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A characterization of linearly repetitive cut and project sets*

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Abstract

For the development of a mathematical theory which can be used to rigorously investigate physical properties of quasicrystals, it is necessary to understand regularity of patterns in special classes of aperiodic point sets in Euclidean space. In one dimension, prototypical mathematical models for quasicrystals are provided by Sturmian sequences and by point sets generated by substitution rules. Regularity properties of such sets are well understood, thanks mostly to well known results by Morse and Hedlund, and physicists have used this understanding to study one dimensional random Schrödinger operators and lattice gas models. A key fact which plays an important role in these problems is the existence of a subadditive ergodic theorem, which is guaranteed when the corresponding point set is linearly repetitive.

In this paper we extend the one-dimensional model to cut and project sets, which generalize Sturmian sequences in higher dimensions, and which are frequently used in mathematical and physical literature as models for higher dimensional quasicrystals. By using a combination of algebraic, geometric, and dynamical techniques, together with input from higher dimensional Diophantine approximation, we give a complete characterization of all linearly repetitive cut and project sets with cubical windows. We also prove

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that these are precisely the collection of such sets which satisfy subadditive ergodic theorems. The results are explicit enough to allow us to apply them to known classical models, and to construct linearly repetitive cut and project sets in all pairs of dimensions and codimensions in which they exist.

Keywords: linear repetitivity, mathematical quasicrystals, cech cohomology of tiling spaces, cut and project sets
Mathematics Subject Classification numbers: Primary 52C23; Secondary 37A45

(Some figures may appear in colour only in the online journal)

1. Introduction

A Delone set $Y \subseteq \mathbb{R}^d$ is linearly repetitive (LR) if there exists a constant $C > 0$ such that, for any $r \geqslant 1$, every patch of size r in Y occurs in every ball of diameter Cr in $\mathbb{R}^d$. This concept was studied by F Durand (for the special case of subshifts) in [16, 17], and by Lagarias and Pleasants in [23], and it has since been explored by many authors (e.g. [1, 2, 10, 11, 14, 15]). It was shown in [23] that linear repetitivity guarantees the existence of strict uniform patch frequencies, equivalently the associated dynamical system on the hull of the point set is strictly ergodic (minimal and uniquely ergodic). Having strict uniform patch frequencies means that every patch occurs across the Delone set with some well-defined positive frequency. The fact that the Dirac comb of a Delone set in $\mathbb{R}^d$ is a translation bounded measure, together with the existence of strict uniform patch frequencies, implies the existence of a unique autocorrelation measure which is also a translation bounded measure (and hence a tempered distribution). This in turn implies that the associated diffraction measure (the Fourier transform of the autocorrelation) is a positive, translation bounded measure. For this reason, aperiodic LR Delone sets are a common source of examples of mathematical models for quasicrystals, and they are sometimes referred to as 'perfectly ordered quasicrystals' [23].

In the direction of potential physical applications, several authors have studied the random Schrödinger operator and lattice gas models on one dimensional quasicrystals, as modeled either by LR Sturmian sequences or point sets constructed using primitive substitution rules [7, 21, 32]. For both of these applications it is necessary to establish the validity of a uniform subadditive ergodic theorem. In the case of point sets constructed using primitive substitutions (in fact, in any dimension), such theorems were established by Geerse and Hof in [18]. For LR Sturmian sequences they were established by Lenz in [25]. The results of our paper are relevant in this context because of the fact, proved by Damanik and Lenz in [15], that linear repetitivity implies the existence of a uniform subadditive ergodic theorem. In the one-dimensional setting, it was shown by Lenz [24] that validity of a subadditive ergodic theorem for a minimal subshift is equivalent to 'uniform positivity of weights' (PQ) (defined here in section 2.5) which, through the results of [11], is easily seen for subshifts to be equivalent to LR. In fact, one part of our main theorem below (theorem 1.1) shows that, for a natural class of cut and project sets which are higher dimensional generalizations Sturmian sequences, LR is equivalent to the existence of a subadditive ergodic theorem. The characterization which we provide in theorem 1.1 thus opens the door for the study of physical properties of a rich collection of higher dimensional point sets, a project which is the focus of our current research on this topic.

In [23, problem 8.3], motivated by many of the connections which we have described, Lagarias and Pleasants asked for a characterization of LR cut and project sets. For 2 to 1 cut and project sets with appropriately chosen windows, such a characterization can be obtained using classical results of Morse and Hedlund [26, 27]. Morse and Hedlund showed that there is a natural bijective correspondence between repetitive Sturmian words and return times, to specially chosen intervals, of orbits of irrational rotations on $\mathbb{R}/\mathbb{Z}$. The latter are precisely given by 2 to 1 cut and project sets. Under this correspondence, a Sturmian word will be LR if and only if the irrational rotation which describes it is determined by a badly approximable real number (i.e. a real number whose continued fraction partial quotients are uniformly bounded). The proof of this fact relies on ideas from Diophantine approximation that are closely connected to the theory of continued fractions. In fact the theory is so robust that, in the special case of Sturmian sequences, one can even derive an exact formula for the repetitivity function (see [3, theorem 11] and [27, p 2]).

For higher dimensional cut and project sets we immediately encounter serious difficulties with generalizing the above mentioned results. One problem is that the underlying dynamical systems which are used for pattern recognition in these sets are, in general, higher rank actions on higher dimensional tori which are more complicated to understand. Another is that there is no known algorithm which can do for higher dimensional cut and project sets what the simple continued fraction algorithm does for Sturmian sequences. Nevertheless, by using a combination of algebraic, geometric, and dynamical tools, together with input from the higher dimensional theory of Diophantine approximation, we are able to obtain the following theorem, which can be seen as a generalization of the classification of Morse and Hedlund, to arbitrary dimensions.

Theorem 1.1. *Let Y be any k to d cubical cut and project set, whose physical space is determined by a collection $\{L_i\}_{i=1}^{k-d}$ of linear forms on d variables. Then the following are equivalent:*

(1) Y is linearly repetitive;
(2) Y satisfies a subadditive ergodic theorem;
(3) $\{L_i\}_{i=1}^{k-d}$ satisfies both (LR1) and (LR2):

(LR1) the sum of the ranks of the kernels of the maps $\mathcal{L}_i : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ defined by $\mathcal{L}_i(n) = L_i(n) \mod 1$ is equal to $d(k-d-1)$;
(LR2) Each L_i is relatively badly approximable.

In the statement of this theorem, a cubical cut and project set is one which is aperiodic, regular and totally irrational, and which is defined using a strip in $\mathbb{R}^k$ whose intersection with $\{0\}^d \times \mathbb{R}^{k-d}$ is the unit cube $\{0\}^d \times [0,1]^{k-d}$. A collection of $k-d$ linear forms $\{L_i\}_{i=1}^{k-d}$ in d variables defines a subspace of $\mathbb{R}^k$, called the physical space, which along with the choice of window naturally determines a collection of associated cut and project sets. Precise definitions of these terms are given in section 2.2.

After proving theorem 1.1 we will explore the relationship between cubical cut and project sets and canonical cut and project sets (formed with a window which is the projection of the unit cube in $\mathbb{R}^k$ to the internal space). It will turn out that in many cases which are commonly cited in the literature, the results we state for cubical cut and project sets also apply to canonical ones. However, what is possibly more interesting is that there are examples of LR cubical cut and project sets which are no longer LR when their windows are replaced by canonical ones.

We give examples later in the paper which seem to indicate that there are two potential sources for this type of behavior. The first is geometric, and arises in the situation when at least two of the linear forms defining the physical space have co-kernels with different ranks. The second (which can occur even in the absence of the geometric situation just described) is Diophantine, and is related to the fact that any number can be written as a product of two badly approximable numbers (this follows from continued fraction Cantor set arguments, see [19]). The complete statements of our results about canonical cut and project sets can be found in section 4, but we summarize what has just been mentioned in the following theorem.

Theorem 1.2. *If Y is a cubical cut and project set which is not LR, then the cut and project set formed from the same data as Y, but with the canonical window in place of the cubical one, is also not LR. However, the converse of this statement is not true, in general.*

It should be pointed out that most of the canonical cut and project sets of specific interest in the literature arise from subspaces defined by linear forms with coefficients in a fixed algebraic number field. In such a case the Diophantine behavior alluded to at the end of the paragraph before theorem 1.2 cannot occur. To illustrate this point, in section 6 we will briefly explain how to prove that Penrose and Ammann–Beenker tilings are LR. This in itself is not a new result, and in fact it is fairly obvious from descriptions of these tilings using substitution rules. What is new is that our proof uses only their descriptions as cut and project sets.

Let us now return to discuss the content of theorem 1.1. In the statement of the theorem, condition (LR1) is necessary and sufficient for Y to have minimal patch complexity. For comparison, in [22, section 5] it was shown that minimal patch complexity is a necessary and sufficient condition for the Čech cohomology (with rational coefficients) of the associated tiling space to be finitely generated. Many of the calculations in the first part of our proof share a common thread with those that arise in the calculation of the cohomology groups. This point of view eventually leads us to equations (3.3)–(3.6), which form the crux of our argument, allowing us to move directly in our proof to a position where we can apply condition (LR2). It should be pointed out that, without the group theoretic arguments that give us the rigid structure imposed by these equations, the proof would fall apart.

Condition (LR2) is a Diophantine condition, which places a strong restriction on how well the subspace defining Y can be approximated by rationals. A linear form is relatively badly approximable if it is badly approximable when restricted to rational subspaces complementary to its kernel. In the next section we will explain this in more detail and prove that it is a well-defined property. Note that in the special case when $k - d = 1$, condition (LR1) is automatically satisfied, and condition (LR2) requires the linear form defining Y to be badly approximable in the usual sense. This observation, together with proposition 4.2, immediately implies the following complete characterization of both cubical and canonical cut and project sets in codimension one.

