Tridispersive thermal convection

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Abstract

We derive the linear instability and nonlinear stability thresholds for a problem of thermal convection in a tridispersive porous medium with a single temperature. Importantly we demonstrate that the nonlinear stability threshold is the same as the linear instability one. The significance of this is that the linear theory is capturing completely the physics of the onset of thermal convection. This result is different to the general theory of thermal convection in a tridispersive porous material where the temperatures in the macropores, mesopores and micropores are allowed be different. In that case the coincidence of the nonlinear stability and linear instability boundaries has not been proved.

Keywords: Tridispersive porous media, Convection in porous media, Tridispersive convection, Nonlinear stability, linear instability

1. Introduction

A tridispersive porous medium is one where the solid skeleton contains three types of pores. One type are the usual pores which are referred to as macro pores. In addition there are pores on a smaller scale referred to as meso pores, and cracks or fissures on a yet smaller scale which are referred to as micro pores. The basic theory for thermal convection in a triple porosity (tridispersive) medium was developed by Nield & Kuznetsov [16]. These writers allowed for distinct velocity, temperature and pressure fields in each of the pore systems, macro, meso and micro.

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The porosity associated with the macropores is denoted by ϕ, i.e. ϕ is the ratio of the volume of the macropores to the total volume of the saturated porous material. Furthermore, the meso pores generate a porosity ϵ which is the ratio of the volume occupied by the mesopores to the volume of the porous body which remains once the macropores are removed. This means the fraction of volume occupied by the mesopores is $\epsilon(1-\phi)$. Then, the micropores generate a porosity η which is the ratio of the volume occupied by the micropores to the volume of the porous body which remains once the macro and mesopores are removed. This yields the fraction of volume occupied by the micropores being $\eta(1-\epsilon)(1-\phi)$ while the fraction of volume occupied by the solid skeleton is $(1-\eta)(1-\epsilon)(1-\phi)$.

Theoretical work on thermal flow in tridispersive porous media commenced with work of Nield and Kuznetsov [16]. Further work on various problems is due to Nield & Kuznetsov [17], Cheng [5], Ghalambaz *et al.* [13], and Straughan [22], chapter 13. Fundamental work on the thermal convection problem was developed by Kuznetsov & Nield [14]. These articles utilize different velocities U_i^f, U_i^p and U_i^c in the macro, meso and micropores, with different temperatures T^f, T^p and T^c.

Undoubtedly the current interest in tridispersive porous media is due to the many applications arising in engineering and in real life. For example, underground oil reservoirs are modelled as triple porosity systems, e.g. Ali *et al.* [1], Deng *et al.* [7], Olusola *et al.* [18], Wang *et al.* [23]. Triple porosity features in modelling methane recovery from coal beds, Wei & Zhang [24], Zou *et al.* [25], and likewise is important in analysing the procurement of drinking water from an aquifer, Zuber & Motyka [26]. In a context important for the present work triple porosity is proving very important in geothermal reservoir modelling, Siratovich *et al.* [19].

In this paper we develop and fully analyse thermal convection in a tridisperse porous medium when only one temperature is employed and the horizontal layer containing the saturated porous medium is heated from below. For many practical situations we believe this is sufficient. To achieve our aim it is first necessary to derive a suitable mathematical model.

2. Basic model

We begin with fields in the solid, fluid in the macro pores, fluid in the meso pores, and fluid in the micro pores, and we denote each phase by s, f, m and c, respectively. As stated earlier the macro, meso and micro porosities are ϕ, ϵ and η. The actual fluid velocities in the macro, meso amd micro pores are denoted by V_i^f, V_i^m and V_i^c and these are connected to the pore averaged velocities in the macro, meso amd micro phases, U_i^f, U_i^m and U_i^c, by the relations

$$U_i^f = \phi V_i^f, \qquad U_i^m = \epsilon(1-\phi)V_i^m, \qquad U_i^c = \eta(1-\phi)(1-\epsilon)V_i^c. \qquad (1)$$

Let ϵ_1, ϵ_2 and ϵ_3 be defined by

$$\epsilon_1 = (1-\phi)(1-\epsilon)(1-\eta), \qquad \epsilon_2 = \epsilon(1-\phi), \qquad \epsilon_3 = \eta(1-\phi)(1-\epsilon). \quad (2)$$

Let the temperatures in the solid, macro, meso and micro phases be denoted by T^s, T^f, T^m and T^c, with $(\rho c)_\alpha$, κ_α, $\alpha = s, f, m$ or c, being the product of the density and specific heat at constant pressure, and the thermal conductivity, in each phase. We wish to write equations for energy balance in each phase in the tridispersive porous medium and to do this we are guided by Kuznetsov & Nield [14], equations (11) - (13) and (18) - (23), and also by the equations for a single porosity porous medium under conditions of local thermal non-equilibrium, cf. Straughan [22], equations (2.9) - (2.12).

