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Resurgence of the dressing phase for $\text{AdS}_5 \times \text{S}^5$

Gleb Arutyunov,^{a,b,1} Daniele Dorigoni^c and Sergei Savin^{a,b}

^a*Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany*

^b*Zentrum für Mathematische Physik, Universität Hamburg,
Bundesstrasse 55, 20146 Hamburg, Germany*

^c*Centre for Particle Theory & Department of Mathematical Sciences, Durham University,
Lower Mountjoy Stockton Road, Durham DH1 3LE, U.K.*

E-mail: gleb.arutyunov@desy.de, daniele.dorigoni@gmail.com,
sergei.savin@desy.de

ABSTRACT: We discuss the resummation of the strong coupling asymptotic expansion of the dressing phase of the $\text{AdS}_5 \times \text{S}^5$ superstring. The dressing phase proposed by Beisert, Eden and Staudacher can be recovered from a modified Borel-Ecalle resummation of this asymptotic expansion only by completing it with new, non-perturbative and exponentially suppressed terms that can be organized into different sectors labelled by an instanton-like number. We compute the contribution to the dressing phase coming from the sum over all the instanton sectors and show that it satisfies the homogeneous crossing symmetry equation. We comment on the semiclassical origin of the non-perturbative terms from the world-sheet theory point of view even though their precise explanation remains still quite mysterious.

KEYWORDS: Nonperturbative Effects, Integrable Field Theories, AdS-CFT Correspondence

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¹Correspondent fellow at Steklov Mathematical Institute, Moscow.

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1 Introduction

In recent years a lot of progress has been achieved in the spectral problem of the gauge-string correspondence by using ideas and methods from the theory of integrable models [1, 2]. For strings on $\text{AdS}_5 \times \text{S}^5$ the corresponding light-cone sigma model is quantum integrable which allows one to obtain its spectrum by means of Thermodynamic Bethe

Ansatz (TBA) [3]–[7] or the modern incarnation of the latter known as Quantum Spectral Curve [8].

We recall that the construction of the TBA is essentially based on the asymptotic S-matrix for scattering of string world-sheet excitations in the uniform light-cone gauge. This S-matrix is determined by symmetries of the $\text{AdS}_5 \times \text{S}^5$ light-cone sigma model up to an overall scalar factor called the dressing factor [9]. Thus, determination of the latter quantity and investigation of its properties constitutes an important part of the spectral problem to which many studies has been devoted in the recent past. In this paper we will undertake an effort to complete the existing considerations and clarify some issues related to a perturbative expansion of the dressing factor (phase) at strong coupling.

Before we pass to the discussion of our approach, we briefly recall what is known about the dressing factor $\sigma = e^{i\theta}$, where θ is the dressing phase. The functional form of σ as a perturbative power series in the inverse string tension g with coefficients written in terms of local conserved charges was conjectured in [9] by discretising equations that encode finite-gap solutions of the classical string sigma model. Since g is related to the 't Hooft coupling λ as $g = \sqrt{\lambda}/2\pi$, from the point of view of gauge theory this power series represents a strong coupling expansion of σ . Further, the asymptotic S-matrix appears to be compatible with crossing symmetry which implies a non-trivial functional equation for the dressing factor — the crossing equation [10]. The found leading (AFS) [9] and sub-leading (HL) [11] terms in the strong coupling expansion of σ were shown to satisfy the crossing equation [12] and an all-order asymptotic solution of the latter was obtained in [13]. The weak coupling expansion for σ was conjectured in [14] (BES) as a sort of analytic continuation of the corresponding strong coupling expansion. In opposite to the latter, the weak coupling expansion of $\theta(x_1, x_2)$ has a finite radius of convergence and defines a function which admits an integral representation (DHM) well defined in a certain kinematical region of particle rapidities x_1, x_2 and for finite values of g [15]. Analytic continuation of the dressing phase to other kinematical regions compatible with crossing symmetry has been constructed in [16], which in fact provides verification of the crossing equation for finite g . Finally, under some assumptions on the analytic structure the minimal solution of the crossing equation has been found and cast precisely in the DHM form [17, 18]. Let us also note that the dressing phase admits a representation in terms of a single integral (rather than double integral representation of DHM) which proved to be useful for numerical construction of solutions of the TBA equations [19].

It was soon realised [20] that a non-perturbative resummation prescription must be implemented if we want to extract the weak coupling expansion of σ from the strong coupling data. After a particular non-perturbative prescription to resum the leading order dressing phase contribution at strong coupling, the authors of [20] were able to expand it in a suitable weak coupling regime. In this way they found a connection between the strong and weak coupling coefficients of the dressing phase, reminiscent of the analytic continuation conjectured in [14]. Similarly the leading contribution to the magnons dressing phase in the strong coupling regime was obtained in [21] via Borel resummation of a particular class of terms in the asymptotic limit. Finally in [22] the authors expanded the dressing phase of [14] reproducing precisely the asymptotic strong coupling coefficients. To obtain the

strong coupling regime, these authors did not use the contour integral type of argument utilized in [14], but rather implemented a suitable ad hoc regularization procedure to expand the integrand of the Beisert-Eden-Staudacher dressing phase and obtained back the formal asymptotic expansion studied in [13]. Although the results of [20, 21] and [22] suggest that the dressing phase proposed in [14] (that we will call BES in what follows) has the correct properties to interpolate between the weak and strong coupling regime, to our mind no rigorous and complete treatment of the resummation procedure of the full strong coupling asymptotic expansion of σ exists so far.

In this paper we would like to present the discussion of the strong coupling expansion of the dressing phase, and its resummation, in the modern context of resurgence [23]. We show how to resum the strong coupling expansion by using a modified version of the well-known Borel transform method. Our main result is that, in order to reproduce the dressing phase of [14], we have to modify the perturbative strong coupling expansion studied in [13] to what is called a transseries expansion by adding new, non-perturbative terms of the form $e^{-4\pi g n}$ with $n \geq 1$ integer. These exponentially suppressed terms can be associated with ambiguities related to the resummation procedure of the purely perturbative expansion. Having modified the purely perturbative coefficients we need to check once again that the new strong coupling dressing phase satisfies the crossing symmetry equation and indeed we show that these new non-perturbative contributions to σ solve the homogenous crossing symmetry equation.

According to our findings the leading non-perturbative correction to the dressing phase comes with an exponentially suppressed factor $e^{-4\pi g}$ multiplied by an infinite perturbative expansion starting from three-loops, i.e. with the factor g^{-2} . From the purely perturbative point of view, the three-loop coefficient is also distinguished because only starting from three-loops the odd coefficients produce contributions to the dressing phase which satisfy the homogeneous crossing equation, while this is not the case for the one-loop perturbative coefficients or the even ones. For the non-perturbative contributions it might be that there is a protection mechanism based on vanishing of the zero mode contributions, forcing perturbation theory on top of these new non-perturbative saddles to start from three-loops, in the same spirit to what has been observed for the case of the instanton corrections for the anomalous dimension of the Konishi operator [24, 25].

The origin of these new, non-perturbative effects in the dressing phase is quite mysterious. This story is analogous to the non-perturbative effects [26, 27] emergent in the strong coupling expansion, $g \rightarrow \infty$, of the cusp anomalous dimension of $\mathcal{N} = 4$. Similarly to the dressing phase, the cusp anomaly has a transseries expansion at strong coupling [28, 29] and, perhaps surprisingly, these exponentially suppressed terms have a semiclassical origin that can be understood from the string theory side. In the dual, weakly coupled description the calculation of the cusp anomaly translates into the computation of the spectrum of folded spinning strings on $\text{AdS}_5 \times \text{S}^5$, the so-called GKP-strings [30]. At low energies we can describe the world-sheet theory in terms of an effective sigma model, containing an $O(6)$ factor [31], with a non-trivial strongly coupled IR dynamics. In a suitable regime [32], this 2-d quantum field theory contains non-perturbative objects, i.e. finite action solutions to the classical equations of motion, that, in the semiclassical approximation, give rise to

exponentially suppressed contributions to the energy levels hence explaining the presence of non-perturbative terms in the cusp anomaly expansion at strong coupling, on the gauge theory side. How precisely these non-perturbative objects translate into the full string theory remains however to be understood.

In the case of the dressing phase the weakly coupled dual side can be most conveniently studied via a different stringy solution: the BMN string [33]. The S-matrix computed from the sigma model perturbation theory has been shown (see e.g. [34, 35]) to reproduce the well known first few orders of the dressing phase expansion. For this reason and from the presence of non-perturbative terms in the dressing phase transseries expansion, we predict the existence of new non-perturbative objects in the world-sheet sigma model theory¹ (or possibly a suitable complexification thereof) that hopefully one can construct more easily in one of the Pohlmeyer reduced versions of the world-sheet theory [36]. Note also that the leading non-perturbative effect presents in the cusp anomalous dimension [26, 31] takes the form $e^{-\pi g}$, or $e^{-\sqrt{\lambda}/2}$ in terms of the 't Hooft coupling, while the leading correction we find in the dressing phase is of the form $e^{-4\pi g}$, or $e^{-2\sqrt{\lambda}}$. This stresses once more that these new non-perturbative corrections we find in the dressing phase should have a different semi-classical origin compared to the cusp anomaly ones. It is interesting to note that the difference between these two types of leading non-perturbative effects is analogous to the relation between the mass gap of the $O(N)$ sigma model and the action of its minimal unton-like saddle which is $N - 2$ times bigger than the mass gap [32], this precisely reproduces a factor of 4 difference in the $O(6)$ case, i.e. the model relevant to our discussion. It is possible that these finite action saddles might be responsible for the non-perturbative effects we found, although it is absolutely not obvious how and why they should be realised in the string description.

From a mathematical point of view we perfectly understand why these non-perturbative terms must be incorporated in order to represent a very particular analytic function, i.e. this BES dressing phase, in terms of a transseries expansion, but from a physical point of view it is a very important question to understand the semi-classical origin of these exponentially suppressed contributions in terms of non-perturbative strings configurations.

Finally, we mention the universality of the methods developed in the present paper. Similar results about non-perturbative sectors of the dressing phase might be expected also for the case of q -deformed theories [37]–[43] and for lower dimensional examples of AdS/CFT, like for instance for AdS₃/CFT₂, see e.g. [44]–[46]. Furthermore similar type of methods can be applied also to different observables within the context of AdS/CFT correspondence, for example it was realized in [47] that the hydrodynamic gradient series for the strongly coupled $\mathcal{N} = 4$ super Yang-Mills plasma is only an asymptotic expansion leading to the works [48]–[50] dealing with resurgence and resummation issues in the fluid context of AdS₅/CFT₄.

The paper is organized as follows. In section 2 we review some known facts about the dressing phase and its strong coupling expansion while in section 3 we introduce a modified version of the Borel transform to resum the perturbative coefficients. We prove in

¹We thank Lorenzo Bianchi for useful discussions on this problem.

section 4 that the Borel-Ecalle resummation of our proposed transseries expansion matches perfectly the BES dressing phase. The exact form of the non-perturbative terms is related to the ambiguity in the resummation of the perturbative expansion, which is computed explicitly in section 5 and then expanded at strong coupling in section 6. In section 7 we use a standard dispersion-like argument to show how the perturbative coefficients of the non-perturbative sectors can be reconstructed from the large order behaviour of the purely perturbative ones and finally, in section 8, we use precisely these coefficients to obtain new, non-perturbative contributions to the dressing phase, solutions to the homogeneous crossing symmetry equation. Because of the involved algebraic manipulations, many of the more technical results obtained in this paper are relegated to the appendices.

2 The dressing phase

Here we collect some known facts about the dressing phase which we need to our further discussion. The S-matrix is determined up to an overall scalar function — the dressing factor $\sigma(x_1^\pm, x_2^\pm)$, which satisfies a non-trivial functional equation — the crossing equation. It turns out to be convenient to write the dressing factor in the exponential form $\sigma(x_1, x_2) = e^{i\theta(x_1, x_2)}$. Here the dressing phase

$$\theta_{12} \equiv \theta(x_1^+, x_1^-, x_2^+, x_2^-) = \sum_{r=2}^{\infty} \sum_{\substack{s>r \\ r+s=\text{odd}}} c_{r,s}(g) [q_r(x_1^\pm)q_s(x_2^\pm) - q_s(x_1^\pm)q_r(x_2^\pm)] \quad (2.1)$$

with

$$q_r(x_k^-, x_k^+) = \frac{i}{r-1} \left[\left(\frac{1}{x_k^+} \right)^{r-1} - \left(\frac{1}{x_k^-} \right)^{r-1} \right], \quad (2.2)$$

where x^\pm are subject to the relation

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}. \quad (2.3)$$

Here g is related to the 't Hooft coupling λ as $g = \sqrt{\lambda}/2\pi$.

The phase θ_{12} can be written as

$$\begin{aligned} \theta_{12} = & +\chi(x_1^+, x_2^+) - \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^+) + \chi(x_1^-, x_2^-) \\ & - \chi(x_2^+, x_1^+) + \chi(x_2^-, x_1^+) + \chi(x_2^+, x_1^-) - \chi(x_2^-, x_1^-), \end{aligned} \quad (2.4)$$

where the function χ obtained from (2.1)–(2.2) is

$$\chi(x_1, x_2) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \frac{-c_{r,s}(g)}{(r-1)(s-1)} \frac{1}{x_1^{r-1} x_2^{s-1}}. \quad (2.5)$$

The coefficients $c_{r,s}(g)$ admit an asymptotic large g expansion

$$c_{r,s}(g) = g^2 \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{-n-1}, \quad g \gg 1, \quad (2.6)$$

where the numerical coefficients are given by

$$c_{r,s}^{(0)} = \frac{1}{2}\delta_{r+1,s}, \quad c_{r,s}^{(1)} = -\frac{1 - (-1)^{r+s}}{\pi} \frac{(r-1)(s-1)}{(s+r-2)(s-r)}, \quad (2.7)$$

and for $n \geq 2$ by

$$c_{r,s}^{(n)} = \frac{(-1)^n \zeta(n)}{2\pi^n \Gamma[n-1]} (r-1)(s-1) \frac{\Gamma[\frac{1}{2}(s+r+n-3)] \Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s+r-n+1)] \Gamma[\frac{1}{2}(s-r-n+3)]}. \quad (2.8)$$

Note that for $n = 0, 1$ this expression is formally $0/0$, but nevertheless (2.7) can easily be recovered from (2.8). At any given order in the asymptotic $1/g$ expansion the double series defining χ is convergent for $|x_{1,2}| > 1$.

The series (2.6) is divergent and of Gevrey-1 type² since the coefficients (2.8) grow as $c_{r,s}^{(n)} \sim n!$, for this reason we can thus perform a Borel resummation of series (2.6).