Corollary 1.3. *A k to $k-1$ cubical or canonical cut and project set defined by a linear form L is linearly repetitive if and only if L is badly approximable.*

2. Definitions and preliminary results

2.1. Summary of notation

For sets A and B, the notation $A \times B$ denotes the Cartesian product. If A and B are subsets of the same Abelian group, then $A + B$ denotes the collection of all elements of the form $a + b$ with $a \in A$ and $b \in B$. If A and B are any two Abelian groups then $A \oplus B$ denotes their direct sum. We write $A \leqslant B$ to mean that A is a subgroup of B.

For $x \in \mathbb{R}$, $\{x\}$ denotes the fractional part of x and $\|x\|$ denotes the distance from x to the nearest integer. For $x \in \mathbb{R}^m$, we set $|x| = \max\{|x_1|, \ldots, |x_m|\}$ and $\|x\| = \max\{\|x_1\|, \ldots, \|x_m\|\}$. We use the symbols $\ll, \gg$ and $\asymp$ for the standard Vinogradov and asymptotic notation. That is, for real-valued functions f, g, typically defined here on $\mathbb{N}$ or $\mathbb{R}_{>0}$, we write $f \ll g$ to mean that $f(x) \leqslant Cg(x)$ for some $C > 0$ and sufficiently large x, $f \gg g$ if $g \ll f$ and $f \asymp g$ if both $f \ll g$ and $f \gg g$.

2.2. Cut and project sets

A cut and project scheme consists of the following:

- the total space $\mathbb{R}^k$;
- the physical space $E \subseteq \mathbb{R}^k$, a d-dimensional proper subspace of $\mathbb{R}^k$;
- the internal space $F_\pi \subseteq \mathbb{R}^k$, a subspace complementary to E in $\mathbb{R}^k$;
- the window $\mathcal{W}_\pi \subseteq F_\pi$.

Write π for the projection from the total space onto E with respect to the decomposition $\mathbb{R}^k = E + F_\pi$. The strip is given by $\mathcal{S} = E + \mathcal{W}_\pi$. For each $s \in \mathbb{R}^k/\mathbb{Z}^k$, we define the cut and project set $Y_s \subseteq E$ by

$$Y_s = \pi(\mathcal{S} \cap (\mathbb{Z}^k + s)).$$

In this situation we refer to Y_s as a **k** to **d** cut and project set.

We adopt the conventional assumption that $\pi|_{\mathbb{Z}^k}$ is injective. For $y \in Y$ we write $\tilde{y}$ for the unique point in $\mathcal{S} \cap \mathbb{Z}^k$ with $\pi(\tilde{y}) = y$. We also assume in much of what follows that E is a totally irrational subspace of $\mathbb{R}^k$, which means that the canonical projection of E into $\mathbb{R}^k/\mathbb{Z}^k$ is dense. This assumption guarantees that the natural linear $\mathbb{R}^d$ action of E on $\mathbb{R}^k/\mathbb{Z}^k$ is uniquely ergodic.

For the problem of studying linear repetitivity, the s in the definition of Y_s plays only a minor role. If we restrict our attention to points s for which $\mathbb{Z}^k + s$ does not intersect the boundary of $\mathcal{S}$ (these are called regular points and the corresponding sets Y_s are called regular cut and project sets) then, as long as E is totally irrational, the sets of finite patches in Y_s do not depend on the choice of s. In particular, the property of being LR does not depend on the choice of s, as long as s is taken to be a regular point. On the other hand, for points s which are not regular, the cut and project set Y_s may contain 'additional' patches coming from points on the boundary, which will make it non-repetitive, and therefore not LR. For this reason, *we will always assume that s is taken to be a regular point, and we will simplify our notation by writing Y instead of* Y_s.

As a point of reference, when allowing E to vary, we also make use of the fixed subspace $F_\rho = \{0\}^d \times \mathbb{R}^{k-d} \subseteq \mathbb{R}^k$, and we define $\rho : \mathbb{R}^k \to E$ and $\rho^* : \mathbb{R}^k \to F_\rho$ to be the projections onto E and F_ρ with respect to the decomposition $\mathbb{R}^k = E + F_\rho$ (assuming, with little loss of generality, that $\mathbb{R}^k$ does in fact decompose in this way). Our notational use of π and ρ is intended to be suggestive of the fact that F_π is the subspace which gives the *projection* defining Y (hence the letter π), while F_ρ is the subspace with which we *reference* E (hence the letter ρ). We write $\mathcal{W} = \mathcal{S} \cap F_\rho$, and for convenience we also refer to this set as the window defining Y. This slight ambiguity should not cause any confusion in the arguments below. See figure 1 for a pictorial description of the various elements in the definition of Y.

As already mentioned, in much of this paper we will focus our attention on the situation where $\mathcal{W}$ is taken to be a cubical window, given by

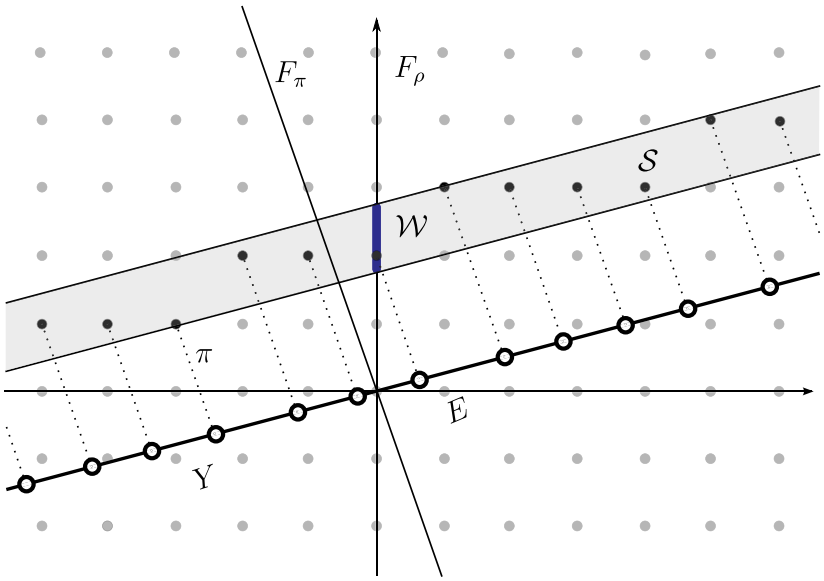

Figure 1. Definition of the cut and project set. In the figure $k = 2$, $d = 1$ and $s = 0$.

$$\mathcal{W} = \left\{ \sum_{i=d+1}^{k} t_i e_i : 0 \leqslant t_i \leqslant 1 \right\}. \tag{2.1}$$

In section 4 we will also consider the case when $\mathcal{W}$ is taken to be a canonical window, which is defined to be the image under ρ^* of the unit cube in $\mathbb{R}^k$.

For any cut and project set, the collection of points $x \in E$ with the property that $Y + x = Y$, forms a group, called the group of periods of Y. We say that Y is aperiodic if the group of periods is $\{0\}$. Finally, as mentioned in the introduction, we say that Y is a cubical (resp. canonical) cut and project set if it is regular, totally irrational, and aperiodic, and if $\mathcal{W}$ is a cubical (resp. canonical) window.

Without loss of generality, by permuting the standard basis vectors if necessary, we will assume that E can be written as

$$E = \{(x, L(x)) : x \in \mathbb{R}^d\},$$

where $L : \mathbb{R}^d \to \mathbb{R}^{k-d}$ is a linear function. For each $1 \leqslant i \leqslant k - d$, we define the linear form $L_i : \mathbb{R}^d \to \mathbb{R}$ by

$$L_i(x) = L(x)_i = \sum_{j=1}^{d} \alpha_{ij} x_j,$$

and we use the points $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$ to parametrize the choice of E.

Our proof of theorem 1.1 gives an explicit correspondence between the collection of k to d LR cubical cut and project sets, and the Cartesian product of the following two sets:

(S1) The set of all $(k - d)$-tuples $(L_1, \ldots, L_{k-d})$, where each L_i is a badly approximable linear form in $m_i \geqslant 1$ variables, with the integers m_i satisfying $m_1 + \cdots + m_{k-d} = d$, and

(S2) The set of all $d \times d$ integer matrices with non-zero determinant.

The fact that the set (S1) is empty when $d < k/2$ implies that, for this range of k and d values, there are no LR cubical (or canonical) cut and project sets. On the other hand, for $d \geqslant k/2$, there are uncountably many, as implied by the following corollary to theorem 1.1.

Corollary 2.1. *For $d < k/2$, there are no LR cubical cut and project sets. For $d \geqslant k/2$, the collection of $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$ which define LR cubical cut and project sets is a set with Lebesgue measure 0 and Hausdorff dimension d. Furthermore, these statements also apply to canonical cut and project sets.*

This corollary will be proved in section 5. We will also show in section 6 how to exhibit specific examples of LR cut and project sets, for any choice of $d \geqslant k/2$.

2.3. Results from diophantine approximation

Let $L : \mathbb{R}^d \to \mathbb{R}^{k-d}$ be a linear map given by a matrix with entries $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$. For any $N \in \mathbb{N}$, there exists an $n \in \mathbb{Z}^d$ with $|n| \leqslant N$ and

$$\|L(n)\| \leqslant \frac{1}{N^{d/(k-d)}}. \tag{2.2}$$

This is a multidimensional analogue of Dirichlet's theorem, which follows from a straightforward application of the pigeonhole principle. We are interested in having an inhomogeneous version of this result, requiring the values taken by $\|L(n) - \gamma\|$ to be small, for all choices of $\gamma \in \mathbb{R}^{k-d}$. For this we will use the following 'transference theorem,' a proof of which can be found in [13, chapter V, section 4].

