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Asymmetric endogenous prize contests

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Abstract We consider a two-player contest in which one contestant has a head-start advantage but both can exert further effort. We allow the prize to depend on total performance in the contest and consider the respective cases in which efforts are productive and destructive of prize value. When the contest success function takes a logit form, and marginal cost is increasing in effort, we show that a Nash equilibrium exists and is unique both in productive and destructive endogenous prize contests. In equilibrium, the underdog expends more resources to win the prize, but still his probability of winning remains below that of the favorite. In a productive contest, the underdog behaves more aggressively and wins the prize more often in comparison to a fixed value contest. Thus, the degree of competitive balance – defined as the level of uncertainty of the outcome – depends upon the (fixed or endogenous) prize nature of the contest.

Keywords: Endogenous prize contests; Productive and destructive effort; Competitive balance

JEL Classifications: D61, D72, D74

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1 Introduction

A variety of economic and social settings can be described as contests in which players exert effort to increase their chance of winning a prize. Most of the literature posits a model in which the prize is exogenously given; classical contributions to contest theory in which players compete for a fixed prize include Tullock (1980), Hirshleifer (1989, 1995, 2001), Dixit (1987), and Nti (1997), among others.

In the fixed prize model, rent-seeking investments represent "the unproductive use of resources to contest, rather than create wealth" (Congleton et al., 2008). That is, contestants expend resources to appropriate or defend pre-existing wealth, rather than to undertake productive activities.

While the fixed prize assumption has been popular and useful due to its analytical tractability, it does not accurately reflect the strategic interaction in many common and important settings in which effort can be either productive or destructive. Labor tournaments to win a promotion, R&D races to win a patent, and sports contests are examples of contests in which effort can be productive. In R&D contests, efforts contribute to higher profits in the event of winning the patent (see, e.g. Cohen et al., 2008; Baik, 1994). In labor tournaments, efforts enhance the profitability of the firm and improve the value of promotion (see, e.g. Shaffer, 2006). In sports contests, larger efforts increase demand for or prize from the contest (see, e.g. Amegashie and Kutsoati, 2005).[1] Military conflicts and lawsuits, on the other hand, are contests in which effort can be destructive. Military combat involves the use of weapons and warfare, causing destruction of infrastructure and resources (see, e.g. Shaffer, 2006; Chang and Luo, 2013; Smith et al., 2014; and Sanders and Walia, 2014). Lawsuits to settle industrial disputes or to dissolve partnerships are settings in which parties often invest in legal representation by expending the very resources they seek to divide.

In this paper we extend the standard fixed prize model to account for prize endogeneity. In particular, we study incentives, equilibrium behavior, and outcomes in contests that combine two features: a prize which depends on aggregate effort, and contestants with different initial strengths – a favorite and an underdog – who are vying for this variable prize. The interaction of these features has largely been ignored as the literature has focused either on endogenous value symmetric settings (see, e.g. Garfinkel and Skaperdas, 2000; Shaffer, 2006; Chang and Luo, 2013; and Chang and Luo, 2016) or on fixed value asymmetric contests (Beviá and Corchón, 2013). Recent advancements in the endogenous prize symmetric contest literature include contributions which model contests as endogenous prize all-pay auctions. Baye et al. (2012) assume

[1] Broader examples of productive contests from the legal profession, financial markets, industrial organization, and the entertainment industry are presented in Chung (1996). A more recent application of an endogenous value contest to advertising is developed in Ridlon (2016).

that this endogeneity takes the form of rank-order spillovers and present a number of contest applications including the dissolution of partnerships, R&D races, litigation, price competition, job tournaments, auctions with regret, etc. These all-pay auctions admit multiple symmetric equilibria both in pure and in mixed strategies. Baye et al. (2012) provide conditions on the payoff functions under which pure and mixed strategy Nash equilibria exist and provide new results which can be used to characterize the equilibria of contests featuring the aforementioned applications.[2] In contrast to this literature, we focus here on Tullock contests rather than on perfectly discriminating ones and show that these contests have a unique pure strategy equilibrium.

It is known from the endogenous prize contest literature that the departure from the fixed value assumption changes the incentives of players to exert effort. In contests with productive effort, there are incentives to expend extra resources because effort increases the size of the rent, yet these incentives are limited because each player does not expect to receive the full return of their own increased effort. Using a symmetric model where prize is concave in effort, Chung (1996) shows that productive effort contests still generate, from a welfare perspective, excessively high aggregate efforts. Our approach allows us draw conclusions regarding the equilibrium winning probabilities and welfare of the stronger and the weaker player.

As a technical matter, when the prize changes with aggregate effort, the monotonicity properties of the payoff functions of players – including the marginal gains and losses from exerting effort – are altered in non-trivial ways. Thus, the extant results on the existence of a Nash equilibrium and the known set of conditions which ensure uniqueness of equilibrium in asymmetric fixed value contests (see, e.g. Skaperdas and Gan, 1995; Szidarovszky and Okuguchi, 1997; Cornes and Hartley, 2003; and Cornes and Hartley, 2012) cannot be applied to contests with endogenous prizes. For the case in which the contest success function takes the popular logit form (Tullock, 1980), and the marginal cost of effort for each player is increasing in the effort level, we show that an equilibirum exists and is unique both in productive and destructive contests. Our general framework captures as special cases the symmetric endogenous prize models by Chung (1996) and Shaffer (2006) and the standard case of a contest with a fixed value which we will use as a reference point in our analysis.

Further, we show that in equilibrium the disadvantaged player (the underdog) exerts more effort, but his probability of winning remains below that of the player with a head-start (the favorite). A comparative statics exercise allows us also to demonstrate that an underdog who faces a weaker favorite expends more effort than an underdog who faces a stronger opponent. These two properties correspond to the equilibrium behavior that would emerge in fixed value contests. We also extend the result by Chung (1996) to an asymmetric setting by showing that in equilibrium total effort exceeds the socially optimal level. The additional insight that we obtain is that, under fairly general conditions,

[2] For a model allowing for multiple prizes and asymmetric payoffs see, e.g. Siegel (2014).

the deviation from socially optimal behavior (as measured by the difference between strategic and welfare optimal effort) is greater for the underdog. We also derive conditions for the endogenous prize function under which, as in the case of fixed value contests, the underdog imposes a greater welfare loss for the favorite.