The crossing equation satisfied by the dressing phase has the form

$$i\theta(x_j, x_k) + i\theta(1/x_j, x_k) = 2 \log h(x_j, x_k), \quad (2.9)$$

where the function h is

$$h(x_j, x_k) = \frac{x_k^- \left(1 - \frac{1}{x_j^- x_k^-}\right) (x_j^- - x_k^+)}{x_k^+ \left(1 - \frac{1}{x_j^+ x_k^+}\right) (x_j^+ - x_k^+)}. \quad (2.10)$$

Here we have chosen to uniformize x^\pm in terms of a single variable x via [12]

$$x^\pm(x) = x \sqrt{1 - \frac{1}{g^2 \left(x - \frac{1}{x}\right)^2}} \pm \frac{i}{g} \frac{x}{x - \frac{1}{x}}. \quad (2.11)$$

3 Modified Borel transform

We start with recalling that the standard Borel transform of a divergent series

$$\sum_{n=0}^{\infty} c_n z^{-n-1} \quad (3.1)$$

with coefficients c_n growing as $n!$ is defined as

$$\mathcal{B}_0 : \sum_{n=0}^{\infty} c_n z^{-n-1} \mapsto \sum_{n=0}^{\infty} \frac{c_n}{n!} \xi^n. \quad (3.2)$$

The standard Borel image is now convergent to some function $\sum_{n=0}^{\infty} \frac{c_n}{n!} \xi^n = \hat{\varphi}(\xi)$ and the initial series can be resummed through the “inverse” of the standard Borel transform which is the Laplace transform

$$\varphi(z) = \mathcal{L}[\hat{\varphi}](z) = \int_0^{\infty} d\xi e^{-z\xi} \hat{\varphi}(\xi) \sim \sum_{n=0}^{\infty} c_n z^{-n-1}, \quad (3.3)$$

²A series $\{c_n\}_{n \in \mathbb{N}}$ is of Gevrey type $1/m$ if the large orders asymptotic terms are bounded by $|c_n| < \alpha C(n!)^m$ for some constants α and C .

where \sim means asymptotic in the standard sense. Typically, $\hat{\varphi}(\xi)$ has singularities which lead to ambiguities in the resummation procedure associated with the choice of integration contour in the Laplace transform as we will discuss in full details later on.

Here, to remove an additional Riemann-zeta factor, we consider a modified (similarly to [51]) Borel transform³

$$\mathcal{B} : \sum_{n=2}^{\infty} c_n z^{-n} \mapsto \sum_{n=2}^{\infty} \frac{c_n}{\zeta(n)\Gamma(n+1)} \xi^n = \hat{\varphi}(\xi), \quad (3.4)$$

which on a monomial acts as

$$\mathcal{B}[z^{-n}] = \frac{\xi^n}{\Gamma(n+1)\zeta(n)}, \quad \text{for } n \geq 2, \quad (3.5)$$

where $\zeta(n)$ denotes the Riemann zeta function. At this stage we introduce the modified Borel transform only to get rid of the Riemann-zeta factor and simplify the expressions to be working with. We will carefully analyse the implications that the use of this transform has with regards to the non-perturbative sectors later in the paper.

This transform can be easily inverted by noticing that the momenta of the measure

$$d\mu = \frac{1}{4 \sinh^2(\xi/2)} d\xi \quad (3.6)$$

are precisely

$$\langle \xi^n \rangle = \int_0^\infty d\mu \xi^n = \int_0^\infty d\xi \frac{\xi^n}{4 \sinh^2(\xi/2)} = \Gamma(n+1)\zeta(n) \quad \text{for } n \geq 2. \quad (3.7)$$

As seen before, the “inverse” can be given via

$$\varphi(z) = z \int_0^\infty \frac{d\xi}{4 \sinh^2(\xi z/2)} \hat{\varphi}(\xi) \sim \sum_{n=2}^{\infty} c_n z^{-n}. \quad (3.8)$$

According to (2.6) the variable z in (3.8) should be identified with g .

Note that by making use of $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ we can easily relate the standard Borel transform to the modified one.⁴ We can rewrite (3.2) in terms of (3.4)

$$\hat{\varphi}_0(\xi) = \sum_{n=2}^{\infty} \frac{\zeta(n) c_n}{\zeta(n)\Gamma(n+1)} \xi^n = \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} \frac{c_n}{\zeta(n)\Gamma(n+1)} \left(\frac{\xi}{k}\right)^n = \sum_{k=1}^{\infty} \hat{\varphi}\left(\frac{\xi}{k}\right), \quad (3.9)$$

where $\hat{\varphi}_0$ denotes the standard Borel transform (3.2), while $\hat{\varphi}$ denotes the modified one, associated with the coefficients $\{c_n\}$. In particular we have the identity

$$\varphi(z) = z \int_0^\infty d\xi e^{-z\xi} \hat{\varphi}_0(\xi) = \sum_{k=1}^{\infty} (k \cdot z) \int_0^\infty d\xi e^{-k \cdot z \xi} \hat{\varphi}(\xi) = z \int_0^\infty \frac{d\xi}{4 \sinh^2(\xi z/2)} \hat{\varphi}(\xi),$$

³Note that the summation extends from $n = 2$ because $\zeta(1) = \infty$.

⁴Equivalently we could have expanded the modified kernel $1/\sinh^2(\xi z/2) = \sum_{k=1}^{\infty} 4k e^{-kz\xi}$, see the discussion around (4.18).

telling us that the modified Borel resummation can be seen as an infinite sum of standard Borel resummations evaluated at shifted couplings $z \mapsto k \cdot z$. This will be important when we will discuss the non-perturbative sectors.

Applying this modified Borel transform to (2.6) we can sum up the modified Borel image

$$\sum_{n=2}^{\infty} \frac{c_{r,s}^{(n)}}{\Gamma(n+1)\zeta(n)} \xi^n = \hat{\varphi}_{r,s}(\xi), \quad (3.10)$$

where

$$\begin{aligned} \hat{\varphi}_{r,s}(\xi) = & \frac{1}{48\pi^3} (r-1)(s-1)\xi^2 \quad (3.11) \\ & \times \left[12\pi {}_4F_3 \left(\left\{ \frac{3}{2} - \frac{r}{2} - \frac{s}{2}, \frac{1}{2} + \frac{r}{2} - \frac{s}{2}, \frac{1}{2} - \frac{r}{2} + \frac{s}{2}, -\frac{1}{2} + \frac{r}{2} + \frac{s}{2} \right\}, \left\{ \frac{1}{2}, \frac{3}{2}, 2 \right\}, \left(\frac{\xi}{4\pi} \right)^2 \right) \right. \\ & \left. + (r-s)(r+s-2) \xi {}_4F_3 \left(\left\{ 2 - \frac{r}{2} - \frac{s}{2}, 1 + \frac{r}{2} - \frac{s}{2}, 1 - \frac{r}{2} + \frac{s}{2}, \frac{r}{2} + \frac{s}{2} \right\}, \left\{ \frac{3}{2}, 2, \frac{5}{2} \right\}, \left(\frac{\xi}{4\pi} \right)^2 \right) \right] \end{aligned}$$

with ${}_pF_q(\{a_1, \dots, a_p\}, \{b_1, \dots, b_q\}, z)$ being the generalised hypergeometric function.

Recalling that $s+r$ must be odd and $r \geq 2$, $s \geq r+1$, we introduce new variables

$$\begin{aligned} p &= \frac{s+r-1}{2}, & s &= p+q+1, \\ q &= \frac{s-r-1}{2}, & r &= p-q, \end{aligned} \quad (3.12)$$

where the integers p, q are restricted to $q \geq 0$, $p \geq q+2$, and, with the definition $\xi \equiv 4\pi x$, the modified Borel transform

$$\hat{\varphi}_{r,s}(\xi) \equiv \hat{\phi}_{p,q}(x) \quad (3.13)$$

takes the form

$$\begin{aligned} \hat{\phi}_{p,q}(x) := & \frac{4}{3} (p-q-1)(p+q)x^2 \quad (3.14) \\ & \times \left[{}_3F_3 \left(\{1-p, p, -q, 1+q\}, \left\{ \frac{1}{2}, \frac{3}{2}, 2 \right\}, x^2 \right) \right. \\ & \left. - (2p-1)(2q+1) x {}_4F_3 \left(\left\{ \frac{3}{2} - p, \frac{1}{2} + p, \frac{1}{2} - q, \frac{3}{2} + q \right\}, \left\{ \frac{3}{2}, 2, \frac{5}{2} \right\}, x^2 \right) \right]. \end{aligned}$$

In terms of the variables p and q the perturbative coefficients $c_{r,s}^{(n)}$ acquire the form

$$c_{p,q}^{(n)} = (-1)^n \zeta(n) \frac{(p+q)(p-q-1)}{2\pi^n \Gamma(n-1)} \frac{\Gamma(\frac{n}{2} + p - 1) \Gamma(\frac{n}{2} + q)}{\Gamma(-\frac{n}{2} + p + 1) \Gamma(-\frac{n}{2} + q + 2)}. \quad (3.15)$$

As discussed earlier in this section, we can naively resum the asymptotic power series with coefficients (3.15) via

$$c_{p,q}(g) = c_{p,q}^{(0)} \cdot g + c_{p,q}^{(1)} + \pi g^2 \int_0^\infty \frac{dx}{\sinh^2(2\pi gx)} \hat{\phi}_{p,q}(x). \quad (3.16)$$

To understand the region of analyticity of the function $c_{p,q}(g)$ in the complex coupling constant g -plane, we need first to understand the analytic properties of the modified Borel transform (3.14) in the complex Borel x -plane.

To begin, we note that the first hypergeometric function in (3.14) is a simple polynomial of degree $2q$ in x . This contribution to the full modified Borel transform is an entire function of x because it is coming from the coefficients $c_{p,q}^{(n)}$ with n even which are only finitely many in number: from the explicit expression (3.15), we see that $c_{p,q}^{(2m)} = 0$ for any $m \geq q + 2$.

The second hypergeometric function in (3.14), which we denote as

$$\Omega(z) := {}_4F_3\left(\left\{\frac{3}{2} - p, \frac{1}{2} + p, \frac{1}{2} - q, \frac{3}{2} + q\right\}, \left\{\frac{3}{2}, 2, \frac{5}{2}\right\}, z\right), \quad (3.17)$$

where $z = x^2$, has a cut along the real interval $(1, +\infty)$. Therefore, the resummation formula (3.16) does not define an analytic function of g , unless we specify a contour of integration that dodges the cut. This introduces an ambiguity in the resummation procedure, related to the particular choice of integration contour, i.e. that is above or below the real line. For the discontinuity of $\Omega(z)$ we found in appendix A the following formula

$$\begin{aligned} \Omega(z + i\epsilon) - \Omega(z - i\epsilon) = & \quad (3.18) \\ & -i \frac{3}{(2p-1)(2q+1)(p+q)!} \\ & \times z \frac{d^q}{\sqrt{z} dz^q} z^{q-1} \frac{d^{p-2}}{dz^{p-2}} \left[(1-z)^{p+q} z^{p-\frac{1}{2}} {}_2F_1\left(\frac{1}{2} + p, \frac{3}{2} + q, p+q+1, 1-z\right) \right], \quad |z| > 1, \end{aligned}$$

so that combining this with (3.14) we find

$$\begin{aligned} \text{Disc } \hat{\phi}_{p,q} = & \quad (3.19) \\ & i \frac{4(p-q-1)}{(p+q-1)!} \\ & \times z \frac{d^q}{dz^q} z^{q-1} \frac{d^{p-2}}{dz^{p-2}} \left[(1-z)^{p+q} z^{p-\frac{1}{2}} {}_2F_1\left(\frac{1}{2} + p, \frac{3}{2} + q, p+q+1, 1-z\right) \right]_{z=x^2}, \quad |z| > 1. \end{aligned}$$

The discontinuity along the cut $(1, +\infty)$ is purely imaginary and for $z = 1$ it vanishes $\text{Disc } \hat{\phi}_{p,q}(1) = 0$. Note also that due to the hypergeometric function the discontinuity itself has a logarithmic branch cut in the directions $\text{Arg } z = \pi$. These facts will shortly be of importance.

As an example we can specify $p = 2, q = 0$ in (3.19) obtaining

$$\begin{aligned} \text{Disc } \hat{\phi}_{2,0}(x) &= \frac{64i}{3\pi} x [(x^2 + 1)E(1 - x^2) - 2x^2K(1 - x^2)] \\ &= 16i(x-1)^2 \left(1 + \frac{3}{2}(x-1) + \frac{15}{32}(x-1)^2 + O((x-1)^3) \right), \quad \Re x > 0 \end{aligned}$$

where $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and second kinds respectively. In $\text{Disc } \hat{\phi}_{p,q}(x)$ the same elliptic functions appear multiplied by two p, q -dependent polynomials in x for which we do not have the generic expression. Since $K(k)$ and $E(k)$ have branch cut discontinuities for $k \in (1, \infty)$ we deduce that $\text{Disc } \hat{\phi}_{2,0}(1+t)$ a branch cut along $t = -1 + i\mathbb{R}$. One can easily check, using the known discontinuities for the elliptic integrals, that the discontinuity of $\text{Disc } \hat{\phi}_{2,0}(x)$ across the cut gives back

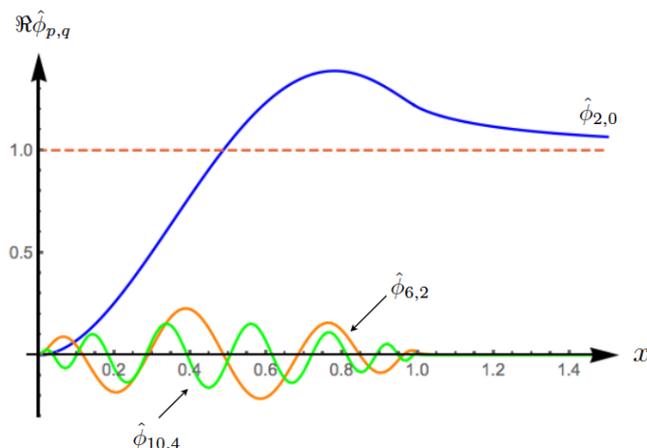


Figure 1. Plot of real part of the function $\hat{\phi}_{p,q}$ for a few values of p and q .

precisely the non-analytic part of $\hat{\phi}_{2,0}(x)$, modulo an overall constant. The fact that the singularity structure of $\hat{\phi}_{p,q}(x)$ and of its discontinuity closes with just these two functions will be of crucial importance later on.

One natural way to fix the ambiguity related to the choice of the integration contour is to demand that $c_{p,q}(g)$ must be real for real g . Further analysis reveals that $\Re \hat{\phi}_{p,q}(x)$ has neither pole nor cut on the real line and for generic p and q is a decreasing function as $x \rightarrow \infty$, see figure 1. We thus can define the manifestly real coefficients by

$$c_{p,q}(g) = c_{p,q}^{(0)} \cdot g + c_{p,q}^{(1)} + \pi g^2 \int_0^\infty \frac{dx}{\sinh^2(2\pi gx)} \Re \hat{\phi}_{p,q}(x), \quad (3.20)$$

whose strong coupling expansion $g \gg 1$ coincide with the original asymptotic formal power series (2.6). This prescription for the resummation procedure seems somehow ad hoc but in the next section we will show that it corresponds in fact to the *median* Borel resummation.

