{y_{u'}\}) \cup \{z'\}$, we find that u is adjacent to $y_{u'}$ and z, but u' is non-adjacent to both $y_{u'}$ and z. Therefore, Claim 1 implies that u and u' must be

adjacent; however then $G[u, u', y_u, y_{u'}]$ is a C_4, a contradiction. Hence, $y_u = y_{u'}$. Since the u-vertices in different sets A_z and $A_{z'}$ share the same y-vertex, and there are at least two such sets (since $\Delta \leq |L_1| - 2$), this immediately implies that u-vertices from the same set A_z also share the same y-vertex. Thus, there exists a vertex $y^* \in N_{L_1}(x)$ such that for every $z \in L_1 \setminus N_{L_1}(x)$ and every $u \in A_z$, we have

$$N_{L_1}(u) = (N_{L_1}(x) \setminus \{y^*\}) \cup \{z\}.$$

Let $A = N_{L_1}(x) \setminus \{y^*\}$. Let A_{y^*} be the set of vertices in L_2 whose neighbourhood in L_1 is $N_{L_1}(x)$ (so $x \in A_{y^*}$). Now for each vertex $z \in L_1 \setminus A$ (including the case where $z = y^*$) and every $u \in A_z$, we have

$$N_{L_1}(u) = A \cup \{z\}.$$

Let X be the set of vertices $u \in L_2 \cup L_3$ whose neighbourhood in L_1 is properly contained in $N_{L_1}(x)$, that is, for which $N_{L_1}(u) \subsetneq N_{L_1}(x) = A \cup \{y^*\}$. Note that, as no vertex in L_3 has a neighbour in L_1, we have $L_3 \subseteq X$. Also note that the sets X and A_z, $z \in L_1 \setminus A$ form a partition of $L_2 \cup L_3$.

Consider two distinct vertices $w_1, w_2 \in L_1 \setminus A$. Note that w_1 and w_2 are not necessarily distinct from y^*, but at least one of w_1, w_2 is distinct from y^*. Also note that if a vertex $u \in X$ is adjacent to w_i $(i = 1, 2)$ then $w_i = y^*$.

Suppose that there is a path P in $G[L_2 \cup L_3]$ from some vertex $t_1 \in A_{w_1}$ to some vertex $t_2 \in A_{w_2}$. We will choose P such that $|V(P)|$ is minimum, where the minimum is taken over all choices of w_1, w_2, t_1, t_2 and P. It follows from the minimality of P that $V(P) \setminus \{t_1, t_2\} \subseteq X$. Moreover, since $N_{L_1}(t_1) = A \cup \{w_1\}$ and $N_{L_1}(t_2) = A \cup \{w_2\}$, it follows that w_1 and w_2 are non-adjacent to t_2 and t_1, respectively. Thus, t_1 and t_2 must be non-adjacent, as otherwise $G[t_1, t_2, w_2, w_1]$ would be a C_4.

Without loss of generality, we may assume that $w_1 \neq y^*$. Since $V(P) \setminus \{t_1, t_2\} \subseteq X$, we find that w_1 must be anticomplete to $V(P) \setminus \{t_1, t_2\}$. Let t_3 be the neighbour of w_2 on $V(P)$ that is nearest to t_1. (If w_2 has no neighbours in $V(P) \setminus \{t_1, t_2\}$, then $t_3 = t_2$.) Note that $t_3 \neq t_1$, since w_2 is not adjacent to t_1. Let P' be the part of the path P from t_1 to t_3. The only neighbour of w_1 in $V(P')$ is t_1. The only neighbour of w_2 in $V(P')$ is t_3. Since w_1 and w_2 are adjacent and P' is an induced path on at least two vertices, it follows that $G[V(P') \cup \{w_1, w_2\}]$ is a cycle on at least four vertices, contradicting the fact that G is chordal. We have so far shown that for any two distinct vertices $w_1, w_2 \in L_1 \setminus A$, there is no path in $G[L_2 \cup L_3]$ from any vertex of A_{w_1} to any vertex of A_{w_2}.

Now suppose that $u \in X$. As $\Delta \leq |L_1| - 2$, there exist two distinct vertices $w_1, w_2 \in L_1 \setminus N_{L_1}(x)$. By Claim 2, there exist two vertices $t_1 \in A_{w_1}$, $t_2 \in A_{w_2}$. Because the sets A_{w_i} are mutually disjoint, t_1 and t_2 are also distinct. It follows from the conclusion above that u can be adjacent to at most one of t_1 and t_2. Without loss of generality, assume that u is non-adjacent to t_1. Note that $w_1 \neq y^*$ by assumption, so u cannot be adjacent to w_1. By Claim 1, u must be adjacent to every vertex of A. Since $N_{L_1}(u) \subsetneq N_{L_1}(x)$, it follows that $N_{L_1}(u) = A$. Since u was an arbitrary vertex in X, together with the observations made earlier, this shows that every vertex in $L_2 \cup L_3$ is adjacent to every vertex of A and at most one other vertex in L_1. Since $\Delta \geq 2$, we have that $|A| \geq 1$, and so L_3 must be empty. Furthermore, since for every pair of distinct $w_1, w_2 \in L_1 \setminus A$ there is no path in $G[L_2 \cup L_3]$ from any vertex of A_{w_1} to any vertex of A_{w_2}, it follows that every component of $G[L_2 \cup L_3]$ has at most one neighbour in $L_1 \setminus A$. By Fact 3, we may apply a bipartite complementation between A and L_2 after which we may apply Lemma 30 to the resulting graph with $V_1 = \{v\} \cup L_1$. This completes the proof of Case 1.

Case 2: $\Delta \geq |L_1| - 1$.
Since $|L_1| \geq |N_{L_1}(x)| = \Delta$, there is at most one vertex in $L_1 \setminus N_{L_1}(x)$. By Fact 1, we may delete this vertex, if it exists. Note that this changes neither the value of Δ nor the choice of x. Therefore, we may assume that $L_1 = N_{L_1}(x)$.
Then $N_{L_1}(w) \subseteq N_{L_1}(x)$ for all $w \in L_2$. If $\Delta = |L_1| \leq 3$, then by deleting at most two vertices of L_1 (which we may do by Fact 1), we obtain a new graph for which we may apply Lemma 30. We may therefore assume without loss of generality that $\Delta \geq 4$.
We distinguish three subcases depending on whether or not x dominates L_2 and whether or not L_2 is a clique.