We note that our results are dependent on the particular logit specification of the contest success function and on our approach to modelling asymmetries. Standard ways to account for asymmetries are either to assume differences in players' (constant) marginal costs (see, e.g. Baik, 1994 or Ridlon, 2016) or in head-starts (see, e.g. Siegel, 2009). We adopt the latter approach here for two reasons. First, the head-start assumption is appropriate for modelling the behavior of contestants who compete by using the same production technology but have different strengths when they enter into the contest. This scenario is relevant for applications such as labor tournaments, R&D races, lawsuits or sports contests. Second, this assumption creates an analytically tractable environment. It allows for a general representation of the cost function but still affords equilibrium and comparative statics analysis. A model of a productive contest with asymmetric, but linear cost functions, is developed by Ridlon (2016) in the context of advertising campaigns. The constant marginal cost assumption could lead to equilibrium outcomes in which only one of the players is advertising (monopoly case) – an equilibrium outcome observed in the case of strong asymmetries. When players are less asymmetric, they both advertise whereby the relationship between their advertising expenditures depends on the level of asymmetry.

Our focus on asymmetric contests allows us to examine how the degree of competitive balance depends on the fixed or endogenous nature of the contest prize. Competitive balance is defined as the level of uncertainty in the outcome of a contest (see, e.g. Owen and King, 2015) and has important implications in sports economics, military conflicts and labor tournaments. The question of competitive balance has so far not been addressed in the literature on endogenous value contests because the extant literature has predominantly focused on symmetric settings (see, e.g. Chung, 1996; and Shaffer, 2006). These settings feature perfectly balanced contests in which all players make the same choices and win the prize with the same probability. In contrast, we explicitly account for asymmetries in players' abilities and compare equilibrium behavior in endogenous versus fixed value contests. We show that, when there is an asymmetry in head-starts, productive contests generate a higher degree of competitive balance (more uncertainty in the outcome) than fixed value contests. That is, in productive contests the underdog wins the prize more often compared to fixed prize contests. Surprisingly, a destructive contest can lead to a higher or lower degree of competitive balance than a fixed prize contest.

The remainder of the paper is organized as follows. In Section 2 we present the model. In Section 3 we discuss the existence, uniqueness, and the properties of the equilibrium. Welfare results are presented in Section 4. In Section 5 we compare productive endogenous value contests with fixed value contests. We conclude with Section 6. All proofs are in the Appendix.

2 The Model

We consider a contest between two parties labeled 1 and 2. The contestants have fixed initial (head-start) allocations denoted by a_i, $i = 1, 2$, which can be viewed as natural or acquired level of strength that players have prior to the contest. Without loss of generality we assume that player 1 is the stronger player, that is, $a_1 \geq a_2 \geq 0$. We refer to player 1 as the "favorite" and to player 2 as the "underdog" hereinafter. Both players can enhance their odds of winning by exerting effort, $e_i \geq 0$, which leads to an overall performance level of $x_i = a_i + e_i$. The cost of effort $c(e_i)$ is strictly increasing and convex, i.e. $c'(e_i) > 0$ and $c''(e_i) > 0$ for $e_i > 0$, and $c'(0) = 0$.

The effort e_i has different interpretation depending on the environment. In sports, e_i can be interpreted as training a player may put forth or some illicit form of effort manipulation such as performance enhancement via the use of a drug or financial expenditure in anticipation of game rigging. In military conflicts, effort might take the form of additional armament and recruitment in excess of defensive measures in open war. In an R&D race, effort can be viewed as building of additional capacity in research. In litigation, the extra effort can be viewed as costly acquisition of additional legal advice or legal representation, etc. We denote the total performance of the players by $z = x_1 + x_2$.

We assume that the prize depends on total performance, i.e., $V(z) > 0$ for all $z > 0$ and $V(0) \geq 0$, and we consider both destructive contests, $V'(z) \leq 0$, and productive contests, $V'(z) \geq 0$. We further assume that $V(z)$ is weakly concave, i.e. $V''(z) \leq 0$ and, for productive contests, $\lim_{z\to\infty} V'(z) = 0$. The probability of winning the contest for any player takes the logit form,

$$p_i(x_1, x_2) := \begin{cases} \frac{x_i}{z} \text{ if } z > 0, \\ \frac{1}{2} \text{ if } z = 0. \end{cases} \tag{1}$$

It is convenient to formulate the strategies in terms of the total performance of each player x_i. For that purpose we express the effort of each player by $e_i = x_i - a_i$. Assuming that both players are risk neutral, the payoff of a player is

$$U_i(x_1, x_2) := p_i(x_1, x_2)\, V(z) - c(x_i - a_i). \tag{2}$$

3 Equilibrium and comparative statics

With these preliminaries, we state our main result which holds true both for contests with productive and destructive effort. Let us define the equilibrium effort of a player by e_i^* and his total performance by x_i^*.

Proposition 1 (Equilibrium) *Asymmetric endogenous prize contests have a unique (interior or corner) Nash equilibrium. In any equilibrium, the underdog exerts (weakly) greater effort, $e_2^* \geq e_1^* \geq 0$, but his performance remains below that of the favorite, $x_2^* \leq x_1^*$.*

We note that while productive contests can have only an interior solution (see proof in Appendix), destructive contests can have either an interior or a corner solution depending on the level of relative head-starts of the underdog and the favorite, a_1/a_2, and the degree of concavity of $V(z)$.[3] Proposition 1 states that the difference in head-starts that exists prior to the contest is partially compensated for by the greater effort exerted by the underdog. However, the strictly increasing marginal cost of effort makes it unprofitable for the underdog to create a fully balanced contest.[4]

Related results to Proposition 1 have been developed in the recent literature on productive contests. Hirai (2012) and Hirai and Szidarovszky (2013) provide an existence and uniqueness result in a setting with linear costs and productive effort but do not analyze equilibrium structure. Chung (1996) and Chowdhury and Sheremata (2011a) provide an equilibrium characterization for the symmetric case.

We examine next how equilibrium effort and performance change for each player with changes in the head-start levels of players. All results hereinafter are stated for the case of an interior equilibrium.[5]

Proposition 2 (Comparative statics) *Equilibrium performance is an increasing function of individual head-start, i.e. $\partial x_i^*(a_1, a_2)/\partial a_i > 0$. The underdog's equilibrium performance is a decreasing function of favorite's head-start i.e. $\partial x_2^*(a_1, a_2)/\partial a_1 < 0$.*

Proposition 2 implies that the performance of the underdog declines with an increase in the head-start of the favorite. That is, an increased gap in the head-start levels of the players weakens the incentive of the underdog to compete.

