Ranking Equilibrium Competition in Auctions with Participation Costs

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January 18, 2017

Abstract

This paper studies the degrees of equilibrium competition in three common forms of auctions with costly participation, and shows that, when bidders' valuation distribution is concave, there is a simple condition to rank the equilibrium competition of those auctions. It also investigates how the results are related to stochastic ordering of bidders' valuation distributions, and provides some illustrative examples.

Keywords: auctions; participation costs; competition; concave distribution; first order stochastic dominance

JEL Classification: D44; D82

1 Introduction

In auctions with participation costs, the degrees of competition, which is measured by the numbers of participating bidders, are endogenously determined. It is a natural and important question to explore the possible difference in equilibrium competition in those auctions, particularly in comparison to social optimum. In this paper, we study three common forms of auctions with costly participation, and complement the current results in the literature by providing some simple conditions that enable us to clearly rank the degrees of equilibrium competition in those auctions.

The literature on auctions with costly participation can roughly be divided into two categories, depending on who pays the costs. One is search auction, denoted by A_s, where a seller incurs costs to attract bidders to the auction (Crémer, et al 2007; Szech, 2011; Li and Xu, 2016). The other is auctions with costly entry, where bidders need to pay entry costs to participate in the auction, which can be further summarized to two branches: in the first branch, denoted by A_u, bidders make entry decisions *before* knowing their true valuations of the product (McAfee and McMillan, 1987; Levin and Smith, 1994); and in

the second, denoted by A_k, bidders make entry decisions only *after* learning their true valuations (Samuelson, 1985; Tan and Yilankaya, 2006; Cao and Tian, 2010).

In the case of search auction A_s, Szech (2011) shows that when bidders' valuation distribution is of increasing failure rate (IFR), a seller will invite more than the socially optimal number of bidders to the auction. This over-invitation result is also reported by Li and Xu (2016) in descending auctions, yet under different assumptions. In the auction of A_u, McAfee and McMillan (1987) show in their seminal paper that the equilibrium number of participating bidders is just equal to the socially optimal one.

In this paper, we complement the above results by investigating the degree of equilibrium competition in the auction of A_k. In this case, the number of participating bidders is a random variable, whose expectation is related to the total number of potential bidders in the market. We show that, when bidders' valuation distribution is concave, the expected number of participating bidders in A_k is strictly increasing in the total number of potential bidders, which enables us to provide a simple condition on ranking the degrees of equilibrium competition between A_k and A_u. Moreover, when bidders' valuation distribution is uniform, there is a robust ranking result of equilibrium competition across A_s, A_u and A_k, regardless of the magnitude of the participation cost and the total number of potential bidders. Finally, we investigate how the results are related to stochastic ordering of bidders' valuation distributions, and provide some illustrative examples.

2   the Model

Consider a standard auction without a reserve price, where there are $N \in [1, \infty)$ potential bidders in the market who may participate. Participation in the auction is costly, and that cost can be paid either by the seller, such as in A_s, or by the bidders, such as in A_u and A_k. We assume there is a unit participation cost of $c \in (0, 1)$ for each bidder. The bidders are *ex ante* homogeneous, whose valuation V conforms to the distribution of F on $[0, 1]$ with density $f > 0$.

When there are $n \leq N$ participating bidders, indexed by $i = 1, 2, ..., n$, let $\{V_i\}_{i=1}^n$ be n independent draws from F, where V_i is bidder i's valuation. The distribution of F is common knowledge, while v_i, the realization of V_i, is privately observed only by bidder i. We denote $V_{k:n}$ the kth highest valuation of the n bidders' such that

$$V_{1:n} \geq V_{2:n} \geq \cdots \geq V_{n:n}.$$

For the order statistics of $V_{k:n}$, let $F_{k:n}$ and $f_{k:n}$ be its cumulative distribution function and probability density function respectively.