Case 2a: *x does not dominate L_2.*
Let $y \in L_2$ be a non-neighbour of x. Recall that $N_{L_1}(x) = L_1$. By Claim 1, we find that y must be adjacent to all but at most one vertex of L_1. If y is not adjacent to some vertex of L_1, we may delete this vertex by Fact 1. We may therefore assume that $\Delta \geq 3$ and that y is complete to L_1.
Suppose that $w \in L_2$ has two non-neighbours $a, b \in N_{L_1}(x)$. As $\{x, y\}$ is complete to L_1, it follows that w is adjacent to both x and y by Claim 1. However, then $G[x, w, y, a]$ is a C_4, contradicting the fact that G is chordal. Therefore, every vertex in L_2 is adjacent to all but at most one vertex of L_1. In particular, as $\Delta \geq 3$, every vertex in L_2 has at least two neighbours in L_1. This fact, together with the fact that no vertex in L_3 has neighbours in L_1 and Claim 1, implies that every vertex of L_2 is adjacent to every vertex of L_3. By applying a bipartite complementation between L_2 and L_3, we separate $G[L_3]$ from $G[V \setminus L_3] = G[\{v\} \cup L_1 \cup L_2]$. Note that $G[L_3]$ is a $\overline{2P_1 + P_2}$-free chordal graph, so it has bounded clique-width by Theorem 25. By Fact 3, we may therefore assume that $L_3 = \emptyset$.
Let X be the set of vertices in L_2 that are complete to L_1. For $z \in L_1$, let U_z be the set of vertices in L_2 that are complete to $L_1 \setminus \{z\}$ and non-adjacent to z. As every vertex in L_2 is adjacent to all but at most one vertex of L_1, we find that the sets X and U_z, $z \in L_1$, form a partition of L_2.
Suppose that there are at most six vertices $z \in L_1$ such that U_z is not empty. By Facts 1 and 3, we may apply a bipartite complementation between L_1 and L_2 and then delete these vertices. In the resulting graph, no vertex of L_2 has a neighbour in L_1 and we can apply Lemma 30. We may therefore assume that there are at least seven vertices $z \in L_1$ such that U_z is not empty.
Consider two distinct vertices $z_1, z_2 \in L_1$. We claim that U_{z_1} must be anticomplete to U_{z_2}. Indeed, if $y_1 \in U_{z_1}$ were adjacent to $y_2 \in U_{z_2}$, then $G[y_1, y_2, z_1, z_2]$ would be a C_4, contradicting the fact that G is chordal.
We will now show that by deleting at most one vertex from L_2 (which we may do by Fact 1), we can make $G[L_2]$ into a P_3-free graph. Indeed, suppose that $G[L_2]$ contains an induced P_3 on vertices v_1, v_2, v_3.

First, consider a vertex $z \in L_1$ such that $v_1, v_2, v_3 \notin U_z$ and U_z is nonempty. Suppose that $y \in U_z$. Then y must have at least one neighbour in $\{v_1, v_2, v_3\}$, otherwise $G[y, v_2, z, v_1, v_3]$ would be a $P_1 + \overline{2P_1 + P_2}$. Since there are at least seven nonempty sets U_z, there must be at least four nonempty sets U_z that do not contain a vertex in $\{v_1, v_2, v_3\}$. Therefore, there must be two sets U_{z_1} and U_{z_2} containing vertices y_1 and y_2, respectively, such that y_1 and y_2 are adjacent to the same vertex in $\{v_1, v_2, v_3\}$, say v_i. Since U_{z_1} and U_{z_2} are anticomplete, y_1 and y_2 are non-adjacent. Hence, $G[y_1, v_i, y_2]$ is a P_3. Also note that $v_i \in X$ since v_i has neighbours in both U_{z_1} and U_{z_2}.

Now let $z_3 \in L_1 \setminus \{z_1, z_2\}$ and suppose that $y_3 \in U_{z_3}$. By the same argument as above, y_3 must have a neighbour in $\{y_1, v_i, y_2\}$. Moreover, as U_{z_3} is anticomplete to both U_{z_1} and U_{z_2}, we find that y_3 is non-adjacent to both y_1 and y_2. Hence, y_3 must be adjacent to v_i. Now choose $z_4, z_5 \in L_1 \setminus \{z_1, z_2\}$ with $y_4 \in U_{z_4}$ and $y_5 \in U_{z_5}$. Such vertices exist by our earlier assumption. By the same argument, $G[y_4, v_i, y_5]$ is a P_3, so v_i is complete to U_{z_3} for every $z_3 \in L_1 \setminus \{z_4, z_5\}$. Hence, v_i is complete to U_z for every $z \in L_1$. This implies that, if $G[L_2]$ contains a P_3, then some vertex of this P_3 is adjacent to every vertex of every set U_z.

Suppose that there exist two vertices $v', v'' \in L_2$ that are both complete to every vertex of every set U_z. Choose $y_1 \in U_{z_1}$ and $y_2 \in U_{z_2}$ with z_1 and z_2 distinct. Note that y_1 and y_2 are non-adjacent and so $y_i \notin \{v', v''\}$ for $i = 1, 2$. So, $\{v', v''\}$ is complete to $\{y_1, y_2\}$ by the assumption on v' and v''. If v' and v'' are non-adjacent, then $G[v', y_1, v'', y_2]$ is a C_4; if v' and v'' are adjacent, then $G[v, v', v'', y_1, y_2]$ is a $P_1 + \overline{2P_1 + P_2}$. In either case we have a contradiction, since G is a $(P_1 + \overline{2P_1 + P_2})$-free chordal graph. We have thus shown that there exists at most one vertex that is complete to all U_z. This implies that if $G[L_2]$ contains an induced P_3, then there is a unique vertex in L_2 that is present in every induced P_3 in $G[L_2]$.

By Fact 1, we may delete the vertex that is on every induced P_3 (if G is not P_3-free already). In this way, we change $G[L_2]$ into a P_3-free graph, which means that each connected component of $G[L_2]$ is now a complete graph.

Consider an arbitrary connected component K of $G[L_2]$. As every vertex in L_2, and thus in $V(K)$, is adjacent to all but at most one vertex in L_1 and as G is C_4-free, we find that either $V(K)$ is complete to L_1 or to $L_1 \setminus \{z\}$ for some $z \in L_1$. Let G' be the graph obtained from G by performing a bipartite complementation between L_1 and L_2. Then every component in $G[L_2]$ has, in G', at most one neighbour in L_1. Case 2a now follows directly from Fact 3 and Lemma 30.

Case 2b: *L_2 is a clique.*

In this case, we may assume that there is a vertex $x' \in L_2 \setminus \{x\}$ that has at least two neighbours in L_1, as otherwise we could delete x (which we may do by Fact 1) and apply Lemma 30. Recall that, by definition, L_3 has no neighbours in L_1. Because both x and x' have at least two neighbours in L_1, Claim 1 tells us that $\{x, x'\}$ is complete to L_3.