[3] See the proof of Proposition 1 in the Appendix.

[4] In a similar head-start model with a fixed prize and constant marginal cost, Froeb and Kobayashi (1996) show that the head start is completely offset resulting in equal performance by the players in equilibrium. Such a result would obtain also in our endogenous prize model under the assumption of constant marginal cost.

[5] As indicated earlier, corner equilibria represent a degenerate case applicable for destructive contests only. The parameter values leading to this case are outlined in the proof of Proposition 1.

Unfortunately, there is no clear cut prediction on the direction of the change in effort of the favorite when the initial allocation of the underdog changes. In Proposition 2′ (given in the Appendix) we show that when $V(z)$ is linear in effort, the favorite's effort increases in the head-start level of the underdog.

4 Welfare analysis

In contests with an exogenous prize, rent seeking expenditures serve the sole purpose of determining the distribution of rents. In these circumstances, and in cases in which effort has a destructive effect on prize, effort is socially wasteful. Chung (1996) argues that in contests with productive effort this result does not necessarily hold because effort not only increases the probability of winning but also the size of the prize. Restricting attention to a symmetric setting, he analyzes these additional effects and concludes that, when the prize is a concave function of total effort, even productive contests are socially wasteful. When contestants are asymmetric in their natural abilities, however, it is known from the fixed value contest literature that they compete less in the contest (see e.g. Baik, 1994 and Konrad, 2009). As discussed in the previous section, this equilibrium behavior also applies to the endogenous prize contests studied here. It is thus a priori unclear whether equilibrium behavior creates social waste in endogenous prize contests with productive effort. Our asymmetric setting allows us to explore not only the social efficiency of aggregate efforts but also the individual effects on efficiency. We first extend Chung's (1996) analysis to asymmetric settings. Let us denote the socially efficient level of effort for each player by x_i^s and the equilibrium level of effort by x_i^*.

Proposition 3 (Welfare effects I) *In endogenous prize contests, the effort of each contestant is greater than the social optimum, $x_i^* > x_i^s$. Moreover, the underdog creates a greater inefficiency, i.e. $x_2^* - x_2^s > x_1^* - x_1^s$.*

Proposition 3 states that in equilibrium there is a misalignment between marginal and social benefits implying that players create a negative welfare effect for each other in equilibrium.[6] As can easily be observed, extra effort by one player reduces the welfare of the other player for all effort levels, i.e., $\partial U_i(x_1, x_2)/\partial x_{-i} = x_i \left[V'(z) - V(z)/z\right]/z < 0$.

To assess these negative welfare effects, we construct a measure that takes into account both the position and the behavior of players in the contest. Using the socially optimal effort level x_i^s as a reference point, we define the negative welfare effect imposed on player i by his opponent as the difference between the equilibrium utility and the utility at the socially optimal effort levels, i.e. $U_i(x_1^*, x_2^*) - U_i(x_1^s, x_2^s)$. Using this definition, we derive the following result.

[6] Our proposition also extends to the corner equilibria in destructive contests except for $e_1^* = e_2^* = 0$, which by definition socially optimal if effort is destructive. See the proof of Proposition 1.

Proposition 4 (Welfare effects II) *In an asymmetric endogenous prize contest the welfare effect that the underdog imposes on the favorite is greater than the effect that the favorite imposes on the underdog when*

$$2\left(V\left(z\right)/z - V'\left(z\right)\right)/z \geq -V''\left(z\right). \tag{3}$$

Note that condition (3) is satisfied by some commonly used (endogenous prize) functions. For instance, it is easy to check that for $V\left(z\right) := z^r$, where $r < 1$, the condition is satisfied. The underlying reason for this result is that the underdog exerts more effort in equilibrium thus reducing to a greater extent the chances of the favorite to win the prize. It is noteworthy, however, that in productive contests the effort of players also serves to increase the prize, creating an effect benefiting to a greater extent the favorite as the more likely winner. Therefore, the favorite sustains a greater harm in equilibrium only under the stated condition on the endogenous prize function.

As we established that productive contests generate excessively high effort levels even when players are asymmetric, we study next how the loss of total welfare depends on the degree of asymmetry between the players.[7] We define the total welfare loss as the combined loss resulting from the inability of players to coordinate their efforts to achieve an efficient performance level, i.e. $U_1\left(x_1^s, x_2^s\right) - U_1\left(x_1^*, x_2^*\right) + U_2\left(x_1^s, x_2^s\right) - U_2\left(x_1^*, x_2^*\right)$. For analytical tractability, in the current analysis and in the subsequent sections, we specialize to a quadratic cost function, $c\left(x_i - a_i\right) = \left(x_i - a_i\right)^2$. To measure the level of asymmetry, we introduce the parameter $k \in [0, 1]$ and assume that $a_1 = (1+k)a/2$, $a_2 = (1-k)a/2$ where $a = a_1 + a_2 > 0$. Thus, the case $k = 0$ corresponds to the symmetric game which we use as a benchmark for our analysis. We thus hold the sum of the aggregate head-start of players constant and equaling to a and explore how welfare changes when players' head-starts are asymmetric. We obtain the following result.

Proposition 5 (Welfare in symmetric vs asymmetric contests) *An asymmetric contest with a quadratic cost function generates a higher level of welfare loss compared to its symmetric counterpart.*

In the proof (see Appendix) we demonstrate that the efficiency loss in the case of asymmetric players is not driven by players' combined level of performance. Indeed, when the cost is quadratic, the aggregate performance level in Nash equilibrium, z^*, and in the socially efficient allocation, z^s, are invariant to the level of asymmetry between the players. It is rather the distribution of total effort across the players that accounts for the inefficiency when asymmetries are present. This inefficiency arises because the underdog expends too much effort and the favorite too little effort compared to the symmetric benchmark which, as we show, leads to a higher total cost of effort provision.

[7] We would like to thank an anonymous referee for raising this question.

5 Endogenous prize vs fixed prize contests: competitive balance

Competitive balance is defined as the level of uncertainty in the outcome of a contest (Owen and King, 2015) and is predominantly used in sports economics because sporting events with a higher competitive balance tend to attract more interest and larger audiences (see, e.g. Forrest and Simmons, 2002; Borland and Macdonald, 2003). The degree of competitive balance is often taken into consideration in the design of regulatory policies toward sports leagues (Szymanski, 2003). However, competitive balance has also implications for other forms of contests such as labor tournaments, litigation, and military conflicts. For example, in military conflicts and international negotiations competitive balance plays a crucial role in peacekeeping and sustained cooperation, respectively.