We denote n^{**} as the socially optimal number of participating bidders, which maximizes the expected social welfare. Therefore,

$$n^{**} \in \arg\max_n \ \mathrm{E}\left[V_{1:n}\right] - nc,$$

where $\mathrm{E}\left[V_{1:n}\right]$ is the expected value of $V_{1:n}$, and n^{**} satisfies

$$\mathrm{E}\left[V_{1:n^{**}} - V_{1:n^{**}-1}\right] \geq c > \mathrm{E}\left[V_{1:n^{**}+1} - V_{1:n^{**}}\right]. \tag{1}$$

The existence of n^{**} is guaranteed by the observation that $\mathrm{E}\left[V_{1:n}\right]$ is increasing and concave in n, with $\lim_{n\to\infty} \mathrm{E}\left[V_{1:n+1} - V_{1:n}\right] = 0$.

3   Ranking Equilibrium Competition

We next consider the degrees of equilibrium competition in the three auctions of A_s, A_u and A_k, where the seller is a revenue-maximizer, yet not imposing a reserve price.

3.1   Search auction A_s

In a search auction A_s, the problem for a revenue-maximizing seller is to

$$\max_n \ \mathrm{E}\left[V_{2:n}\right] - nc. \tag{2}$$

It is known that when F is of IFR, (2) is a well-defined convex problem. Its solution, denoted by n_s^*, is given by

$$\mathrm{E}\left[V_{2:n_s^*} - V_{2:n_s^*-1}\right] \geq c > \mathrm{E}\left[V_{2:n_s^*+1} - V_{2:n_s^*}\right].$$

Lemma 1 (Sczeh, 2011) *If F is of IFR, then in auction A_s, a revenue maximizing seller invites more than the socially optimal number of bidders to the auction, that is,*

$$n_s^* \geq n^{**}.$$

The intuition is that, bidders' winning rent is decreasing in n, and inviting an extra bidder will then reduce the expected total surplus of the bidders, which is ignored by the seller but is taken into account when computing the expected social welfare.

3.2   Auction with entry cost I: A_u

There are two stages in the auction of A_u: in the second stage, it is a standard auction among the participating bidders; in the first stage, knowing what will follow, each bidder decides whether or not to incur c and enter the auction. In the auction of A_u, the bidders make entry decisions *before* learning their true valuations.

When there are n participating bidders, the expected profit for a bidder is

$$\mathrm{E}\pi\left(n\right) = \frac{1}{n}\mathrm{E}\left[V_{1:n} - V_{2:n}\right] - c = \mathrm{E}\left[V_{1:n} - V_{1:n-1}\right] - c. \tag{3}$$

The equilibrium number of participating bidders, denoted by n_u^*, is therefore given by

$$\mathrm{E}\left[V_{1:n_u^*} - V_{1:n_u^*-1}\right] \geq c > \mathrm{E}\left[V_{1:n_u^*+1} - V_{1:n_u^*}\right],$$

which is the same as (1). We then have the following result.

Lemma 2 (McAfee and McMillan, 1987) *In auction A_u where bidders make entry decisions before learning their valuations, equilibrium entry is efficient, that is,*

$$n_u^* = n^{**}.$$

This happens because the bidders make entry decisions before knowing their valuations, and therefore the *ex ante* information rent is zero for the bidders. As the expected social welfare is equal to the sum of the expected auction revenue and the expected payoff of the participating bidders, which is equal to zero, then revenue maximization for the seller is equivalent to welfare maximization.

3.3 Auction with entry cost II: A_k

The setup of the auction of A_k is the same as that of A_u, except that now the bidders make entry decisions *after* learning their true valuations. It is well known that there exists a unique symmetric equilibrium with cutoff valuation $\hat{v}$ such that

$$\hat{v} \cdot F_{1:N-1}(\hat{v}) - c = 0, \tag{4}$$

where N is the total number of potential bidders. Moreover, when F is concave, there does not exist asymmetric equilibrium (Tan and Yilankaya, 2006). From (4), the expected number of participating bidders, denoted by $n_k^*(N)$, is

$$n_k^*(N) = N\left[1 - F(\hat{v}(N))\right], \tag{5}$$

from the property of binomial distribution.

Lemma 3 $\hat{v}(N)$ *is increasing in* N, *and* $\lim_{N\to\infty} \hat{v}(N) = 1$.