If $y \in L_2$ is non-adjacent to $z \in L_3$ then $y \notin \{x, x'\}$, so $G[v, x, x', y, z]$ is a $P_1 + \overline{2P_1 + P_2}$, since L_2 is a clique. So, L_2 is complete to L_3. By Fact 3, we may apply a bipartite complementation between L_2 and L_3, after which $G[L_3]$ will be disconnected from the rest of the graph (since $V = \{v\} \cup L_1 \cup L_2 \cup L_3$ and L_3 is anticomplete to $\{v\} \cup L_1$). Since $G[L_3]$ is a $\overline{2P_1 + P_2}$-free chordal graph, it has bounded clique-width. So, it remains to show that $G[V \setminus L_3] =$

$G[\{v\} \cup L_1 \cup L_2]$ has bounded clique-width. Now $G[\{v\} \cup L_1]$ and $G[L_2]$ are complete graphs. Moreover, as G is chordal, G is C_4-free. Applying a complementation to the whole graph (which we may do by Fact 2) gives a $2P_2$-free bipartite graph, which has bounded clique-width by Lemma 15.

Case 2c: *x dominates L_2, but L_2 is not a clique.*
Since $G[L_2]$ is $\overline{2P_1 + P_2}$-free, $G[L_2 \setminus \{x\}]$ must be P_3-free. In other words, each connected component of $G[L_2 \setminus \{x\}]$ is a complete graph.
Since L_2 is not a clique, $L_2 \setminus \{x\}$ must contain at least two cliques, so deleting x from G (which we may do by Fact 1) means that $G[L_2]$ no longer has a dominating vertex. Note that this deletion may change the value of Δ. By the same arguments as at the start of the proof, we may assume that $\Delta \geq 2$ and so $V(G) = \{v\} \cup L_1 \cup L_2 \cup L_3$. Again, by Claim 2, we may assume that every vertex of L_1 has a neighbour in L_2 in G. Then if $\Delta \leq |L_1| - 2$, we may apply Case 1. We may therefore assume that $\Delta \geq |L_1| - 1$. By the same arguments as at the start of Case 2, we may assume that $|L_1| = \Delta$ and $\Delta \geq 4$. To make this assumption, we may have to delete vertices from L_1, which could cause vertices that were in L_2 previously to now be in L_3 for this modified graph. However, at no point above do we add vertices to L_2, so it is still the case that every component of $G[L_2]$ is a complete graph. Therefore Case 2b or Case 2a applies, depending on whether $G[L_2]$ now contains one or more components, respectively. This completes the proof of Theorem 31.

■

D. The Case $H = \overline{S_{1,1,2}}$

We now show that the clique-width of $\overline{S_{1,1,2}}$-free chordal graphs is bounded. Switching to the complement, we study the $S_{1,1,2}$-free co-chordal graphs, which form a subclass of $(2P_2, C_5, S_{1,1,2})$-free graphs. First, in Lemma 32, we show that prime $(2P_2, C_5, S_{1,1,2})$-free graphs are thin spiders if they contain an induced net. We then use this lemma in combination with the two prime extension lemmas from Section 2 (Lemmas 3 and 4) to provide, in Lemma 33, a structural description of prime $\overline{S_{1,1,2}}$-free chordal graphs. Finally, in Theorem 34, we use this structural description to show boundedness of the clique-width of $\overline{S_{1,1,2}}$-free chordal graphs.

Lemma 32. *If a prime $(2P_2, C_5, S_{1,1,2})$-free graph G contains an induced subgraph isomorphic to the net (see Fig. 5), then G is a thin spider.*

Proof. Suppose that G is a prime $(2P_2, C_5, S_{1,1,2})$-free graph and suppose that G contains a net, say N, with vertices $a_1, a_2, a_3, b_1, b_2, b_3$ such that $\{a_1, a_2, a_3\}$ is an independent set (the *end-vertices* of N), $\{b_1, b_2, b_3\}$ is a clique (the *mid-vertices* of N), and the only edges between a_1, a_2, a_3 and b_1, b_2, b_3 are $a_ib_i \in E(G)$ for $i \in \{1, 2, 3\}$.

Let $M = V(G) \setminus V(N)$. We partition M as follows: For $i \in \{1, \ldots, 5\}$, let M_i be the set of vertices in M with exactly i neighbours in $V(N)$. Let U be the set of vertices in M adjacent to every vertex of $V(N)$. Let Z be the set of vertices in M with no neighbours in $V(N)$. Note that Z is an independent set in G, since G is $2P_2$-free.

We now analyze the structure of G through a series of claims.

Claim 1. $M_1 \cup M_2 \cup M_5 = \emptyset$.

First, suppose that $x \in M_1 \cup M_2$. By symmetry, we may assume that x is adjacent to at least one vertex in $\{a_1, b_1\}$ and anticomplete to $\{a_2, b_2\}$. If x is adjacent to a_1 then $G[x, a_1, a_2, b_2]$ is a $2P_2$. Therefore x is adjacent to b_1, but not to a_1. However, this means that $G[b_1, a_1, x, b_2, a_2]$ is an $S_{1,1,2}$. We conclude that $M_1 \cup M_2 = \emptyset$.

Now suppose that $x \in M_5$. We may assume by symmetry that x is non-adjacent to a_1 or b_1. Then $G[x, a_2, a_3, b_1, a_1]$ is an $S_{1,1,2}$. It follows that $M_5 = \emptyset$, completing the proof of Claim 1.

Next, we prove that the vertices in M_3 and M_4 have a restricted type of neighbourhood in $V(N)$:

Claim 2. *Every $x \in M_3$ is adjacent to either exactly one end-vertex a_i and its two opposite mid-vertices b_j and b_k ($j \neq i$, $k \neq i$) or to all three mid-vertices of N.*

Suppose that $x \in M_3$ is non-adjacent to at least one mid-vertex. If x is adjacent to at least two end-vertices, say a_1 and a_2, then x must be adjacent to b_1 or b_2, otherwise $G[x, a_1, b_1, b_2, a_2]$ would be a C_5. By symmetry, we may assume that x is adjacent to b_1. As $x \in M_3$, this means that $G[x, a_1, b_2, b_3]$ is a $2P_2$. Hence, by symmetry, x must be adjacent to exactly one end-vertex, say a_1, and two mid-vertices. If x is non-adjacent to b_2 then $G[a_1, x, a_2, b_2]$ is a $2P_2$. By symmetry, x must therefore be adjacent to b_2 and b_3, completing the proof of Claim 2.

The situation for M_4 is similar to that of M_3, as shown in the following claim.

Claim 3. *If $x \in M_4$ then x is adjacent to exactly one end-vertex and all mid-vertices.*

Let $x \in M_4$. Without loss of generality, x must be adjacent to an end-vertex, say a_1. If x is adjacent to all three end-vertices a_1, a_2, a_3 and, say, b_1 then $G[x, a_2, b_2, b_3, a_3]$ is a C_5. If x is adjacent to exactly two end-vertices, say a_1 and a_2, then $G[x, a_1, a_2, b_3, a_3]$ is an $S_{1,1,2}$ unless x is non-adjacent to b_3. However, if x is non-adjacent to b_3 then $G[a_1, x, b_3, a_3]$ is a $2P_2$. Hence, x must be adjacent to exactly one end-vertex. Consequently, as $x \in M_4$, we find that x is adjacent to all three mid-vertices of N. This completes the proof of Claim 3.