Theorem 2.2 [13, chapter V, theorem VIII]. *Given a linear map L as above, the following statements are equivalent:*

(T1) There exists a constant $C_1 > 0$ such that

$$\|L(n)\| \geqslant \frac{C_1}{|n|^{d/(k-d)}},$$

for all $n \in \mathbb{Z}^d \setminus \{0\}$.

(T2) There exists a constant $C_2 > 0$ such that, for all $\gamma \in \mathbb{R}^{k-d}$, the inequalities

$$\|L(n) - \gamma\| \leqslant \frac{C_2}{N^{d/(k-d)}}, \quad |n| \leqslant N,$$

are soluble, for all $N \geqslant 1$, with $n \in \mathbb{Z}^d$.

Next, with a view towards applying this theorem, let $\mathcal{B}_{d,k-d}$ denote the collection of numbers $\alpha \in \mathbb{R}^{d(k-d)}$ with the property that there exists a constant $C = C(\alpha) > 0$ such that, for all nonzero integer vectors $n \in \mathbb{Z}^d$,

$$\|L(n)\| \geqslant \frac{C}{|n|^{d/(k-d)}}.$$

The Khintchine–Groshev theorem (see [8] for a detailed statement and proof) implies that the Lebesgue measure of $\mathcal{B}_{d,k-d}$ is 0. However in terms of Hausdorff dimension these sets are large. It is a classical result of Jarnik that $\dim \mathcal{B}_{1,1} = 1$, and this was extended by Wolfgang Schmidt, who showed in [31, theorem 2] that, for any choices of $1 \leqslant d < k$,

$$\dim \mathcal{B}_{d,k-d} = d(k-d).$$

Finally, we introduce the definition of relatively badly approximable linear forms. As mentioned in the introduction, these are linear forms that are badly approximable when restricted to rational subspaces complementary to their kernels. To be precise, suppose that $L : \mathbb{R}^d \to \mathbb{R}$ is a single linear form in d variables, and define $\mathcal{L} : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ by $\mathcal{L}(n) = L(n) \bmod 1$. Let $S \leqslant \mathbb{Z}^d$ be the kernel of $\mathcal{L}$, and write $r = \mathrm{rk}(S)$ and $m = d - r$. We say that L is relatively badly approximable if there exists a constant $C > 0$ and a group $\Lambda \leqslant \mathbb{Z}^d$ of rank m, with $\Lambda \cap S = \{0\}$ and

$$\|\mathcal{L}(\lambda)\| \geqslant \frac{C}{|\lambda|^m} \quad \text{for all} \quad \lambda \in \Lambda \setminus \{0\}.$$

Now suppose that L is relatively badly approximable and let Λ be a group satisfying the condition in the definition. Let $F \subseteq \mathbb{Z}^d$ be a complete set of coset representatives for $\mathbb{Z}^d/(\Lambda + S)$. We have the following lemma.

Lemma 2.3. *Suppose that L is relatively badly approximable, with Λ and F as above. Then there exists a constant $C' > 0$ such that, for any $\lambda \in \Lambda$ and $f \in F$, with $\mathcal{L}(\lambda + f) \neq 0$, we have that*

$$\|\mathcal{L}(\lambda + f)\| \geqslant \frac{C'}{1 + |\lambda|^m}.$$

Proof. Any element of F has finite order in $\mathbb{Z}^d/(\Lambda + S)$. Therefore, for each $f \in F$ there is a positive integer u_f, and elements $\lambda_f \in \Lambda$ and $s_f \in S$, for which

$$f = \frac{\lambda_f + s_f}{u_f}.$$

If $\mathcal{L}(\lambda + f) \neq 0$ then either $\lambda + f = s_f/u_f \neq 0$, or $\lambda + u_f^{-1}\lambda_f \neq 0$. The first case only pertains to finitely many possibilities, and in the second case we have that

$$\begin{aligned} \|\mathcal{L}(\lambda + f)\| &\geqslant u_f^{-1} \cdot \|\mathcal{L}(u_f\lambda + \lambda_f + s_f)\| \\ &= u_f^{-1} \cdot \|\mathcal{L}(u_f\lambda + \lambda_f)\| \\ &\geqslant \frac{C}{u_f|u_f\lambda + \lambda_f|^m}. \end{aligned}$$

Therefore, replacing C by an appropriate constant $C' > 0$, and using the fact that F is finite, finishes the proof. □

We can also deduce that if L is relatively badly approximable, then the group Λ in the definition may be replaced by any group $\Lambda' \leqslant \mathbb{Z}^d$ which is complementary to S. This is the content of the following lemma.

Lemma 2.4. *Suppose that L is relatively badly approximable. Then, for any group $\Lambda' \leqslant \mathbb{Z}^d$ of rank m, with $\Lambda' \cap S = \{0\}$, there exists a constant $C' > 0$ such that*

$$\|\mathcal{L}(\lambda')\| \geqslant \frac{C'}{|\lambda'|^m} \quad \text{for all} \quad \lambda' \in \Lambda' \setminus \{0\}.$$

Proof. Let Λ be the group in the definition of relatively badly approximable. Choose a basis $v_1, \ldots, v_m$ for Λ', and for each $1 \leqslant j \leqslant m$ write

$$v_j = \frac{\lambda_j + s_j}{u_j},$$

with $\lambda_j \in \Lambda, s_j \in S$, and $u_j \in \mathbb{N}$.

Each $\lambda' \in \Lambda'$ can be written in the form

$$\lambda' = \sum_{j=1}^{m} a_j v_j,$$

with integers $a_1, \ldots, a_m$, and we have that

$$\|\mathcal{L}(\lambda')\| \geqslant (u_1 \cdots u_m)^{-1} \left\| \mathcal{L}\left(\sum_{j=1}^{m} b_j \lambda_j \right) \right\|,$$

with $b_j = a_j u_1 \cdots u_m / u_j \in \mathbb{Z}$ for each j. If the integers a_j are not identically 0 then, since $\Lambda' \cap S = \{0\}$, it follows that

$$\lambda := \sum_{j=1}^{m} b_j \lambda_j \neq 0.$$

Using the relatively badly approximable hypothesis gives that

$$\|\mathcal{L}(\lambda')\| \geqslant \frac{C}{u_1 \cdots u_m \cdot |\lambda|^m}.$$

Finally since $|\lambda| \ll |\lambda'|$, we have that

$$\frac{C}{u_1 \cdots u_m \cdot |\lambda|^m} \geqslant \frac{C'}{|\lambda'|^m},$$

for some constant $C' > 0$. □

2.4. Patterns in cut and project sets

Let F_ρ, ρ, ρ^*, and $\tilde{y}$ be defined as in section 2.2. Assume that we are given a bounded convex set $\Omega \subseteq E$ which contains a neighborhood of 0 in E. Then, for each $r \geqslant 0$, define the patch of size $\mathbf{r}$ at $\mathbf{y}$, by

$$P(y, r) := \{y' \in Y : \rho(\tilde{y'} - \tilde{y}) \in r\Omega\}.$$

In other words, $P(y, r)$ consists of the projections (under π) to Y of all points of $\mathcal{S}$ whose first d coordinates are in a certain neighborhood of the first d coordinates of $\tilde{y}$.

We remark that there are several different definitions of 'patches of size r' in the literature. For example, from the point of view of Y being contained in E, it is more natural to define a patch of size r at y to be the collection of points of Y which lie within distance r of y. In [20] we considered this definition of patch (what we called there type 1 patches), together with the definition that we have given above (type 2 patches). In fact, the two definitions of patches agree except possibly on a constant neighborhood of their boundaries

(see [20, equation (4.1)]). Therefore if Y is LR for one definition of 'patch of size r', it will be LR for the other, and similarly for other reasonable definitions.

For $y_1, y_2 \in Y$, we say that $P(y_1, r)$ and $P(y_2, r)$ are equivalent if

$$P(y_1, r) = P(y_2, r) + y_1 - y_2.$$

This defines an equivalence relation on the collection of patches of size r. We denote the equivalence class of the patch of size r at y by $\mathcal{P}(y, r)$.

As indicated in the introduction, a cut and project set Y is linearly repetitive (LR) if there is a $C > 0$ such that, for every $r > 0$, every ball of size Cr in E contains a representative from every equivalence class of patches of size r. There are two technical points which will ease our discussion below. First of all, since the points of Y are relatively dense, in the above definition we are free to restrict our attention, without loss of generality (by increasing C if necessary), to balls of size Cr centered at points of Y. Secondly, the property of being LR does not depend on the choice of Ω used to define the patches. This follows from the fact that, if $\Omega' \subseteq E$ is any other bounded convex set which contains a neighborhood of 0, then there are dilations of Ω' which contain, and which are contained in, Ω.

Let $\mathcal{W} = \mathcal{S} \cap F_\rho$. There is a natural action of $\mathbb{Z}^k$ on F_ρ, given by

$$n \cdot w = \rho^*(n) + w = w + (0, n_2 - L(n_1)),$$

for $n = (n_1, n_2) \in \mathbb{Z}^k = \mathbb{Z}^d \times \mathbb{Z}^{k-d}$ and $w \in F_\rho$. For each $r \geqslant 0$ we define the **r**-singular points of $\mathcal{W}$ by

$$\mathrm{sing}(r) := \mathcal{W} \cap \left((-\rho^{-1}(r\Omega) \cap \mathbb{Z}^k) \cdot \partial\mathcal{W} \right),$$

and the **r**-regular points by

$$\mathrm{reg}(r) := \mathcal{W} \setminus \mathrm{sing}(r).$$

The following result follows from the proof of [20, lemma 3.2] (see also [22]).