To straightforwardly integrate $\Re \hat{\phi}_{p,q}$ is rather difficult because it contains a separate polynomial part. Also the first two terms in (3.20) come apart which suggests that they originate from contour integrals around isolated points, as was explained in [14]. Therefore, to proceed, we show that $\Re \hat{\phi}_{p,q}$ admits another but alternative representation through the function

$$\begin{aligned} \hat{\Phi}_{p,q}(x) = & \delta_{q,0} + (-1)^{p+q} 2^{5-4p} (p-q-1)(p+q) \frac{\Gamma(2p-2)}{\Gamma(p-q)\Gamma(p+q+1)} \\ & \times x^{2-2p} \cdot {}_4F_3 \left(\left\{ p-1, p-\frac{1}{2}, p, p+\frac{1}{2} \right\}; \{2p, p-q, p+q+1\}; x^{-2} \right). \end{aligned} \quad (3.21)$$

Namely, both functions $\hat{\phi}_{p,q}$ and $\hat{\Phi}_{p,q}$ share the same real part

$$\Re \hat{\phi}_{p,q}(x) = \Re \hat{\Phi}_{p,q}(x), \quad x > 0, \quad (3.22)$$

a statement which is analytically proven in appendix B in two different ways.⁵ At this point it is gratifying to see that (3.21) is essentially the same formula as equation (3.25) in [14], which has been proposed there to describe a sort of analytic continuation of the coefficients c_{rs} from strong to weak coupling. Note that, contrary to $\hat{\phi}_{p,q}$, the new function $\hat{\Phi}_{p,q}$ is an even function of x and this property will be crucial in the next section to extend the integration over the whole real line to implement a Cauchy-like argument .

4 Non-perturbative resummation of the coefficients $c_{r,s}(g)$

In this section we prove that the manifestly real resummation (3.20) proposed in the previous section does indeed coincide with the coefficients for the BES dressing phase introduced in [14]. Furthermore we show that the proposed real resummation (3.20) can be understood as the Borel-Ecalle resummation of a particular transseries expansion, generalization of the formal power series (2.6) that we started with.

4.1 From the Borel sum to the BES dressing phase

According to the discussion in the previous section, the coefficients $c_{p,q}$ can be represented as

$$c_{p,q}(g) = c_{p,q}^{(0)} \cdot g + c_{p,q}^{(1)} + \frac{1}{2}\pi g^2 \int_{-\infty}^{\infty} \frac{dx}{\sinh^2(2\pi gx)} \Re \hat{\Phi}_{p,q}(x), \quad (4.1)$$

where the integration was extended to the whole real line since $\Re \hat{\Phi}_{p,q}(x)$ is an even function of x . The rest of the computation follows the same steps as in [14] but now for arbitrary values of r and s and, therefore, we outline it here for completeness.

The starting point is to pass from integration of $\hat{\Phi}_{p,q}$ over the real line to integration of $\hat{\Phi}_{p,q}$ along the contour depicted on figure 2. The function $\hat{\Phi}_{p,q}$ has a cut on the interval $(-1, 1)$ and the integration contour C_1 runs just above this cut. Since the kernel $f(z) = \frac{\hat{\Phi}_{p,q}(z)}{\sinh^2(2\pi gz)}$ is symmetric with respect to $z \rightarrow -z$, the contribution from two points symmetric around zero amounts to $f(-\bar{z}) + f(z) = f(\bar{z}) + f(z) = \overline{f(z)} + f(z) = 2\Re f(z)$, because $f(z)$ is real analytic. Thus, integration of $f(z)$ above the cut is equivalent to the integration of $\Re f(z)$ over the interval $(-1, 1)$. One has however to take into account that $f(z)$ has a residue at infinity and at $z = 0$ which lead to additional contributions. In particular, the two isolated terms entering (4.1) can be treated (similarly to [14]) as the following contour integrals:

1. For the integral around the contour C_3 , where $|x| \rightarrow \infty$ with $\epsilon < \arg x < \pi - \epsilon$, a non-trivial contribution occurs only due to the leading term in $\hat{\Phi}_{p,q}(x) = \delta_{q,0} + \dots$, which is present for $q = 0$ only. One gets

$$\frac{1}{2}\pi g^2 \int_{C_3} dx \frac{\hat{\Phi}_{p,q}(x)}{\sinh^2(2\pi gx)} = \delta_{q,0} \lim_{|x| \rightarrow \infty} \left[-\frac{1}{4} \coth(2\pi g|x|e^{i\theta}) \Big|_{\theta=0}^{\theta=\pi} \right] = \frac{1}{2}g\delta_{q,0} = c_{r,s}^{(0)}. \quad (4.2)$$

⁵We warn the reader that to verify the coincidence of the real parts of the above functions numerically, for instance, by using *Mathematica*, one needs to apply first to the function $\hat{\Phi}$ the command “FunctionExpand” which renders the answer in terms of complete elliptic integrals of the first and second kind. After that a numerical comparison can be straightforwardly performed.

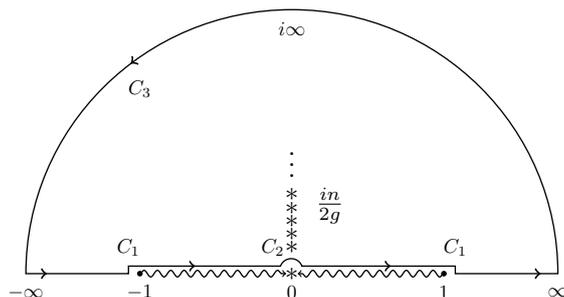


Figure 2. Integration contour for the coefficients $c_{r,s}$.

2. To compute the integral around the contour C_2 , we have to expand $\hat{\Phi}_{p,q}(x)$ around zero and we find for the leading behaviour

$$\hat{\Phi}_{p,q}(x) = -\frac{16i}{\pi} \frac{(p+q)(p-q-1)}{(2p-1)(2q+1)} x + \dots, \quad (4.3)$$

i.e. it is purely imaginary for real x . Note that the case $q = 0$ should be treated with care which results in the absence of the leading $\delta_{q,0}$ when $q = 0$ in the small x expansion for $\hat{\Phi}_{p,q}(x)$. Hence, the contribution from the contour C_2 is

$$\begin{aligned} \frac{1}{2}\pi g^2 \int_{C_2} dx \frac{\hat{\Phi}_{p,q}(x)}{\sinh^2(2\pi g x)} &= \pi g^2 \frac{16i}{\pi} \frac{(p+q)(p-q-1)}{(2p-1)(2q+1)} \int_{C_2} dx \frac{1}{4g^2\pi^2 x} \\ &= -\frac{2}{\pi} \frac{(p+q)(p-q-1)}{(2p-1)(2q+1)} = c_{p,q}^{(1)}. \end{aligned} \quad (4.4)$$

Thus, the resummation formula can be written as the following contour integral

$$c_{p,q}(g) = \frac{1}{2}\pi g^2 \int_C dx \frac{\hat{\Phi}_{p,q}(x)}{\sinh^2(2\pi g x)}, \quad (4.5)$$

with contour $C = C_1 \cup C_2 \cup C_3$ from figure 2. This is evaluated by Cauchy theorem as

$$c_{p,q}(g) = i(\pi g)^2 \sum_{n=1}^{\infty} \text{Res}_{in/2g} \frac{\hat{\Phi}_{p,q}(x)}{\sinh^2(2\pi g x)} = \frac{i}{4} \sum_{n=1}^{\infty} \left. \frac{d}{dx} \hat{\Phi}_{p,q}(x) \right|_{x=\frac{in}{2g}}. \quad (4.6)$$

The derivative is given by

$$\begin{aligned} \frac{d}{dx} \hat{\Phi}_{p,q}(x) &= 2^3 (-1)^{p+q+1} \frac{\Gamma(2p-1)}{\Gamma(p+q)\Gamma(p-q-1)} \\ &\quad \times (4x)^{1-2p} {}_4F_3 \left(\left\{ p - \frac{1}{2}, p, p, p + \frac{1}{2} \right\}; \{2p, p-q, p+q+1\}; x^{-2} \right). \end{aligned} \quad (4.7)$$

Thus, for the coefficients $c_{p,q}$ we find

$$\begin{aligned} c_{p,q}(g) &= (-1)^q 2^{2-2p} \frac{\Gamma(2p-1)}{\Gamma(p+q)\Gamma(p-q-1)} \\ &\quad \times \sum_{n=1}^{\infty} (n/g)^{1-2p} {}_4F_3 \left(\left\{ p - \frac{1}{2}, p, p, p + \frac{1}{2} \right\}; \{2p, p-q, p+q+1\}; \left(\frac{2i}{n/g} \right)^2 \right). \end{aligned} \quad (4.8)$$

Next, we apply the following identity

$$z^{-\alpha} {}_{m+k}F_n \left(\left\{ a_1, \dots, a_m, \frac{\alpha}{k}, \frac{\alpha+1}{k}, \dots, \frac{\alpha+k-1}{k} \right\}; \{b_1, \dots, b_n\}; \left(\frac{k\lambda}{z} \right)^k \right) = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-zt} {}_mF_n(\{a_1, \dots, a_m\}; \{b_1, \dots, b_n\}; (\lambda t)^k), \quad (4.9)$$

where we identify $z = n/g$, $\alpha = 2p - 1$, $k = 2$ and $\lambda = i$. Hence,

$$c_{p,q}(g) = \frac{(-1)^q 2^{2-2p}}{\Gamma(p+q)\Gamma(p-q-1)} \times \sum_{n=1}^\infty \int_0^\infty dt t^{2p-2} e^{-nt/g} {}_2F_3 \left(\left\{ p, p + \frac{1}{2} \right\}; \{2p, p-q, p+q+1\}; -t^2 \right) \quad (4.10)$$

or, going back to the (r, s) -representation

$$c_{r,s}(g) = 2(-1)^{(s-r-1)/2} (s-1)(r-1) \frac{1}{\Gamma(r)\Gamma(s)} \times \sum_{n=1}^\infty \int_0^\infty dt t^{r+s-3} e^{-2nt/g} {}_2F_3 \left(\left\{ \frac{s+r}{2}, \frac{s+r-1}{2} \right\}; \{r, s, r+s-1\}; -4t^2 \right). \quad (4.11)$$

Here one can recognise the well-known formula

$${}_0F_1(r, -t^2) {}_0F_1(s, -t^2) = {}_2F_3 \left(\left\{ \frac{r+s}{2}, \frac{r+s-1}{2} \right\}; \{r, s, r+s-1\}; -4t^2 \right) \quad (4.12)$$

and use the representation of the Bessel function $J_\nu(t)$ via the hypergeometric one

$$J_\nu(2t) = \frac{t^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1, -t^2) \quad (4.13)$$

to get

$$c_{r,s}(g) = 2(-1)^{(s-r-1)/2} (s-1)(r-1) \sum_{n=1}^\infty \int_0^\infty \frac{dt}{t} e^{-2nt/g} J_{r-1}(2t) J_{s-1}(2t). \quad (4.14)$$

Summing a geometric series up, one finally gets

$$c_{r,s}(g) = 2(-1)^{(s-r-1)/2} (s-1)(r-1) \int_0^\infty \frac{dt}{t(e^t-1)} J_{r-1}(gt) J_{s-1}(gt). \quad (4.15)$$

This formula proves that the median Borel resummed formula for $c_{r,s}$ coincides with the coefficients of the BES dressing phase that first appeared in [14].

4.2 Non-perturbative ambiguities and median resummation

Let us go back to the initial problem of going from the modified Borel transform to a suitable analytic continuation (3.16) of the original asymptotic formal power series (2.6).

To properly define the inverse transform (3.16), we need to integrate over a contour where the modified Borel transform $\hat{\phi}_{p,q}(x)$ is not singular. As shown above, the singular directions in the complex x Borel plane, also called *Stokes directions*, for the case under considerations are $\text{Arg } x = 0$ and $\text{Arg } x = \pi$.

We can thus introduce the directional Borel resummation via

$$\mathcal{S}_\theta[c_{p,q}](g) = c_{p,q}^{(0)} \cdot g + c_{p,q}^{(1)} + \pi g^2 \int_0^{e^{i\theta} \infty} \frac{dx}{\sinh^2(2\pi gx)} \hat{\phi}_{p,q}(x), \quad (4.16)$$

which defines an analytic function in the wedge of the complex coupling constant plane given⁶ by $D_\theta = \{g \in \mathbb{C} \mid \Re(e^{i\theta}g) > 0\}$, provided that θ is a regular direction, i.e. $\theta \notin \{0, \pi\}$.

For every θ for which the above integral exists, if we expand for $g \gg 1$ we obtain precisely the original asymptotic, formal power series expansion (2.6). Furthermore when $\{0, \pi\} \notin [\theta_1, \theta_2]$ we have that $\mathcal{S}_{\theta_2}[c_{p,q}]$ is the analytic continuation of $\mathcal{S}_{\theta_1}[c_{p,q}]$, i.e. $\mathcal{S}_{\theta_1}[c_{p,q}](g) = \mathcal{S}_{\theta_2}[c_{p,q}](g)$ for every $g \in D_{\theta_1} \cap D_{\theta_2}$. This allows us to analytically continue the function $\mathcal{S}_{\theta_1}[c_{p,q}](g)$ on a wider wedge of the complex g -plane, i.e. on the union of the two domains $D_{\theta_1} \cup D_{\theta_2}$.

Due to the presence of singularities in the Borel plane, if we keep on increasing $\text{Arg } g$, or equivalently θ , we will necessarily encounter branch cut singularities for the analytic continuation of the purely perturbative asymptotic power series (2.6). To understand the reason for that, we pick $\epsilon > 0$ and small, and consider the two *lateral resummations* across the Stokes line $\theta = 0$ given by $\mathcal{S}_{\pm\epsilon}[c_{p,q}](g)$, a similar story holds for the other Stokes line $\theta = \pi$. These two analytic functions, although having the same asymptotic expansion (2.6), differ from one another on the intersection of their domains of analyticity. Their difference (related to the so called Stokes automorphism) can be written as an integration over the Hankel contour \mathcal{C} shown in figure 3, originating from infinity below the positive real axis, circling the origin and then going back to infinity above the positive real axis:

$$\begin{aligned} \Delta \mathcal{S}_{p,q}(g) &\equiv \mathcal{S}_{+\epsilon}[c_{p,q}](g) - \mathcal{S}_{-\epsilon}[c_{p,q}](g) = \pi g^2 \int_{\mathcal{C}} \frac{dx}{\sinh^2(2\pi gx)} \hat{\phi}_{p,q}(x) \\ &= \pi g^2 \int_1^\infty \frac{dx}{\sinh^2(2\pi gx)} \text{Disc } \hat{\phi}_{p,q}(x) \\ &= \sum_{n=1}^\infty (4\pi n g^2) e^{-4\pi n g} \int_0^\infty dt e^{-4\pi n g t} \text{Disc } \hat{\phi}_{p,q}(t+1), \end{aligned} \quad (4.17)$$

⁶The integral (4.16) is well-defined for $g \in \mathbb{C}$ such that $|\sinh^2(2\pi gx)| > 1$ for $|x|$ large enough. This leads to two disjoint domains of analyticity separated by the line $\Re(e^{i\theta}g) = 0$. We decided to restrict our attention to the upper domain but one could have directly worked with the union of the two disjoint domains analyzing the discontinuity across them.

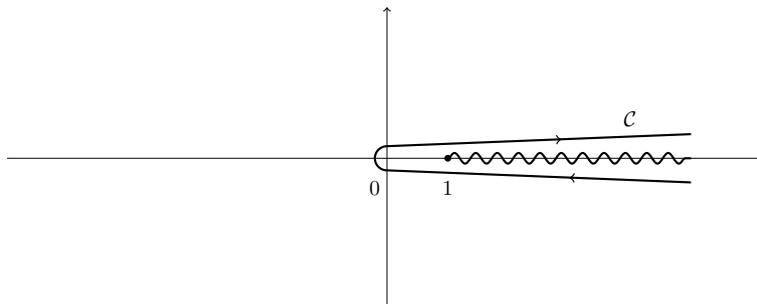


Figure 3. Integration contour in the Borel plane used to compute the difference between lateral resummations.

where we used the fact that the discontinuity (3.19) starts at $x = 1$ and, in the last step, we expanded the \sinh for $g \gg 1$

$$\frac{1}{\sinh^2(2\pi gx)} = \sum_{n=1}^{\infty} 4ne^{-4\pi gn x}, \quad (4.18)$$

to make explicit the exponentially suppressed factor $e^{-4\pi ng}$, benchmark of non-perturbative physics.