In the context of the present model, we study whether endogenous value contests result in a higher or lower degree of competitive balance in comparison to fixed value contests. Such an analysis would help develop intuition of whether predictions based on fixed prize models overestimate or underestimate the competitive balance in endogenous value contests. In particular, we measure competitive balance as the difference in the winning probabilities of the two players, $(p_1 - p_2)$.[8] A competitive balance of zero corresponds to a fully balanced contest, and a competitive balance of one represents the polar case of a fully unbalanced contest in which the winner of the contest can be predicted with certainty.

To compare endogenous prize with fixed prize contests, we represent the prize in endogenous prize contests in the form

$$V(z) = V + g(z), \tag{4}$$

where $V > 0$ is the fixed component of the prize and $g(z)$ is the variable component of the prize such that $g(\overline{z}) = 0$ for $\overline{z} = a_1 + a_2$.[9] For fixed prize contests we assume $g(z) = 0$, for productive effort contests we assume $g'(z) > 0$, and for destructive effort contests we assume $g'(z) < 0$. We also assume $g''(z) \leq 0$ for both productive and destructive contests. As indicated, for simplicity we will work with quadratic cost of effort, $c(x_i - a_i) = (x_i - a_i)^2$.[10] The first order conditions for a Nash equilibrium of the players $i = 1, 2$ can be expressed in terms of winning probabilities and total performance as follows:

$$\frac{\partial U_i(x_1, x_2)}{\partial x_i} = 0 \Leftrightarrow \frac{V(z)}{z} + p_i\left[V'(z) - \frac{V(z)}{z}\right] - c'(p_i z - a_i) = 0. \tag{5}$$

[8] As player 1 is the more likely winner in our model (see Proposition 1), the competitive balance is non-negative.

[9] Similar structure has been analyzed by Cohen et al. (2008) who look at a contest design problem in which the designer's objective is to maximize either the highest effort or total effort of contestants.

[10] Our results holds for any $c(e_i)$ such that $c'''(e_i) = 0$.

Adding the left hand-sides of this equation for the two players, using the condition $p_1 + p_2 = 1$, and rearranging terms, we obtain that the total performance of the players in equilibrium, z^*, is determined by the equation

$$V(z)/z + V'(z) = 2z - 2\,(a_1 + a_2)\,. \tag{6}$$

Combining equations (4) and (6) we obtain:

$$V/z + [g\,(z)/z + g'\,(z)] = 2z - 2\,(a_1 + a_2)\,. \tag{7}$$

Let us denote the aggregate effort in equilibrium of a productive contest by z_p, of a fixed prize contest by z_f, and of a destructive contest by z_d. Note that the term in the squared brackets is positive in productive effort contests, zero in fixed value contests, and negative in destructive effort contests. Therefore, from (7) follows that contestants cut down on their effort in destructive contests and increase their effort in productive contests compared to the fixed prize case, i.e. $z_d < z_f < z_p$.

We turn now to the analysis of winning probabilities. Subtracting the expression (5) for player 2 from that for player 1, we obtain

$$(p_1 - p_2)\,[V/z + g\,(z)/z - g'\,(z)] = 2\,(p_2 - p_1)\,z + 2\,(a_1 - a_2)\,,$$

which rearranging results in

$$(p_1 - p_2) = \frac{2(a_1 - a_2)}{V/z + g\,(z)/z - g'\,(z) + 2z}. \tag{8}$$

Expression (8) allows us to compare the probabilities of winning of the players in fixed and endogenous prize contests. Our main result is given as follows.

Proposition 6 (Productive vs fixed prize contests) *A productive contest with a quadratic cost function generates a higher degree of competitive balance than the corresponding fixed value contest. That is, the underdog is more likely to win a productive contest than the corresponding fixed prize contest.*

We demonstrate that the right hand-side of equation (8) is smaller for productive contests. To gain an intuition for the reason why a productive contest leads to a higher competitive balance, consider a situation in which both players expand their effort by the same amount. Under the initial asymmetry in head-starts, and considered Tullock contest success function, this will lead to a higher degree of competitive balance. As both players have greater incentives to expand effort in a productive contest (compared to a fixed value contest), and a unit increase in own effort increases the winning probability of the underdog more than that of the favorite, the underdog has incentives to narrow the gap in winning probabilities in a productive contest.

Next we turn to the analysis of destructive contests. Our last result demonstrates that no ultimate ranking can be established for destructive contests with respect to competitive balance.

Proposition 7 (Destructive vs fixed prize contests) *In a destructive contest with a prize that diminishes linearly in effort, i.e.* $V(z) = V + \overline{z} - z$, *the degree of competitive balance is higher than in the corresponding fixed value contest if and only if the inequality* $z_d < z_f - 1/2$ *holds. That is, the underdog is more likely to win a destructive contest compared to a fixed value contest under the stated condition.*

Our results show that in contests that are highly destructive, so that players sufficiently cut down on effort compared to the fixed prize benchmark, the degree of competitive balance is higher. In these contests the favorite reduces effort in order to preserve the value of the prize leaving the underdog with better chances of winning compared to the fixed prize benchmark. Conversely, in less destructive contests the favorite cuts down on effort to a smaller extent and ends up with better chances of winning compared to the fixed prize benchmark.

6 Conclusion

This paper examines strategic behavior in contests in which the size of the prize is endogenous and depends on the total effort of contestants. Our model allows for asymmetry in the players' initial strengths and explores the equilibrium behavior of the favorite and the underdog. We showed that for the class of contests based on Tullock's (1980) classical model, in which the contest success function has a logit form and the marginal cost of effort is increasing, the contest has a unique Nash equilibrium. In equilibrium, the underdog exerts more effort but not at a sufficiently high level so as to compensate for his disadvantage in head-start (Proposition 1).

While some of the properties of fixed value contests extend to endogenous value contests, others change in an intriguing way. Our comparative statics indicate that the directional changes in equilibrium efforts in endogenous value contests with head-starts are similar to the fixed prize benchmark: an underdog facing a stronger opponent competes less aggressively (Proposition 2). Under an additional restriction on the contest value function, we find that a favorite facing a stronger opponent competes more aggressively (Proposition 2′). Further, as in fixed prize contests, endogenous prize contests lead to an excessively high effort from an economic efficiency standpoint even in the case of productive contests. While both players exert more effort than is socially optimal, the distortion caused by the underdog is greater (Proposition 3). Further, the harm that the underdog imposes on the favorite is also greater, although this particular result holds under a specific condition on the endogenous prize function (Proposition 4). Compared to symmetric productive contests, we observe that asymmetric contests lead to a greater welfare loss for the players (Proposition 5). This welfare loss is observed not because players increase aggregate effort, but because the underdog exerts too much effort and the favorite too little effort compared to more efficient symmetric player Nash allocation.