The equations of balance of energy in the solid, macro pore, meso pore, and micro pore phases are then

$$\epsilon_1(\rho c)_s T^s_{,t} = \epsilon_1\kappa_s\Delta T^s + s_1(T^f - T^s) + s_2(T^m - T^s) + s_3(T^c - T^s), \quad (3)$$

$$\phi\big[(\rho c)_f T^f_{,t} + (\rho c)_f V_i^f T^f_{,i}\big] = \phi\kappa_f\Delta T^f + h_{12}(T^m - T^f) + s_1(T^s - T^f), \quad (4)$$

$$\begin{aligned}\epsilon_2\big[(\rho c)_m T^m_{,t} + (\rho c)_m V_i^m T^m_{,i}\big] =& \epsilon_2\kappa_m\Delta T^m + h_{12}(T^f - T^m)\\ &+ h_{23}(T^c - T^m) + s_2(T^s - T^m),\end{aligned} \quad (5)$$

and

$$\epsilon_3\big[(\rho c)_c T^c_{,t} + (\rho c)_c V_i^c T^c_{,i}\big] = \epsilon_3\kappa_c\Delta T^c + h_{23}(T^m - T^c) + s_3(T^s - T^c). \quad (6)$$

Equations (4) - (6) follow Kuznetsov & Nield [14] and assume interactions in temperature between the macro and meso phases, and between the meso and micro phases. The coefficients s_1, s_2, s_3, h_{12} and h_{23} represent interaction terms. The model with different temperatures is completed by adding equations (18) - (23) of Kuznetsov & Nield [14] together with the incompressibility conditions for V_i^f, V_i^m and V_i^c.

In this work we are interested in a model in which $T^s = T^f = T^m = T^c = T$, where T is a common temperature. Since it is the same fluid in the macro, meso and micro phases we also assume ρc is the same for the f, m and c terms. Thus, we may add (3) - (6) and employ (1) to derive a single energy balance equation for the tridispersive porous medium. We put

$$(\rho c)_a = \epsilon_1(\rho c)_s + (\rho c)_f$$

and

$$k_a = \phi\kappa_f + \epsilon_2\kappa_m + \epsilon_3\kappa_c + \epsilon_1\kappa_s$$

to derive the energy balance equation

$$(\rho c)_a T_{,t} + (\rho c)_f (U_i^f + U_i^m + U_i^c)T_{,i} = k_a \Delta T. \tag{7}$$

The model is completed by adjoining the momentum equations (18) - (23), and the mass balance equations (14) and (16), of Kuznetsov & Nield [14], which become for a single temperature T,

$$\begin{aligned}
0 &= -\frac{\mu}{K_f}U_i^f - p_{,i}^f - \zeta_{12}(U_i^f - U_i^m) + g\rho_F\alpha k_i T, \\
0 &= -\frac{\mu}{K_m}U_i^m - p_{,i}^m + \zeta_{12}(U_i^f - U_i^m) - \zeta_{23}(U_i^m - U_i^c) + g\rho_F\alpha k_i T, \\
0 &= -\frac{\mu}{K_c}U_i^c - p_{,i}^c + \zeta_{23}(U_i^m - U_i^c) + g\rho_F\alpha k_i T,
\end{aligned} \tag{8}$$

and

$$U_{i,i}^f = 0, \qquad U_{i,i}^m = 0, \qquad U_{i,i}^c = 0. \tag{9}$$

Thus, the complete model is described by equations (8), (9) and (7), making thirteen equations in total for the thirteen variables $U_i^f, U_i^m, U_i^c, T, p^f, p^m$ and p^c. We now investigate the problem of thermal convection in a tridispersive porous medium according to the above system of equations.