Lemma 2.5. *Suppose that $\mathcal{W}$ is a parallelotope generated by integer vectors. For every equivalence class $\mathcal{P} = \mathcal{P}(y, r)$, there is a unique connected component U of* $\mathrm{reg}(r)$ *with the property that, for any $y' \in Y$,*

$$\mathcal{P}(y', r) = \mathcal{P}(y, r) \text{ if and only if } \rho^*(\tilde{y'}) \in U.$$

For each equivalence class $\mathcal{P} = \mathcal{P}(y, r)$ we define $\xi_\mathcal{P}$, the frequency of $\boldsymbol{\mathcal{P}}$, by

$$\xi_\mathcal{P} := \lim_{R \to \infty} \frac{\#\{y' \in Y : |y'| \leqslant R,\ \mathcal{P}(y', r) = \mathcal{P}(y, r)\}}{\#\{y' \in Y : |y'| \leqslant R\}}.$$

It is not difficult to show that, in our setup, the limit defining $\xi_\mathcal{P}$ always exists. Lemma 2.5, combined with the Birkhoff Ergodic theorem, proves that, for totally irrational E, the frequencies of equivalence classes of patches are given by the volumes of connected components of $\mathrm{reg}(r)$.

Lemma 2.6. *If E is totally irrational then for any $r > 0$ and any equivalence class $\mathcal{P} = \mathcal{P}(y, r)$, the frequency $\xi_\mathcal{P}$ is equal to the volume of the connected component U in the statement of lemma 2.5.*

This lemma is the full content of [20, lemma 3.2] (this idea is also implicit in [9]).

2.5. Subadditive ergodic theorems and (PQ)

As discussed in the introduction, it is often of relevance to understand when a Delone set has sufficient regularity to satisfy a subadditive ergodic theorem. We shall not define this notion here (we refer the reader to the references in the introduction), and instead will work the condition *positivity of quasiweights*, shown in [11] to be equivalent to validity of a subadditive ergodic theorem for repetitive Delone sets.

Let $\mathcal{P}$ be an equivalence class of patches of size $r > 0$ and B be a bounded subset of E. Write $\#'_{\mathcal{P}} B$ for the maximum number of disjoint patches of size r in B which are in the equivalence class $\mathcal{P}$, and define

$$\nu'(\mathcal{P}) = r^d \cdot \liminf_{|C|\to\infty} \frac{\#'_{\mathcal{P}} C}{|C|},$$

where C runs over all cubes in E. We say that Y satisfies condition (PQ) if

$$\inf_{\mathcal{P}} \nu'(\mathcal{P}) > 0,$$

where the infimum is taken over all equivalence classes of patches, for all $r > 0$.

So the condition (PQ) is essentially a condition on the densities of r-patches within the pattern. A condition called '(U)' is introduced in [11] which ensures some regularity in the spacing between occurrences of patches. It is shown in [11, theorem 2], that LR is equivalent to the conditions (PQ) and (U) both being satisfied. One consequence of our results is that, for cubical cut and project sets, condition (U) can be omitted from this criteria. In other words, for cubical cut and project sets, condition (PQ) by itself is a necessary and sufficient condition for LR. Results of this type for analogous problems in symbolic dynamics were previously discovered by other authors (see in particular Lenz [24]), and they appear to have been anticipated to hold in greater generality (see [11, remark 5]).

3. Proof of theorem 1.1

We assume that $\mathcal{W}$ is given by (2.1) and we identify $\mathcal{W}$ with a subset of $\mathbb{R}^{k-d}$, in the obvious way. Recall that if Y is LR with respect to one convex patch shape Ω, then it is LR with respect to all convex patch shapes. The precise shape Ω which we will use will be specified later in the proof, but until then everything we say will apply to any fixed choice of such a shape.

For $r > 0$ let $c(r)$ denote the number of equivalence classes of patches of size r. If Y is LR or satisfies (PQ) then clearly there exists a constant $C > 0$ such that $c(r)$ is bounded above by Cr^d, for all $r > 0$. For the first part of the proof of theorem 1.1 we will show that condition (LR1) is necessary and sufficient for a bound of this type to hold.

For each $1 \leqslant i \leqslant k-d$, let $S_i \leqslant \mathbb{Z}^d$ denote the kernel of the map $\mathcal{L}_i$, and let r_i be the rank of S_i. Furthermore, for each subset $I \subseteq \{1, \ldots, k-d\}$ let

$$S_I = \bigcap_{i\in I} S_i,$$

and let r_I be the rank of S_I. For convenience, set $S_\emptyset = \mathbb{Z}^d$ and $r_\emptyset = d$. For any pair $I, J \subseteq \{1, \ldots, k-d\}$, the sum set $S_I + S_J$ is a subgroup of $\mathbb{Z}^d$, and it therefore has rank at most d. On the other hand we have that

$$\mathrm{rk}(S_I + S_J) = \mathrm{rk}(S_I) + \mathrm{rk}(S_J) - \mathrm{rk}(S_I \cap S_J),$$

which gives the inequality

$$r_I + r_J \leqslant d + r_{I\cup J}. \tag{3.1}$$

As one application of this inequality we see immediately that

$$\begin{aligned} r_1 + r_2 + \cdots + r_{k-d} &\leqslant d + r_{12} + r_3 + \cdots + r_{k-d} \\ &\leqslant 2d + r_{123} + r_4 + \cdots + r_{k-d} \\ &\vdots \\ &\leqslant d(k-d-1) + r_{12\ldots(k-d)} \\ &= d(k-d-1). \end{aligned} \tag{3.2}$$

The last equality here uses the assumption that Y is aperiodic.

From lemma 2.5, we know that $c(r)$ is equal to the number of connected components of $\mathrm{reg}(r)$. Let the map $\mathcal{C} : \mathbb{Z}^{d(k-d)} \to \mathcal{W}$ be defined by

$$\mathcal{C}(n^{(1)}, \ldots, n^{(k-d)}) = (\{L_1(n^{(1)})\}, \ldots, \{L_{k-d}(n^{(k-d)})\}),$$

for $n^{(1)}, \ldots, n^{(k-d)} \in \mathbb{Z}^d$. Identify $\mathbb{Z}^d$ with the set $\mathcal{Z} = \mathbb{Z}^k \cap \langle e_1, \ldots, e_d\rangle_{\mathbb{R}}$, and for each $r > 0$ let $\mathcal{Z}_r \subseteq \mathbb{Z}^d$ be defined by

$$\mathcal{Z}_r = -\rho^{-1}(r\Omega) \cap \mathcal{Z}.$$

Since our window $\mathcal{W}$ is a fundamental domain for the integer lattice in F_ρ, there is a one to one correspondence between points of Y and elements of $\mathcal{Z}$. This correspondence is given explicitly by mapping a point $y \in Y$ to the vector in $\mathcal{Z}$ given by the first d coordinates of $\tilde{y}$. Also, notice that if $n \in \mathbb{Z}^k$ and $-n \cdot 0 \in \mathcal{W}$, then it follows that

$$-n \cdot 0 = (\{L_1(n_1, \ldots, n_d)\}, \ldots, \{L_{k-d}(n_1, \ldots, n_d)\}).$$

These observations together imply that the collection of all vertices of connected components of $\mathrm{reg}(r)$ is precisely the set $\mathcal{C}(\mathcal{Z}_r^{k-d})$, which in turn implies that

$$c(r) \asymp |\mathcal{C}(\mathcal{Z}_r^{k-d})|.$$

The values of the function $\mathcal{C}$ define a natural $\mathbb{Z}^{d(k-d)}$ action on $\mathcal{W}$. Therefore we may regard the set $\mathcal{C}(\mathbb{Z}^{d(k-d)})$ as a group, isomorphic to

$$\mathbb{Z}^{d(k-d)}/\ker(\mathcal{C}) \cong \mathbb{Z}^d/S_1 \oplus \cdots \oplus \mathbb{Z}^d/S_{k-d}.$$

If (LR1) holds then we have that

$$\mathrm{rk}(\mathcal{C}(\mathbb{Z}^{d(k-d)})) = d(k-d) - \sum_{i=1}^{k-d} r_i = d,$$

and from this it follows that

$$|\mathcal{C}(\mathcal{Z}_r^{k-d})| \asymp r^d.$$

On the other hand, if (LR1) does not hold then by (3.2) we have that

$$\mathrm{rk}(\mathcal{C}(\mathbb{Z}^{d(k-d)})) > d,$$

which implies that

$$|\mathcal{C}(\mathcal{Z}_r^{k-d})| \gg r^{d+1}.$$

We conclude that $c(r) \ll r^d$ if and only if condition (LR1) holds, so (LR1) is a necessary condition for LR or (PQ).

Next we assume that (LR1) holds and we prove that, under this assumption, condition (LR2) is necessary and sufficient in order for Y to be LR. First of all, suppose that I and J were disjoint, nonempty subsets of $\{1, \ldots, k-d\}$ for which

$$r_I + r_J < d + r_{I\cup J}.$$

Then, by the same argument used in (3.2), we would have that

$$\sum_{i=1}^{k-d} r_i \;\leqslant\; d(k-d-3) + r_{(I\cup J)^c} + r_I + r_J \;<\; d(k-d-1).$$

This clearly contradicts (LR1). Therefore if (LR1) holds then, by (3.1), we have that

$$r_I + r_J = d + r_{I\cup J},$$

whenever I and J are disjoint and nonempty.

For each $1 \leqslant i \leqslant k-d$, define $J_i = \{1, \ldots, k-d\} \setminus \{i\}$, and let $\Lambda_i = S_{J_i}$. Write $m_i = r_{J_i}$ for the rank of Λ_i. Then, by what was established in the previous paragraph, we have that

$$m_i + r_i = d.$$

If n is any nonzero vector in Λ_i, then n is in S_j for all $j \neq i$. Since Y is aperiodic, this means that $n \notin S_i$, which gives that

$$\mathrm{rk}(\Lambda_i + S_i) = m_i + r_i - \mathrm{rk}(\Lambda_i \cap S_i) = d.$$

Furthermore, for any $j \neq i$, the fact that $\Lambda_j \subseteq S_i$ implies that $\Lambda_j \cap \Lambda_i = \{0\}$, so

$$\mathrm{rk}(\Lambda_1 + \cdots + \Lambda_{k-d}) = \sum_{i=1}^{k-d} \mathrm{rk}(\Lambda_i) = \sum_{i=1}^{k-d} (d - r_i) = d.$$

For each i, let $F_i \subseteq \mathbb{Z}^d$ be a complete set of coset representatives for $\mathbb{Z}^d/(\Lambda_i + S_i)$. Also, write $\Lambda = \Lambda_1 + \cdots + \Lambda_{k-d}$, and let $F \subseteq \mathbb{Z}^d$ be a complete set of representatives for $\mathbb{Z}^d/\Lambda$. What we have shown so far implies that all of the sets $F_1, \ldots, F_{k-d}$, and F are finite.

