Noncooperative Oligopoly in Markets with a Continuum of Traders and a Strongly Connected Set of Commodities*

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Abstract

We show the existence of a Cournot-Nash equilibrium for a mixed version of the Shapley window model, where large traders are represented as atoms and small traders are represented by an atomless part. Previous existence theorems for the Shapley window model, provided by Sahi and Yao (1989) in the case of economies with a finite number of traders and by Busetto, Codognato, and Ghosal (2011) in the case of mixed exchange economies, are essentially based on the assumption that there are at least two atoms with strictly positive endowments and indifference curves contained in the strict interior of the commodity space. Our result does not require this restriction. It relies on the characteristics of the atomless part of the economy and exploits the fact that traders belonging to the atomless part have an endogenous "Walrasian" behavior.

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1 Introduction

Busetto, Codognato, and Ghosal (2011) proved the existence of a Cournot-Nash equilibrium for the Shapley window model in mixed exchange economies à la Shitovitz, i.e., in exchange economies with large traders, represented as atoms, and small traders, represented by an atomless part (see Shitovitz (1973)).[1] The Shapley window model belongs to a very fruitful line of research on noncooperative market games, initiated by Lloyd S. Shapley and Martin Shubik (for a survey of this literature, see Giraud (2003)). It was proposed informally by Shapley and subsequently formalized by Sahi and Yao (1989) in the case of exchange economies with a finite number of traders. For this case, they proved the existence of a Cournot-Nash equilibrium.

Codognato and Ghosal (2000) studied the Shapley window model in the case of exchange economies with an atomless continuum of traders and they proved an equivalence theorem à la Aumann between the set of Cournot-Nash and Walras allocations (see Aumann (1964)).[2] This result, together with the existence theorem of a Walras equilibrium in markets with a continuum of traders proved in Aumann (1966), implies the existence of a Cournot-Nash equilibrium also in this limit framework.

The proof provided by Busetto et al. (2011) for the mixed market case is based on the same assumptions as the proof provided by Sahi and Yao (1989) for the finite case. In particular, it requires that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space. These restrictions are stated by Busetto et al. (2011) in their Assumption 4.

Codognato and Julien (2013) replaced this assumption on atoms' endowments and preferences with a different restriction requiring that the atomless part holds, in the aggregate, each commodity and that preferences of the traders belonging to the atomless part are represented by Cobb-Douglas utility functions. Under these assumptions, they showed the existence of

[1]Mixed exchange economies were systematically analyzed, in the line opened by Shitovitz (1973), using the core as a solution concept (for a survey of this literature, see Gabszewicz and Shitovitz (1992)). Nevertheless, the idea of mixing large players and small players in a game theoretical framework was first introduced by John W. Milnor and Lloyd S. Shapley in two Rand research memoranda written in the early 1960s, then merged into a single article by Milnor and Shapley (1978).

[2]Codognato and Ghosal (2000) actually extended to the Shapley window model some results connecting Cournot-Nash and Walras allocations in strategic market games with an atomless continuum of traders previously proved by Dubey and Shapley (1994).

a Cobb-Douglas-Cournot-Nash equilibrium for the Shapley window model, i.e., a Cournot-Nash equilibrium where the strategies of the traders belonging to the atomless part depend on the parameters of their Cobb-Douglas utility functions.

In this paper, we develop Codognato and Julien's idea in a more general context: Our main result consists in an existence proof for the mixed version of the Shapley window model proposed by Busetto et al. (2011) which is essentially based on restrictions on endowments and preferences of the atomless part of the economy rather than of atoms. In particular, we remove their Assumption 4 and we use the fact, proved by Codognato and Ghosal (2000), that traders belonging to the atomless part have an endogenous "Walrasian" behavior: Their best reply attains indeed a commodity bundle which maximizes their utility subject to their budget constraint at the prevailing market clearing prices, which they are not able to manipulate.

More precisely, we exploit this property of the atomless part's behavior to show a preliminary price convergence theorem, under the assumption that each commodity is held, in the aggregate, by the atomless part and that traders' utility functions are continuous, strongly monotone, quasi-concave, and measurable. This result establishes that any sequence of prices corresponding to a sequence of Cournot-Nash equilibria has a subsequence which converges to a strictly positive price vector and it has an autonomous relevance since it can be employed to show existence theorems for mixed exchange economies under different sets of assumptions, as argued in the paper.