Proof. Suppose N can take real value, and by simple differentiation,

$$\frac{\partial \hat{v}}{\partial N} = -\frac{\ln F(\hat{v})}{\frac{1}{\hat{v}} + (N-1)\frac{f(\hat{v})}{F(\hat{v})}} > 0. \tag{6}$$

Second, from monotone convergence theorem, we know that the sequence of $\hat{v}(N)$ converges to its supreme, denoted by $\bar{v}$. If $\bar{v} < 1$, then $\lim_{N\to\infty} \hat{v}(N) F_{1:N-1}(\hat{v}(N)) = 0 < c$, which results in a contradiction. ∎

We are more interested in the properties of $n_k^*(N)$, and have the following result.

Lemma 4 *If* $F(x)$ *is concave, then* $n_k^*(N)$ *is increasing in* N, *and* $\bar{n}_k^* = \lim_{N\to\infty} n_k^*(N) = -\ln c$.

Proof. If $F(x)$ is concave, then $f(x) \leq F(x)/x$. From (5) and (6), we have

$$\begin{aligned}
\frac{\partial n_k^*}{\partial N} &= [1 - F(\hat{v})] + Nf(\hat{v})\frac{\ln F(\hat{v})}{\frac{1}{\hat{v}} + (N-1)\frac{f(\hat{v})}{F(\hat{v})}} \\
&\geq [1 - F(\hat{v})] + Nf(\hat{v})\frac{\ln F(\hat{v})}{\frac{f(\hat{v})}{F(\hat{v})} + (N-1)\frac{f(\hat{v})}{F(\hat{v})}} \\
&= [1 - F(\hat{v})] + F(\hat{v})\ln F(\hat{v}) \geq 0.
\end{aligned}$$

Second, applying l'Hôpital's rule,

$$\begin{aligned}
\lim_{N\to\infty} n_k^*(N) &= \lim_{N\to\infty}\left(\frac{c}{\hat{v}(N)}\right)^{\frac{1}{N-1}}\left\{-\frac{N^2[\ln c - \ln\hat{v}(N)]}{(N-1)^2} + \frac{N^2}{(N-1)}\frac{\ln F(\hat{v})}{1 + (N-1)\frac{f(\hat{v})\hat{v}}{F(\hat{v})}}\right\} \\
&= -\ln c,
\end{aligned}$$

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because

$$\lim_{N\to\infty} \hat{v}(N) = 1; \quad \lim_{N\to\infty} \left(\frac{c}{\hat{v}(N)}\right)^{\frac{1}{N-1}} = 1; \quad \lim_{N\to\infty} \frac{N}{(N-1)} \frac{N \ln F(\hat{v})}{1 + (N-1)\frac{f(\hat{v})\hat{v}}{F(\hat{v})}} = 0.$$

∎

Lemma 4 then enables us to provide a simple condition on the comparison between $n_k^*(N)$ and n_u^*, as follows.

Proposition 5 *When $F(v)$ is concave,*

- *If $\bar{n}_k^* \leq n_u^*$, then for any given finite N, $n_k^*(N) < n_u^*$;*

- *If $\bar{n}_k^* > n_u^*$, then there exists a finite N_0 such that, $n_k^*(N) < n_u^*$ iff $N < N_0$.*

Moreover, when V conforms to uniform distribution, there is a robust ranking result on equilibrium competition across the auctions of A_s, A_u and A_k.

Lemma 6 *If $V \sim U[0,1]$, then for any $c \in (0,1)$ and any finite $N \in [1,\infty)$,*

$$n_k^*(N) < n_u^* = n^{**} \leq n_s^*.$$

Proof. As $V \sim U[0,1]$, then F is both concave and of IFR. From Lemma 4, $n_k^*(N)$ is increasing in N, and $\lim_{N\to\infty} n_k^*(N) = \bar{n}_k^* = -\ln c$. When $V \sim U[0,1]$, the condition for n_u^* is

$$n_u^*(n_u^* + 1) \leq c^{-1} < (n_u^* + 1)(n_u^* + 2).$$

When $n = \bar{n}_k^*$, we have $\bar{n}_k^*(\bar{n}_k^* + 1) = -\ln c(1 - \ln c) < c^{-1}$, and therefore $n_u^* \geq \bar{n}_k^* > n_k^*(N)$ for any finite N. The parts of $n_u^* = n^{**} \leq n_s^*$ is already proved as above in Lemma 1. ∎

The above ranking results on equilibrium competition provide interesting implications on public regulations. For example, in auction A_k, if there is insufficient entry in the auction, according to Proposition 5, then a regulator may encourage competition by subsidizing bidders' entry costs, which may induce more efficient allocations in equilibrium. Similar argument also applies for the case of excessive competition, such as in auction A_s.