Marked differences between endogenous prize and fixed prize contests exist with respect to competitive balance. A productive contest generates a higher degree of competitive balance compared to a fixed prize contest (Proposition 6). That is, in a productive contest the underdog behaves more aggressively and wins the prize more often compared to a fixed prize contest. This finding implies that approaches to measure competitive balance relying on the fixed prize framework with head-starts would tend to underestimate the degree of competitive balance.

The result regarding the degree of competitive balance in destructive contests is more nuanced. Destructive contests can have a higher or a lower degree of competitive balance compared to fixed prize contests (Proposition 7). In very destructive contests in which players cut down on effort substantially compared to the fixed prize benchmark, the favorite competes less aggressively in order to limit the destruction of the prize. The result may be applicable to the analysis of armed conflict in that it predicts that the weaker party is more likely to resort to additional armament compared to what would be rational behavior in a fixed prize contest. Less destructive contests, however, could lead to a higher effort by the favorite and have a lower degree of competitive balance.

Appendix

Proof of Proposition 1. We proceed in three steps in which we demonstrate (I) the existence, (II) the uniqueness, and (III) the stated structure of the equilibrium.

(I) Existence

We first show that the effort levels of each player that are above a certain threshold $\bar{x}_i$ are strictly dominated. We then restrict the equilibrium analysis to a game in which players' strategies are defined over a compact and convex set $[a_i, \bar{x}_i]$. For destructive contests it is straightforward that effort levels exceeding $\bar{x}_i$ where $c(\bar{x}_i - a_i) = V(a_1 + a_2)$ lead to a negative payoff for each e_{-i} and are thus strictly dominated by the strategy of zero effort. For productive contests, the First Order Conditions (FOC) for an interior maximum are given by

$$\begin{aligned}\frac{\partial U_i(x_1, x_2)}{\partial x_i} &= \frac{\partial p_i(x_1, x_2)}{\partial x_i} V(z) + p_i(x_1, x_2) V'(z) - c'(x_i - a_i) \\ &= \frac{x_{-i}}{z^2} V(z) + \frac{x_i}{z} V'(z) - c'(x_i - a_i) = 0. \end{aligned} \tag{FOC}$$

It follows that $\partial U_i(x_1, x_2)/\partial x_i < V(z)/z + V'(z) - c'(x_i - a_i)$. Applying L'Hospital's rule we obtain $\lim_{z\to\infty} V(z)/z = \lim_{z\to\infty} V'(z) = 0$. Note that $\lim_{x_i\to\infty} c'(x_i - a_i) > 0$. Thus, there exists $\bar{x}_i$ such that $\partial U_i(x_1, x_2)/\partial x_i < 0$

for all $x_i > \bar{x}_i$ and all $x_{-i} \geq a_{-i}$. Then, strategies exceeding $\bar{x}_i$ are strictly dominated. Next we show that the payoff functions are concave in the own performance of players, i.e.,

$$\begin{aligned}\frac{\partial^2 U_i(x_1, x_2)}{\partial x_i^2} =& -\frac{2x_{-i}}{z^3} V(z) + \frac{2x_{-i}}{z^2} V'(z) + \frac{x_i}{z} V''(z) - c''(x_i - a_i) \\ =& 2x_{-i}\Phi(z) + x_i \Psi(z) - c''(x_i - a_i) < 0, \qquad \text{(CONC)}\end{aligned}$$

where we simplify notation by $\Phi(z) = \left[V'(z) - \frac{V(z)}{z}\right]/z^2 < 0$, and $\Psi(z) = V''(z)/z < 0$ hereinafter. (CONC) holds by the concavity of $V(z)$, i.e. $V''(z) < 0$ and $V'(z) < V(z)/z$. Observe that when $a_1 > 0$ we have $z = a_1 + a_2 > 0$. In this case the contest success function presented in (2) and the expected utility function presented in (3) are continuous and we can apply Glicksberg's (1952) and Fan's (1952) extension of Kakutani's (1941) fixed point theorem which guarantees the existence of a pure strategy Nash equilibrium in games with a compact strategy space and concave payoff functions. When $a_1 = a_2 = 0$, the expected utility function is discontinuous at the origin. In this case, however, the contest is symmetric and the equilibrium can explicitly be derived. Using the (FOC) and the condition $x_1^* = x_2^* = z^*/2$, we obtain the symmetric equilibrium aggregate performance z^* as the solution to $V(z)/2z + V'(z)/2 = c'(z/2)$. Note that this equation has a unique solution as the left hand-side is monotonically decreasing and right hand-side is monotonically increasing in z.

(II) Uniqueness

We denote the winning probability of each player by $p_i = x_i/z$.[11] The aggregate effort of each player can be expressed as $x_i = p_i \cdot z$. In an interior Nash equilibrium, the probability shares add to one and the FOCs are satisfied. Hence, p_1, p_2, and z satisfy $p_1 + p_2 = 1$ and

$$\frac{\partial U_i(x_1, x_2)}{\partial x_i} = G_i(p_i, p_{-i}, z) = p_{-i}\frac{V(z)}{z} + p_i V'(z) - c'(p_i z - a_i) = 0, \quad (9)$$

as $V(z)$ is concave, $V'(z)$ and $V(z)/z$ are decreasing in z. Moreover, the function $-c'(p_i z - a_i)$ is also decreasing in z by the convexity of $c(\cdot)$. Using the relationship $p_1 + p_2 = 1$, we rearrange equation (9) as

$$G_i(p_i, z) = V(z)/z + p_i[V'(z) - V(z)/z] - c'(p_i z - a_i) = 0. \quad (10)$$

As $V'(z) < V(z)/z$, $G_i(p_i, z)$ is decreasing in both arguments. Let us assume now that there are (at least) two equilibria with shares for player i given by p_i^1 and p_i^2 where $p_i^1 > p_i^2$. By (10) it follows that $z^1 < z^2$ as $G_i(p_i, z)$ decreases in p_i and z. As $G_{-i}(p_{-i}, z)$ is also decreasing in both arguments, it follows that $p_{-i}^1 > p_{-i}^2$, a contradiction to the condition that in equilibrium $p_i^1 + p_{-i}^1 = p_i^2 + p_{-i}^2 = 1$. A similar argument shows that the model cannot have both an interior and a corner solution where equation (10) is satisfied

[11] The approach of using the winning probabilities to provide equilibrium existence and uniqueness results for fixed prize contests was developed by Cornes and Hartley (2005).

for at least one player with an inequality (i.e. the marginal utility is negative) thus ensuring that the Nash equilibrium is unique.