Let Mid_3 denote the set of vertices in M_3 that are adjacent to all three mid-vertices of N (and non-adjacent to any end-vertex of N).

Claim 4. *U is complete to $(M_3 \cup M_4)$.*

Suppose that $u \in U$ and $x \in (M_3 \cup M_4)$ are not adjacent. If $x \in \text{Mid}_3$ then $G[u, a_2, a_3, b_1, x]$ is an $S_{1,1,2}$. If $x \in (M_3 \cup M_4) \setminus \text{Mid}_3$, then without loss of generality x is adjacent to a_1 and $G[u, a_2, a_3, a_1, x]$ is an $S_{1,1,2}$. This completes the proof of Claim 4.

Let Z_1 denote the set of vertices in Z that have a neighbour in $M_3 \cup M_4$, and let $Z_0 = Z \setminus Z_1$.

Claim 5. *Z_1 is anticomplete to $((M_3 \cup M_4) \setminus \text{Mid}_3)$.*

Suppose that $z \in Z_1$ and $x \in (M_3 \cup M_4) \setminus \text{Mid}_3$ are adjacent. Without loss of generality, we may assume that x is adjacent to a_1 and b_3. Then $G[x, a_1, z, b_3, a_3]$ is an $S_{1,1,2}$. This completes the proof of Claim 5.

Thus, the only possible neighbours of Z_1 vertices in $M_3 \cup M_4$ are the vertices in Mid_3.

Claim 6. *U is complete to Z_1.*

Suppose that $u \in U$ and $z \in Z_1$ are non-adjacent. By the definition of Z_1, the vertex z has a neighbour $x \in M_3 \cup M_4$. By Claim 5, it follows that $x \in \text{Mid}_3$. By Claim 4, x must be adjacent to u. Then $G[u, a_2, a_3, x, z]$ is an $S_{1,1,2}$. This completes the proof of Claim 6.

Recall that $M_1 \cup M_2 \cup M_5 = \emptyset$ and let $X = V(N) \cup M_3 \cup M_4 \cup Z_1$. Then X is a module: every vertex in U is complete to X (due to the definition of U, together with Claims 4 and 6) and every vertex in Z_0 is anticomplete to X (due to the definitions of Z, Z_0 and Z_1, together with the fact that Z is an independent set). Since G is prime, X must be a trivial module. Since X contains more than one vertex, it follows that $V(G) = X = V(N) \cup M_3 \cup M_4 \cup Z_1$. Hence, $U \cup Z_0 = \emptyset$.

It remains to show that $G = G[V(N) \cup M_3 \cup M_4 \cup Z_1]$ is a thin spider. For $i \in \{1, 2, 3\}$ let $M'_i = (M_3 \cup M_4) \cap N(a_i)$. Note that $M_3 \cup M_4 = \text{Mid}_3 \cup M'_1 \cup M'_2 \cup M'_3$. The next two claims show how each M'_i is connected to other subsets of $V(G)$.

Claim 7. *For $i \neq j$, M'_i is complete to M'_j.*

By symmetry, we may assume that $i = 1$ and $j = 2$. If $x \in M'_1$ is non-adjacent to $y \in M'_2$ then, by Claims 2 and 3, we find that $G[x, a_1, y, a_2]$ is a $2P_2$. This completes the proof of Claim 7.

Claim 8. *For every $i = 1, 2, 3$, M'_i is complete to* Mid_3.

By symmetry we may assume that $i = 1$. If $x \in M'_1$ is non-adjacent to $y \in \text{Mid}_3$ then, by Claims 2 and 3, we find that $G[b_2, a_2, y, x, a_1]$ is an $S_{1,1,2}$. This completes the proof of Claim 8.

By Claims 2, 3, 5, 7, and 8 we find that, for every $i \in \{1, 2, 3\}$, $M'_i \cup \{b_i\}$ is a module, so $M'_i = \emptyset$ (since G is prime). Consequently, $V(G) = V(N) \cup \text{Mid}_3 \cup Z_1$. Next, we show the following.

Claim 9. Mid_3 *is a clique.*

Suppose that Mid_3 is not a clique. Let Q be the vertex set of a component of $\overline{G[\text{Mid}_3]}$, such that $\overline{G[Q]}$ contains an edge (so $G[Q]$ contains a nonedge). Since G is prime, Q cannot be a module in G. Note that, in G, the set $\text{Mid}_3 \setminus Q$ is complete to Q. Moreover, every vertex in $Q \subseteq \text{Mid}_3$ is adjacent to every mid-vertex of N and non-adjacent to every end-vertex of N (by definition). Hence, there must be vertices $x, y \in Q$ and $z \in Z_1$ such that z distinguishes x and y, say z is adjacent to x in G, but not to y. Because $\overline{G[Q]}$ is connected, we may assume that x and y are adjacent in $\overline{G}$, in which case x and y are non-adjacent in G. However, then $G[b_3, a_3, y, x, z]$ is an $S_{1,1,2}$. This completes the proof of Claim 9.

By Claim 9 and the definition of Mid_3, we find that $\{b_1, b_2, b_3\} \cup \text{Mid}_3$ is a clique. By the definition of Z and the fact that Z is independent, $\{a_1, a_2, a_3\} \cup Z_1$ is an independent set. Therefore, G is a split graph. By Lemma 9, since G is prime and $S_{1,1,2}$-free, it must be a spider. Since G contains an induced net, it must be a thin spider. ■

Lemma 33. *If G is a prime $\overline{S_{1,1,2}}$-free chordal graph, then it is either a $\overline{2P_1 + P_2}$-free graph or a thick spider.*

Proof. Let G be a prime $\overline{S_{1,1,2}}$-free chordal graph. Note that since G is $\overline{S_{1,1,2}}$-free, it cannot contain d-$\mathbb{A}$ or d-domino as an induced subgraph (see also Fig. 4). If G is $\overline{P_1 + P_4}$-free then, by Lemma 3, it must therefore be $\overline{2P_1 + P_2}$-free.

Now suppose that G contains an induced copy of $\overline{P_1 + P_4}$. Since G is prime, $\overline{G}$ is also prime. Furthermore, $\overline{G}$ is $(2P_2, C_5, S_{1,1,2})$-free. By Lemma 4, $\overline{G}$ must contain one of the graphs in Figure 5. The only graph in Figure 5 which is $(2P_2, C_5, S_{1,1,2})$-free is the net, so $\overline{G}$ must contain a net. By Lemma 32, $\overline{G}$ is a thin spider, so G is a thick spider, completing the proof. ■

As a corollary of the above lemma, we get the following.