3. Thermal convection

We shall assume the porous medium is contained in the horizontal layer $0 < z < d$ with the temperature at $z = 0$ held fixed at $T_L{}^\circ C$ while the temperature at $z = d$ is kept fixed at $T_U{}^\circ C$ with $T_L > T_U$.

To investigate thermal convection we study stability of the steady solution

$$\bar{U}_i^f \equiv 0\,, \quad \bar{U}_i^m \equiv 0\,, \quad \bar{U}_i^c \equiv 0\,, \quad \bar{T} = T_L - \beta z\,, \tag{10}$$

where β is the temperature gradient

$$\beta = \frac{T_L - T_U}{d} \,. \tag{11}$$

Now let u_i^f, u_i^m, u_i^c, θ, π^f, π^m, π^c be perturbations to the steady solution and then non-dimensionalize the resulting perturbation equations with the substitutions

$$x_i = x_i^* d \,, \quad t = t^* \mathcal{T} \,, \quad \frac{\mu}{\zeta_{12} K_f} = \mu_f \,, \quad \frac{\mu}{\zeta_{12} K_m} = \mu_m, \quad \frac{\mu}{\zeta_{12} K_c} = \mu_c, , \tag{12}$$

where

$$\mathcal{T} = \frac{d^2 (\rho c)_a}{k_a} \tag{13}$$

and where the velocity scale U and temperature scale $T^\sharp$, and the Rayleigh number Ra are given by

$$U = \frac{k_a}{d(\rho c)_f} \,, \qquad T^\sharp = U \sqrt{\frac{(\rho c)_f \beta d^2 \zeta_{12}}{\rho_F g \alpha k_a}} \tag{14}$$

and

$$Ra = R^2 = \frac{(\rho c)_f \beta d^2 \rho_F g \alpha}{k_a \zeta_{12}} \,. \tag{15}$$

The resulting perturbation equations have form

$$\begin{aligned}
&\mu_f u_i^f + (u_i^f - u_i^m) = -\pi_{,i}^f + R\theta k_i, \qquad u_{i,i}^f = 0 \,, \\
&\mu_m u_i^m - (u_i^f - u_i^m) + \zeta_R (u_i^m - u_i^c) = -\pi_{,i}^m + R\theta k_i, \qquad u_{i,i}^m = 0 \,, \\
&\mu_c u_i^c + \zeta_R (u_i^c - u_i^m) = -\pi_{,i}^c + R\theta k_i, \qquad u_{i,i}^c = 0 \,, \\
&\theta_{,t} + (u_i^f + u_i^m + u_i^c)\theta_{,i} = R(w^f + w^m + w^c) + \Delta\theta \,,
\end{aligned} \tag{16}$$

where $\mathbf{u}^f = (u^f, v^f, w^f)$, $\mathbf{u}^m = (u^m, v^m, w^m)$ and $\mathbf{u}^c = (u^c, v^c, w^c)$, and $\zeta_R = \zeta_{23}/\zeta_{12}$. These equations hold in the domain $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\} \times \{t > 0\}$. The boundary conditions are

$$u_i^f n_i = 0, \quad u_i^m n_i = 0, \quad u_i^c n_i = 0, \quad \theta = 0, \quad \text{on } z = 0, 1 \,, \tag{17}$$

where n_i is the unit outward normal to $z = 0$ or $z = 1$, and u_i^f, u_i^m, u_i^c, θ, π^f, π^m, π^c satisfy a plane tiling shape in the (x, y) plane. The periodic convection cell which arises will be denoted by V.

4. Linear instability

To determine the linear instability boundary one discards the nonlinear terms in equations (16) and one seeks a solution in which u_i^f, u_i^m, u_i^c, θ, π^f, π^m, π^c have a time dependence like $e^{\sigma t}$. This leads to a system of equations of form

$$\begin{aligned}
&\mu_f u_i^f + (u_i^f - u_i^m) = -\pi^f_{,i} + R\theta k_i, \qquad u^f_{i,i} = 0\,,\\
&\mu_m u_i^m - (u_i^f - u_i^m) + \zeta_R(u_i^m - u_i^c) = -\pi^m_{,i} + R\theta k_i, \qquad u^m_{i,i} = 0\,,\\
&\mu_c u_i^c - \zeta_R(u_i^m - u_i^c) = -\pi^c_{,i} + R\theta k_i, \qquad u^c_{i,i} = 0\,,\\
&\sigma\theta = R(w^f + w^m + w^c) + \Delta\theta.
\end{aligned} \tag{18}$$