We use it to prove our main existence theorem under the assumption that the set of commodities is strongly connected through traders' characteristics. It imposes a joint restriction on the endowments and preferences of the atomless part and is a variant of a hypothesis proposed by Codognato and Ghosal (2000). This assumption, combined with the continuity properties of the Walrasian correspondence generated by the atomless part's behavior, guarantees that the aggregate matrix of the bids obtained as the limit of a sequence of perturbed Cournot-Nash equilibria is irreducible.

Finally, we show that our price convergence theorem can be used to prove a further existence result, which differs both from our main theorem and the one proposed by Busetto et al. (2011). Like this latter theorem, it requires that all commodities are held by at least two atoms and, in the aggregate, by the atomless part but, in contrast with it, it does not require any further condition on traders' utility functions beyond continuity, strong monotonicity, quasi-concavity, and measurability.

The paper is organized as follows. In Section 2, we introduce the mathematical model. In Section 3, we state and prove the general price convergence theorem. In Section 4, we state and prove our main existence theorem. In Section 5, we discuss the model and we provide a further existence theorem. In Section 6, we draw some conclusions from our analysis.

2 The mathematical model

We consider an exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < +\infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and $T_0 = T \setminus T_1$ the atomless part of T. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are l different commodities. A commodity bundle is a point in R^l_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^l_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}(t) > 0$, *for each* $t \in T$, $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^l_+ \to R$, satisfying the following assumptions.

Assumption 2. $u_t : R^l_+ \to R$ *is continuous, strongly monotone, and quasi-concave, for each* $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of R^l_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by all the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times R^l_+ \to R$, *given by* $u(t, x) = u_t(x)$, *for each* $t \in T$ *and for each* $x \in R^l_+$, *is* $\mathcal{T} \bigotimes \mathcal{B}$*-measurable.*

We finally impose an assumption on endowments and preferences of the atomless part, which is a reformulation of a hypothesis introduced by Codognato and Ghosal (2000). It requires that the set of commodities is strongly connected through traders' characteristics.[3] In order to formalize this feature, we need some preliminary definitions: We denote by L the set of commodities $\{1, \ldots, l\}$. We say that two commodities $i, j \in L$ stand in relation C if there is a measurable set T^i, with $\mu(T^i) > 0$, such that $T^i = \{t \in T_0 : \mathbf{w}^i(t) > 0, \mathbf{w}^r(t) = 0,$ for each $r \in L \setminus \{i\}\}$, $u_t(\cdot)$ is differentiable, additively separable in commodity j, i.e., $u_t(x) = v_t^j(x^j) + v_t(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^l)$, for each $x \in R^l_+$, and $\frac{dv_t^j(0)}{dx^j} = +\infty$, for each $t \in T^i$.[4]

Then, the concept of a set of commodities strongly connected through traders' characteristics can be defined as follows.

Definition 1. *The set of commodities L is said to be strongly connected through traders' characteristics if $\{(i, j) : iCj\} \neq \emptyset$ and the directed graph $D_L(L, C)$ is strongly connected, i.e., any ordered pair of distinct vertices, i and j, of $D_L(L, C)$ is connected by a path.*

We can now state our last assumption.

Assumption 4. *The set of commodities L is strongly connected through traders' characteristics.*

A price vector is a nonnull vector $p \in R^l_+$. Let $\mathbf{X}^0 : T_0 \times R^l_{++} \to \mathcal{P}(R^l)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R^l_{++}$, $\mathbf{X}^0(t, p) = \operatorname{argmax}\{u(x) : x \in R^l_+ \text{ and } px \leq p\mathbf{w}(t)\}$. It is well-known that the previous assumptions guarantee that the correspondence $\mathbf{X}^0(t, \cdot)$ is upper hemicontinuous, for each $t \in T_0$. For each $p \in R^l_{++}$, let $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \{\int_{T_0} \mathbf{x}^0(t, p)\, d\mu : \mathbf{x}^0(\cdot, p)$ is integrable and $\mathbf{x}^0(t, p) \in \mathbf{X}^0(t, p)$, for each $t \in T_0\}$. Finally, let $\mathbf{Z}^0 : R^l_{++} \to \mathcal{P}(R^l)$ be a correspondence which associates with each $p \in R^l_{++}$ the Minkowski difference between the set $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu$ and the set $\{\int_{T_0} \mathbf{w}(t)\, d\mu\}$.[5]

We define now the strategic market game associated with the exchange economy described above. It is a slightly reformulated version of the Shapley window model for mixed economies proposed by Busetto et al. (2011).

[3]Codognato and Ghosal (2000) used the concept of a "net" to characterize the set of commodities in a similar way. We refer to them for further details.