Theorem 34. *Every $\overline{S_{1,1,2}}$-free chordal graph has clique-width at most 4.*

Proof. Let G be an $\overline{S_{1,1,2}}$-free chordal graph. By Lemma 13, we may assume that G is prime. If G is $\overline{2P_1 + P_2}$-free then it has clique-width at most 3 by Lemma 19. By Lemma 33, we may therefore assume that G is a thick spider, in which case it has clique-width at most 4 by Lemma 10. ■

Note that the bound in the above theorem is tight. Indeed, consider the thick spider consisting of a clique K on four vertices and an independent set I on four vertices, where every vertex in K has exactly one non-neighbour in I and vice versa. It is easy to check that this graph is $\overline{S_{1,1,2}}$-free and chordal. Using the software of Heule and Szeider [42], one can verify that it has clique-width 4.

4. THE CLASSIFICATIONS

In this section we first prove our main result, Theorem 1, which was presented in Section 1. Recall that F_1 and F_2 are the graphs shown in Figure 3.

Theorem 1 **(restated).** *Let H be a graph with $H \notin \{F_1, F_2\}$. The class of H-free chordal graphs has bounded clique-width if and only if*

(a) $H = K_r$ *for some* $r \geq 1$;
(b) $H \subseteq_i$ bull;
(c) $H \subseteq_i P_1 + P_4$;
(d) $H \subseteq_i \overline{P_1 + P_4}$;
(e) $H \subseteq_i \overline{K_{1,3} + 2P_1}$;
(f) $H \subseteq_i P_1 + \overline{P_1 + P_3}$;
(g) $H \subseteq_i P_1 + \overline{2P_1 + P_2}$; *or*
(h) $H \subseteq_i \overline{S_{1,1,2}}$.

Proof. Let H be a graph with $H \notin \{F_1, F_2\}$. If $H = K_r$ for some $r \geq 1$, then we use Lemma 16. If H is an induced subgraph of a graph in {bull, $P_1 + P_4$, $\overline{P_1 + P_4}$}, then we use Lemmas 17, 18, or 19, respectively. If H is an induced subgraph of a graph

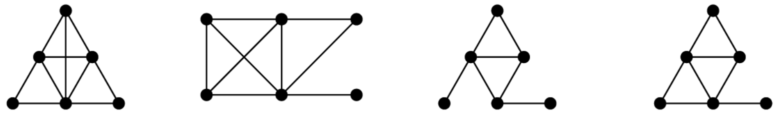

FIGURE 9. The graphs $\overline{\text{bull} + P_1}$, F_3, Q and $\overline{Q}$ from Claim 2.

TABLE I. The maximal $K_{1,3}$-free induced subgraphs of $\overline{\text{bull} + P_1}$, F_3, Q and $\overline{Q}$

H	Maximal $K_{1,3}$-free induced subgraphs of H
$\overline{\text{bull} + P_1}$	bull, $\overline{P_1 + P_4}$, $\overline{2P_1 + P_3}$
F_3	$\overline{K_{1,3} + P_1}$, $P_1 + \overline{P_1 + P_3}$, $\overline{2P_1 + P_3}$
Q	bull, $P_1 + P_4$, $P_1 + \overline{P_1 + P_3}$, $\overline{S_{1,1,2}}$
$\overline{Q}$	bull, $\overline{P_1 + P_4}$, $P_1 + \overline{P_1 + P_3}$, $\overline{S_{1,1,2}}$

in $\{\overline{K_{1,3} + 2P_1}, P_1 + \overline{P_1 + P_3}, P_1 + \overline{2P_1 + P_2}, \overline{S_{1,1,2}}\}$, then we use Theorems 25, 29, 31, or 34, respectively.

We now prove the reverse direction of the theorem. Let $H \notin \{F_1, F_2\}$ be a graph such that the class of H-free chordal graphs has bounded clique-width. We first prove two useful claims, which show that we are done in some special cases.

Claim 1. *If H is a proper induced subgraph of F_1 or F_2, then H is an induced subgraph of a graph in* $\{\text{bull}, \overline{K_{1,3} + 2P_1}, P_1 + \overline{P_1 + P_3}, P_1 + \overline{2P_1 + P_2}, \overline{S_{1,1,2}}\}$.

We prove Claim 1 as follows. Note that F_1 and F_2 are six-vertex graphs. The five-vertex induced subgraphs of F_1 are bull, $\overline{K_{1,3} + P_1}$, and $P_1 + \overline{P_1 + P_3}$. The five-vertex induced subgraphs of F_2 are bull, $\overline{K_{1,3} + P_1}$, $P_1 + \overline{2P_1 + P_2}$, $\overline{2P_1 + P_3}$, and $\overline{S_{1,1,2}}$. Since $\overline{K_{1,3} + P_1}$ and $\overline{2P_1 + P_3}$ are induced subgraphs of $\overline{K_{1,3} + 2P_1}$, this completes the proof of the Claim 1.

Claim 2. *If H is an induced subgraph of a graph in* $\{\overline{\text{bull} + P_1}, F_3, Q, \overline{Q}\}$ *(see Fig. 9), then H must be an induced subgraph of a graph in* $\{\text{bull}, P_1 + P_4, \overline{P_1 + P_4}, \overline{K_{1,3} + 2P_1}, P_1 + \overline{P_1 + P_3}, \overline{S_{1,1,2}}\}$.

We prove Claim 2 as follows. If $H \in \{\overline{\text{bull} + P_1}, F_3, Q, \overline{Q}\}$, then H contains an induced $K_{1,3}$. By Lemma 21, since the class of H-free chordal graphs has bounded clique-width, H must be $K_{1,3}$-free. Hence, H must be a $K_{1,3}$-free induced subgraph of $\overline{\text{bull} + P_1}$, F_3, Q, or $\overline{Q}$. We list the maximal $K_{1,3}$-free induced subgraphs of $\overline{\text{bull} + P_1}$, F_3, Q, and $\overline{Q}$, respectively, in Table I. Since $\overline{K_{1,3} + P_1}$ and $\overline{2P_1 + P_3}$ are induced subgraphs of $\overline{K_{1,3} + 2P_1}$, this completes the proof of Claim 2.

Due to Claims 1 and 2, if H is an induced subgraph of a graph in $\{\overline{\text{bull} + P_1}, F_1, F_2, F_3, Q, \overline{Q}\}$, then we are done.