Again thinking of $\mathbb{Z}^d$ as being identified with the set $\mathcal{Z}$, let

$$\mathcal{Z}_{r,\Lambda} = \mathcal{Z}_r \cap \Lambda, \quad \mathcal{Z}_{r,\Lambda_i} = \mathcal{Z}_r \cap \Lambda_i, \quad \text{and} \quad \mathcal{Z}_{r,S_i} = \mathcal{Z}_r \cap S_i.$$

For each i, choose a basis $\{v_j^{(i)}\}_{j=1}^{m_i}$ for Λ_i, and define

$$\Omega_i' = \left\{ \sum_{j=1}^{m_i} t_i v_j^{(i)} : -1/2 \leqslant t_i < 1/2 \right\},$$

and

$$\Omega' = \Omega_1' + \cdots + \Omega_{k-d}',$$

so that Ω' is a fundamental domain for $\mathbb{R}^d/\Lambda$. We now specify Ω to be the subset of points in E whose first d coordinates lie in Ω'. In other words,

$$\Omega = E \cap \rho^{-1}(\Omega').$$

Notice that every $n \in \Lambda$ has a unique representation of the form

$$n = \sum_{i=1}^{k-d} \sum_{j=1}^{m_i} a_{ij} v_j^{(i)}, \quad a_{ij} \in \mathbb{Z}.$$

Using this representation, we have that

$$\mathcal{L}(n) = \mathcal{C}((n^{(i)})_{i=1}^{m_i}),$$

where, for each i, the vector $n^{(i)} \in \mathbb{Z}^d$ is given by

$$n^{(i)} = \sum_{j=1}^{m_i} a_{ij} v_j^{(i)}.$$

This gives a one to one correspondence between elements of $\mathcal{L}(\Lambda)$ and elements of the set

$$\mathcal{C}(\Lambda_1 \times \cdots \times \Lambda_{k-d}) = \mathcal{L}_1(\Lambda_1) \times \cdots \times \mathcal{L}_{k-d}(\Lambda_{k-d}).$$

We will combine this observation with the facts that

$$\mathcal{L}(\mathbb{Z}^d) = \mathcal{L}(\Lambda + F)$$

and

$$\mathcal{C}(\mathbb{Z}^{d(k-d)}) = \mathcal{C}((\Lambda_1 + F_1) \times \cdots \times (\Lambda_{k-d} + F_{k-d})),$$

in order to study the spacings between points of the sets $\mathcal{L}(\mathcal{Z}_r)$ and $\mathcal{C}(\mathcal{Z}_r^{k-d})$.

First of all, it is clear that

$$\mathcal{L}(\mathcal{Z}_r) \supseteq \mathcal{L}_1(\mathcal{Z}_{r,\Lambda_1}) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r,\Lambda_{k-d}}), \tag{3.3}$$

and that

$$\mathcal{C}(\mathcal{Z}_r^{k-d}) \supseteq \mathcal{L}_1(\mathcal{Z}_{r,\Lambda_1}) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r,\Lambda_{k-d}}). \tag{3.4}$$

Since all of the sets $F_1, \ldots, F_{k-d}$, and F are finite, there is a constant $\kappa > 0$ with the property that, for all sufficiently large r,

$$\begin{aligned} \mathcal{Z}_r &\subseteq \mathcal{Z}_{r+\kappa,\Lambda} + F, \quad \text{and} \\ \mathcal{Z}_r &\subseteq \mathcal{Z}_{r+\kappa,\Lambda_i} + \mathcal{Z}_{r+\kappa,S_i} + F_i, \end{aligned}$$

for each $1 \leqslant i \leqslant k-d$. For the second inclusion here we are using the definition of Ω and the fact that $\Lambda_j \subseteq S_i$ for all $j \neq i$. These inclusions imply that

$$\begin{aligned} \mathcal{L}(\mathcal{Z}_r) &\subseteq \mathcal{L}(\mathcal{Z}_{r+\kappa,\Lambda}) + \mathcal{L}(F) \\ &\subseteq \mathcal{L}_1(\mathcal{Z}_{r+\kappa,\Lambda_1} + F) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r+\kappa,\Lambda_{k-d}} + F), \end{aligned} \tag{3.5}$$

and that

$$\begin{aligned} \mathcal{C}(\mathcal{Z}_r^{k-d}) &\subseteq \mathcal{C}\left((\mathcal{Z}_{r+\kappa,\Lambda_1} + F_1) \times \cdots \times (\mathcal{Z}_{r+\kappa,\Lambda_{k-d}} + F_{k-d})\right) \\ &= \mathcal{L}_1(\mathcal{Z}_{r+\kappa,\Lambda_1} + F_1) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r+\kappa,\Lambda_{k-d}} + F_{k-d}). \end{aligned} \tag{3.6}$$

Now we are positioned to make our final arguments.

Suppose first of all that (LR2) holds. Let U be any connected component of $\mathrm{reg}(r)$. Then U is a $(k-d)$-dimensional box, with faces parallel to the coordinate hyperplanes, and with vertices in the set $\mathcal{C}(\mathcal{Z}_r^{k-d})$. Therefore we can write U in the form

$$U = \{x \in \mathcal{W} : \ell_i < x_i < r_i\}, \tag{3.7}$$

where for each i, the values of ℓ_i and r_i are elements of the set $\mathcal{L}_i(\mathcal{Z}_r)$. By equation (3.6), together with lemma 2.3, there is a constant $c_1 > 0$ such that, for every i,

$$r_i - \ell_i \geqslant \frac{c_1}{r^{m_i}}.$$

Next we will show that there is a constant $c_2 > 0$ such that, for all sufficiently large r, the orbit of every point in $F_\rho/\mathbb{Z}^{k-d}$ under the action of $\mathcal{Z}_{c_2 r}$ intersects every connected component of $\mathrm{reg}(r)$. Then lemma 2.5 will imply that Y is LR. To show that there is such a constant c_2, we use (3.3) and theorem 2.2. Each one of the linear forms L_i is a badly approximable linear form in m_i variables, when restricted to Λ_i. Therefore, by (T2) of theorem 2.2, there is a constant $\eta > 0$ with the property that, for all sufficiently large r and for each i, the collection of points $\mathcal{L}_i(\mathcal{Z}_{c_2 r,\Lambda_i})$ is $\eta/(c_2 r^{m_i})$-dense in $\mathbb{R}/\mathbb{Z}$. Choosing $c_2 > 3c_1/\eta$ completes the proof of this part of the theorem, verifying that (LR1) and (LR2) together imply linear repetitivity. By the results of [11], we also have that condition (PQ) is satisfied and thus that the cut and project set satisfies a subadditive ergodic theorem.

For the final part, suppose that (LR1) holds and (LR2) does not. Then one of the linear forms L_i is not relatively badly approximable, and we assume without loss of generality that it is L_1. Let $\epsilon > 0$ be arbitrary. By (3.4) and our hypothesis on L_1, we can choose r large enough so that there is a connected component U of $\mathrm{reg}(r)$, given as in (3.7), with

$$r_1 - \ell_1 < \frac{\epsilon}{r^{m_1}}.$$

By a simple counting argument, we may also choose U so that its other sides have lengths

$$r_i - \ell_i \leqslant \frac{C}{r^{m_i}}.$$

for some constant $C > 0$ depending only on the ranks of the kernels of the $\mathcal{L}_i$. Thus U has volume at most ϵ/r^d and so its corresponding patch of size r, by lemma 2.6, has only this frequency. Since $\epsilon > 0$ was arbitrary, clearly neither linear repetitivity nor condition (PQ) is satisfied.

4. Canonical cut and project sets

In this section we turn our attention to canonical cut and project sets. In order to gain a broader perspective on our results, we will use the notion of local derivability for point sets, first introduced in [6]. Suppose that Y_1 and Y_2 are two cut and project sets formed from a common physical space E, and suppose (without loss of generality for the purposes of all of our results) that Y_1 and Y_2 are both uniformly discrete and relatively dense. We say that $\mathbf{Y_1}$ is locally derivable from $\mathbf{Y_2}$ if there exists a constant $c > 0$ with the property that, for all $x \in E$ and for all sufficiently large r, the equivalence class of the patch of size r centered at x in Y_2 uniquely determines the patch of size $r - c$ centered at x in Y_1. There is a minor technical issue here, that x may not belong to Y_1 or Y_2. However, since Y_1 and Y_2 are relatively dense, this can be rectified by requiring that x be moved, when necessary in the definition above, to a nearby point of the relevant cut and project set. Finally, we say that Y_1 and Y_2 are mutually locally derivable (MLD) if each set is locally derivable from the other.

The argument in [6, appendix] (see also [4] and [5, remark 7.6]) provides us with the following characterization of MLD cut and project sets Y_1 and Y_2 as above.