[4]In this definition, differentiability is to be understood as continuous differentiability and it includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58). Moreover, it can be proved that the separable utility function used in the definition is the representation of separable preferences (see, for instance, Kreps (2012), p. 42).

[5]For a discussion of the properties of the correspondences introduced above and their proofs see, for instance, Debreu (1982), Section 4.

A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R_+^{l^2})$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R_+^{l^2} : \sum_{j=1}^{l} b_{ij} \leq \mathbf{w}^i(t),\ i = 1, \ldots, l\}$. With some abuse of notation, we denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t, where $b_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy selection is an integrable function $\mathbf{b} : T \to R_+^{l^2}$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}$ to be the matrix such that $\bar{\mathbf{b}}_{ij} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$, $i, j = 1, \ldots, l$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$, and by $\bar{\mathbf{B}} \setminus b(t)$ the corresponding aggregate matrix.

The following definitions are borrowed from Sahi and Yao (1989).

Definition 2. *A nonnegative square matrix A is said to be irreducible if, for every pair (i, j), with $i \neq j$, there is a positive integer k such that $a_{ij}^{(k)} > 0$, where $a_{ij}^{(k)}$ denotes the ij-th entry of the k-th power A^k of A.*

Definition 3. *Given a strategy selection* $\mathbf{b}$*, a price vector p is said to be market clearing if*

$$p \in R_{++}^l,\ \sum_{i=1}^{l} p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^{l} \bar{\mathbf{b}}_{ji}),\ j = 1, \ldots, l. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique, up to a scalar multiple, price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise. We can assume that prices are normalized in such a way that $p(\mathbf{b}) \in \Delta$, where $\Delta = \{p \in R_+^l : \sum_{i=1}^{l} p^i = 1\}$, when $p(\mathbf{b}) \gg 0$.

Given a strategy selection $\mathbf{b}$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{b}_{ji}(t) + \sum_{i=1}^{l} \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in R_{++}^l, \\
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise},
\end{aligned}$$

$j = 1, \ldots, l$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, the traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$.[6] It is straightforward to show that this assignment is an allocation.

We are now able to introduce a notion of Cournot-Nash equilibrium for this reformulation of the Shapley window model (see Codognato and Ghosal (2000) and Busetto et al. (2011)).

Definition 4. *A strategy selection $\hat{\mathbf{b}}$ such that $\hat{\bar{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.[7]

Finally, we define the notion of a perturbation of the strategic market game (it was already used by Sahi and Yao (1989) and Busetto et al. (2011) in their existence proofs).

Given $\epsilon > 0$ and a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}_\epsilon$ to be the matrix such that $\bar{\mathbf{b}}_{\epsilon ij} = (\bar{\mathbf{b}}_{ij} + \epsilon)$, $i, j = 1, \ldots, l$. Clearly, the matrix $\bar{\mathbf{B}}_\epsilon$ is irreducible. The interpretation is that an outside agency places fixed bids of ϵ for each pair of commodities (i, j).

Given $\epsilon > 0$, we denote by $p^\epsilon(\mathbf{b})$ the function which associates, with each strategy selection $\mathbf{b}$, the unique, up to a scalar multiple, price vector satisfying

$$\sum_{i=1}^{l} p^i(\bar{\mathbf{b}}_{ij} + \epsilon) = p^j(\sum_{i=1}^{l}(\bar{\mathbf{b}}_{ji} + \epsilon)), \; j = 1, \ldots, l. \quad (2)$$

As already said, prices belong to the unit simplex.

Definition 5. *Given $\epsilon > 0$, a strategy selection $\hat{\mathbf{b}}^\epsilon$ is an ϵ-Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}^\epsilon(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon))) \geq u_t(\mathbf{x}(t, b(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.

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[6]In order to save in notation, with some abuse we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b}))$.

[7]Let us notice that, as this definition of a Cournot-Nash equilibrium explicitly refers to irreducible matrices, it applies only to active equilibria (on this point, see Sahi and Yao (1989)).

3 The price convergence theorem

In order to prove their existence theorem, Busetto et al. (2011) used a result, proved as Lemma 9 in Sahi and Yao (1989), which states that there exists a constant $\eta > 0$ such that $p^{\epsilon j}(\hat{\mathbf{b}}^{\epsilon}) \geq \eta$, $j = 1, \ldots, l$, for each strategy selection $\hat{\mathbf{b}}^{\epsilon}$ with $\epsilon \leq 1$. By applying this result, Busetto et al. (2011) showed that any convergent sequence of normalized prices corresponding to a sequence of ϵ-Cournot-Nash equilibria has a convergent subsequence whose limit is a strictly positive price vector. Sahi and Yao's Lemma 9, and consequently Busetto et al.'s convergence result, are essentially based on the assumption that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space.[8] This restriction is stated by Busetto et al. (2011) in their Assumption 4.