3.4 Further discussion

We next investigate how $n_k^*(N)$ is related to the stochastic ordering of bidders' valuation distributions. Suppose bidders' valuations are now independent draws from the distribution of G on $[0,1]$, with $G \succ_{FOSD} F$ in terms of first order stochastic dominance (FOSD). If we denote the expected number of participating bidders under G by $\tilde{n}_k^*(N)$, we then have the following result.

Lemma 7 *If $G \succ_{FOSD} F$, then $\tilde{n}_k^*(N) \geq n_k^*(N)$.*

Proof. If $G \succ_{FOSD} F$, then $G_{1:N} \succ_{FOSD} F_{1:N}$. From (4), we have $\hat{v} \cdot F_{1:N-1}(\hat{v}) = c = \tilde{v} \cdot G_{1:N-1}(\tilde{v})$, where $\tilde{v}$ is the new cutoff valuation under G, which implies $\hat{v} \leq \tilde{v}$ and $F(\hat{v}) \geq G(\tilde{v})$. The result then follows from (5). ∎

Finally, let us consider a family of concave distributions in the form of $F_\alpha(v) = v^\alpha$, indexed by $\alpha \leq 1$ and ordered in FOSD. We are interested in how the ranking of equilibrium competition between A_u and A_k is related to α. First, in the auction of A_k, from (4) and (5), the expected number of participating bidders is

$$n_k^*(N, \alpha) = N \left[1 - c^{\frac{\alpha}{\alpha(N-1)+1}}\right],$$

which is increasing in α, with its limit $\bar{n}_k^* = \lim_{N \to \infty} n_k^*(N, \alpha) = -\ln c$. Second, in the auction of A_u, from (3), we have

$$\mathrm{E}\pi(n; \alpha) = \frac{\alpha}{[\alpha(n-1)+1][\alpha n + 1]} - c,$$

and the equilibrium number of bidders, $n_u^*(\alpha)$, is given by $\mathrm{E}\pi(n_u^*; \alpha) \geq 0 > \mathrm{E}\pi(n_u^* + 1; \alpha)$. If $\mathrm{E}\pi(\bar{n}_k^*; \alpha) \geq 0$, then $n_u^*(\alpha) \geq \bar{n}_k^* > n_k^*(N, \alpha)$ for any finite N (Proposition 5).

Figure 1 below provides an illustration of the ranking result in the (c, α) space. In the diagram, the blue curve plots the set of all points (c, α) such that $\mathrm{E}\pi(\bar{n}_k^*; \alpha) = 0$, which define an implicit function of $\alpha = \eta(c)$. In Figure 1, for each c,

- If $\alpha \geq \eta(c)$, then $\mathrm{E}\pi(\bar{n}_k^*; \alpha) \geq 0$, and therefore $n_k^*(N, \alpha) < n_u^*(\alpha)$ for any finite N;

- If $\alpha < \eta(c)$, then $\mathrm{E}\pi(\bar{n}_k^*; \alpha) < 0$, then there exists a finite N_c such that $n_k^*(N, \alpha) < n_u^*(\alpha)$ iff $N < N_c$.
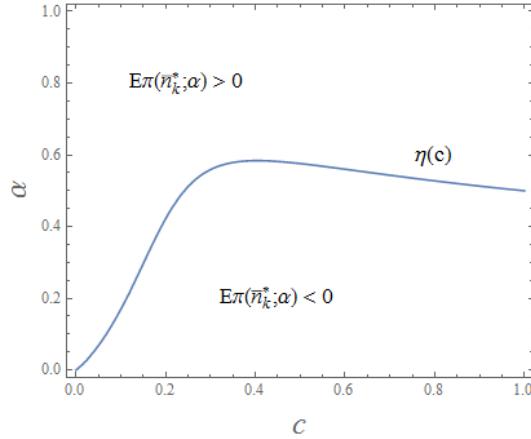

Figure 1: $n_k^*(N, \alpha)$ vs. $n_u^*(\alpha)$

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