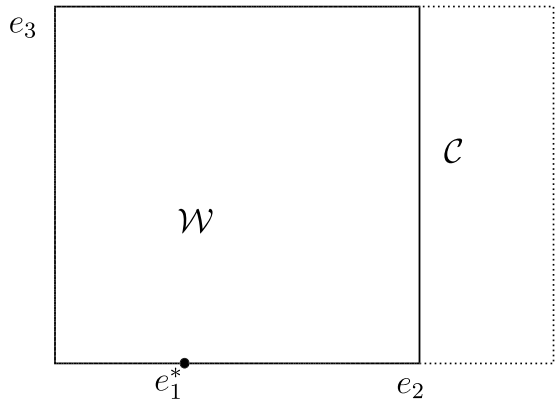

Figure 2. In the figure $k = 3$, $d = 1$ and $e_1^* \in \mathbb{R}e_2$. The canonical window $\mathcal{C}$ is a rectangle.

Lemma 4.1. *Let Y_1 and Y_2 be regular, totally irrational, aperiodic k to d cut and project sets, constructed with the same physical and internal spaces and with windows $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. Then Y_1 is locally derivable from Y_2 if and only if $\mathcal{W}_1$ is a finite union of sets each of which is a finite intersection of translates of $\mathcal{W}_2$ (or of the closure of its complement), with translations taken from $\rho^*(\mathbb{Z}^k)$.*

From this lemma we may deduce the following result relating cubical and canonical cut and project sets.

Proposition 4.2. *Let Y_1 be a cubical cut and project set, and let Y_2 be the cut and project set formed from the same data as Y_1, but with the canonical window. Then Y_1 is locally derivable from Y_2. Furthermore, Y_2 is locally derivable from Y_1 if and only if, for each $1 \leqslant i \leqslant d$, the point $\rho^*(e_i)$ lies on a line of the form $\mathbb{R}e_j$, for some $d+1 \leqslant j \leqslant k$.*

Proof. Let us temporarily write $\mathcal{C}$ for the canonical window and continue using $\mathcal{W}$ for the cubical window. We may write

$$\mathcal{C} = \left\{ \sum_{i=1}^{k} x_i e_i^* : x_i \in [0,1] \right\} \tag{4.1}$$

where the $e_i^* = \rho^*(e_i)$ are the projections to F_ρ of the k standard basis vectors of $\mathbb{R}^k$. We claim that

$$\mathcal{W} = \bigcap_{v \in V} \mathcal{C} - v,$$

where V is the set of projected vertices of the unit-hypercube contained in the first d coordinate directions, that is

$$V = \left\{ \rho^*\Big(\sum_{i=1}^{d} \epsilon_i e_i\Big) : \epsilon_i \in \{0,1\} \right\}.$$

Notice that $e_j^* = e_j$ for $j \in \{d+1, \ldots, k\}$, so an element $x \in \mathcal{W}$ can be written (uniquely) as $x = \sum_{j=d+1}^{k} x_j e_j^*$ where the $x_j \in [0,1]$. So clearly from (4.1) we have that $x + v \in \mathcal{C}$ for

any $v \in V$ and hence $\mathcal{W} \subseteq \bigcap \mathcal{C} - v$. For the opposite inclusion, suppose that $x \in \bigcap \mathcal{C} - v$. For $j \in \{d+1, \ldots, k\}$, consider the set P_j of e_i^* with $1 \leqslant i \leqslant d$ and strictly positive coordinate in the direction e_j, and analogously the set N_j of the e_i^* with strictly negative coordinate in the direction e_j. Then the maximum value of any $y \in \mathcal{C}$ in the e_j direction is attained by $e_j + p_j$, where $p_j = \sum_{v \in P_j} v$; the minimum is attained by $n_j = \sum_{v \in N_j} v$ (if an empty sum occurs then consider it as zero). Since $p_j \in V$ we have that $x + p_j \in \mathcal{C}$ and so has magnitude at most that of $e_j + p_j$ in the j direction. It follows that x has magnitude at most 1 in the e_j direction and, by an analogous argument considering $x + n_j$, has magnitude at least 0 in the e_j direction. Repeating this argument for each coordinate direction of F_ρ we see that $x \in \mathcal{W}$, proving the inclusion $\bigcap \mathcal{C} - v \subseteq \mathcal{W}$. By lemma 4.1, since $V \subseteq \rho^*(\mathbb{Z}^k)$, we have that Y_1 is locally derivable from Y_2.

For the second statement, suppose firstly that for each $1 \leqslant i \leqslant d$ the points e_i^* lie on a line $\mathbb{R}e_j$ for $d+1 \leqslant j \leqslant k$. Then it is not hard to see that $\mathcal{C}$ is a box with coordinates aligned with the standard basis vectors of F_ρ, see figure 2. Take a coordinate direction e_j with $d+1 \leqslant j \leqslant k$. Then the points of $\mathcal{C}$ have their jth coordinate in an interval $[l_j, u_j]$, where $l_j, u_j \in \rho^*(\mathbb{Z}^k)$. So $\mathcal{C} \cap (\mathcal{C}^c - l_j - e_j) \cap (\mathcal{C}^c - u_j)$ is still a coordinate aligned box but now has jth coordinate ranging in $[0, 1]$. Here, $\mathcal{C}^c$ denotes the closure of the complement of $\mathcal{C}$ in F_ρ. Repeating for each coordinate direction and applying lemma 4.1, we see that Y_2 is locally derivable from Y_1.

Finally, suppose that for some $1 \leqslant i \leqslant d$ we have that $e_i^* \notin \mathbb{R}e_j$ for any $d+1 \leqslant j \leqslant k$. Notice that the boundaries of $\mathcal{C}$ and $\mathcal{W}$ are each contained in a finite set of translates of a finite collection of hyperplanes. The same is true under taking finite unions of finite intersections of translates of the windows, or closures of their complements, and applying such an operation cannot add new hyperplanes to the collection (they can be removed though, as is typical for the local derivation from the canonical to the cubical window). So we need only find a hyperplane necessary to define the boundary of $\mathcal{C}$ which is not needed for $\mathcal{W}$. By assumption, e_i^* is non-zero in at least two of the standard coordinate directions for F_ρ. Define a set S of vectors consisting of e_i^* and $k-d-2$ other of the vectors $e_j^* = e_j$, with $d+1 \leqslant j \leqslant k$, chosen so that every coordinate direction of F_ρ has non-zero entry in at least one element of S. Then the hyperplane H spanned by the vectors of S is a $(k-d-1)$-dimensional hyperplane which is not in the collection of hyperplanes defining the boundary of $\mathcal{W}$. Indeed by our choice of vectors in S, we may choose a point of H with non-zero entry in each coordinate direction, whereas the hyperplanes defining the boundary of $\mathcal{W}$ are all orthogonal to one such coordinate direction. Moreover, H is necessary to define the boundary of $\mathcal{C}$. To see this, assign to H an orientation (i.e. a choice of orthonormal vector) and consider the sum p of vectors e_n^*, with $1 \leqslant n \leqslant k$, which lie strictly above H. Then there is a face

$$F = \left\{ p + \sum_{v_i \in S} \lambda_i v_i : \lambda_i \in [0, 1] \right\}$$

contained in the boundary of $\mathcal{C}$, see figure 3. Indeed, it obviously follows from (4.1) that $F \subseteq \mathcal{C}$, and no element of $\mathcal{C}$ lies above $p + H$ or else there would be a vector e_n^* missing in the definition of p, so in fact F is a subset of the boundary of $\mathcal{C}$. It follows that the hyperplane H is necessary to define the boundary of $\mathcal{C}$, and so Y_2 cannot be locally derivable from Y_1 by lemma 4.1. □

If a Delone set Y_1 is locally derivable from Y_2, and if Y_2 is LR, then it follows directly from the definitions that Y_1 is also LR. Therefore, proposition 4.2 implies the first part of theorem 1.2, that if a cubical cut and project set is not LR, then neither is the corresponding canonical cut and project set.

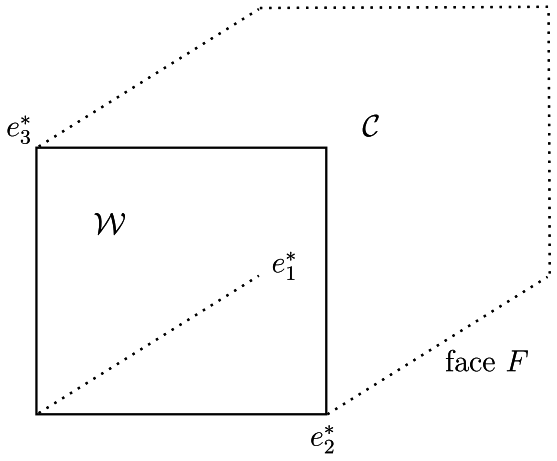

Figure 3. In the figure $k = 3$, $d = 1$ and $e_1^* \notin \mathbb{R}e_1 \cup \mathbb{R}e_2$. The canonical window $\mathcal{C}$ is a hexagon with a slanted face F.

For the second part of theorem 1.2, we observe that the same arguments applied to the cubical window above could equally have been applied to analogues of the cubical window but chosen in different reference spaces. More precisely, consider a collection ρ' of $k-d$ coordinate directions e_j of $\mathbb{R}^k$ and the corresponding 'reference space' $F_{\rho'}$ spanned by them, along with window $\mathcal{W}_{\rho'} \subseteq F_{\rho'}$ given by

$$\mathcal{W}_{\rho'} = \left\{ \sum_{e_j \in \rho'} t_i e_i : 0 \leqslant t_i \leqslant 1 \right\}.$$

By a permutation of the basis vectors, this is identical to our standard cubical setting, and so we may directly apply theorem 1.1 to see whether or not cut and project sets Y' defined by E and $\mathcal{W}_{\rho'}$ are LR or not. Similarly, we may deduce from proposition 4.2 that Y' is still locally derivable from the canonical cut and project set Y_2. Hence, to finish the proof of theorem 1.2, we need only find a cubical cut and project set which is LR with respect to the standard reference space but is not with respect to another. The existence of such cut and project schemes is proved by the following example:

Example 4.3. Suppose that α_1, α_2, and β are positive real numbers with $(\alpha_1, \alpha_2) \in \mathcal{B}_{2,1}$, $\beta \in \mathcal{B}_{1,1}$ and $\{1, \alpha_1, \alpha_2, \beta^{-1}\}$ rationally independent. Let E be the three dimensional subspace of $\mathbb{R}^5$ defined by

$$E = \{(x, \alpha_1 x_1 + \alpha_2 x_2 + x_3, \beta x_3) : x \in \mathbb{R}^3\}.$$

We claim that the standard cubical cut and project sets defined by E are LR, but those defined by cubical windows in the reference space $F_{\rho'}$ with $\rho' = \{e_3, e_5\}$ are not LR, from which it follows that the canonical cut and projects associated to E are also not LR.