In this section, we provide a different price convergence theorem, obtained by removing Busetto et al.'s Assumption 4 and focusing on restrictions concerning endowments and preferences of the atomless part of the economy rather than of atoms. More precisely, we exploit the property of small traders, proved by Codognato and Ghosal (2000), of being "Walrasian" at a Cournot-Nash equilibrium. Our price convergence theorem establishes that any sequence of normalized prices corresponding to a sequence of Cournot-Nash equilibria has a convergent subsequence whose limit is a strictly positive price vector. We use it to show our main existence theorem, but it can be more generally employed to show other existence theorems for mixed exchange economies where Busetto et al.'s Assumption 4 is relaxed. We will give an example in Section 5 by means of the existence result stated in Theorem 3.

We can now state and prove this general price convergence result.

Theorem 1. *Under Assumptions 1, 2, and 3, let* $\{\hat{p}^n\}$ *be a sequence of normalized prices such that* $\{\hat{p}^n\} = p(\hat{\mathbf{b}}^n)$ *where* $\hat{\mathbf{b}}^n$ *is a Cournot-Nash equilibrium, for each* $n = 1, 2, \ldots$. *Then, there exists a subsequence* $\{\hat{p}^{k_n}\}$ *of the sequence* $\{\hat{p}^n\}$ *which converges to a price vector* $\hat{p} \gg 0$.

Proof. Let $\{\hat{p}^n\}$ be a sequence of normalized prices such that $\{\hat{p}^n\} = p(\hat{\mathbf{b}}^n)$ where $\hat{\mathbf{b}}^n$ is a Cournot-Nash equilibrium, for each $n = 1, 2, \ldots$. Then, there is a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to a price

[8] Formally, this assumption requires that there are at least two traders in T_1 for whom $\mathbf{w}(t) \gg 0$, $u_t(\cdot)$ is continuously differentiable in R^l_{++}, and $\{x \in R^l_+ : u_t(x) = u_t(\mathbf{w}(t))\} \subset R^l_{++}$.

vector $\hat{p} \in \Delta$ as the unit simplex Δ is a compact set. Suppose that $\hat{p} \in \partial\Delta$, where $\partial\Delta$ denotes the boundary of the unit simplex. Following Debreu (1982), let $|x| = \sum_{i=1}^{l} |x^i|$, for each $x \in R^l$, and let $d[0, S] = \inf_{x \in S} |x|$, for each $S \subset R^l$. Then, the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{k_n})]\}$ diverges to $+\infty$ since $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$ and $\hat{p} \in \partial\Delta$, as pointed out by Debreu (1982) in Property (iv).[9] Let $\hat{\mathbf{x}}^n(t) = \mathbf{x}(t, \hat{\mathbf{b}}^n(t), p(\hat{\mathbf{b}}^n))$, for each $t \in T$, and for each $n = 1, 2, \ldots$. Then, $\hat{\mathbf{x}}^n(t) \in \mathbf{X}^0(t, \hat{p}^n)$, for each $t \in T_0$, and for each $n = 1, 2, \ldots$, by the same argument used by Codognato and Ghosal (2000) to prove their Theorem 2.[10] But then, $(\int_{T_0} \hat{\mathbf{x}}^n(t)\, d\mu - \int_{T_0} \mathbf{w}(t)\, d\mu) \in \mathbf{Z}^0(\hat{p}^n)$, for each $n = 1, 2, \ldots$. We have that

$$\int_{T_0} \hat{\mathbf{x}}^n(t)\, d\mu \leq \int_{T_0} \mathbf{w}(t)\, d\mu + \int_{T_1} \mathbf{w}(t)\, d\mu$$

as $\int_T \hat{\mathbf{x}}^n(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$, for each $n = 1, 2, \ldots$. Then,

$$\left| \int_{T_0} \hat{\mathbf{x}}^{in}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu$$

as $-\int_{T_1} \mathbf{w}^i(t)\, d\mu \leq \int_{T_0} \hat{\mathbf{x}}^{in}(t)\, d\mu \leq 2 \int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu$, $i = 1, \ldots, l$, for each $n = 1, 2, \ldots$. But then,

$$\sum_{i=1}^{l} \left| \int_{T_0} \hat{\mathbf{x}}^{in}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu \right),$$