(III) Structure

We first note that productive contests cannot have a corner solution, i.e., $e_i^* > 0$ for $i = 1, 2$, as $\partial U_i(a_i, x_{-i})/\partial x_i > 0$ for any $x_{-i} \geq a_{-i}$ and $c'(0) = 0$. Restricting attention to interior equilibria, we will show that for $a_1 > a_2$ we have $x_1^* > x_2^*$ and $e_1^* < e_2^*$. Indeed, using (FOC) for both players, we obtain

$$\frac{x_2^*}{z^{*2}} V\left(z^*\right) + \frac{x_1^*}{z^*} V'\left(z^*\right) = c'\left(x_1^* - a_1\right) \text{ and } \frac{x_1^*}{z^{*2}} V\left(z^*\right) + \frac{x_2^*}{z^*} V'\left(z^*\right) = c'\left(x_2^* - a_2\right).$$

Subtracting the first equation from the second one we obtain

$$\frac{x_2^* - x_1^*}{z^*} \left[\frac{V\left(z^*\right)}{z^*} - V'\left(z^*\right)\right] = c'\left(x_1^* - a_1\right) - c'\left(x_2^* - a_2\right). \tag{11}$$

As the term in the squared brackets is positive by the concavity of $V(z)$, the differences $x_2^* - x_1^*$ and $c'\left(x_1^* - a_1\right) - c'\left(x_2^* - a_2\right)$ must have the same sign. From $c'\left(x_1^* - a_1\right) > c'\left(x_2^* - a_2\right)$ it follows that $x_1^* - a_1 > x_2^* - a_2$ as $c(\cdot)$ is convex. Thus, $x_2^* - x_1^* < a_2 - a_1 < 0$, a contradiction. The only possibility remaining is $x_1^* - a_1 < x_2^* - a_2$ and $x_1^* > x_2^*$ as stated in the proposition.[12]

Note that destructive contests ($V'(z) < 0$) may have corner equilibria. For these equilibria we will show that it cannot be the case that the underdog exerts no effort while the favorite does, i.e., $e_1^* > 0$ and $e_2^* = 0$ cannot hold true. Proceeding by contradiction, note that $\partial U_1(a_1, a_2)/\partial x_1 > 0 > \partial U_2(a_1, a_2)/\partial x_2$. Denoting $a = a_1 + a_2$ we obtain

$$\frac{a_2}{a^2} V(a) + \frac{a_1}{a} V'(a) > 0 > \frac{a_1}{a^2} V(a) + \frac{a_2}{a} V'(a) \Leftrightarrow \frac{a_2}{a_1} > -a \frac{V'(a)}{V(a)} > \frac{a_1}{a_2},$$

which contradicts $a_1 \geq a_2$. Similar reasoning shows that a corner equilibrium with $e_i^* = 0$ for $i = 1, 2$ exists if and only if $a_1/a_2 < -aV'(a)/V(a)$, and a corner equilibrium with $e_1^* = 0$ and $e_2^* > 0$ exists if and only if $a_2/a_1 < -aV'(a)/V(a) < a_1/a_2$.[13] For the former case, we obtain that $x_1^* \geq x_2^*$ as stated in the proposition. For the latter case assume that $x_1^* < x_2^*$. Then using (FOC),

$$\frac{x_2^*}{z^{*2}} V(z^*) + \frac{x_1^*}{z^*} V'(z^*) < \frac{x_1^*}{z^{*2}} V(z^*) + \frac{x_2^*}{z^*} V'(z^*) - c(x_2^* - a_2) = 0,$$

which contradicts $x_1^* < x_2^*$ as $V(z) \geq 0$ and $V'(z) < 0$.

■

[12] Note that in a model with a constant marginal cost the right hand-side of equation (11) would be zero, from which would follow that the left hand-side must also be zero, i.e. $x_1^* = x_2^*$.

[13] Note that $-2aV'(2a)/V(2a) > 1$ does not necessarily hold as for any destructive contest $V(0) > 0$ is a necessary condition.

Proof of Proposition 2. The partial derivatives $\partial x_i^* (a_1, a_2)/\partial a_i$, and $\partial x_{-i}^* (a_1, a_2)/\partial a_i$ are given by the solution to the following system of equations defined by the total differential of the (FOC) (see, e.g. Chiang and Wainwright (2005), p. 201) resulting in

$$\begin{bmatrix} \frac{\partial^2 U_i(x_1,x_2;a_i)}{\partial x_i^2} & \frac{\partial^2 U_i(x_1,x_2;a_i)}{\partial x_i \partial x_{-i}} \\ \frac{\partial^2 U_{-i}(x_1,x_2;a_i)}{\partial x_i \partial x_{-i}} & \frac{\partial^2 U_{-i}(x_1,x_2;a_i)}{\partial x_{-i}^2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_i^*(a_1,a_2)}{\partial a_i} \\ \frac{\partial x_{-i}^*(a_1,a_2)}{\partial a_i} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 U_i(x_1,x_2;a_i)}{\partial x_i \partial a_i} \\ -\frac{\partial^2 U_{-i}(x_1,x_2;a_i)}{\partial x_{-i} \partial a_i} \end{bmatrix}. \qquad (12)$$

Denoting the determinant of coefficient matrix in (12) by $|J|$, and by $|J_i|$ ($|J_{-i}|$) the determinant of the matrix whose i^{th} ($-i^{th}$) column is replaced by the vector on the right hand side of (12), we obtain by Cramer's Rule

$$\frac{\partial x_i^* (a_1, a_2)}{\partial a_i} = \frac{|J_i|}{|J|} \text{ and } \frac{\partial x_{-i}^* (a_1, a_2)}{\partial a_i} = \frac{|J_{-i}|}{|J|}.$$

The diagonal elements of matrix J are negative by (CONC). On the other hand,

$$\partial^2 U_i (x_1, x_2; a_i)/\partial x_i \partial x_{-i} = (x_{-i} - x_i)\Phi(z) + x_i \Psi(z). \qquad (13)$$