Since the class of split graphs is contained in the class of chordal graphs, the class of H-free split graphs must also have bounded clique-width. By Lemma 23, the graph H must therefore be a complete graph, an edgeless graph, or an induced subgraph of a graph in $\{F_4, \overline{F_4}, F_5, \overline{F_5}\}$ (see Fig. 8). If H is a complete graph, then we are done. If H is an edgeless graph, then Lemma 20 tells us that H can have at most three vertices, in which case H is an induced subgraph of the bull and we are done. We may therefore assume that H is an induced subgraph of a graph in $\{F_4, \overline{F_4}, F_5, \overline{F_5}\}$ and we will consider each of these possibilities in turn. Furthermore, H must be $4P_1$-free and $K_{1,3}$-free, otherwise the clique-width of H-free chordal graphs would be unbounded (by Lemmas 20 and 21, respectively).

Case 1: $H \subseteq_i F_4$.
Since F_4 contains an independent set on five vertices and H is $4P_1$-free, two of these vertices must be absent from H. Therefore, H must be an induced subgraph of bull, $P_1 + P_4$, $P_1 + \overline{P_1 + P_3}$, $P_1 + \overline{2P_1 + P_2}$, or $\overline{P_1 + \overline{P_1 + P_3}}$. In the first four cases, we are done immediately. The graph $\overline{P_1 + \overline{P_1 + P_3}}$ (also known as the dart) is an induced subgraph of F_3, so in the fifth case we are done by Claim 2.

Case 2: $H \subseteq_i \overline{F_4}$.
The graph $\overline{F_4}$ contains two induced copies of $K_{1,3}$ (which are not vertex-disjoint). Since H is $K_{1,3}$-free, it follows that H is an induced subgraph of F_1, $\overline{K_{1,3} + 2P_1}$, or $\overline{P_1 + P_4}$. In the first case, we are done by Claim 1. In the other two cases, we are done immediately.

Case 3: $H \subseteq_i F_5$.
Since F_5 contains an independent set on four vertices, one of these vertices must be absent from H. Therefore, H must be an induced subgraph of F_1, F_2, F_3, or Q. In the first two cases, we apply Claim 1 and in the other two we apply Claim 2.

Case 4: $H \subseteq_i \overline{F_5}$.
Since $\overline{F_5}$ contains an independent set on four vertices, one of these vertices must be absent from H. Therefore, H must be an induced subgraph of $\overline{\text{bull} + P_1}$, F_2, F_3, or $\overline{Q}$. In each of these cases, we are done by Claim 1 or 2.

This completes the proof of Theorem 1. ■

We now prove our dichotomy for H-free weakly chordal graphs, which we recall next.

Theorem 2 (restated). *Let H be a graph. The class of H-free weakly chordal graphs has bounded clique-width if and only if H is an induced subgraph of P_4.*

Proof. Let H be a graph. First, suppose that H is an induced subgraph of P_4. Then the class of H-free weakly chordal graphs is contained in the class of P_4-free graphs, which have bounded clique-width by Lemma 14. Now, suppose that H is not an induced subgraph of P_4. Next, we show that the class of H-free weakly chordal graphs has unbounded clique-width.

Suppose that H is not a split graph. Then the class of H-free weakly chordal graphs contains the class of split graphs, which has unbounded clique-width by Lemma 22 (or Lemma 23). From now on assume that H is a split graph. Suppose that H contains a cycle C. As H is a split graph, it is $(C_4, C_5, 2P_2)$-free by Lemma 8. Hence, C is isomorphic to C_3. Then the class of H-free weakly chordal graphs contains the class of bipartite weakly chordal graphs, which contains the class of bipartite permutation graphs,

K_4 $2P_1 + P_3$

FIGURE 10. The graphs K_4 and $2P_1 + P_3$.

which has unbounded clique-width by Lemma 24. From now on assume that H contains no cycle.

We claim that H has an induced $3P_1$. For contradiction, suppose that H is $3P_1$-free. Then every connected component of H is a path. As H is $3P_1$-free, H has at most two connected components, each of which is a path on at most four vertices. Because H is not an induced subgraph of P_4, this means that H has exactly two connected components. As H is $3P_1$-free, each of these components is a path on at most two vertices. As H is $2P_2$-free, at most one of the components contains an edge. However, then H is an induced subgraph of P_4, a contradiction. Now, as H has an induced $3P_1$, the class of complements of H-free weakly chordal graphs contains the class of C_3-free weakly chordal graphs, which has unbounded clique-width, as shown above. Applying Fact 2 completes the proof. ■

5. AN APPLICATION

In this section, we give an application of Theorem 1 by showing how to use it to prove that the class of $(K_4, 2P_1 + P_3)$-free graphs has bounded clique-width (see also Fig. 10). Combining this result with five cases recently solved by Dabrowski et al. [24], this means that there are only eight (nonequivalent) classes of (H_1, H_2)-free graphs for which it is not known whether the clique-width is bounded [24, 29].

Theorem 35. *The class of* $(K_4, 2P_1 + P_3)$*-free graphs has bounded clique-width.*

Proof. Suppose that G is a $(K_4, 2P_1 + P_3)$-free graph. If G is chordal then it is a K_4-free chordal graph, in which case it has bounded clique-width by Lemma 16. We may therefore assume that G contains an induced cycle C with vertices $v_1, v_2, \ldots, v_k$ in that order, such that $k \geq 4$. We may also assume that this induced cycle is chosen such that k is minimal. Note that $k \leq 7$, otherwise $G[v_1, v_3, v_5, v_6, v_7]$ would be a $2P_1 + P_3$.

We partition the vertices not on the cycle C as follows. For $S \subseteq \{1, \ldots, k\}$, let V_S contain those vertices $x \in V(G) \setminus C$ such that $N_C(x) = \{v_i \mid i \in S\}$. We say that a set V_S is *large* if it contains at least seven vertices, otherwise we say that it is *small*. We now prove some useful properties about these sets.

Claim 1. *Suppose that* $S, T \subseteq \{1, \ldots, k\}$ *with* $S \neq T$. *If* $x, x' \in V_S$ *and* $y, y' \in V_T$ *then* $G[x, x', y, y']$ *is not a* $4P_1$.

Indeed, suppose that $G[x, x', y, y']$ is a $4P_1$. Without loss of generality, we may assume that $i \in T \setminus S$. Then $G[x, x', y, v_i, y']$ is a $2P_1 + P_3$.

Claim 2. *If* v_i *and* v_j *are consecutive vertices of the cycle and* $\{i, j\} \subseteq S \subseteq \{1, \ldots, k\}$ *then* V_S *is an independent set.*

Indeed, if $x, x' \in S$ were adjacent then $G[x, x', v_i, v_j]$ would be a K_4.

Claim 3. *Suppose that $S \subseteq \{1, \ldots, k\}$. If $|S| \leq 1$ then V_S is small.*

Indeed, suppose that $S = \emptyset$ or $S = \{1\}$. If $x, y \in V_S$ are distinct then they must be adjacent, otherwise $G[x, y, v_2, v_3, v_4]$ would be a $2P_1 + P_3$. Therefore, V_S is a clique in G. Since G is K_4-free, $|V_S| \leq 3$.