for each $n = 1, 2, \ldots$. Moreover, there exists an n_0 such that

$$d[0, \mathbf{Z}^0(\hat{p}^{k_n})] > \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu \right),$$

for each $n \geq n_0$, as the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{k_n})]\}$ diverges to $+\infty$. Then,

$$\sum_{i=1}^{l} \left| \int_{T_0} \hat{\mathbf{x}}^{ik_n}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| > \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu \right)$$

as $\sum_{i=1}^{l} | \int_{T_0} \hat{\mathbf{x}}^{ik_n}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu | \geq d[0, \mathbf{Z}^0(\hat{p}^{k_n})]$, for each $n \geq n_0$, a contradiction. Hence, $\hat{p} \gg 0$. ■

[9] See Debreu (1982), p. 728.

[10] See Codognato and Ghosal (2000), p. 49.

4 The existence theorem

In this section, we state and prove our main existence theorem for the mixed version of the Shapley window model, which differs from that proved by Busetto et al. (2011) in that it replaces their Assumption 4, on endowments and preferences of atoms, with the assumption that the set of commodities is strongly connected through traders' characteristics, imposing restrictions on endowments and preferences of the atomless part. Our existence result crucially rests on the price convergence theorem proved in the previous section.

Theorem 2. *Under Assumptions 1, 2, 3, and 4, there exists a Cournot-Nash equilibrium* $\hat{\mathbf{b}}$.

Proof. To show Theorem 2, we first need to prove the existence of an ϵ-Cournot-Nash equilibrium. The following lemma, which was proved by Busetto et al. (2011) applying the Kakutani-Fan-Glicksberg theorem, states that such an equilibrium exists.

Lemma 1. *For each* $\epsilon > 0$*, there exists an* ϵ*-Cournot-Nash equilibrium* $\hat{\mathbf{b}}^{\epsilon}$.

Proof. See the proof of Lemma 3 in Busetto et al. (2011). ■

We now show that the sequence of ϵ-Cournot-Nash equilibria has a limit and that this limit is a Cournot-Nash equilibrium. Following Busetto et al. (2011), in this part of the proof, we apply a generalization of the Fatou's lemma in several dimensions provided by Artstein (1979). Let $\epsilon_n = \frac{1}{n}$, $n = 1, 2, \ldots$. By Lemma 1, there is an ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_n}$, for each $n = 1, 2, \ldots$. The fact that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_n}\}$ belongs to the compact set $\{(b_{ij}) \in R^{l^2}_{+} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \ldots, l\}$ and the sequence $\{\hat{p}^{\epsilon_n}\}$, where $\hat{p}^{\epsilon_n} = p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})$, belongs to the unit simplex Δ, for each $n = 1, 2, \ldots$, implies that there are a subsequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ of the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_n}\}$ which converges to an element of the set $\{(b_{ij}) \in R^{l^2}_{+} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \ldots, l\}$, and a subsequence $\{\hat{p}^{\epsilon_{k_n}}\}$ of the sequence $\{\hat{p}^{\epsilon_n}\}$ which converges to a price vector $\hat{p} \in \Delta$, with $\hat{p} \gg 0$, by Theorem 1. Since the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}\}$ satisfies the assumptions of Theorem A in Artstein (1979), by this theorem there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}(t)\}$, for each $t \in T$ and such that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$. Moreover, $\hat{p}$ and $\bar{\hat{\mathbf{B}}}$ satisfy (1), since $\hat{p}^{\epsilon_{k_n}}$ and $\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}_{\epsilon_{k_n}}$ satisfy (2), for each $n = 1, 2, \ldots$, the sequence