Note that although we used the modified Borel transform (3.4) to resum the purely perturbative series, we expanded the $1/\sinh^2$ kernel to express the discontinuity $\Delta S_{p,q}(g)$ as an infinite sum of usual instanton factors, i.e. exponentially suppressed terms $e^{-4\pi ng}$, multiplied by the standard Borel resummation (3.3) of a single function $\hat{\phi}_{p,q}(t+1)$ evaluated at shifted coupling $4\pi ng$.

Upon integration, we can see that each of these non-perturbative contributions is multiplied by a formal power series $\tilde{\Phi}_{p,q}^{\text{NP}}(4\pi gn)$ whose standard Borel transform (3.2) can be extracted easily from (4.17)

$$\hat{\Phi}_{p,q}^{\text{NP}}(t) = \mathcal{B}_0[\tilde{\Phi}_{p,q}^{\text{NP}}](t) = +i \text{Disc } \hat{\phi}_{p,q}(t+1), \quad (4.19)$$

so we can represent $\Delta S_{p,q}(g)$ as

$$\begin{aligned} \Delta S_{p,q}(g) &= -i \sum_{n=1}^{\infty} (4\pi ng^2) e^{-4\pi ng} \mathcal{L}[\hat{\Phi}_{p,q}^{\text{NP}}](4\pi gn) \\ &= -i \sum_{n=1}^{\infty} (4\pi ng^2) e^{-4\pi ng} \tilde{\Phi}_{p,q}^{\text{NP}}(4\pi gn), \end{aligned} \quad (4.20)$$

where in the second line, with a slight abuse of notation, we used a formal transseries (see [53]) representation for $\Delta S_{p,q}(g)$. Note that the formal power series $\tilde{\Phi}_{p,q}^{\text{NP}}(4\pi gn)$ associated with the non-perturbative sector is also asymptotic since we already know that $\hat{\Phi}_{p,q}^{\text{NP}}(x)$ has a singularity at $x = -1$, see (3.19). However its Laplace transform, $\mathcal{L}[\hat{\Phi}_{p,q}^{\text{NP}}](4\pi gn)$, is perfectly well defined since $\hat{\Phi}_{p,q}^{\text{NP}}(x)$ has no singularities along \mathbb{R}^+ . We will study the asymptotic nature of the formal power series $\tilde{\Phi}_{p,q}^{\text{NP}}(4\pi gn)$ at the end of section 6.

At this point we specify one particular determination of the analytic continuation of (2.6) as follows

$$c_{p,q}^P(g) = \begin{cases} \mathcal{S}_{-\epsilon}[c_{p,q}](g), & 0 < \text{Arg } g < \pi, \\ \mathcal{S}_{+\epsilon}[c_{p,q}](g), & -\pi < \text{Arg } g < 0, \end{cases} \quad (4.21)$$

where the suffix P reminds us that this is only the resummation of the perturbative power series (2.6). This analytic function has two branch cuts, one for $\text{Arg } g = 0$ and the other for $\text{Arg } g = \pi$. The two discontinuities are easy to obtain using the formula (4.17)

$$\text{Disc}_0 c_{p,q}^P(g) = -\Delta S_{p,q}(g), \quad (4.22)$$

$$\text{Disc}_\pi c_{p,q}^P(g) = -\Delta S_{p,q}(-g), \quad (4.23)$$

where for the discontinuity along the direction $\text{Arg } g = \pi$, we used the results, proven in the previous section, that the discontinuity of the modified Borel transform is a function of x^2 over the Borel plane. We will study in detail these discontinuity in the following section.

Note that the particular analytic continuation $c_{p,q}^P(g)$ of the perturbative power series (2.6) has two different and complex limits as $\text{Arg } g \rightarrow 0^\pm$. This ‘‘ambiguity’’ in defining our resummation procedure for real coupling suggests that despite (4.21) has the correct asymptotic power series expansion it misses nonetheless crucial non-perturbative contributions and leads to the wrong (i.e. non-physical) analytic continuation.

To obtain an analytic continuation that is real for real coupling we make use of the median resummation [52], i.e. the appropriate, unambiguous, analytic continuation that is real for real coupling. In the present case the median resummation is very simple and ultimately consists in taking the real part of the modified Borel transform of the purely perturbative expansion.

To show this we first rewrite the perturbative coefficients (2.8) by expanding the Riemann zeta as $\zeta(k) = \sum_{n=1}^\infty n^{-k}$ and then use the Laplace integral (3.3) to resum the standard Borel transform (3.2). This let us rewrite the resummation (3.16) of the purely perturbative coefficients as an infinite sum of standard Borel resummations evaluated at shifted couplings $4\pi gn$

$$c_{p,q}(g) = c_{p,q}^{(0)} \cdot g + c_{p,q}^{(1)} + \sum_{n=1}^\infty (4\pi n g^2) \mathcal{L}[\hat{\phi}_{p,q}](4\pi gn), \quad (4.24)$$

as discussed below equation (3.8). Note that this is completely identical to expanding the $\sinh^2(2\pi gx)$ kernel via (4.18) in the modified resummation (3.16) as previously mentioned.

We already know that each of these Laplace transforms is ill-defined because of the logarithmic branch cut discontinuity (3.19) of the integrand $\hat{\phi}_{p,q}(x)$. In Ecalle’s language we have that the only non-trivial alien derivative in the direction $\text{Arg } x = 0$ is given by $\hat{\Delta}_1 \hat{\phi}_{p,q}(x)$, directly related to (3.19). Further applications of the alien derivative would give rise to $\hat{\Delta}_1 \hat{\Phi}_{p,q}^{\text{NP}}(x)$ but we know already that $\hat{\Delta}_1 \hat{\Phi}_{p,q}^{\text{NP}}(x) = 0$ since the direction $\text{Arg } x = 0$ is a regular direction for the discontinuity $\text{Disc } \hat{\phi}_{p,q}(x)$, see (3.19). This fact simplifies dramatically the fractional Stokes automorphism that becomes simply $\mathfrak{S}_0^{\pm 1/2} = \text{Id} \pm \frac{1}{2} \hat{\Delta}_1$. Had we been interested in the direction $\text{Arg } x = \pi$ we would have found a singular behaviour

for the non-perturbative sector, i.e. $\hat{\Delta}_{-1}\hat{\Phi}_{p,q}^{\text{NP}}(x) \neq 0$, entirely captured by $\hat{\phi}_{p,q}(x)$, see the discussion below (3.19), but since we focus only on the wedge $\Re g > 0$ we can completely forget about the singularity structure of $\hat{\Phi}_{p,q}^{\text{NP}}(x)$. Since we will not make use of these more sophisticated objects we refer to [52] for all the details.

To simplify the discussion one can think that the relevant singular behaviour of $\hat{\phi}_{p,q}(x)$, captured entirely by (3.19), is the only discontinuity in play for the wedge $\Re g > 0$ under consideration, then the median resummation [52] consists in subtracting from the directional Borel resummation (4.16) the associated ambiguity. In practice this means that for each term of the form $(4\pi n g^2)\mathcal{L}[\hat{\phi}_{p,q}](4\pi g n)$ in (4.24) we have to subtract the corresponding non-perturbative ambiguity, easily read from the full ambiguity $\Delta S_{p,q}(g)$ (4.20), which takes the form $\mp \frac{i}{2}(4\pi n g^2) e^{-4\pi n g} \tilde{\Phi}_{p,q}^{\text{NP}}(4\pi g n)$. The sign is chosen accordingly to the contour of the ambiguous Laplace transform $\mathcal{L}[\hat{\phi}_{p,q}](4\pi g n)$ being understood as above or below the cut, similarly to (4.16).

This means that in the sector $\Re g > 0$ under consideration the median resummation transseries is given by

$$\begin{aligned} c_{p,q}^{\text{TS}}(g) &= c_{p,q}^{(0)} \cdot g + c_{p,q}^{(1)} + \sum_{n=1}^{\infty} (4\pi n g^2) [\tilde{\phi}_{p,q}(4\pi g n) + \mathfrak{s} e^{-4\pi n g} \tilde{\Phi}_{p,q}^{\text{NP}}(4\pi g n)] \\ &= c_{p,q}(g) + \mathfrak{s} \sum_{n=1}^{\infty} (4\pi n g^2) e^{-4\pi n g} \tilde{\Phi}_{p,q}^{\text{NP}}(4\pi g n), \end{aligned} \tag{4.25}$$

where $\tilde{\phi}_{p,q}(g)$ denotes the formal power series obtained from the strong coupling expansion of $\mathcal{L}[\hat{\phi}_{p,q}](g)$. The first term of the second line denotes the collection of all the purely perturbative power series as in (4.24). The parameter \mathfrak{s} is linked with the lateral resummation procedure and called the transseries parameter $\mathfrak{s} = -i/2$ for $0 < \text{Arg } g < \pi/2$ and $\mathfrak{s} = +i/2$ for $-\pi/2 < \text{Arg } g < 0$. Note that although we have infinitely many instanton sectors they are all multiplied by the same and only transseries parameter \mathfrak{s} . The reason for this is that the one under consideration is not a one-(or multi-)parameter transseries (see for example [54]) but rather an infinite sum of two terms transseries of the form $[\tilde{\phi}_{p,q}(4\pi g n) + \mathfrak{s} e^{-4\pi n g} \tilde{\Phi}_{p,q}^{\text{NP}}(4\pi g n)]$, at least in the wedge under consideration.

One can perform the infinite sum (4.24) of all the purely perturbative contributions evaluated at shifted couplings reproducing the modified Borel transform (i.e. we repeat backwards the argument presented above) while the non-perturbative sectors remain as an infinite sum. This generates an apparent mismatch between perturbative and non-perturbative sectors. Nonetheless, (4.25) remains a very simple infinite sum of two-terms transseries in disguise. This transseries (4.25) is a formal representation of a unique analytic function that can be explicitly obtained via Borel-Ecalle resummation. First we notice that with the median resummation choice for the transseries parameter \mathfrak{s} we have that the two lateral resummations of the transseries coincide for real and positive coupling g

$$c_{p,q}^{\text{TS}}(g) = \mathcal{S}_{-\epsilon}[c_{p,q}](g) - \frac{i}{2} \sum_{n=1}^{\infty} (4\pi n g^2) e^{-4\pi n g} \mathcal{L}[\hat{\Phi}_{p,q}^{\text{NP}}](4\pi g n) \tag{4.26}$$

$$= \mathcal{S}_{+\epsilon}[c_{p,q}](g) + \frac{i}{2} \sum_{n=1}^{\infty} (4\pi n g^2) e^{-4\pi n g} \mathcal{L}[\hat{\Phi}_{p,q}^{\text{NP}}](4\pi g n), \tag{4.27}$$

where $\mathcal{L}[\hat{\Phi}_{p,q}^{\text{NP}}](g)$ denotes the standard Laplace transform (3.3) of the perturbative series associated with the non-perturbative sector (4.19) which is completely regular, i.e. it does not have any Stokes line in the wedge under consideration.

Furthermore we can make use of (4.20) to show that in the case at hand the median resummation of (2.6) is simply given by

$$c_{p,q}^{\text{TS}}(g) = \mathcal{S}_{\text{med}}[c_{p,q}](g) = \begin{cases} \mathcal{S}_{-\epsilon}[c_{p,q}](g) + \frac{1}{2} \Delta S_{p,q}(g), & 0 < \text{Arg } g < \pi/2, \\ \mathcal{S}_{+\epsilon}[c_{p,q}](g) - \frac{1}{2} \Delta S_{p,q}(g), & -\pi/2 < \text{Arg } g < 0. \end{cases} \quad (4.28)$$

The superscript is to remind us that, upon expansion of the analytic function $c_{p,q}^{\text{TS}}(g)$ for $g \gg 1$, we do not obtain just the power series (2.6) but rather the transseries (see [53]) representation.⁷ The function $c_{p,q}^{\text{TS}}(g)$, when expanded at strong coupling gives precisely (4.25), containing infinitely many exponentially suppressed, i.e. non-perturbative, terms of the form $e^{-4\pi n g}$. Each of these non-perturbative contributions is multiplied by a formal power series $\tilde{\Phi}_{p,q}^{\text{NP}}(4\pi g n)$.

Finally, by combining (3.19), (4.17) and (4.28), the Borel-Ecalle resummation of the transseries gives

$$c_{p,q}^{\text{TS}}(g) = c_{p,q}^{(0)} \cdot g + c_{p,q}^{(1)} + \pi g^2 \int_0^\infty \frac{dx}{\sinh^2(2\pi g x)} \Re \hat{\phi}_{p,q}(x), \quad (4.29)$$

which is precisely the integral form (3.20) used in the previous section that we proved coinciding with the coefficients (4.15) of the BES dressing phase. So we learn that the correct strong coupling expansion of the BES coefficients (4.15) is not simply given by the asymptotic power series (2.6) but rather from the transseries (4.25) which coincides with (2.6) perturbatively but it contains infinitely many new exponentially suppressed terms.

However, although the transseries representations (4.25) contains infinitely many terms, it remains a two-terms transseries in disguise as noted above. Generically, physical observables are represented with multiple parameter transseries and the actual implementation of the median resummation can be very complicated. Even in a one-parameter transseries there are very intricate set of relations between the different instanton sectors and the median resummation is not as straightforward, while in our case the presence of infinite many instantons is a red-herring and is ultimately connected to the expansion of the Riemann zeta function $\zeta(n)$ as noted before. We refer to [54] for a comprehensive discussion on the cancellation of non-perturbative ambiguities and the construction of the median resummation in one- and two-parameters transseries, relevant for more general physical observables than the one discussed in the present paper.

Another important thing to keep in mind is that the problem under consideration is indeed a linear problem which roughly means that each instanton sector does not “communicate” in an intricate way with all the others. This is a very lucky case which simplifies

⁷With a slight abuse of notation we denote the median resummation (4.28) with the same symbol as its transseries representation (4.25) having in mind that they both uniquely define the one and the same analytic function.

dramatically the Borel-Ecalle resummation procedure. One of the central points in Ecalle's works [23] is precisely the decodification of the complicated set of relations connecting the different perturbative coefficients in different sectors and the deep intertwining between all sectors: perturbative and non-perturbative. In the present case this could go underappreciated due to the linearity of the problem and perhaps one of the nicest illustrations where the full power of Ecalle's work can be better appreciated is shown in a nonlinear case [55] within the context of large- N dualities where the authors are also able to obtain a very explicit strong-weak coupling interpolation similar to the one described in our paper.

As already shown in the previous section, equation (4.29) coincides with physical answer given by the coefficients of the BES dressing phase (4.15), but in order to obtain (4.29) we had to pass from the formal power series (2.6) to the transseries (4.25). This amounted to introduce infinitely many non-perturbative contributions and ultimately means that the initial purely perturbative formal power series (2.6) is not enough to reconstruct the physical answer.