Note that using expressions (CONC) and (13),

$$\begin{aligned} |J| &= \frac{\partial^2 U_i (x_1, x_2; a_i)}{\partial x_i^2} \frac{\partial^2 U_{-i} (x_1, x_2; a_{-i})}{\partial x_{-i}^2} - \frac{\partial^2 U_i (x_1, x_2; a_i)}{\partial x_i \partial x_{-i}} \frac{\partial^2 U_{-i} (x_1, x_2; a_{-i})}{\partial x_i \partial x_{-i}} \\ &= \Gamma(x_i, x_{-i}, z) + (x_i + x_{-i})^2 (\Phi^2(z) + \Phi(z)\Psi(z)) > 0, \end{aligned}$$

where

$$\begin{aligned} \Gamma(x_i, x_{-i}, z) = &- c'' (x_{-i} - a_{-i}) (2x_{-i}\Phi(z) + \Psi(z)) - c'' (x_i - a_i) (2x_i\Phi(z) + \Psi(z)) \\ &+ c'' (x_{-i} - a_{-i}) c'' (x_i - a_i) > 0. \end{aligned}$$

For the derivatives on the right hand-side of (12) we obtain

$$\frac{\partial^2 U_i (x_1, x_2; a_i)}{\partial x_i \partial a_i} = c'' (x_i - a_i) > 0 \text{ and } \frac{\partial^2 U_{-i} (x_1, x_2; a_{-i})}{\partial x_{-i} \partial a_i} = 0.$$

Hence, as stated in the first part of the proposition $\partial x_i^* (a_1, a_2) \partial a_i > 0$.

For the second part of the proposition note that

$$|J_i| = -\frac{\partial^2 U_i (x_1, x_2; a_i)}{\partial x_{-i}^2} \frac{\partial^2 U_i (x_1, x_2; a_i)}{\partial x_i \partial a_i} = -\frac{\partial^2 U_{-i} (x_1, x_2; a_i)}{\partial x_{-i}^2} c'' (x_i - a_i) > 0,$$

and

$$\begin{aligned} |J_{-i}| = &\frac{\partial^2 U_i (x_1, x_2; a_i)}{\partial x_i \partial x_{-i}} \frac{\partial^2 U_i (x_1, x_2; a_i)}{\partial x_i \partial a_i} \\ = &\{(x_i - x_{-i})\Phi(z) + x_{-i}\Psi(z)\} c'' (x_i - a_i). \qquad (14) \end{aligned}$$

Thus, for the underdog we obtain $\partial x_2^* (a_1, a_2)/\partial a_1 = |J_2|/|J| < 0$. This expression is negative as $x_1^* > x_2^*$.

■

Proposition 2′ (Linear contests) *In contests in which $V(z)$ is linear in z, a favorite who faces a stronger underdog exerts more effort in equilibrium.*

Proof of Proposition 2′. From the analysis in Proposition 2 we observe that the direction of change in the equilibrium effort of the favorite depends on the sign of the expression $\partial x_1^* (a_1, a_2)/\partial a_2 = |J_1|/|J|$.

We have already established that $|J| > 0$ yet $|J_1| = (x_2^* - x_1^*)\Phi(z) + x_1^* \Psi(z)$ is an expression that entails the equilibrium efforts of players. When $V(z)$ is linear in z we have $V''(z) = 0$ and $\Phi(z) \leq 0$. As $x_2^* - x_1^* < 0$, it follows that $\partial x_1^* (a_1, a_2)/\partial a_2 > 0$.

■

Proof of Proposition 3. The socially optimal level of effort x_i^s for each player for productive contests is determined by the equation

$$V'(z^s) = c'(x_i^s - a_i). \tag{15}$$

The (FOC) characterizing the equilibrium, x_i^*, and z^* require that the marginal benefit of a player equals the marginal cost of his effort, i.e.,

$$B_i(x_1^*, x_2^*) := \frac{x_{-i}^*}{z^{*2}} V(z^*) + \frac{x_i^*}{z^*} V'(z^*) = c'(x_i^* - a_i).$$

It is easy to see that for the equilibrium effort levels the marginal private benefit of effort of each player exceeds the social marginal benefit. Indeed,

$$B_i(x_1^*, x_2^*) = \frac{x_{-i}^*}{z^*} \frac{V(z^*)}{z^*} + \frac{x_i^*}{z^*} V'(z^*) > \frac{x_i^* + x_{-i}^*}{z^*} V'(z^*) = V'(z^*).$$

As $c'(x_i - a_i)$ is increasing in x_i it follows that $x_i^* > x_i^s$.

To prove the second property, observe first that in equilibrium the marginal private benefit of the underdog exceeds that of the favorite, i.e.,

$$B_2(x_1^*, x_2^*) - B_1(x_1^*, x_2^*) = \frac{x_1^* - x_2^*}{z^*} \left[\frac{V(z^*)}{z^*} - V'(z^*) \right] > 0.$$

Therefore, $c'(x_1^* - a_1) < c'(x_2^* - a_2) \Leftrightarrow x_1^* - a_1 < x_2^* - a_2$, which implies $x_1^* - x_2^* < a_1 - a_2$. From (15), it follows that $c'(x_1^s - a_1) = c'(x_2^s - a_2)$ and hence $x_1^s - x_2^s = a_1 - a_2$. Thus,

$$x_1^* - x_2^* < x_1^s - x_2^s \Leftrightarrow x_2^* - x_2^s > x_1^* - x_1^s.$$

For destructive contests, it is clear that the socially optimal level of effort is $x_i^s = 0$ for all $i = 1, 2$. Thus, in the interior equilibrium, $x_i^* > x_i^s$. In the corner solution $e_2^* > e_1^* = 0$, $x_1^* = x_1^s$ but $x_2^* > x_2^s$, so the underdog still creates greater inefficiency. Only for the corner solution $e_1^* = e_2^* = 0$, there is no efficiency loss.