Firstly, it is easily checked that E is totally irrational and that the cubical cut and project sets defined using E are aperiodic. The linear forms defining E are

$$L_1(x_1, x_2, x_3) = x_1\alpha_1 + x_2\alpha_2 + x_3; \;\; L_2(x_1, x_2, x_3) = x_3\beta.$$

In the notation of the proof of theorem 1.1, the kernels of the linear forms $\mathcal{L}_1$ and $\mathcal{L}_2$ have ranks $r_1 = 1$ and $r_2 = 2$. Therefore theorem 1.1 allows us to conclude that any cubical cut and project set formed using E is LR.

To see that replacing the standard cubical window with the cubical window in $F_{\rho'}$ results in non-LR cut and project sets, first note that it is essentially equivalent to make a change of basis and consider the resulting setup in the original setting of theorem 1.1. We make the change of basis which sends e_3 to e_5, and vice versa, leaving all other basis vectors fixed. Then E is transformed to the new subspace E' spanned by $(1, 0, 0, \alpha_1, 0)$, $(0, 1, 0, \alpha_2, 0)$ and $(0, 0, \beta, 1, 1)$. With some simple linear manipulations of these vectors, it is easily seen that E' is the graph of the linear forms

$$L'_1(x_1, x_2, x_3) = x_1\alpha_1 + x_2\alpha_2 + x_3\beta^{-1}; \;\; L'_2(x_1, x_2, x_3) = \beta^{-1}x_3.$$

By assumption, $L'_1(n) \notin \mathbb{Z}$ for all $n \in \mathbb{Z}^3$ and so $\mathcal{L}'_1$ has trivial kernel. As before, $\mathcal{L}'_2$ has kernel of rank 2. So the sum of the ranks of the kernels is now only 2, not 3 as required by (LR1). Hence the cubical cut and project sets defined by E', or equivalently by E and window $\mathcal{W}_{\rho'}$, are not LR.

In looser but more geometric terms, this example works as follows. The canonical window in F_ρ is a hexagon, with two opposite sides coordinate aligned with e_4, two aligned with e_5 and a final two 'diagonal' ones, translates of an arc from 0 to $(1, \beta)$. In defining the acceptance domains for the canonical window, new vertical cuts are added by varying just the first two coordinates of $\mathcal{L}_1$, since $L_1(0, 0, 1) = 1$. New horizontal cuts arise from varying the third coordinate of $\mathcal{L}_2$. However, in the canonical window, whilst asymptotically r horizontal cuts are made in defining acceptance domains for r-patches as before, around r^3 diagonal cuts are made, making the complexity too high for the canonical cut and project sets to be LR. This is recognised by passing to suitable cubical cut and project scheme where the counterparts of these horizontal and diagonal cuts are also made in defining the acceptance domains for r-patches.

The approach above yields examples of LR cubical cut and project sets whose corresponding canonical ones are not LR. However, in this approach examples are manufactured so that, whilst the cubical cut and project sets in the standard direction F_ρ are LR, there is reparametrization of the scheme so that the resulting cubical cut and project sets are no longer LR. This naturally leads us to the following:

Problem 4.4. Is it true that a canonical cut and project set will be LR if and only if all of the cubical cut and project sets obtained from taking different parametrizations of E, with respect to different orderings of the standard basis vectors, are also LR?

At the moment we do not know the answer to this question. However, if the answer is yes, it means that theorem 1.1 gives a complete characterization of all canonical as well as cubical cut and project sets. Even if the answer is no, the condition we have established—that a canonical cut and project set can only be LR if each of linear forms defining E satisfies (LR1) and (LR2) for every parametrization—is highly restrictive. We leave this as an open problem for future research.

Interestingly, it appears that there could be a different type of behavior which can cause canonical cut and project sets to fail to be LR, demonstrated in the proof below. This behavior is related to Diophantine approximation, and occurs because of the fact that two subspaces defined by relatively badly approximable linear forms can still intersect in a subspace which is not definable using relatively badly approximable forms. A one dimensional realization of

this fact is the famous theorem of Marshall Hall [19, theorem 3.2], which implies that every non-zero real number can be written as a product of two badly approximable numbers. This provides the basis for the following example with $k = 4$ and $d = 2$, the smallest possible choices of k and d for which 'cubical LR but canonical not' can occur.

Lemma 4.5. *Suppose that α and β are positive badly approximable real numbers with the property that*

$$\inf_{n\in\mathbb{N}} n \cdot \left\{\frac{5\alpha\beta}{2}n\right\} = 0. \tag{4.2}$$

If E is the two dimensional subspace of $\mathbb{R}^4$ defined by

$$E = \{(x, -(2/5)x_1 - \alpha x_2, -\beta(x_1 + (5/2)x_2)) : x \in \mathbb{R}^2\},$$

then every cubical cut and project set defined using E is LR, but no canonical cut and project sets defined using E are LR.

Proof. First of all we remark that, by a general version of Khintchine's theorem (see [30, theorem 1]), almost every real number γ has the property that

$$\inf_{n\in\mathbb{N}} n\{n\gamma\} = 0.$$

For such a γ, it follows from Hall's theorem that there are badly approximable α and β satisfying $5\alpha\beta/2 = \gamma$, and therefore (assuming $\gamma > 0$) the hypotheses of the lemma.

It is easy to see that E is totally irrational and that cubical and canonical cut and project sets formed using E will be aperiodic. By theorem 1.1, cubical cut and project sets formed using E will be LR.

The canonical window in F_ρ is an octagon which includes, on its boundary, the line segment from e_4 to $(2/5)e_3 + (1 + \beta)e_4$. Each integer $n \in \mathbb{Z}^4$ acts on this line segment, moving it to a line segment which we denote by ℓ_n. The initial point of ℓ_n is the point

$$((2/5)n_1 + \alpha n_2 + n_3, \beta n_1 + (5\beta/2)n_2 + 1 + n_4)$$

in the e_3e_4-plane. For any choice of n_2, n_3, and n_4, there is a unique choice of n_1 with the property that ℓ_n intersects the line $e_3 + \mathbb{R}e_4$, and it is clear that $|n_1|$ is bounded above by a constant (depending at most on α and β) times the maximum of $|n_2|, |n_3|$, and $|n_4|$. The intersection point just described is $e_3 + ye_4$, where $y = y(n_2, n_3, n_4)$ is given by

$$y = \frac{-5\alpha\beta}{2}n_2 + 1 + n_4 + \frac{5\beta}{2}(1 + n_2 - n_3).$$

Since (4.2) is satisfied, for any $\epsilon > 0$ there is a number $r > 1$ and integers n_2 and n_4 with $|n_2|, |n_4| \leqslant r$, such that

$$\left|\frac{5\alpha\beta}{2}n_2 - (1 + n_4) + 1\right| < \frac{\epsilon}{r}.$$

For such a choice of n_2 and n_4, and with n_1 selected as above, we take $n_3 = 1 + n_2$. Then the line segment ℓ_n, together with the lines $e_4 + \mathbb{R}e_3$ and $e_3 + \mathbb{R}e_4$, bound a triangle of area $\ll \epsilon r^{-2}$. Such a triangle bounds an acceptance domain for a patch of radius $\gg r$ of the canonical cut and project set, using analogous arguments to the proof of lemma 2.5 (see also [22]).

Since this means that the frequency of such a patch is $\ll \epsilon r^{-2}$ and ϵ can be taken arbitrarily small, the resulting cut and project set cannot be LR. □

5. Hausdorff dimension results

In this section we will prove corollary 2.1. Our proof of theorem 1.1 demonstrates how, to each LR cubical cut and project set, we may associate a subgroup $\Lambda \leqslant \mathbb{Z}^d$ of finite index, with decomposition

$$\Lambda = \Lambda_1 + \cdots + \Lambda_{k-d},$$

so that each L_i is badly approximable, when viewed as a linear form in m_i variables, restricted to Λ_i. The first part of corollary 2.1 clearly follows from the fact that the integers $m_i \geqslant 1$ have sum equal to d.

In the other direction, suppose that $d \geqslant k-d$. If we start with $k-d$ positive integers m_i, with sum equal to d, and a collection of badly approximable linear forms $L_i : \mathbb{R}^{m_i} \to \mathbb{R}$ then, thinking of

$$\mathbb{R}^d = \mathbb{R}^{m_1} + \cdots + \mathbb{R}^{m_{k-d}},$$

any cubical cut and project set arising from the subspace

$$E = \{(x, L_1(x), \ldots, L_{k-d}(x)) : x \in \mathbb{R}^d\}$$

is LR, by the proof of theorem 1.1. It follows that the collection of $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$ which define LR cubical cut and project sets is a countable union (over all allowable choices of Λ_i above) of sets of Lebesgue measure 0 and Hausdorff dimension at most

$$\dim \mathcal{B}_{m_1,1} + \cdots + \dim \mathcal{B}_{m_{k-d},1} = m_1 + \cdots + m_{k-d} = d.$$

Since the cubical cut and project sets corresponding to $\Lambda_i = \mathbb{Z}^{m_i}$ are all LR, the Hausdorff dimension of this set is equal to d.

The part of corollary 2.1 about canonical cut and project sets follows from the same arguments just given, together with proposition 4.2.