It is worth emphasizing that, due to its asymptotic nature, the strong coupling transseries representation (4.25) is only a formal object but its Borel-Ecalle resummation defines a perfectly good analytic function in a wedge of the complex g -plane. In particular, this means that the weak coupling expansion coefficients, obtainable from the gauge theory side, must be encoded in some intricate way in the strong coupling transseries coefficients (4.25). We do not know how to read this weak coupling expansion directly from the strong coupling transseries, but as proven above, the median resummation of the strong coupling coefficients yields precisely the coefficients of the BES dressing phase (4.15), which directly allow for a weak coupling expansion that matches precisely the gauge theory results as shown in [14].

5 Ambiguity of the Borel resummation

As we have just seen, the ambiguity in the Borel resummation procedure comes from the discontinuity of the integrand of the Laplace transform (4.17). In this section we therefore compute this ambiguity explicitly by analyzing

$$\Delta S_{p,q}(g) = \pi g^2 \int_1^\infty \frac{dx}{\sinh^2(2\pi gx)} \text{Disc } \hat{\phi}_{p,q}(x), \tag{5.1}$$

where

$$\begin{aligned} \text{Disc } \hat{\phi}_{p,q} &= i \frac{4(p-q-1)}{(p+q-1)!} \\ &\times z \frac{d^q}{dz^q} z^{q-1} \frac{d^{p-2}}{dz^{p-2}} \left[(1-z)^{p+q} z^{p-\frac{1}{2}} {}_2F_1 \left(\frac{1}{2} + p, \frac{3}{2} + q, p+q+1, 1-z \right) \right]_{z=x^2}, \quad |z| > 1. \end{aligned} \tag{5.2}$$

The important property of the discontinuity is that $\text{Disc } \hat{\phi}_{p,q}(1) = 0$.

Since we have

$$\frac{1}{\sinh^2(2\pi gx)} = \sum_{n=1}^\infty 4n e^{-4\pi gn x}, \tag{5.3}$$

we can write

$$\Delta S_{p,q}(g) = -g \sum_{n=1}^{\infty} \int_1^{\infty} dx (-4\pi ng) e^{-4\pi ngx} \text{Disc } \hat{\phi}_{p,q}(x). \quad (5.4)$$

Substituting here the explicit formula for the discontinuity we get

$$\begin{aligned} \Delta S_{p,q}(g) &= -2ig \frac{(p-q-1)}{(p+q-1)!} \sum_{n=1}^{\infty} \int_1^{\infty} dz (-h_n \sqrt{z}) e^{-h_n \sqrt{z}} \\ &\times \frac{d^q}{dz^q} z^{q-1} \frac{d^{p-2}}{dz^{p-2}} z^{p-\frac{1}{2}} \left[(1-z)^{p+q} {}_2F_1 \left(\frac{1}{2} + p, \frac{3}{2} + q, p+q+1, 1-z \right) \right], \end{aligned} \quad (5.5)$$

where we have introduced a concise notation

$$h_n = 4\pi ng. \quad (5.6)$$

We proceed integrating by parts and noting that boundary terms always vanish we arrive at the following expression

$$\Delta S_{p,q}(g) = -2ig \frac{(p-q-1)}{(p+q-1)!} \sum_{n=1}^{\infty} \int_1^{\infty} dz Q_n(z) (z-1)^{p+q} {}_2F_1(1-z), \quad (5.7)$$

where for conciseness we omitted the parameters of ${}_2F_1$ and introduce the following function

$$Q_n(z) = \sum_{k=0}^{\infty} \frac{(-h_n)^{k+1}}{k!} z^{p-\frac{1}{2}} \frac{d^{p-2}}{dz^{p-2}} z^{q-1} \frac{d^q}{dz^q} z^{\frac{1}{2}(k+1)}. \quad (5.8)$$

In appendix C we show that $Q_n(z)$ has the following representation as a double sum

$$\begin{aligned} Q_n(z) &= e^{-h_n \sqrt{z}} \sum_{k=0}^q \sum_{m=0}^{p-k-2} (-1)^{1+k+p} 2^{1-k-m-p} (h_n \sqrt{z})^{p+k-m} \\ &\times \frac{\sqrt{\pi} q! \Gamma(p+m-k-1)}{h_n k! m! (q-k)! \Gamma(p-m-k-1) \Gamma(\frac{3}{2} + k - q)}. \end{aligned} \quad (5.9)$$

Since

$$(-1)^{p+k-m} h_n^{p+k-m} \frac{\partial^{p+k-m}}{\partial h_n^{p+k-m}} e^{-h_n \sqrt{z}} = (h_n \sqrt{z})^{p+k-m} e^{-h_n \sqrt{z}}, \quad (5.10)$$

we have

$$\begin{aligned} Q_n(z) &= \frac{1}{h_n} \sum_{k=0}^q \sum_{m=0}^{p-k-2} \frac{(-1)^{m+1} \sqrt{\pi} 2^{1-k-m-p} q! \Gamma(p+m-k-1)}{k! m! (q-k)! \Gamma(p-m-k-1) \Gamma(\frac{3}{2} + k - q)} \\ &\times h_n^{p+k-m} \frac{\partial^{p+k-m}}{\partial h_n^{p+k-m}} e^{-h_n \sqrt{z}}. \end{aligned} \quad (5.11)$$

Thus, we can represent $Q_n(z)$ as a certain differential operator acting on $e^{-h_n \sqrt{z}}$:

$$Q_n(z) = \hat{Q}_n e^{-h_n \sqrt{z}}, \quad (5.12)$$

with the whole z dependence just sitting in the exponent. Then further computation reduces to the following integral

$$\Delta S_{p,q}(g) = -2ig \frac{(p-q-1)}{(p+q-1)!} \sum_{n=1}^{\infty} \hat{Q}_n \int_1^{\infty} dz e^{-h_n \sqrt{z}} (z-1)^{p+q} {}_2F_1(a, b, c, 1-z), \quad (5.13)$$

where $a = p + \frac{1}{2}$, $b = q + \frac{3}{2}$ and $c = p + q + 1$. Thus, we are led to compute the integral

$$f(h) = \int_1^{\infty} dz e^{-h\sqrt{z}} (z-1)^{c-1} {}_2F_1(a, b, c, 1-z). \quad (5.14)$$

For generic values of a, b, c this integral is given in [56]. Keeping for the moment p and q generic (non-integer), the answer is given by the following formula

$$\begin{aligned} \frac{f(h)}{h} &= \frac{1}{2\pi} \Gamma(-p) \Gamma(-1-q) \Gamma(1+p+q) {}_1F_2\left(\left\{\frac{1}{2}\right\}; \{1+p, 2+q\}; t\right) \\ &+ \frac{2\Gamma(2p-1)\Gamma(p-q-1)\Gamma(1+p+q)}{4^p \Gamma(\frac{1}{2}+p)\Gamma(-\frac{1}{2}+p)} \frac{1}{t^p} {}_1F_2\left(\left\{\frac{1}{2}-p\right\}; \{1-p, 2-p+q\}; t\right) \\ &+ \frac{2\Gamma(1-p+q)\Gamma(1+2q)\Gamma(1+p+q)}{4^{1+q} \Gamma(\frac{1}{2}+q)\Gamma(\frac{3}{2}+q)} \frac{1}{t^{1+q}} {}_1F_2\left(\left\{-\frac{1}{2}-q\right\}; \{p-q, -q\}; t\right), \end{aligned} \quad (5.15)$$

where we have introduced a concise notation $t = h^2/4$. This formula can be obtained by using the Mellin transform technique, see e.g. [57]. While well-defined for generic p and q , the above expression becomes nonsensical for p and q being positive integers. In the latter case the answer can still be found from (5.15) by using the continuity principle — first one starts from generic p, q close to integer values by introducing a kind of regularisation and then takes a limit to these values. A regularisation parameter controls the apparent singularities which are supposed to cancel in the final expression.

To proceed, we introduce the following shorthand notation

$$H_1 \equiv {}_1F_2\left(\left\{\frac{1}{2}\right\}; \{1+p, 2+q\}; t\right) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)\Gamma(1+p)\Gamma(2+q)}{\sqrt{\pi}\Gamma(1+k)\Gamma(1+k+p)\Gamma(2+p+k)} t^k \quad (5.16)$$

and consider the power series expansion for the second hypergeometric function

$$\begin{aligned} H_2 &\equiv t^{-p} {}_1F_2\left(\left\{\frac{1}{2}-p\right\}; \{1-p, 2-p+q\}; t\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(1-p)\Gamma(\frac{1}{2}+k-p)\Gamma(2-p+q)}{\Gamma(1+k)\Gamma(\frac{1}{2}-p)\Gamma(1+k-p)\Gamma(2+k-p+q)} t^{k-p}. \end{aligned} \quad (5.17)$$

Denote by \bar{p} and \bar{q} positive integers to which p and q are close by. Then this sum can be split into three parts

$$\begin{aligned} H_2 &= \frac{1}{\Gamma(\frac{1}{2}-p)} \sum_{k=0}^{\bar{p}-\bar{q}-2} \frac{\Gamma(\frac{1}{2}+k-p)}{\Gamma(1+k)} \frac{\Gamma(p-k)}{\Gamma(p)} \frac{\Gamma(p-q-k-1)}{\Gamma(p-q-1)} t^{k-p} \\ &+ \frac{\Gamma(2-p+q)}{\Gamma(\frac{1}{2}-p)\Gamma(p)} \sum_{k=0}^{\bar{q}} \frac{(-1)^{k-1+\bar{p}-\bar{q}} \Gamma(-\frac{1}{2}+k-p+\bar{p}-\bar{q}) \Gamma(1-k+p-\bar{p}+\bar{q})}{\Gamma(k+\bar{p}-\bar{q})\Gamma(1+k+\bar{p}-p+q-\bar{q})} t^{k-1-p+\bar{p}-\bar{q}} \\ &+ \frac{\Gamma(1-p)\Gamma(2-p+q)}{\Gamma(\frac{1}{2}-p)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k-p+\bar{p})}{\Gamma(1+k+\bar{p})\Gamma(1+k-p+\bar{p})\Gamma(2+k-p+q+\bar{p})} t^{k-p+\bar{p}}. \end{aligned} \quad (5.18)$$

Here to obtain the second line we made a shift of the original summation label k as $k \rightarrow k + \bar{p} - \bar{q} - 1$, while to get the third line we shifted as $k \rightarrow k + \bar{p}$. Note that the first line of H_2 is finite in the limit $p \rightarrow \bar{p}$, $q \rightarrow \bar{q}$, while the second and the third lines are “linearly” and “quadratically” divergent, respectively, cf. the factors in front of the corresponding sums.

Analogously, we consider

$$\begin{aligned} H_3 &\equiv \frac{1}{t^{1+q}} {}_1F_2\left(\left\{-\frac{1}{2}-q\right\}; \{p-q, -q\}; t\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(-\frac{1}{2}+k-q)\Gamma(p-q)\Gamma(-q)}{\Gamma(1+k)\Gamma(-\frac{1}{2}-q)\Gamma(k-q)\Gamma(k+p-q)} t^{k-q-1} \end{aligned} \quad (5.19)$$

and split the sum into two parts

$$\begin{aligned} H_3 &= \frac{\Gamma(p-q)}{\Gamma(1+q)\Gamma(-\frac{1}{2}-q)} \sum_{k=0}^{\bar{q}} (-1)^k \frac{\Gamma(-\frac{1}{2}+k-q)\Gamma(1-k+q)}{\Gamma(1+k)\Gamma(k+p-q)} t^{k-q-1} \\ &+ \frac{\Gamma(-q)\Gamma(p-q)}{\Gamma(-\frac{1}{2}-q)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k-q+\bar{q})}{\Gamma(2+k+\bar{q})\Gamma(1+k-q+\bar{q})\Gamma(1+k+p-q+\bar{q})} t^{k-q+\bar{q}}. \end{aligned} \quad (5.20)$$

To obtain the second line we made a shift of the original summation label as $k \rightarrow k + \bar{q} + 1$. The first line in the expression above is finite in the limit $p \rightarrow \bar{p}$, $q \rightarrow \bar{q}$, while the second one is “linearly” divergent.

Now we put everything together and simplify the factors in front of the sums

$$\begin{aligned} \frac{f(h)}{h} &= -\frac{\sqrt{\pi}\Gamma(1+p+q)}{2\sin(\pi p)\sin(\pi q)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(1+k)\Gamma(1+k+p)\Gamma(2+p+k)} t^k \\ &+ \cos(\pi p)\Gamma(1+p+q) \sum_{k=0}^{\bar{p}-\bar{q}-2} \frac{\Gamma(\frac{1}{2}+k-p)\Gamma(p-k)\Gamma(p-q-k-1)}{2\pi^{3/2}\Gamma(1+k)} t^{k-p} \\ &+ \frac{\cos(\pi p)\Gamma(1+p+q)}{2\sqrt{\pi}\sin\pi(p-q)} \sum_{k=0}^{\bar{q}} \frac{(-1)^{k+\bar{p}-\bar{q}}\Gamma(-\frac{1}{2}+k-p+\bar{p}-\bar{q})\Gamma(1-k+p-\bar{p}+\bar{q})}{\Gamma(k+\bar{p}-\bar{q})\Gamma(1+k+\bar{p}-p+q-\bar{q})} t^{k-1-p+\bar{p}-\bar{q}} \\ &- \frac{\sqrt{\pi}\cot(\pi p)\Gamma(1+p+q)}{2\sin\pi(p-q)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k-p+\bar{p})}{\Gamma(1+k+\bar{p})\Gamma(1+k-p+\bar{p})\Gamma(2+k-p+q+\bar{p})} t^{k-p+\bar{p}} \\ &- \frac{\cos(\pi q)\Gamma(1+p+q)}{2\sqrt{\pi}\sin\pi(p-q)} \sum_{k=0}^{\bar{q}} (-1)^k \frac{\Gamma(-\frac{1}{2}+k-q)\Gamma(1-k+q)}{\Gamma(1+k)\Gamma(k+p-q)} t^{k-q-1} \\ &+ \frac{\sqrt{\pi}\cot(\pi p)\Gamma(1+p+q)}{2\sin\pi(p-q)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k-q+\bar{q})}{\Gamma(2+k+\bar{q})\Gamma(1+k-q+\bar{q})\Gamma(1+k+p-q+\bar{q})} t^{k-q+\bar{q}}. \end{aligned} \quad (5.21)$$

The second line in this expression is finite (it comes from the first line of H_2) and we can therefore put there $p = \bar{p}$, $q = \bar{q}$. This gives the first contribution I_1 to $f(h)$ corresponding to integer values of p, q

$$I_1 = (-1)^p \Gamma(1+p+q) \sum_{k=0}^{p-q-2} \frac{\Gamma(\frac{1}{2}+k-p)\Gamma(p-k)\Gamma(p-q-k-1)}{2\pi^{3/2}\Gamma(1+k)} t^{k-p}. \quad (5.22)$$

Obviously, I_1 contains inverse powers of t from t^{-p} up to t^{-q-2} .