$\{\hat{p}^{\epsilon_{k_n}}\}$ converges to $\hat{p}$, the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$, and the sequence $\{\epsilon_{k_n}\}$ converges to 0. We now show that, if two commodities $i, j \in L$ stand in the relation C, then $\bar{\hat{\mathbf{b}}}_{ij} > 0$. Suppose that $\bar{\hat{\mathbf{b}}}_{ij} = 0$. Then, $\int_{T^i} \hat{\mathbf{b}}_{ij}(t)\, d\mu = 0$ as $\mu(T^i) > 0$. Consider a trader $\tau \in T^i$. We can suppose that $\hat{\mathbf{b}}_{ij}(\tau) = 0$ as we ignore null sets. Since $\hat{\mathbf{b}}(\tau)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}(\tau)\}$, there is a subsequence $\{\hat{\mathbf{b}}^{\epsilon_{h_{k_n}}}(\tau)\}$ of this sequence which converges to $\hat{\mathbf{b}}(\tau)$. Let $\hat{\mathbf{x}}^{\epsilon_n}(\tau) = \mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))$, for each $n = 1, 2, \ldots$, and $\hat{\mathbf{x}}(\tau) = \mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), \hat{p})$. Then, the subsequence $\{\hat{\mathbf{x}}^{\epsilon_{h_{k_n}}}(\tau)\}$ of the sequence $\{\hat{\mathbf{x}}^{\epsilon_n}(\tau)\}$ converges to $\hat{\mathbf{x}}(\tau)$ as the sequence $\{\hat{\mathbf{b}}^{\epsilon_{h_{k_n}}}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$ and the sequence $\{\hat{p}^{\epsilon_{h_{k_n}}}\}$ converges to $\hat{p}$, with $\hat{p}^{\epsilon_{h_{k_n}}} \gg 0$, for each $n = 1, 2, \ldots$, and $\hat{p} \gg 0$. But then, $\hat{\mathbf{x}}^j(\tau) = 0$ as $\hat{\mathbf{b}}_{ij}(\tau) = 0$ and $\hat{\mathbf{x}}(\tau) \in \mathbf{X}^0(\tau, \hat{p})$ as $\hat{\mathbf{x}}^{\epsilon_{h_{k_n}}}(\tau) \in \mathbf{X}^0(\tau, \hat{p}^{\epsilon_{h_{k_n}}})$, for each $n = 1, 2, \ldots$, and the correspondence $\mathbf{X}^0(\tau, \cdot)$ is upper hemicontinuous. Therefore, we have that $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^j} = +\infty$ as $i, j \in L$ stand in the relation C and $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^j} \leq \lambda \hat{p}^j$, by the necessary conditions of the Kuhn-Tucker Theorem. Moreover, there must be a commodity h such that $\hat{\mathbf{x}}^h(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $\hat{p}\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^h} = \lambda \hat{p}^h$, by the necessary conditions of the Kuhn-Tucker Theorem. But then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^h} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, if two commodities $i, j \in L$ stand in the relation C, then $\bar{\hat{\mathbf{b}}}_{ij} > 0$. This implies that the matrix $\bar{\hat{\mathbf{B}}}$ is irreducible by our Assumption 4 and by the argument used by Codognato and Ghosal (2000) in the proof of their Theorem 2.[11] Consider a trader $\tau \in T_1$. The matrix $\bar{\hat{\mathbf{B}}} \setminus b(\tau)$ is irreducible as $\bar{\hat{\mathbf{b}}}_{ij} \setminus b(\tau) > 0$ for any pair of commodities $i, j \in L$ which stand in the relation C, by the previous argument. Consider a trader $\tau \in T_0$. The matrix $\bar{\hat{\mathbf{B}}} \setminus b(\tau)$ is irreducible as $\bar{\hat{\mathbf{B}}} = \bar{\hat{\mathbf{B}}} \setminus b(\tau)$. Then, the matrix $\bar{\hat{\mathbf{B}}} \setminus b(t)$ is irreducible, for each $t \in T$. But then, from the same argument used by Busetto et al. (2011) in their existence proof (Cases 1 and 3), it follows that $u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t))))$, for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.[12] Hence, $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium. ■

The role played by Assumption 4 in the proof of Theorem 2 can be further made clear by using the following example, originally provided by Busetto et al. (2011). It considers a mixed economy where only Assumptions 1, 2, and 3 are satisfied, and shows that in this economy a Cournot-Nash equilibrium

[11]See Codognato and Ghosal (2000), p. 50.

[12]See Busetto et al. (2011), p. 43.

may not exist.

Example. *Consider the following exchange economy:* $l = 2$, $T = T_1 \cup T_0$, *where* $T_1 = \{2, 3\}$ *and* $T_0 = [0, 1]$, $\mathbf{w}(2) = (1, 0)$, $\mathbf{w}(3) = (0, 1)$, $u_2(\cdot)$ *and* $u_3(\cdot)$ *satisfy our Assumption 2,* $\mathbf{w}(t) = (1, 0)$, *for each* $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1)$, *for each* $t \in [\frac{1}{2}, 1]$, $u_t(\cdot) = kx^1 + x^2$, *for each* $t \in [0, \frac{1}{2}]$, $u_t(\cdot) = x^1 + kx^2$, *for each* $t \in [\frac{1}{2}, 1]$, $k > 1$. *This exchange economy does not admit any Cournot-Nash equilibrium.*