Claim 4. *Suppose that $S, T \subseteq \{1, \ldots, k\}$ with $S \neq T$. If V_S and V_T are independent sets in G and V_T is large, then at most one vertex of V_S has more than one non-neighbour in V_T.*

Indeed, since $|V_T| \geq 7 \geq 4$, by Claim 1 for any pair of vertices $x, x' \in V_S$, at least one of these vertices must have at least two neighbours in V_T. Therefore, every vertex of V_S except perhaps one has at least two neighbours in V_T. Consider a vertex $x \in V_S$ that has two neighbours $y, y' \in V_T$. The vertex x cannot have two non-neighbours $z, z' \in V_T$, otherwise $G[z, z', y, x, y']$ would be a $2P_1 + P_3$. Therefore, every vertex of V_S except perhaps one has at most one non-neighbour in V_T. This completes the proof of the claim.

Claim 5. *Suppose that $S, T, U \subseteq \{1, \ldots, k\}$ are pairwise distinct. If V_S, V_T and V_U are independent sets in G then $G[V_S \cup V_T \cup V_U]$ has bounded clique-width.*

Indeed, if any set in $\{V_S, V_T, V_U\}$ is small, then by Fact 1 we may assume that it is empty. By Claim 4 and Fact 1, we may delete at most two vertices from each of V_S, V_T, V_U after which every vertex in each of these sets will have at most one non-neighbour in each of the other two sets. In other words, every vertex in one of these sets will have at most two non-neighbours in total in the other two sets. Applying a bipartite complementation between each pair of sets (which we may do by Fact 3) yields a graph of maximum degree at most 2. This graph has bounded clique-width by Lemma 12.

Claim 6. *Suppose that $R, S, T, U \subseteq \{1, \ldots, k\}$ are pairwise distinct. If V_R, V_S, V_T, V_U are all independent sets in G, then at least one of V_R, V_S, V_T, V_U is small.*

Indeed, suppose, for contradiction, that all of V_R, V_S, V_T, V_U are large. Let V'_R, V'_S, V'_T, and V'_U be the sets of those vertices in V_R, V_S, V_T, and V_U, respectively, which do not have two non-neighbours in any of the three other sets. By Claim 4, each of V'_R, V'_S, V'_T, and V'_U has at least $7 - 3 = 4$ vertices. Let $r \in V'_R$. Since $|V'_S| \geq 2$, there must be a vertex $s \in V'_S$ adjacent to r. Since $|V'_T| \geq 3$, there must be a vertex $t \in V'_T$ adjacent to r and s. Since $|V'_U| \geq 4$, there must be a vertex $u \in V'_U$ adjacent to r, s, and t. Now $G[r, s, t, u]$ is a K_4, a contradiction.

If any set V_S is small then, by Fact 1, we may assume that it is empty. We may therefore assume that every set V_S is either large or empty. Furthermore, we may assume that some large set V_S is not an independent set, otherwise we can apply Claim 6, to find that at most three sets V_S are nonempty and then, after deleting the $k \leq 7$ vertices of C (which we may do by Fact 1), we can apply Claim 5 to find that the clique-width of G is bounded.

We claim that $k = 4$. For contradiction, suppose that $5 \leq k \leq 7$. Let $S \subseteq \{1, \ldots, k\}$ be a set such that V_S is large and not independent. By Claim 3, it follows that $|S| \geq 2$. By Claim 2, the vertices of V_S cannot be adjacent to two consecutive vertices of C. Without loss of generality, assume that $1 \in S$, which implies that $2, k \notin S$. Then there must be a number $j \in \{3, \ldots, k-1\}$ such that $j \in S$, and $2, \ldots, j-1 \notin S$. If $j \leq k-2$, then choosing $x \in V_S$ we find that $G[x, v_1, \ldots, v_j]$ is a C_{j+1}, contradicting the minimality of k.

If $j = k - 1$, then choosing $x \in V_S$ we find that $G[v_{k-1}, v_k, v_1, x]$ is a C_4, contradicting the minimality of k. Hence, we conclude that indeed $k = 4$.

Again, let $S \subseteq \{1, \ldots, k\}$ be a set such that V_S is large and not independent. By Claims 2 and 3, we find that $S = \{1, 3\}$ or $S = \{2, 4\}$. If there exist vertices $x, y, z \in V_{\{1,3\}}$ that induce a P_3 then $G[v_2, v_4, x, y, z]$ would be a $2P_1 + P_3$, which is not possible. Therefore $G[V_{\{1,3\}}]$ must be P_3-free, so it must be a disjoint union of cliques. If $G[V_{\{1,3\}}]$ contained a K_3 on vertices x, y, z then $G[v_1, x, y, z]$ would be a K_4, which is not possible. Thus, every component of $G[V_{\{1,3\}}]$ and (by symmetry) $G[V_{\{2,4\}}]$ must be isomorphic to either P_1 or P_2.

If $G[V_{\{1,3\}}]$ and $G[V_{\{2,4\}}]$ each contain at most one edge then, by deleting at most one vertex from each of $V_{\{1,3\}}$ and $V_{\{2,4\}}$ (which we may do by Fact 1), we obtain a graph in which every set V_S is independent, in which case we find that G has bounded clique-width by proceeding as before: we first apply Claim 6, then delete the vertices of C by Fact 1 and finally apply Claim 5. Without loss of generality, we may therefore assume that $G[V_{\{1,3\}}]$ contains two edges xx' and yy' (which together induce a $2P_2$).

We claim that every set V_T other than $V_{\{1,3\}}$ and $V_{\{2,4\}}$ is empty. Indeed, for contradiction, suppose that such a set V_T is nonempty. Then, as stated above, V_T must be independent and large. By Claim 3, $|T| \geq 2$. By symmetry we may therefore assume that $\{1, 2\} \subseteq T$. If $z \in V_T$ is adjacent to both x and x' then $G[x, x', v_1, z]$ would be a K_4, which is not possible. Therefore, any vertex in V_T can be adjacent to at most one vertex in each of $\{x, x'\}$ and $\{y, y'\}$. Since $|V_T| \geq 7 \geq 5$, we find that V_T contains two vertices z, z', which are not adjacent to each other (as V_T is independent) and which are both non-adjacent to the same vertex in $\{x, x'\}$ and to the same vertex in $\{y, y'\}$. By Claim 1, this is a contradiction, so V_T must indeed be empty.