The rest of (5.21) is divergent. To proceed, we introduce the following regularisation

$$p = \bar{p} + \frac{1}{2}\epsilon, \quad q = \bar{q} - \frac{1}{2}\epsilon. \quad (5.23)$$

To take the limit, we need the formulae

$$\sin \pi(\epsilon + m) = \sin(\pi\epsilon)(-1)^m, \quad \cot \pi\left(\pm \frac{\epsilon}{2} + m\right) = \pm \cot \frac{\pi}{2}\epsilon, \quad (5.24)$$

valid for any integer m . The second contribution to $f(h)$ comes therefore from finite sums

$$I_2 = \lim_{\epsilon \rightarrow 0} \left[\frac{\cos(\pi p)\Gamma(1+p+q)}{2\sqrt{\pi}\sin\pi(p-q)} \sum_{k=0}^{\bar{q}} \frac{(-1)^{k+\bar{p}-\bar{q}}\Gamma(-\frac{1}{2}+k-p+\bar{p}-\bar{q})\Gamma(1-k+p-\bar{p}+\bar{q})}{\Gamma(k+\bar{p}-\bar{q})\Gamma(1+k+\bar{p}-p+q-\bar{q})} t^{k-1-p+\bar{p}-\bar{q}} \right. \\ \left. - \frac{\cos(\pi q)\Gamma(1+p+q)}{2\sqrt{\pi}\sin\pi(p-q)} \sum_{k=0}^{\bar{q}} (-1)^k \frac{\Gamma(-\frac{1}{2}+k-q)\Gamma(1-k+q)}{\Gamma(1+k)\Gamma(k+p-q)} t^{k-q-1} \right]. \quad (5.25)$$

Substituting here the formulae (5.23) and taking the limit $\epsilon \rightarrow 0$, we find

$$I_2 = -\Gamma(1+p+q) \sum_{k=0}^q (-1)^{k+p} \frac{\Gamma(-\frac{1}{2}+k-q)\Gamma(1-k+q)}{2\pi^{3/2}\Gamma(1+k)\Gamma(k+p-q)} t^{k-q-1} \quad (5.26) \\ \times \left[\log t + \psi\left(-\frac{1}{2}+k-q\right) - \psi(1+k) - \psi(k+p-q) - \psi(1-k+q) \right],$$

where after the computation we replaced $\bar{p} \rightarrow p$ and $\bar{q} \rightarrow q$. Note that this term contains inverse powers of t from t^{-q-1} up to t^{-1} .

Finally, the third contribution comes from infinite sums

$$I_3 = \Gamma(1+p+q) \lim_{\epsilon \rightarrow 0} \left[-\frac{\sqrt{\pi}}{2\sin(\pi p)\sin(\pi q)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(1+k)\Gamma(1+k+p)\Gamma(2+p+k)} t^k \right. \\ \left. - \frac{\sqrt{\pi}\cot(\pi p)}{2\sin\pi(p-q)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k-p+\bar{p})}{\Gamma(1+k+\bar{p})\Gamma(1+k-p+\bar{p})\Gamma(2+k-p+q+\bar{p})} t^{k-p+\bar{p}} \right. \\ \left. + \frac{\sqrt{\pi}\cot(\pi p)}{2\sin\pi(p-q)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k-q+\bar{q})}{\Gamma(2+k+\bar{q})\Gamma(1+k-q+\bar{q})\Gamma(1+k+p-q+\bar{q})} t^{k-q+\bar{q}} \right].$$

The expression I_3 delivers the most complicated contribution which upon taking the limit and renaming $\bar{p} \rightarrow p$ and $\bar{q} \rightarrow q$ reads

$$I_3 = -(\log t)^2 \sum_{k=0}^{\infty} \frac{(-1)^{p-q}\Gamma(\frac{1}{2}+k)\Gamma(1+p+q)}{4\pi^{3/2}\Gamma(1+k)\Gamma(1+k+p)\Gamma(2+k+q)} t^k \quad (5.27) \\ - \log t \sum_{k=0}^{\infty} \frac{(-1)^{p-q}\Gamma(\frac{1}{2}+k)\Gamma(1+p+q)}{2\pi^{3/2}\Gamma(1+k)\Gamma(1+k+p)\Gamma(2+k+q)} t^k \\ \times \left[\psi\left(\frac{1}{2}+k\right) - \psi(1+k) - \psi(1+k+p) - \psi(2+k+q) \right] \\ - \sum_{k=0}^{\infty} \frac{(-1)^{p-q}\Gamma(\frac{1}{2}+k)\Gamma(1+p+q)}{4\pi^{3/2}\Gamma(1+k)\Gamma(1+k+p)\Gamma(2+k+q)} t^k \\ \times \left[\left(\psi(1+k) + \psi(1+k+p) + \psi(2+k+p) - \psi\left(\frac{1}{2}+k\right) \right)^2 \right. \\ \left. + \psi^{(1)}\left(\frac{1}{2}+k\right) - \psi^{(1)}(1+k) - \psi^{(1)}(1+k+p) + \psi^{(1)}(2+k+p) \right].$$

In this way we have found that the original integral is given by the sum of three terms

$$f(h) = h(I_1 + I_2 + I_3). \quad (5.28)$$

In fact, the whole expression I_3 can be written as

$$\begin{aligned} I_3 &= \frac{d^2}{d\epsilon^2} \sum_{k=0}^{\infty} \frac{(-1)^{p+q+1} \Gamma(\frac{1}{2} + k + \epsilon) \Gamma(1 + p + q)}{4\pi^{3/2} \Gamma(1 + k + \epsilon) \Gamma(1 + k + p + \epsilon) \Gamma(2 + k + q + \epsilon)} t^{k+\epsilon} \Big|_{\epsilon=0} \\ &= \frac{d^2}{d\epsilon^2} \frac{t^\epsilon (-1)^{p+q+1} \Gamma(\frac{1}{2} + \epsilon) \Gamma(1 + p + q)}{4\pi^{3/2} \Gamma(1 + \epsilon) \Gamma(1 + \epsilon + p) \Gamma(2 + \epsilon + q)} {}_2F_3 \left(\left\{ 1, \frac{1}{2} + \epsilon \right\}; \{1 + \epsilon, 1 + \epsilon + p, 2 + \epsilon + q\}; t \right) \Big|_{\epsilon=0}. \end{aligned} \quad (5.29)$$

6 Strong coupling expansion of the discontinuity

Here we show how to obtain an asymptotic expansion at large g starting from the exact answer for the difference $\Delta S_{p,q}(g)$. To this end we have to analyse the expansion of I_3 when $t \rightarrow \infty$. The simplest way to proceed is to use the formula (5.29), where we keep ϵ finite and send $t \rightarrow \infty$. The corresponding expansion of ${}_2F_3$ is known to be

$${}_2F_3(\{a_1, a_2\}; \{b_1, b_2, b_3\}; t) = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3, \quad (6.1)$$

where

$$\begin{aligned} \mathcal{F}_1 &= \frac{\Gamma(b_1) \Gamma(b_2) \Gamma(b_3) \Gamma(a_2 - a_1)}{\Gamma(a_2) \Gamma(b_1 - a_1) \Gamma(b_2 - a_1) \Gamma(b_3 - a_1)} \\ &\quad \times (-t)^{-a_1} {}_4F_1 \left(\{a_1, a_1 - b_1 + 1, a_1 - b_2 + 1, a_1 - b_3 + 1\}; \{a_1 - a_2 + 1\}; \frac{1}{t} \right), \end{aligned} \quad (6.2)$$

$$\begin{aligned} \mathcal{F}_2 &= \frac{\Gamma(b_1) \Gamma(b_2) \Gamma(b_3) \Gamma(a_1 - a_2)}{\Gamma(a_1) \Gamma(b_1 - a_2) \Gamma(b_2 - a_2) \Gamma(b_3 - a_2)} \\ &\quad \times (-t)^{-a_2} {}_4F_1 \left(\{a_2, a_2 - b_1 + 1, a_2 - b_2 + 1, a_2 - b_3 + 1\}; \{a_2 - a_1 + 1\}; \frac{1}{t} \right), \end{aligned} \quad (6.3)$$

and \mathcal{F}_3 will be discussed later.

For the case at hand we identify

$$a_1 = 1, \quad a_2 = \frac{1}{2} + \epsilon, \quad b_1 = 1 + \epsilon, \quad b_2 = 1 + \epsilon + p, \quad b_3 = 2 + \epsilon + q. \quad (6.4)$$

We start with analysis of the contribution of \mathcal{F}_1 into I_3 , which we denote as $I_3^{(1)}$. We have

$$\begin{aligned} I_3^{(1)} &= \frac{d^2}{d\epsilon^2} \left[\frac{t^{\epsilon-1} (-1)^{p+q}}{4\pi^{3/2}} \frac{\Gamma(1 + p + q) \Gamma(-\frac{1}{2} + \epsilon) \Gamma(\frac{3}{2} - \epsilon)}{\Gamma(\epsilon) \Gamma(1 - \epsilon) \Gamma(1 - \epsilon - p) \Gamma(-\epsilon - q) \Gamma(p + \epsilon) \Gamma(1 + \epsilon + q)} \right. \\ &\quad \left. \times \sum_{k=0}^{\infty} \frac{\Gamma(1 - \epsilon + k) \Gamma(1 - \epsilon + k - p) \Gamma(-\epsilon + k - q)}{\Gamma(\frac{3}{2} - \epsilon + k)} \frac{1}{t^k} \right] \Big|_{\epsilon=0}. \end{aligned} \quad (6.5)$$

Further simplification gives

$$I_3^{(1)} = \sum_{k=0}^{\infty} \frac{d^2}{d\epsilon^2} \left[\frac{\Gamma(1 + p + q) \Gamma(1 - \epsilon + k) \Gamma(1 - \epsilon + k - p) \Gamma(-\epsilon + k - q) \sin^2(\pi\epsilon) \tan(\pi\epsilon)}{4\pi^{7/2} \Gamma(\frac{3}{2} - \epsilon + k)} \frac{1}{t^{k+1-\epsilon}} \right] \Big|_{\epsilon=0}.$$

Now it is important to realise that the expression in the brackets above has different behaviour in the limit $\epsilon \rightarrow 0$ depending on the value of the summation variable k . If $k \geq p$ then due to the factor $\sin^2(\pi\epsilon) \tan(\pi\epsilon)$ the expansion starts from ϵ^3 and therefore it does not produce any contribution at order ϵ^2 . This means that we can cut the infinite sum at $k = p - 1$. Then we naturally spilt it into two parts

$$I_3^{(1)} = \sum_{k=q+1}^{p-1} \frac{d^2}{d\epsilon^2} \left[\frac{\Gamma(1+p+q)\Gamma(1-\epsilon+k)\Gamma(1-\epsilon+k-p)\Gamma(-\epsilon+k-q) \sin^2(\pi\epsilon) \tan(\pi\epsilon)}{4\pi^{7/2}\Gamma(\frac{3}{2}-\epsilon+k)} \frac{1}{t^{k+1-\epsilon}} \right] \Big|_{\epsilon=0} \\ + \sum_{k=0}^q \frac{d^2}{d\epsilon^2} \left[\frac{\Gamma(1+p+q)\Gamma(1-\epsilon+k)\Gamma(1-\epsilon+k-p)\Gamma(-\epsilon+k-q) \sin^2(\pi\epsilon) \tan(\pi\epsilon)}{4\pi^{7/2}\Gamma(\frac{3}{2}-\epsilon+k)} \frac{1}{t^{k+1-\epsilon}} \right] \Big|_{\epsilon=0}. \quad (6.6)$$

Then we make a replacement in both sums

$$\Gamma(1-\epsilon+k-p) \sin(\pi\epsilon) = \frac{\pi(-1)^{p+k}}{\Gamma(\epsilon+p-k)}, \quad (6.7)$$

and in the second one we also replace

$$\Gamma(-\epsilon+k-q) \sin(\pi\epsilon) = \frac{\pi(-1)^{1+k+q}}{\Gamma(1+\epsilon+q-k)}. \quad (6.8)$$

This gives

$$I_3^{(1)} = \sum_{k=q+1}^{p-1} \frac{d^2}{d\epsilon^2} \left[\frac{(-1)^{p+k}\Gamma(1+p+q)\Gamma(1-\epsilon+k)\Gamma(-\epsilon+k-q) \sin(\pi\epsilon) \tan(\pi\epsilon)}{4\pi^{5/2}\Gamma(\frac{3}{2}-\epsilon+k)\Gamma(\epsilon+p-k)} \frac{1}{t^{k+1-\epsilon}} \right] \Big|_{\epsilon=0} \\ + \sum_{k=0}^q \frac{d^2}{d\epsilon^2} \left[\frac{\Gamma(1+p+q)\Gamma(1-\epsilon+k) \tan(\pi\epsilon)}{4\pi^{3/2}\Gamma(\frac{3}{2}-\epsilon+k)\Gamma(\epsilon+p-k)\Gamma(1+\epsilon+q-k)} \frac{1}{t^{k+1-\epsilon}} \right] \Big|_{\epsilon=0}. \quad (6.9)$$

In the first sum $\sin(\pi\epsilon) \tan(\pi\epsilon) \sim \pi^2\epsilon^2$ in the limit $\epsilon \rightarrow 0$ which allows one to immediately find the corresponding contribution. To proliferate a comparison with the finite contributions delivered by I_1 and I_2 , it is convenient to implement in the first sum the change of the summation variable $k \rightarrow -k+p-1$, while in the second one $k \rightarrow -k+q$, correspondingly. This gives

$$I_3^{(1)} = -\Gamma(1+p+q) \sum_{k=0}^{p-q-2} \frac{(-1)^k \Gamma(p-k)\Gamma(p-q-k-1)}{2\sqrt{\pi}\Gamma(\frac{1}{2}-k+p)\Gamma(1+k)} t^{k-p} \\ + \sum_{k=0}^q \frac{d^2}{d\epsilon^2} \left[\frac{\Gamma(1+p+q)\Gamma(1-\epsilon-k+q) \tan(\pi\epsilon)}{4\pi^{3/2}\Gamma(\frac{3}{2}-\epsilon-k+q)\Gamma(\epsilon+p-q+k)\Gamma(1+\epsilon+k)} t^{k-q-1+\epsilon} \right] \Big|_{\epsilon=0}. \quad (6.10)$$

Since

$$\frac{1}{\Gamma(\frac{1}{2}-k+p)} = \frac{1}{\pi} (-1)^{k+p} \Gamma\left(\frac{1}{2}+k-p\right), \quad (6.11)$$

we observe that the first sum just becomes $-I_1$, while differentiation over ϵ in the second one leaves us with the following answer

$$I_3^{(1)} = -I_1 - \Gamma(1+p+q) \sum_{k=0}^q \frac{(-1)^{p+q}\Gamma(1-k+q)}{2\sqrt{\pi}\Gamma(1+k)\Gamma(k+p-q)\Gamma(\frac{3}{2}-k+q)} t^{k-q-1} \\ \times \left[\log t - \psi(1+k) - \psi(k+p-q) - \psi(1-k+q) + \psi\left(\frac{3}{2}-k+q\right) \right]. \quad (6.12)$$

Now taking into account eq. (6.11) as well as the fact that $\psi(\frac{3}{2} - k + q) = \psi(-\frac{1}{2} + k - q)$, we see that the second sum is nothing else but $-I_2$. Thus, we have found, that

$$I_3^{(1)} = -I_1 - I_2, \quad (6.13)$$

that is in the strong coupling expansion the contribution of $I_3^{(1)}$ cancels out against the sum $I_1 + I_2$.