Proof. See the proof of Example 2 in Busetto et al. (2011). ■

Assumptions 1, 2, and 3 are sufficient to prove Theorem 1, as shown in Section 3. They are also sufficient to prove the existence of a Walras equilibrium in atomless exchange economies, as shown by Debreu (1982).[13] By establishing that they are not sufficient to show the existence of a Cournot-Nash equilibrium for a mixed version of the Shapley window model, the example above emphasizes the fact that, in the proof of Theorem 2, continuity properties of the Walrasian correspondence generated by the atomless part's behavior must be combined with Assumption 4, which requires that the set of commodities is strongly connected through traders' characteristics, in order to assure that the aggregate matrix of the bids obtained as the limit of a sequence of perturbed Cournot-Nash equilibria is irreducible.

5 Discussion of the model

Busetto et al. (2011) showed their existence theorem for the mixed version of Shapley window model under the assumption that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space. Our Theorem 2 provides a different existence proof, which replaces this assumption on atoms' endowments and preferences with other restrictions, on endowments and preferences of the atomless part, expressed by the assumption that the set of commodities is strongly connected through traders' characteristics. The crucial role played in our proof by this assumption has been stressed through the example proposed in the previous section. The other fundamental element in the proof of this result is represented by Theorem 1, which holds without any further assumption beyond Assumptions 1, 2, and 3. We prove now that this price convergence result can

[13]Debreu (1982)'s result is a generalization of the existence theorem of a Walras equilibrium for exchange economies with a continuum of traders proved by Aumann (1966).

be used to show a further existence theorem for a mixed exchange economy: Like the existence result in Busetto et al. (2011), it imposes that all commodities are held by at least two atoms and, in the aggregate, by the atomless sector, but differs from that result since it does not require any further condition on traders' utility functions beyond continuity, strong monotonicity, quasi-concavity, and measurability.

Let us now replace our Assumption 4 with the following.

Assumption 4′. *There are at least two traders in T_1 for whom $\mathbf{w}(t) \gg 0$.*

This assumption is less restrictive than Assumption 4 in Busetto et al. (2011) as it removes the restriction that the two atoms with strictly positive endowments also have continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space.

We now state and prove the new existence theorem.

Theorem 3. *Under Assumptions 1, 2, 3, and 4′, there exists a Cournot-Nash equilibrium $\hat{\mathbf{b}}$.*

Proof. Following Sahi and Yao (1989), we define the notion of a δ-positive ϵ-Cournot-Nash equilibrium. Let $\bar{T}_1 \subset T_1$ be a set consisting of two traders in T_1 for whom Assumption 4′ holds. Moreover, let $\delta = \min_{t \in \bar{T}_1}\{\frac{1}{l}\min\{\mathbf{w}^1(t), \ldots, \mathbf{w}^l(t)\}\}$. We say that the correspondence $\mathbf{B}^\delta : T \to \mathcal{P}(R_+^{l^2})$ is a δ-positive strategy correspondence if $\mathbf{B}^\delta(t) = \mathbf{B}(t) \cap \{(b_{ij}) \in R_+^{l^2} : \sum_{i \notin J}\sum_{j \in J}(b_{ij} + b_{ji}) \geq \delta,\ \text{for each } J \subseteq \{1, \ldots, l\}\}$, for each $t \in \bar{T}_1$, and if $\mathbf{B}^\delta(t) = \mathbf{B}(t)$, for each $t \in T \setminus \bar{T}_1$. Moreover, we say that a strategy selection $\mathbf{b}$ is δ-positive if $\mathbf{b}(t) \in \mathbf{B}^\delta(t)$, for each $t \in T$. Finally, we say that an ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^\epsilon$ is δ-positive if $\hat{\mathbf{b}}^\epsilon$ is a δ-positive strategy selection. The following lemma is a strengthening of Lemma 1.

Lemma 2. *For each $\epsilon > 0$, there exists a δ-positive ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^\epsilon$.*

Proof. See the proof of Lemma 4 in Busetto et al. (2011). ■

We show now that the sequence of δ-positive ϵ-Cournot-Nash equilibria has a limit and that this limit is a δ-positive ϵ-Cournot-Nash equilibrium. Following Busetto et al. (2011), in this part of the proof we apply again the generalization of the Fatou's lemma in several dimensions provided by Artstein (1979). Let $\epsilon_n = \frac{1}{n}$, $n = 1, 2, \ldots$. By Lemma 2, for each $n = 1, 2, \ldots$, there is a δ-positive ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_n}$. The fact that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_n}\}$ belongs to the compact set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq$