Theorem 1
The strong form of the principle of exchange of stabilities holds for system (18), in the sense that $\sigma \in \mathbb{R}$.
Proof.
Multiply equations (18) by $u_i^{f*}, u_i^{m*}, u_i^{c*}$ and θ^*, the complex conjugates of $u_i^f, u_i^m, u_i^c, \theta$. Then integrate each resulting equation over the period cell V. Denote by $(\cdot,\cdot)$ and $\parallel \cdot \parallel$ the inner product and norm on the complex Hilbert space $L^2(V)$ and then one may show after some integration by parts and use of the boundary conditions,

$$\begin{aligned}
&(\mu_f + 1) \parallel \mathbf{u}^f \parallel^2 - (u_i^m, u_i^{f*}) = R(\theta, w^{f*}),\\
&(\mu_m + 1 + \zeta_R) \parallel \mathbf{u}^m \parallel^2 - (u_i^f, u_i^{m*}) - \zeta_R(u_i^c, u_i^{m*}) = R(\theta, w^{m*}),\\
&(\mu_c + \zeta_R) \parallel \mathbf{u}^c \parallel^2 - \zeta_R(u_i^m, u_i^{c*}) = R(\theta, w^{c*}),\\
&\sigma \parallel \theta \parallel^2 = - \parallel \nabla\theta \parallel^2 + R(w^f, \theta^*) + R(w^m, \theta^*) + +R(w^c, \theta^*).
\end{aligned} \tag{19}$$

Add the four equations in (19) to derive the following equation

$$\begin{aligned}
\sigma \parallel \theta \parallel^2 = &- \parallel \nabla\theta \parallel^2 - (\mu_f + 1) \parallel \mathbf{u}^f \parallel^2\\
&- (\mu_m + 1 + \zeta_R) \parallel \mathbf{u}^m \parallel^2 - (\mu_c + \zeta_R) \parallel \mathbf{u}^c \parallel^2\\
&+ (u_i^m, u_i^{f*}) + (u_i^f, u_i^{m*}) + \zeta_R\big[(u_i^c, u_i^{m*}) + (u_i^m, u_i^{c*})\big]\\
&+ R\big[(\theta, w^{f*}) + (w^f, \theta^*)\big] + R\big[(\theta, w^{m*}) + (w^m, \theta^*)\big]\\
&+ R\big[(\theta, w^{c*}) + (w^c, \theta^*)\big]\,.
\end{aligned} \tag{20}$$

Rearrange the functions $\mathbf{u}^f$, $\mathbf{u}^m$, $\mathbf{u}^c$ and θ in their real and imaginary parts and put $\sigma = \sigma_r + i\sigma_1$. Take the imaginary part of equation (20) to obtain

$$\sigma_1 \parallel \theta \parallel^2 = 0. \tag{21}$$

We require $\| \theta \| \neq 0$ and so $\sigma_1 = 0$. Thus, $\sigma \in \mathbb{R}$ and the theorem is proved.

This result is important as it shows oscillatory convection does not hold. Therefore, to find the linear instability boundary we analyze system (19) with $\sigma = 0$.

To determine the linear instability critical Rayleigh number we remove $u^f, u^m, u^c, v^f, v^m, v^c$ from equations (19) by taking the double curl of the momentum equations in (19) and we retain only the third components of the resulting equations. This leaves one needing to solve

$$\begin{aligned}
&\mu_f \Delta w^f + \Delta w^f - \Delta w^m = R\Delta^* \theta\,, \\
&\mu_m \Delta w^m - (\Delta w^f - \Delta w^m) + \zeta_R(\Delta w^m - \Delta w^c) = R\Delta^* \theta\,, \\
&\mu_c \Delta w^c - \zeta_R(\Delta w^m - \Delta w^c) = R\Delta^* \theta\,, \\
&\Delta\theta + R(w^f + w^m + w^c) = 0\,,
\end{aligned} \tag{22}$$

where $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian.