Now we analyse the contribution of the terms \mathcal{F}_2 , which we denote as $I_3^{(2)}$. We have

$$I_3^{(2)} = \frac{i(-1)^{p+q}\Gamma(1+p+q)}{4\pi\sqrt{t}\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)} {}_3F_0\left(\left\{\frac{1}{2}, \frac{1}{2}-p, -\frac{1}{2}-q\right\}; \{0\}, \frac{1}{t}\right) \frac{d^2}{d\epsilon^2} \frac{e^{-i\pi\epsilon}}{\cos(\pi\epsilon)} \Big|_{\epsilon=0} = 0. \quad (6.14)$$

Finally, the contribution \mathcal{F}_3 is given by the following formula

$$\mathcal{F}_3 = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{2\sqrt{\pi}\Gamma(a_1)\Gamma(a_2)} t^{\frac{\nu}{2}} \left(e^{i\pi\nu-2\sqrt{t}} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{2^{-\ell} c_\ell}{(\sqrt{t})^\ell} + e^{2\sqrt{t}} \sum_{\ell=0}^{\infty} \frac{2^{-\ell} c_\ell}{(\sqrt{t})^\ell} \right), \quad (6.15)$$

where $\nu = a_1 + a_2 - b_1 - b_2 - b_3 + \frac{1}{2} = -2 - p - q - 2\epsilon$. Determination of the asymptotic coefficients c_ℓ represents a rather non-trivial task which we undertake in appendix D. There we show that the coefficients c_ℓ do not depend on ϵ and are given by the following explicit formula

$$c_\ell = \frac{\Gamma(2+\ell+p+q) {}_3F_2(\{-\ell, \frac{1}{2}+p, \frac{3}{2}+q\}, \{1-\frac{\ell}{2}+\frac{p}{2}+\frac{q}{2}, \frac{3}{2}-\frac{\ell}{2}+\frac{p}{2}+\frac{q}{2}\}, 1)}{(-2)^\ell \Gamma(\ell+1) \Gamma(2-\ell+p+q)}. \quad (6.16)$$

Hence we have the following contribution of \mathcal{F}_1 which we denote $I_3^{(3)}$,

$$I_3^{(3)} = \frac{d^2}{d\epsilon^2} \frac{(-1)^{p+q+1} \Gamma(1+p+q)}{8\pi^2 t (\sqrt{t})^{p+q}} \left((-1)^{p+q} e^{-2\pi i \epsilon} e^{-2\sqrt{t}} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{2^{-\ell} c_\ell}{(\sqrt{t})^\ell} + e^{2\sqrt{t}} \sum_{\ell=0}^{\infty} \frac{2^{-\ell} c_\ell}{(\sqrt{t})^\ell} \right) \Big|_{\epsilon=0}.$$

Differentiating over ϵ and taking the limit $\epsilon \rightarrow 0$ leaves us with the following expression

$$I_3^{(3)} = \frac{\Gamma(1+p+q)}{2t(\sqrt{t})^{p+q}} e^{-2\sqrt{t}} \sum_{k=0}^{\infty} (-1)^k \frac{2^{-k} c_k}{(\sqrt{t})^k}. \quad (6.17)$$

Note that the growing exponent $e^{2\sqrt{t}}$ does not enter the asymptotic expansion. Recalling that $t = h^2/4$ we arrive at the following strong coupling asymptotic expansion of the integral (5.14)

$$f(h) = e^{-h} \frac{2^{p+q+1} \Gamma(p+q+1)}{h^{p+q+1}} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{c_\ell}{h^\ell}. \quad (6.18)$$

With this expression at hand we can now find the asymptotic expansion of $\Delta S_{p,q}$. According to eq. (5.13) we have

$$\begin{aligned} \Delta S_{p,q}(g) = & -ig(p-q-1)(p+q) \sum_{k=0}^q \sum_{m=0}^{p-k-2} \frac{(-1)^{m+1} \sqrt{\pi} 2^{3-k-m+q} q! \Gamma(p+m-k-1)}{k! m! (q-k)! \Gamma(p-m-k-1) \Gamma(\frac{3}{2}+k-q)} \\ & \times \sum_{\ell=0}^{\infty} (-1)^\ell c_\ell \sum_{n=1}^{\infty} h_n^{p+k-m-1} \frac{\partial^{p+k-m}}{\partial h_n^{p+k-m}} \left[\frac{e^{-h_n}}{h_n^{\ell+p+q+1}} \right]. \end{aligned} \quad (6.19)$$

Performing differentiations we get

$$\begin{aligned} \Delta S_{p,q}(g) = & \tag{6.20} \\ & ig(p-q-1)(p+q) \sum_{k=0}^q \sum_{m=0}^{p-k-2} \frac{(-1)^{p+k} \sqrt{\pi} 2^{3-k-m+q} q! \Gamma(p+m-k-1)}{k! m! (q-k)! \Gamma(p-m-k-1) \Gamma(\frac{3}{2}+k-q)} \\ & \times \sum_{\ell=0}^{\infty} (-1)^{\ell} c_{\ell} \sum_{s=q+m-k}^{p+q} \frac{\Gamma(p-m+k+1) \Gamma(1+k+p+\ell+s-m)}{\Gamma(1+\ell+p+q) \Gamma(1+s+k-m-q) \Gamma(1+p+q-s)} \sum_{n=1}^{\infty} \frac{e^{-h_n}}{h_n^{2+\ell+s}}. \end{aligned}$$

Due to the gamma function standing in the middle of the denominator in the second line of the above formula, the sum over s can be extended down to zero. Next we introduce a “loop” parameter $L = \ell + s + 3$ and change the order of summation arranging the sum over L to precede the one over ℓ :

$$\begin{aligned} \Delta S_{p,q}(g) = & \tag{6.21} \\ & ig(p-q-1)(p+q) \sum_{k=0}^q \sum_{m=0}^{p-k-2} \frac{(-1)^{p+k} \sqrt{\pi} 2^{3-k-m+q} q! \Gamma(p+m-k-1)}{k! m! (q-k)! \Gamma(p-m-k-1) \Gamma(\frac{3}{2}+k-q)} \\ & \times \left[\sum_{L=3}^{p+q+2} \sum_{\ell=0}^{L-3} \frac{(-1)^{\ell} c_{\ell} \Gamma(p-m+k+1) \Gamma(k+p+L-m-2)}{\Gamma(1+\ell+p+q) \Gamma(L-2-\ell+k-m-q) \Gamma(4+p+q-L+\ell)} \sum_{n=1}^{\infty} \frac{e^{-h_n}}{h_n^{L-1}} \right. \\ & \left. + \sum_{L=p+q+3}^{\infty} \sum_{\ell=L-3-p-q}^{L-3} \frac{(-1)^{\ell} c_{\ell} \Gamma(p-m+k+1) \Gamma(k+p+L-m-2)}{\Gamma(1+\ell+p+q) \Gamma(L-2-\ell+k-m-q) \Gamma(4+p+q-L+\ell)} \sum_{n=1}^{\infty} \frac{e^{-h_n}}{h_n^{L-1}} \right]. \end{aligned}$$

Here in the last line the lower integration bound $L - 3 - p - q \geq 0$ of the variable ℓ can be extended down to zero without changing the answer because of the gamma function $\Gamma(4+p+q-L+\ell)$. This allows one to combine two sums over L and obtain a formula

$$\begin{aligned} \Delta S_{p,q}(g) = & \tag{6.22} \\ & ig(p-q-1)(p+q) \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} \\ & \times \sum_{\ell=0}^{L-3} \frac{(-1)^{\ell} c_{\ell}}{\Gamma(1+\ell+p+q) \Gamma(4+p+q-L+\ell) \Gamma(L-2-\ell+k-m-q)} \\ & \times \sum_{k=0}^q \sum_{m=0}^{p-k-2} \frac{(-1)^{p+k} \sqrt{\pi} 2^{3-k-m+q} q! \Gamma(p+m-k-1)}{k! m! (q-k)! \Gamma(p-m-k-1) \Gamma(\frac{3}{2}+k-q) \Gamma(p-m+k+1) \Gamma(k+p+L-m-2)}, \end{aligned}$$

where we have taken into account that

$$\sum_{n=1}^{\infty} \frac{e^{-h_n}}{h_n^{L-1}} = \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}}. \tag{6.23}$$

In appendix D by using the explicit form (6.16) of the coefficients c_{ℓ} we bring the expression for discontinuity $\Delta S_{p,q}(g)$ found above to the following form

$$\Delta S_{p,q}(g) = (4ig)(p-q-1)(p+q) \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} c_L(p, q), \tag{6.24}$$

where the coefficients $c_L(p, q)$ are given by

$$c_L(p, q) = \sum_{k=0}^{L-3} \frac{\Gamma(p + \frac{1}{2} + k)\Gamma(q + \frac{3}{2} + k)}{\Gamma(p + \frac{1}{2})\Gamma(q + \frac{3}{2})\Gamma(k + 1)} \tag{6.25}$$

$$\times \sum_{n=0}^{k+p+q} \frac{(-1)^n 2^{2n+3} (n+1)}{\sqrt{\pi}} \frac{\Gamma(p + \frac{1}{2} + n)\Gamma(q + \frac{3}{2} + n)}{\Gamma(p + q + 1 + k - n)\Gamma(n + \frac{3}{2})\Gamma(2n + 5 - L)}.$$

In appendix D we also provide an alternative method to compute the discontinuity $\Delta S_{p,q}(g)$ and find the same expression (D.37).

Note that not only the original perturbative coefficients (3.15), but also these new perturbative coefficients of the non-perturbative sector are factorially growing with L . This is absolutely not obvious from the above equation but it is ultimately related to the fact that these coefficients were derived from the strong coupling expansion of the Laplace integral (5.5) of the discontinuity $\text{Disc } \hat{\phi}_{p,q}(x)$ which has a singular behaviour at $x = -1$, hence the asymptoticity of this expansion. As mentioned below (3.19), the discontinuity of $\text{Disc } \hat{\phi}_{p,q}(x)$ is related to the non-entire part of the original Borel transform itself $\hat{\phi}_{p,q}(x)$. This is also the reason why the transseries (4.25) constructed via the median resummation is just a two-term transseries. Although neither the current expression for $c_L(p, q)$ nor the one we will derive shortly, see equation (7.21), are particularly amenable to extract analytically the asymptotic form valid for $L \gg 1$, we can nonetheless perform a numerical study to obtain the asymptotic form

$$c_L(p, q) \sim \frac{(-1)^L \Gamma(L-2)}{\pi (p-q-1)(p+q)} \left(\frac{(4\pi)^3 c_{p,q}^{(3)}}{\zeta(3)} + \frac{(4\pi)^5 c_{p,q}^{(5)}}{\zeta(5) (L-3)(L-4)} \right. \tag{6.26}$$

$$\left. + \frac{(4\pi)^7 c_{p,q}^{(7)}}{\zeta(7) (L-3)(L-4)(L-5)(L-6)} + O(L^{-6}) \right)$$

where $c_{p,q}^{(n)}$ are precisely the perturbative coefficients given in equation (3.15). Note that only the perturbative coefficients with n odd appear in the asymptotic formula because the even ones, being finite in number, give rise to a modified Borel transform which is entire and cannot possibly be captured by the strong coupling coefficients of the discontinuity analysed in this section. As a check we can compute the difference between the coefficients obtained via (6.25) and the first term in the asymptotic expansion (6.26)

$$d_L = \left(c_L(p, q) \times \frac{\pi (p-q-1)(p+q)}{(-1)^L \Gamma(L-2)} - \frac{(4\pi)^3 c_{p,q}^{(3)}}{\zeta(3)} \right) \times (L-3)(L-4) \sim \frac{(4\pi)^5 c_{p,q}^{(5)}}{\zeta(5)} + O(L^{-2})$$

and in figure 4 we plot d_L as a function of L for $p = 3$ and $q = 1$. Using the perturbative coefficients given in equation (3.15) one obtains $(4\pi)^5 c_{3,1}^{(5)}/\zeta(5) = -8400$ matching perfectly with the predicted asymptotic expansion as seen in the figure.

7 Dispersion relation and the non-perturbative sector

Having computed the discontinuities of the modified Borel transform across the two Stokes directions 0 and π , we can obtain the asymptotic expansion of the perturbative coefficients

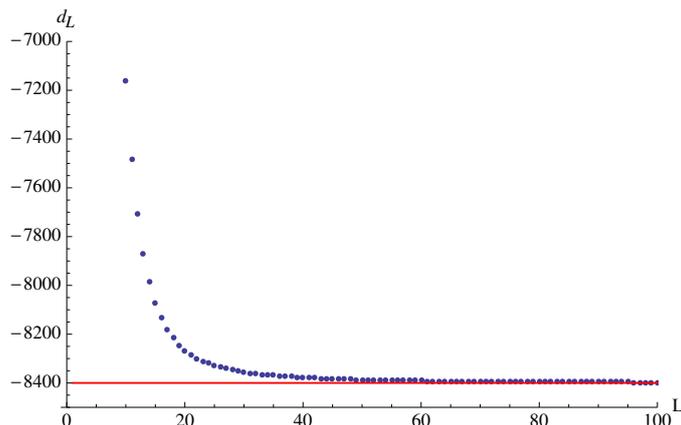


Figure 4. Plot of the value for d_L as a function of L when $p = 3$ and $q = 1$. Asymptotically it reaches the expected value (red line) $(4\pi)^5 c_{3,1}^{(5)} / \zeta(5) = -8400$.

$c_{p,q}^{(n)}$ for $n \gg 1$, via a standard dispersion-like type of argument [58, 59]. The way to proceed is to consider the Cauchy integral for the analytic continuation (4.21) of the purely perturbative series $c_{p,q}^P(g)$:

$$c_{p,q}^P(g) = \frac{1}{2\pi i} \oint dw \frac{c_{p,q}^P(w)}{w - g}, \tag{7.1}$$

where the contour is around the complex point g .

Making use of

$$\frac{1}{w - g} = - \sum_{n=0}^{\infty} w^n g^{-n-1}, \tag{7.2}$$

valid for $g \rightarrow \infty$, we can read the perturbative coefficients $c_{p,q}^{(n)}$ from the contour integral

$$\begin{aligned} c_{p,q}^{(n)} &= - \frac{1}{2\pi i} \oint dw c_{p,q}^P(w) w^{n-2} \\ &= - \frac{1}{2\pi i} \int_0^{\infty} dw \text{Disc}_0 c_{p,q}^P(w) w^{n-2} - \frac{1}{2\pi i} \int_0^{-\infty} dw \text{Disc}_{\pi} c_{p,q}^P(w) w^{n-2}, \end{aligned} \tag{7.3}$$

where we pushed the contour of integration to infinity as depicted in figure 5, under that assumption that the residue at infinity of $c_{p,q}^P$ vanishes.

We know the discontinuities across the singular directions 0 and π from (4.22)–(4.23) so

$$c_{p,q}^{(n)} = \frac{1}{2\pi i} \int_0^{\infty} dw \Delta S_{p,q}(w) w^{n-2} + \frac{1}{2\pi i} \int_0^{-\infty} dw \Delta S_{p,q}(-w) w^{n-2}.$$

To compute these two integral we make use of the perturbative expansion (6.24), and, by analyzing loop order, L , by loop order we simply need to evaluate

$$\int_0^{\infty} dw \frac{w \text{Li}_{L-1}(e^{-4\pi w})}{(4\pi w)^{L-1}} w^{n-2}, \tag{7.4}$$

that, for $n \gg 1$, gives

$$\frac{\zeta(n)\Gamma(n + 1 - L)}{(4\pi)^n}. \tag{7.5}$$

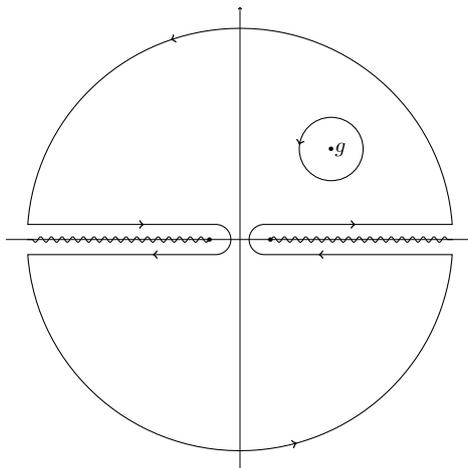


Figure 5. The Cauchy contour around the point g can be closed outward as a sum over Hankel contours.