Recall that by Fact 1, we may delete the four vertices of C. We are therefore reduced to proving that $G[V_{\{1,3\}} \cup V_{\{2,4\}}]$ has bounded clique-width. Note that if $x \in V_{\{1,3\}}$ is non-adjacent to two vertices y and y' in $V_{\{2,4\}}$ then y and y' must be adjacent, otherwise $G[y, y', v_1, x, v_3]$ would be a $2P_1 + P_3$ (which is not possible). This, together with the fact that every component of $G[V_{\{1,3\}}]$ and $G[V_{\{2,4\}}]$ is isomorphic to P_1 or P_2, implies that any vertex in $V_{\{1,3\}}$ has at most two non-neighbours in $V_{\{2,4\}}$, and vice versa. Let G' be the graph obtained from $G[V_{\{1,3\}} \cup V_{\{2,4\}}]$ by applying a bipartite complementation between $V_{\{1,3\}}$ and $V_{\{2,4\}}$. Then G' has maximum degree at most 3. By Fact 3, it remains to show that every connected component of G' has bounded clique-width.

Consider a connected component D of G'. We first prove that D contains at most four vertices of degree 3. Let $x \in D$ be a vertex that has degree 3 in D. Without loss of generality assume that $x \in V_{\{1,3\}}$. Then x has two neighbours $y, y' \in V_{\{2,4\}}$ and one neighbour $x' \in V_{\{1,3\}}$. Recall that y is adjacent to y' in G (and hence in D) due to the fact that G is $(2P_1 + P_3)$-free. For the same reason and because $G[V_{\{1,3\}}]$ only has connected components isomorphic to P_1 or P_2, we find that y and y' are adjacent to x' in D if they have degree 3 in D. Hence, either $V(D) = \{x, x', y, y'\}$ or y, y' each have degree 2 in D and x' is a cut-vertex of D. In the first case, D has at most four vertices of degree 3. In the second case, we note that x' is adjacent to neither y nor y' in D (otherwise, for the same reason as before, x' would be adjacent to both of them if it had degree 3 in D, so $V(D)$ would only contain the vertices x, x', y, y'). We then find that D is either obtained by identifying a vertex of a triangle and the end-vertex of a path, meaning that D has only one vertex of degree 3 (namely x), or else by connecting two vertex-disjoint triangles via a path between one vertex of one triangle and one of the other, meaning that D has exactly two vertices of degree 3.

Because D has at most four vertices of degree 3, we may remove these vertices by Fact 1 and then apply Lemma 12 to find that D has bounded clique-width. This completes the proof of Theorem 35. ■

6. CONCLUDING REMARKS

In our main result, we characterized all but two graphs H for which the class of H-free chordal graphs has bounded clique-width. In particular, we identified four new graph classes of bounded clique-width, namely the classes of H-free chordal graphs with $H \in \{\overline{K_{1,3}+2P_1}, P_1+\overline{P_1+P_3}, P_1+\overline{2P_1+P_2}, \overline{S_{1,1,2}}\}$. We also showed that the restriction from H-free graphs to H-free weakly chordal graphs does not yield any new classes of bounded clique-width. Moreover, we determined a new class of (H_1, H_2)-free graphs, namely the class of $(K_4, 2P_1+P_3)$-free graphs, which has bounded clique-width via a reduction to chordal graphs. Combining the latter with five cases recently solved by Dabrowski et al. [24] means that only the following eight cases, up to an equivalence relation,[5] are open in the classification for (H_1, H_2)-free graphs (see [24, 29]).

(1) $H_1 = 3P_1, \overline{H_2} \in \{P_1+S_{1,1,3}, P_2+P_4, S_{1,2,3}\}$;
(2) $H_1 = 2P_1+P_2, \overline{H_2} \in \{P_1+P_2+P_3, P_1+P_5\}$;
(3) $H_1 = P_1+P_4, \overline{H_2} \in \{P_1+2P_2, P_2+P_3\}$; or
(4) $H_1 = \overline{H_2} = 2P_1+P_3$.

We identify the following three main directions for future work.

(1) *Determine whether or not the class of H-free chordal graphs has bounded clique-width when $H \in \{F_1, F_2\}$.*
For this purpose, we recently managed to show that the class of H-free split graphs has bounded clique-width in both these cases [8] and we are currently exploring whether it is possible to generalize the proof of this result to the class of H-free chordal graphs. This seems to be a challenging task, as clique-width has a subtle transition from bounded to unbounded even if the class of graphs under consideration has a "slight" enlargement. For instance, we showed that the class of $(P_1+\overline{P_1+P_3})$-free chordal graphs has bounded clique-width, whereas the class of $(P_1+\overline{2P_1+P_3})$-free chordal graphs, or even $(2P_1+\overline{3P_1})$-free split graphs (see Lemma 23) already has unbounded clique-width.
(2) *Exploit the techniques developed in this article to attack some of the other open cases in the classification for (H_1, H_2)-free graphs.*
In particular, the case $H_1 = 2P_1+P_3, H_2 = \overline{2P_1+P_3}$ seems a good candidate for a possible proof of bounded clique-width via a reduction to $\overline{2P_1+P_3}$-free chordal graphs (this subclass of chordal graphs has bounded clique-width by Theorem 1).

[5]For graphs $H_1, \ldots, H_4$, the classes of (H_1, H_2)-free graphs and (H_3, H_4)-free graphs are equivalent if $\{H_3, H_4\}$ can be obtained from $\{H_1, H_2\}$ by some combination of the two operations: complementing both graphs in the pair, or if one of the graphs in the pair is K_3, replacing it with $\overline{P_1+P_3}$ or vice versa. If two classes are equivalent, then one has bounded clique-width if and only if the other one does (see, e.g. [29]).

Indeed, some partial results in this case are known [2]. For this direction, we also note that it may be worthwhile to more closely examine the relationship between our study and the one on the computational complexity of the graph isomorphism problem (GI) for classes of (H_1, H_2)-free graphs, which was initiated by Kratsch and Schweitzer [47]. Recently, Schweitzer [59] proved that for this study the number of open cases is finite and pointed out similarities between classifying boundedness of clique-width and solving GI for special graph classes. Indeed, Grohe and Schweitzer [39] recently proved that graph isomorphism is polynomial-time solvable on graphs of bounded clique-width.

(3) *Determine whether or not the class of H-free split graphs has bounded clique-width when $H \in \{F_4, F_5\}$.*
The fact that the (un) boundedness of the clique-width of the class of H-free split graphs is known for so many graphs, H raises the question whether we can obtain a full classification of all graphs H for which the class of H-free split graphs has bounded clique-width. We recently reduced [8] this to two problematic cases, namely the graphs F_4 and F_5 displayed in Figure 8.

Finally, we pose the question of whether it is possible to extend the four newly found classes of H-free chordal graphs (when $H \in \{\overline{K_{1,3} + 2P_1}, P_1 + \overline{P_1 + P_3}, P_1 + \overline{2P_1 + P_2}, \overline{S_{1,1,2}}\}$) to larger classes of graphs for which DOMINATING SET and HAMILTON CYCLE are polynomial-time solvable.

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