Theorem 2
The critical Rayleigh number is given by

$$Ra_{crit} = R^2_{crit} = 4\pi^2 \frac{\big[(\mu_f + 1)(\mu_m\mu_c + \zeta_R\mu_m + \zeta_R\mu_c) + \mu_f(\mu_c + \zeta_R)\big]}{\big[\zeta_R(\mu_m + \mu_c + 9) + 4\mu_c + \mu_m\mu_c + \mu_m\big]}\,. \tag{23}$$

Proof
Employ normal modes in (22) and then one may reduce the calculation for the critical Rayleigh number to minimizing R in a and n, where

$$R^2 = \frac{\Lambda_n^2}{a^2} \frac{\big[(\mu_f + 1)(\mu_m\mu_c + \zeta_R\mu_m + \zeta_R\mu_c) + \mu_f(\mu_c + \zeta_R)\big]}{\big[\zeta_R(\mu_m + \mu_c + 9) + 4\mu_c + \mu_m\mu_c + \mu_m\big]}\,, \tag{24}$$

and where $\Lambda_n = n^2\pi^2 + a^2$. One shows that the minimum in (24) is achieved for $n = 1$ and for $a = \pi$. Thus, the critical Rayleigh number is given by (23) and the theorem is proved.

5. Nonlinear stability

Equation (23) delivers the linear instability threshold but it yields no information on stability. In the absence of a nonlinear stability result theorem 2 is limited in use. One really needs to calculate a threshold for the nonlinear stability critical Rayleigh number and if this is close to the value in (23), then the linear result is likely to be useful, cf. Straughan [20]. We now address the nonlinear problem and achieve an optimal result. In order to achieve this we employ inequalities involving an appropriate energy for system (16). Results of asymptotic stability in this vein have been the subject of several recent articles using a judicious choice of an energy functional, cf. Amendola & Fabrizio [2], Amendola *et al.* [3], Amendola *et al.* [4], Deepika & Narayana [6], Deseri *et al.* [8], Fabrizio & Lazzari [9], Fabrizio *et al.* [10], Fabrizio *et al.* [11], Franchi & Morro [12], Nandal & Mahajan [15], Straughan [20–22].

Theorem 3
The critical Rayleigh number (23) of linear instability theory is the same as the critical Rayleigh number of nonlinear stability theory, and thus, a Rayleigh number below this ensures global asymptotic stability of the base solution.

Proof
The proof begins by multiplying the u_i^f equation in (16) by u_i^f, the u_i^m equation in (16) by u_i^m, the u_i^c equation in (16) by u_i^c, the θ equation in (16) by θ, and one integrates each over the period cell V. After some integration by parts one may obtain the following identities

$$\begin{aligned}
&\mu_f \parallel \mathbf{u}^f \parallel^2 + \Big(u_i^f, \{u_i^f - u_i^m\}\Big) = R(\theta, w^f),\\
&\mu_m \parallel \mathbf{u}^m \parallel^2 - \Big(u_i^m, \{u_i^f - u_i^m\}\Big) + \zeta_R\,(u_i^m, \{u_i^m - u_i^c\}) = R(\theta, w^m),\\
&\mu_c \parallel \mathbf{u}^c \parallel^2 + \zeta_R\,(u_i^c, \{u_i^c - u_i^m\}) = R(\theta, w^c),\\
&\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 = R(w^f + w^m + w^c, \theta) - \parallel \nabla\theta \parallel^2,
\end{aligned} \tag{25}$$

where now $(\cdot,\cdot)$ and $\parallel \cdot \parallel$ denote the inner product and norm on the real Hilbert space $L^2(V)$. Add the four equations in (25) to obtain

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 = RI - D\,, \tag{26}$$

where

$$I = 2(w^f + w^m + w^c, \theta) \tag{27}$$

and the dissipation term is given by

$$\begin{aligned}
D = &\parallel \nabla\theta^2 \parallel + \mu_f \parallel \mathbf{u}^f \parallel^2 + \mu_m \parallel \mathbf{u}^m \parallel^2 + \mu_c \parallel \mathbf{u}^c \parallel^2\\
&+ \parallel \mathbf{u}^f - \mathbf{u}^m \parallel^2 + \zeta_R \parallel \mathbf{u}^m - \mathbf{u}^c \parallel^2\ .
\end{aligned} \tag{28}$$

Define the term R_E by

$$\frac{1}{R_E} = \max_H \frac{I}{D} \tag{29}$$

where H consists of L^2 functions for u_i^f, u_i^m and u_i^c and H^1 functions for θ. Then from (26) one obtains

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 \leq -D\left(1 - \frac{R}{R_E}\right). \tag{30}$$

If $R < R_E$, say $1 - R/R_E = a > 0$, then by using Poincaré's inequality in (30) one sees that

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 \leq -a\pi^2 \parallel \theta \parallel^2 . \tag{31}$$

This inequality may be integrated to yield

$$\parallel \theta(t) \parallel^2 \leq \parallel \theta(0) \parallel^2 \exp(-2a\pi^2 t). \tag{32}$$

Thus, inequality (32) shows that $\parallel \theta(t) \parallel$ decays exponentially provided $R < R_E$.