$\int_T \mathbf{w}^i(t)d\mu$, $i,j = 1,\ldots,l$, $\sum_{i\notin J}\sum_{j\in J}(b_{ij}+b_{ji}) \geq \int_{\bar{T}_1}\delta\, d\mu$, for each $J \subseteq \{1,\ldots,l\}\}$ and the sequence $\{\hat{p}^{\epsilon_n}\}$, where $\hat{p}^{\epsilon_n} = p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})$, belongs to the unit simplex Δ, for each $n = 1,2,\ldots$, implies that there is a subsequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ of the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_n}\}$ which converges to an element of the set $\{(b_{ij}) \in R^{l^2}_+ : b_{ij} \leq \int_T \mathbf{w}^i(t)d\mu$, $i,j = 1,\ldots,l$, $\sum_{i\notin J}\sum_{j\in J}(b_{ij}+b_{ji}) \geq \int_{\bar{T}_1}\delta\, d\mu$, for each $J \subseteq \{1,\ldots,l\}\}$ and a subsequence $\{\hat{p}^{\epsilon_{k_n}}\}$ of the sequence $\{\hat{p}^{\epsilon_n}\}$ which converges to a price vector $\hat{p} \in \Delta$, with $\hat{p} \gg 0$, by Theorem 1. Since the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}(t)\}$, for each $t \in T$, and such that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$. Moreover, $\hat{p}$ and $\bar{\hat{\mathbf{B}}}$ satisfy (1), since $\hat{p}^{\epsilon_{k_n}}$ and $\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}_{\epsilon_{k_n}}$ satisfy (2), for each $n = 1,2,\ldots$, the sequence $\{\hat{p}^{\epsilon_{k_n}}\}$ converges to $\hat{p}$, the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$, and the sequence $\{\epsilon_{k_n}\}$ converges to 0. Then, the matrix $\bar{\hat{\mathbf{B}}}$ is completely reducible, by Lemma 1 in Sahi and Yao (1989), as $\hat{p} \gg 0$. But then, $\bar{\hat{\mathbf{B}}}$ must be irreducible as $\hat{\mathbf{b}}$ is δ-positive, by Remark 3 in Sahi and Yao (1989). Moreover, the same remark implies that the matrix $\bar{\hat{\mathbf{B}}} \setminus b(t)$ is irreducible, whenever it is completely reducible, for each $t \in T_1$, as there are at least two traders in T_1 for whom $\mathbf{w}(t) \gg 0$, by Assumption $4'$. Then, from the same argument used in the proof of Cases 1, 2, and 3 in Busetto et al. (2011), it follows that $u_t(\mathbf{x}(t,\hat{\mathbf{b}}(t),p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t,b(t),p(\hat{\mathbf{b}}\setminus b(t))))$, for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$. Hence, $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium. ■

6 Conclusion

In this paper, we have reconsidered the issues concerning the existence of a Cournot-Nash equilibrium for mixed exchange economies. In particular, we have shown a general price convergence theorem and an existence theorem for the mixed version of the Shapley window model which are both based on the Walrasian properties of atomless part's behavior. Our Theorem 2 provides an existence result which can be applied to economic structures left uncovered by the existence theorem proved by Busetto et al. (2011): In our theorem, all traders may indeed have corner endowments, and indifference curves which touch the boundary of the consumption set. In particular, it guarantees the existence of a Cournot-Nash equilibrium for the case of bilateral oligopoly with a competitive fringe for each commodity, thereby contributing to the growing literature on this type of economic structure, initiated by Gabszewicz and Michel (1997) and further analyzed by Bloch and Ghosal (1997), Bloch and Ferrer (2001), Dickson and Hartley (2008), Amir and Bloch (2009), among others.

We leave for further research the problem of proving the existence of a Cournot-Nash equilibrium for a bilateral oligopoly configuration without a competitive fringe, a case which violates Assumption 1 of Theorem 2. We have also exhibited the generality of our price convergence theorem, by proving another existence result, Theorem 3, less requiring than the existence theorem proved by Busetto et al. (2011), since it assumes that all commodities are held by at least two atoms and, in the aggregate, by the atomless part without imposing further assumptions on traders' utility functions beyond continuity, strong monotonicity, quasi-concavity, and measurability. Moreover, we conjecture that, under the same assumptions, the price convergence theorem could be used to prove an asymptotic equivalence between Cournot-Nash and Walras equilibria similar to that obtained by Busetto, Codognato, and Ghosal (2017): we also leave this proof for future research.

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