■

Proof of Proposition 4. The underdog imposes a greater welfare effect on the favorite when $U_1(x_1^*, x_2^*) - U_1(x_1^s, x_2^s) > U_2(x_1^*, x_2^*) - U_2(x_1^s, x_2^s)$, which implies

$$D(x_1^*, x_2^*; x_1^s, x_2^s) := \int_{x_1^s}^{x_1^*}\int_{x_2^s}^{x_2^*}\left(\frac{\partial^2 U_1(x_1, x_2)}{\partial x_1 \partial x_2} - \frac{\partial^2 U_2(x_1, x_2)}{\partial x_1 \partial x_2}\right) dx_2 dx_1 > 0. \quad (16)$$

Substituting the cross derivative of the utility function from (13) into expression (16),

$$D(x_1^*, x_2^*; x_1^s, x_2^s) = \int_{x_1^s}^{x_1^*}\int_{x_2^s}^{x_2^*} (x_1 - x_2)(2\Phi(z) + \Psi(z))\, dx_2 dx_1.$$

Note that, because $a_1 \geq a_2$, we have $x_1^s \geq x_2^s$ and in equilibrium $x_1^* \geq x_2^*$. We divide the area $(C) = [x_1^s, x_1^*] \times [x_2^s, x_2^*]$ into the areas $(A) = [\min\{x_2^*, x_1^s\}, \max\{x_2^*, x_1^s\}] \times [\min\{x_2^*, x_1^s\}, \max\{x_2^*, x_1^s\}]$ and $(B) = (C)/(A)$. For the case $x_1^s < x_2^*$ these areas are presented in Figure 1.

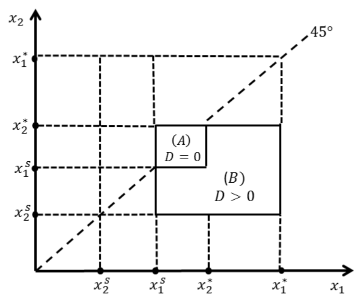

Fig. 1 In Area (A) players impose equal welfare effect on each other. In Area (B) the underdog creates a greater welfare effect for the favorite.

Due to the symmetry of cross derivatives, we have

$$\iint_A (x_1 - x_2)(2\Phi(z) + \Psi(z))\, dx_2 dx_1 = 0.$$

That is, in area (A) the harm that the underdog imposes on the favorite equals to the harm that the favorite imposes on the underdog. Thus, in area (A), $D(\cdot) = 0$ as illustrated in Figure 1. Consequently,

$$D\left(x_1^*, x_2^*; x_1^s, x_2^s\right) = \iint_B \left(x_1 - x_2\right)\left(2\Phi(z) + \Psi(z)\right) dx_2 dx_1.$$

Notice that for all (x_1, x_2) in area (B) illustrated in Figure 1, we have $x_1 > x_2$. Therefore, the sign of $D\left(x_1^*, x_2^*; x_1^s, x_2^s\right)$ is determined by the sign of the expression

$$2\Phi(z) + \Psi(z) = 2\frac{V(z)}{z^2} - 2\frac{V'(z)}{z} + V''(z), \tag{17}$$

which, when positive, implies that the underdog imposes a greater harm on the favorite.

■

Proof of Proposition 5. For the assumed quadratic cost function, the equations (FOC) reduce to

$$x_{-i}V(z)/z^2 + x_i V'(z)/z = 2\left(x_i - a_i\right). \tag{18}$$

Combining (18) for $i = 1, 2$ and using the identities $z^* = x_1^* + x_2^*$ and $a = a_1 + a_2$ we obtain that total effort z^* in Nash equilibrium satisfies $V(z)/z + V'(z) = 2(z - a)$. Analogously, using equation (15) we obtain that the socially optimal effort level z^s satisfies $V'(z) = (z - a)$. Hence, total effort in Nash equilibrium and in social optimum is independent of the level of asymmetry. From Proposition 1 we obtain for the case of strict asymmetry $(k > 0)$ that $x_1^* > x_2^*$ and hence $x_1^* > z/2$ and $x_2^* < z/2$. Note that in the case of symmetry $(k = 0)$ we have $x_1^* = x_2^* = z/2$. From the inequality $V(z^*)/z^* > V'(z^*)$ and equation (18) it follows that for $k > 0$ the inequality

$$x_2(k)V(z)/z^2 + x_1(k)V'(z)/z > V(z)/2z + V'(z)/2$$

holds. Hence, $2(x_2^*(k) - (1-k)a/2) > z - a$ for $k > 0$. It follows that $x_2^*(k) - (1-k)a/2 > (z^* - a)/2$ and $x_1^*(k) - (1+k)a/2 < (z^* - a)/2$ for $k > 0$. From this result and the convexity of $c(x_i - a_i)$ it follows that $c(x_1^* - (1+k)a/2) + c(x_2^* - (1-k)a/2) > 2c(z/2 - a/2)$. Therefore, for $k > 0$ we obtain $U_1\left(x_1^*(k), x_2^*(k)\right) + U_2\left(x_1^*(k), x_2^*(k)\right) < 2U_1\left(z/2, z/2\right)$. That is, the utility loss is greater in the case of asymmetric players.

■

Proof of Proposition 6. Let $\Delta p^f = p_1^f - p_2^f$, denote the competitive balance (the difference in the winning probability of the favorite and the underdog) in fixed prize contests. Similarly, let $\Delta p^p = p_1^p - p_2^p$ denote the competitive balance in productive effort contests. We need to show that $\Delta p^f > \Delta p^p$. Using equation (8), we obtain

$$\Delta p^f > \Delta p^p \Leftrightarrow \left(z_f - z_p\right)\left[2 - \frac{V}{z_f z_p}\right] < \frac{g\left(z_p\right)}{z_p} - g'\left(z_p\right).$$

As $g(z)$ is concave, the right hand side of the inequality is positive. Further, equation (7) implies that for fixed value contests the inequality $V/z_f^2 < 2$ holds. Since $z_p > z_f$ it follows that $V/z_p z_f < V/z_f^2 < 2$. Thus, the left hand side of the above inequality is negative and the right hand side is positive.

■

Proof of Proposition 7. Using equations (7) and (8) we obtain that the competitive balance in a fixed value contest is given by

$$\Delta p^f = 2\frac{a_1 - a_2}{V/z_f + 2z_f},$$

where $V/z_f = 2z_f - 2(a_1 + a_2)$. In a destructive contest the competitive balance is given by

$$\Delta p^d = 2\frac{a_1 - a_2}{\frac{V}{z_d} + 2z_d + \frac{a_1+a_2}{z_d}},$$

where $V/z_d + (a_1 + a_2)/z_d - 2 = 2z_d - 2(a_1 + a_2)$. Hence,

$$\Delta p^d > \Delta p^f \Leftrightarrow 4z_d + 2 - 2(a_1 + a_2) < 4z_f - 2(a_1 + a_2) \Leftrightarrow z_d < z_f - 1/2.$$

■

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