Next, form $(25)_1$+$(25)_2$+$(25)_3$ to derive

$$\begin{aligned} \mu_f \parallel \mathbf{u}^f \parallel^2 + \mu_m \parallel \mathbf{u}^m \parallel^2 + \mu_c \parallel \mathbf{u}^c \parallel^2 + \parallel \mathbf{u}^f - \mathbf{u}^m \parallel^2 \\ + \zeta_R \parallel \mathbf{u}^m - \mathbf{u}^c \parallel^2 = R(\theta, w^f + w^m + w^c). \end{aligned} \tag{33}$$

Employ the arithmetic-geometric mean inequality on the right hand side of (33) to obtain

$$\mu_f \parallel \mathbf{u}^f \parallel^2 + \mu_m \parallel \mathbf{u}^m \parallel^2 + \mu_c \parallel \mathbf{u}^c \parallel^2 \leq R^2 A \parallel \theta \parallel^2 , \tag{34}$$

where

$$A = \frac{1}{\mu_f} + \frac{1}{\mu_m} + \frac{1}{\mu_c} .$$

Thus, from (32) and (34) one deduces that $R < R_E$ also guarantees decay of $\mathbf{u}^f, \mathbf{u}^m$ and $\mathbf{u}^c$. Hence, the condition $R < R_E$ represents a global (for all initial data) nonlinear stability threshold.

To complete the proof of the theorem we need to solve the maximum problem (29). For Lagrange multipliers λ^f, λ^m and λ^c, the Euler-Lagrange equations which arise from (29) are found to be

$$\begin{aligned}
&R_E\theta k_i - \mu_f u_i^f - (u_i^f - u_i^m) = \lambda^f_{,i}\,,\\
&R_E\theta k_i - \mu_m u_i^m + (u_i^f - u_i^m) + \zeta_R(u_i^m - u_i^c) = \lambda^m_{,i}\,,\\
&R_E\theta k_i - \mu_c u_i^c - \zeta_R(u_i^c - u_i^m) = \lambda^c_{,i}\,,\\
&(w^f + w^m + w^c)R_E + \Delta\theta = 0.
\end{aligned} \tag{35}$$

We observe that equations (35) have the same form as equations (18) when $\sigma = 0$. Thus the critical value of R_E^2 has the same value as Ra_{crit} in theorem 2 and the theorem is proved.

6. Conclusions

In this article we have produced a mathematical model for non-isothermal flow in a saturated porous medium with a triple porosity system. The triple porosity arises because the material has pores on a macro scale, meso pores on a lesser length scale, and micro pores which are typically cracks or fissures.

We have analysed thermal convection in a horizontal layer of tridisperse porous medium and have found the critical Rayleigh number, Ra_{crit}, given by (23), which determines when convective motion will commence. This critical Rayleigh number depends on the permeabilities associated to the macro, meso and micro porosity systems, and it also depends on the interaction coefficients for the flows between the same systems. We have shown that the critical Rayleigh number is optimal in the sense that if the actual Rayleigh number exceeds Ra_{crit} then convective motion will commence via the mechanism of stationary convection, whereas if the Rayleigh number is less than Ra_{crit} then perturbations to the conduction solution (10) will decay to zero exponentially in L^2 norm.

For interpretation of results and comparision with other porous media literature it may be convenient to employ an alternative Rayleigh number which does not directly involve the interaction coefficient ζ_{12}. This is straightforward to do and so if we define an alternative Rayleigh number Ra_1 by

$$Ra_1 = \frac{(\rho c)_f}{k_a}\,\beta d^2 g\rho_F\alpha\,\frac{K_f}{\mu} \tag{36}$$

then one finds

$$Ra = \mu_f Ra_1 \,. \tag{37}$$

Values of the critical numbers for Ra_1 may be easily obtained from (37) and (23).

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