6. Examples

6.1. Explicit examples for all $d \geqslant k-d$

For $d \geqslant k/2$ it is easy to give examples of subspaces E satisfying the hypotheses of theorem 1.1. Write $d = m_1 + \cdots + m_{k-d}$, with positive integers m_i, and for each i let K_i be an algebraic number field, of degree $m_i + 1$ over $\mathbb{Q}$. Suppose that the numbers $1, \alpha_{i1}, \ldots, \alpha_{im_i}$ form a $\mathbb{Q}$-basis for K_i, and define $L_i : \mathbb{R}^{m_i} \to \mathbb{R}$ to be the linear form with coefficients $\alpha_{i1}, \ldots, \alpha_{im_i}$. Then, using the decomposition $\mathbb{R}^d = \mathbb{R}^{m_1} + \cdots + \mathbb{R}^{m_{k-d}}$, let

$$E = \{(x, L_1(x), \ldots, L_{k-d}(x)) : x \in \mathbb{R}^d\}.$$

The collection of points

$$\{(L_1(n), \ldots, L_{k-d}(n)) : n \in \mathbb{Z}^d\}$$

is dense in $\mathbb{R}^{k-d}/\mathbb{Z}^{k-d}$, and it follows from this that the subspace E is totally irrational. The intersection of the kernels of the corresponding maps $\mathcal{L}_i$ is $\{0\}$, so any cubical cut and project set formed from E will be aperiodic. Condition (LR1) of theorem 1.1 is clearly satisfied.

Furthermore, by a result of Perron [28], each of the linear forms L_i is badly approximable. Therefore (LR2) is also satisfied, and any cubical cut and project set formed from E is LR. Furthermore, the hypotheses in the second part of proposition 4.2 are also satisfied, so any canonical cut and project set formed using E is also LR.

6.2. Ammann–Beenker tilings

There are many situations in practice where one would like to deal with specific examples of cut and project sets arising from canonical windows. Our main theorem, as stated, does not appear to apply in this important setting. However, we would like to indicate how the ideas in the proof can be easily adapted to prove that many common examples of canonical cut and project sets in the literature are also LR. In very heuristic terms, in order for our proof of theorem 1.1 to carry through, two observations are important: Firstly, that the connected components of $\mathrm{reg}(r)$ are large, and secondly, that the collection of points $\mathcal{L}(\mathcal{Z}_r)$ is dense, relative to how large the connected components are. After these two key properties have been established, it is possible to appeal to the proof of [20, lemma 3.1] and deduce LR. We will revisit these key ideas below, where we sketch how they can be applied to Ammann–Beenker and Penrose tilings.

In this subsection we will consider the cut and project set which gives rise to the Ammann–Beenker tiling, which is one of the simplest non-trivial examples that illustrates the types of issues which can arise. In the next subsection we consider a 5 to 2 cut and project set corresponding to a Penrose tiling, with the purpose of demonstrating the changes needed in order to accommodate a non-totally irrational physical space. The fact that both of these point sets are LR is not a new result, but our purpose is to indicate how our machinery (which applies in many situations in which previously used techniques do not) can be used to prove this, in a setting which is hopefully familiar to the reader. For illustrations of these sets and the window which defines Ammann–Beenker tiling see [5, figures 6.41 and 6.44] and [22, figure 4].

Collections of vertices of Ammann–Beenker tilings can be obtained as canonical cut and project sets, using the two dimensional subspace E of $\mathbb{R}^2$ defined by

$$E = \{(x, L_1(x), L_2(x)) : x \in \mathbb{R}^2\},$$

with

$$L_1(x) = \frac{\sqrt{2}}{2}(x_1 + x_2) \quad \text{and} \quad L_2(x) = \frac{\sqrt{2}}{2}(x_1 - x_2).$$

Although we cannot directly appeal to either theorem 1.1 or proposition 4.2, we will explain how the machinery we have developed can be used to easily show that these sets are LR. The canonical window $\mathcal{W}$ in F_ρ is the regular octagon with vertices at

$$\left(\frac{1+\sqrt{2}}{2} \pm \frac{1+\sqrt{2}}{2}, \frac{1}{2} \pm \frac{1}{2}\right) \text{ and } \left(\frac{1+\sqrt{2}}{2} \pm \frac{1}{2}, \frac{1}{2} \pm \frac{1+\sqrt{2}}{2}\right).$$

By the proof of [20, lemma 3.1], every patch of size r corresponds (in the sense of the statement of lemma 2.5) to a finite collection of connected components of $\mathrm{reg}(r)$. Therefore to demonstrate that a canonical cut and project set formed using E is LR, it is enough to show that the there is a constant $C > 0$ with the property that, for all sufficiently large r, the orbit of any regular point $w \in F_\rho$, under the action of the collection of integers

$$\rho^{-1}(Cr\Omega) \cap \mathbb{Z}^k,$$

intersects every connected component of $\mathrm{reg}(r)$.

We claim that every connected component of $\mathrm{reg}(r)$ contains a square with side length $\gg r^{-1}$. This follows from elementary considerations, by writing down the equations of the line segments that form the boundary of $\mathcal{W}$, considering the action of

$$-\rho^{-1}(r\Omega) \cap \mathbb{Z}^k$$

on these line segments, and then computing all possible intersection points. Since all of our algebraic operations take place in the field $\mathbb{Q}(\sqrt{2})$, it is not difficult to show that every connected component of $\mathrm{reg}(r)$ must contain a right isosceles triangle of side length $\gg r^{-1}$. The claim about squares follows immediately.

Finally, the linear forms L_1 and L_2 are relatively badly approximable, and the sum of the ranks of $\mathcal{L}_1$ and $\mathcal{L}_2$ is equal to 2. Therefore our study of the orbits of points towards the end of the proof of theorem 1.1 applies as before, allowing us to conclude that the Cr-orbit of any regular point in F_ρ intersects every connected component of $\mathrm{reg}(r)$.

6.3. Penrose tilings

This example is similar to the previous one, but it also gives an indication of how to apply our techniques in cases when the physical space is not totally irrational. Let $\zeta = \exp(2\pi i/5)$ and let Y be a canonical cut and project set defined using the two dimensional subspace E of $\mathbb{R}^5$ generated by the vectors

$$(1, \mathrm{Re}(\zeta), \mathrm{Re}(\zeta^2), \mathrm{Re}(\zeta^3), \mathrm{Re}(\zeta^4))$$

and

$$(0, \mathrm{Im}(\zeta), \mathrm{Im}(\zeta^2), \mathrm{Im}(\zeta^3), \mathrm{Im}(\zeta^4)).$$

Well known results of de Bruijn [12] and Robinson [29] show that the set Y is the image under a linear transformation of the collection of vertices of a Penrose tiling, and in fact that all Penrose tilings can be obtained in a similar way from cut and project sets. The fact that Y is LR can be deduced directly from the definition of the Penrose tiling as a primitive substitution. However, as in the previous example, we will indicate how to prove this starting from the definition of Y as a cut and project set.

The subspace E is contained in the rational subspace orthogonal to $(1, 1, 1, 1, 1)$. In this case theorem 1.1 does not apply directly, but the proof in section 3 is still robust enough to allow us to draw the desired conclusions. Set

$$\alpha_1 = \cos(2\pi/5), \alpha_2 = \cos(4\pi/5), \beta_1 = \sin(2\pi/5), \text{and} \beta_2 = \sin(4\pi/5),$$

so that

$$E = \{(x, x\alpha_1 + y\beta_1, x\alpha_2 + y\beta_2, x\alpha_2 - y\beta_2, x\alpha_1 - y\beta_1) : x, y \in \mathbb{R}\}.$$

After making the change of variables $x_1 = x$ and $x_2 = x\alpha_1 + y\beta_1$, we can write E as

$$E = \{(x, L_1(x), L_2(x), L_3(x)) : x = (x_1, x_2) \in \mathbb{R}^2\}.$$

The functions L_i are linear forms which (using the fact that $4\alpha_1^2 + 2\alpha_1 - 1 = 0$) are given by

$$L_1(x) = -x_1 + 2\alpha_1 x_2,$$
$$L_2(x) = -2\alpha_1 x_1 - 2\alpha_1 x_2, \text{ and}$$
$$L_3(x) = 2\alpha_1 x_1 - x_2.$$

Write $\mathcal{L}_i : \mathbb{Z}^2 \to \mathbb{R}/\mathbb{Z}$ for the restriction of L_i to $\mathbb{Z}^2$, modulo 1, and notice that $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 = 0$. This means that the orbit of 0 under the natural $\mathbb{Z}^2$-action of E on $F_\rho/\mathbb{Z}^3$ is contained in the two dimensional rational subtorus with equation $x + y + z = 0$. The kernels of the forms $\mathcal{L}_i$ are all rank 1 subgroups of $\mathbb{Z}^2$, and it follows that the number of connected components of $\mathrm{reg}(r)$ which intersect the rational subtorus is $\asymp r^2$.

Since the forms are linearly dependent, we can understand the orbit of a point in $F_\rho/\mathbb{Z}^3$ under the $\mathbb{Z}^2$-action by considering only the values of $\mathcal{L}_1$ and $\mathcal{L}_3$. In other words, we can consider the projection of the problem onto the e_1e_3-plane. Consider the intersection of a connected component of $\mathrm{reg}(r)$ with the subspace $x + y + z = 0$. This is a two dimensional region, bounded by the intersections of the subspace with translates (by the $\mathbb{Z}^5$ action) of the hyperplanes forming the boundary of the canonical window. Computing the vertices of the region is an operation which takes place in $\mathbb{Q}(\sqrt{5})$. As in the previous example, this leads to the conclusion that the intersection of any connected component of $\mathrm{reg}(r)$ with the subspace $x + y + z = 0$, when projected to the e_1e_3-plane, contains a square of side length $\gg r^{-1}$. The remainder of the proof follows exactly as before, allowing us to conclude that Y is LR.

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