So, by putting everything together, we obtain the asymptotic expansion valid for $n \gg 1$

$$c_{p,q}^{(n)} \sim \frac{2(1 - (-1)^n)}{\pi} \frac{\zeta(n)\Gamma(n-2)}{(4\pi)^n} (p+q)(p-q-1) \times \left(c_3(p,q) + \frac{c_4(p,q)}{n-3} + \frac{c_5(p,q)}{(n-3)(n-4)} + \mathcal{O}(n^{-3}) \right), \quad (7.6)$$

where the first three coefficients are

$$\begin{aligned} c_3(p,q) &= 4(-1)^{(p-q)}, \\ c_4(p,q) &= 4(-1)^{(p-q)} \times \left(2p(p-1) + \frac{1}{2}(2q+1)^2 \right), \\ c_5(p,q) &= 4(-1)^{(p-q)} \times \frac{1}{8} \left(-3 + 4p(p-1) + 4q(q+1) \right) \left(4p(p-1) + (2q+1)^2 \right). \end{aligned} \quad (7.7)$$

Note that the n even coefficients completely disappear from this analysis because, as explained before, the $c_{p,q}^{(n)}$ with n even are non-vanishing only for a finite number of terms. The large order behaviour of the perturbative coefficients captures precisely the lower order perturbative coefficients on top of the non-perturbative contributions in the transseries (4.25), i.e. the coefficients for the strong coupling expansion of $\tilde{\Phi}_{p,q}^{\text{NP}}(g)$. In figure 6 we show how well, at large n , the perturbative coefficients $c_{p,q}^{(n)}$ can be approximated by even their leading asymptotic expansion

$$c_{p,q}^{(n)\text{As}} = \frac{2(1 - (-1)^n)}{\pi} \frac{\zeta(n)\Gamma(n-2)}{(4\pi)^n} (p+q)(p-q-1)c_3(p,q), \quad (7.8)$$

and also how one could numerically extrapolate even the subleading corrections, $c_4(p,q)$, $c_5(p,q)$, etc., shown in (7.6).

The formula (7.6) allows us to obtain an explicit formula for the polynomials $c_L(p,q)$ by comparing the large n asymptotic expansion of the coefficients $c_{p,q}^{(n)}$ with the right hand

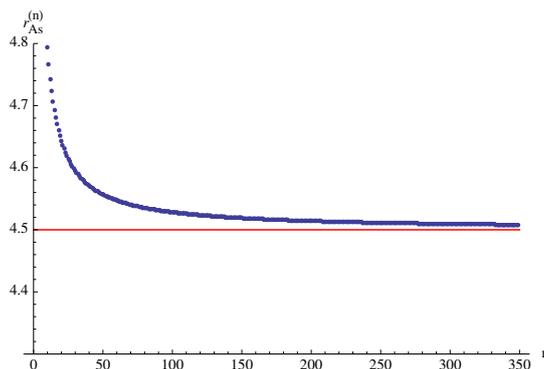


Figure 6. We plot $r_{As}^{(n)} = (R^{(n)}(2, 0) - 1) \times (n - 3) \sim \frac{9}{2} + O(n^{-1})$ which measures the deviation from 1 of the ratio $R^{(n)}(p, q)$ (7.9) for the particular case $p = 2, q = 0$.

side of (7.6). Hence, we need to asymptotically expand formula (3.15) for large n . To this end we consider the ratio between $c_{p,q}^{(n)}$ and its leading asymptotic coefficient which for n odd takes the form

$$R^{(n)}(p, q) = \frac{c_{p,q}^{(n)}}{c_{p,q}^{(n)As}} = \frac{c_{p,q}^{(n)} \pi (4\pi)^n}{4\zeta(n)\Gamma(n-2)(p+q)(p-q-1)c_3(p, q)} \sim 1 + \frac{c_4(p, q)}{c_3(p, q)} \frac{1}{(n-3)} + \frac{c_5(p, q)}{c_3(p, q)} \frac{1}{(n-3)(n-4)} + O(n^{-3}). \quad (7.9)$$

In what follows it appears advantageous to use the change of variables $n = 2(m + 1)$ where m is half-integer and replace $q \rightarrow q - 1$. Then for $R^{(n)}(p, q - 1)$ we get

$$R^{(n)}(p, q - 1) = \frac{2^{4m} m}{\pi} B(m + 1 - p, m + p) B(m + 1 - q, m + q), \quad (7.10)$$

where $B(a, b)$ is the Euler beta integral

$$B(a, b) = \int_0^1 dv v^{b-1} (1-v)^{a-1}. \quad (7.11)$$

We observe that in the formula (7.10) contribution of p and q completely factorises and comes in a symmetric fashion. Therefore, our task now is to find an asymptotic expansion of the Euler integral when $m \rightarrow \infty$. First we compute the integral by means of the saddle point method. Consider

$$B \equiv B(m + 1 - p, m + p) = \int_0^1 dv v^{m+p-1} (1-v)^{m-p} = \int_0^1 dv v^{p-1} (1-v)^{-p} e^{m \log(v(1-v))}. \quad (7.12)$$

For large m the dominant contribution to this integral comes from the critical point $v = \frac{1}{2}$ for which the “action” is $\log(v(1-v))|_{v=1/2} = -\log 4$. This motivates to perform the following change of integration variable

$$t = \log(4) - \log(v(1-v)) \quad (7.13)$$

which converts the integral to

$$B = 2^{-2m} \int_0^\infty dt \frac{e^{-mt}}{2\sqrt{e^t-1}} \left[(e^{t/2} + \sqrt{e^t-1})^{2p-1} + (e^{t/2} - \sqrt{e^t-1})^{2p-1} \right]. \quad (7.14)$$

Now using binomial expansions twice we rewrite the integrand as a double sum

$$B = 2^{-2m} \sum_{s=0}^{p-1} \sum_{r=0}^s \binom{2p-1}{2s} \binom{s}{r} (-1)^r \int_0^\infty dt \frac{e^{(-m+p-r-1/2)t}}{\sqrt{e^t-1}}. \quad (7.15)$$

Evaluating this integral in the regime $m \gg 1$ and further performing one summation we arrive at

$$B = -2^{-2m} \sqrt{\pi} \sum_{r=0}^{p-1} (-1)^{p-r} 4^r \frac{(2p-1)\Gamma(p+r)}{\Gamma(2r+2)\Gamma(p-r)} \times \frac{\Gamma(m-r)}{\Gamma(m+\frac{1}{2}-r)}. \quad (7.16)$$

The ratio of two gamma function has the known asymptotic expansion in the limit $m \rightarrow \infty$, namely

$$\frac{\Gamma(m-r)}{\Gamma(m+\frac{1}{2}-r)} \sim m^{-1/2} \sum_{l=0}^{\infty} (-1)^l (1/2)_l \frac{B_l^{(1/2)}(-r)}{l!} \frac{1}{m^l}, \quad (7.17)$$

where $B_l^{(1/2)}(-r)$ are the generalised Bernoulli polynomials also known as Norlund polynomials, see e.g. [60].

Using this result we can obtain the asymptotic expansion of the Euler beta for $m \gg 1$:

$$B(m+1-p, m+p) \sim -2^{-2m} \sqrt{\frac{\pi}{m}} \times \sum_{l=0}^{\infty} \frac{1}{m^l} d_l(p), \quad (7.18)$$

where

$$d_l(p) = (-1)^{p+l} \frac{(2p-1)(1/2)_l}{l!} \sum_{r=0}^{p-1} (-1)^r 4^r B_l^{(1/2)}(-r) \frac{\Gamma(p+r)}{\Gamma(2r+2)\Gamma(p-r)}. \quad (7.19)$$

The function $R_n(p, q-1)$ can be expanded for large n , using the variable $n = 2(m+1)$, via the convolution of the above coefficients

$$R_n(p, q-1) \sim \sum_{l=0}^{\infty} \frac{1}{m^l} \left(\sum_{k=0}^l d_k(p) d_{l-k}(q) \right). \quad (7.20)$$

This is not quite the expansion we sought for, as shown in equation (7.9) we want to express this ratio as

$$R_n(p, q-1) \sim 1 + \frac{c_4(p, q-1)}{c_3(p, q-1)} \frac{1}{2m-1} + \frac{c_5(p, q-1)}{c_3(p, q-1)} \frac{1}{(2m-1)(2m-2)} + \mathcal{O}(m^{-3}).$$

We can easily relate one expansion to the other via

$$c_L(p, q) = -4 (-1)^{p+q} \sum_{l=0}^{L-3} S_{L-3}^{(l)} 2^l \left(\sum_{k=0}^l d_k(p) d_{l-k}(q+1) \right), \quad (7.21)$$

where $S_L^{(l)}$ denotes the Stirling number of the first kind. This expression looks completely different from the one previously obtained in equation (6.25), nonetheless one can check numerically that the two expressions match perfectly. Even in this alternative derivation the factorial growth with L of the coefficients $c_L(p, q)$ is not manifest.

As we will shortly see, these coefficients $c_L(p, q)$, polynomials in p and q , will give rise to important non-perturbative contribution to the dressing phase (2.1).

8 Non-perturbative contributions to the dressing phase

We have shown in section 4, that the purely perturbative asymptotic power series expansion (2.6), is not enough to reconstruct the coefficients of the BES dressing phase (4.15). We need to replace the perturbative expansion by the transseries representation (4.25) whose Borel-Ecalle resummation (4.29) matches precisely the non-perturbative result (4.15). In this section we see the effects that our transseries expansion produces to the dressing phase.

8.1 Effects of the non-perturbative sector to the dressing phase

Our replacement from the perturbative power series (2.6) to the transseries (4.25) is not without consequences. In [13] the authors showed that if we restrict the sum (2.6) to even n , we obtain a strong coupling solution to the crossing symmetry equation (2.9). This particular solution does not have the correct weak coupling limit and for this reason the authors considered the analytic continuation of the series (2.6) by summing over all the integers n . This amounts to adding to the dressing phase a solution to the homogeneous crossing symmetry equation

$$i\theta(x_j, x_k) + i\theta(1/x_j, x_k) = 0. \tag{8.1}$$

The BES coefficients (4.15) proposed in [14] thus interpolates between the formal power series expansion (2.6) at strong coupling and the correct gauge theory weak coupling limit. The crucial point is that this integral representation for the BES coefficients is not quite equivalent to the formal power series (2.6), but rather it is obtained via the Borel-Ecalle resummation of the transseries (4.25). This means that the non-perturbative terms we added must lead to additional contributions to the dressing phase, solutions to the homogeneous crossing symmetry equation (8.1).

Let us compute this additional non-perturbative contributions to the dressing phase. Since we know from [13] that the formal power series (2.6) solves the full crossing symmetry equation (2.9) we can just focus on the purely non-perturbative terms (6.24) of our transseries ansatz (4.25). The non-perturbative contributions to the function $\chi(x_1, x_2)$, given by equation (2.5) and written using p, q variables is

$$\chi_{\text{NP}}(x_1, x_2) = \mathfrak{s} \sum_{p=2}^{\infty} \sum_{q=0}^{p-2} \frac{i\Delta S_{p,q}(g)}{(p-q+1)(p+q)} \frac{1}{x_1^{p-q-1} x_2^{p+q}}, \tag{8.2}$$

where \mathfrak{s} is once again the transseries parameter discussed in section 4.2, i.e. $\mathfrak{s} = -i/2$ for $0 < \text{Arg } g < \pi/2$ and $\mathfrak{s} = +i/2$ for $-\pi/2 < \text{Arg } g < 0$.

We do not know how to perform this double sum (8.2) using the exact integral representation (5.5) for $\Delta S_{p,q}(g)$, but we can easily compute it loop order by loop order using the strong coupling expansion (6.24).

Using (6.24) we can rewrite the above equation in the form of the loop expansion

$$\chi_{\text{NP}}(x_1, x_2) = -4\mathfrak{s} \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} \chi_{\text{NP}}^{(L)}(x_1, x_2), \quad (8.3)$$

with the definition

$$\chi_{\text{NP}}^{(L)}(x_1, x_2) = \sum_{p=2}^{\infty} \sum_{q=0}^{p-2} \frac{c_L(p, q)}{x_1^{p-q-1} x_2^{p+q}}, \quad (8.4)$$

where the coefficients $c_L(p, q)$ are given for example by (6.25) (see also the other equivalent forms presented in equation (7.21) and appendix D). Note that the coupling constant, g , only appears in front of the series (8.3). It is surprising that the L^{th} loop contribution to χ_{NP} coming from *all* instantons sectors can be fully resummed giving rise to the exponentially suppressed factor $\text{Li}_{L-1}(e^{-4\pi g})$. From (2.3) one could think that we have to expand $x_{1,2}$ as functions of g , but to do so one has to specify how the momenta of the scattered particles scale with g and this give rise to many different regimes to consider, for instance, BMN/spinning strings or “giant magnon”. Picking up a certain expansion of $x_{1,2}$ identifies the corresponding quasi-particle excitations and in this respect reflects the physics of the problem, however the additional non-perturbative terms we predicted must still give rise to solutions to the homogeneous crossing symmetry equation (8.1) which effectively means that we can treat $x_{1,2}^{\pm}$ as independent of g for the purpose of this section.

From the explicit coefficients (6.25)–(7.7) it is a straightforward calculation to obtain the first few

$$\begin{aligned} \chi_{\text{NP}}^{(3)}(x_1, x_2) &= \frac{4x_2}{(1+x_1x_2)(x_2^2-1)}, \\ \chi_{\text{NP}}^{(4)}(x_1, x_2) &= \frac{2x_2(1-6x_1x_2+14x_2^2+x_1^2x_2^2+36x_1x_2^3+x_2^4+6x_1^2x_2^4+2x_1x_2^5+9x_1^2x_2^6)}{(1+x_1x_2)^3(x_2^2-1)^3}, \\ \chi_{\text{NP}}^{(5)}(x_1, x_2) &= \frac{x_2P(x_1, x_2)}{(1+x_1x_2)^5(x_2^2-1)^5}, \end{aligned} \quad (8.5)$$

where $P(x_1, x_2)$ is a certain polynomial of degree 4 and 12 respectively in x_1 and x_2 . The reader can easily develop higher order contributions to $\chi_{\text{NP}}^{(L)}$ from the formula (8.4).

At this point it is simply a matter of calculation to plug these non-perturbative contributions $\chi_{\text{NP}}^{(L)}$ into the dressing phase (2.4) and show that these new terms are solutions to the homogeneous crossing symmetry equations (8.1). Note that the full series (8.3) is a solution to the homogenous equation because every order in the g^{-1} expansion solves (8.1): i.e. the coefficient $\chi_{\text{NP}}^{(L)}(x_1, x_2)$ of the g^{-L+1} term is already on its own a solution to the crossing symmetry equation coming from the resummation of infinitely many instanton sectors.

Note that the first non-perturbative contribution is given by $\chi_{\text{NP}}^{(3)}$, which corresponds to a three-loop perturbative correction g^{-2} , on top of a non-perturbative background.

As mentioned in the Introduction, the vanishing of the tree level, one- and two-loops contributions might be explained by a protection mechanism based on vanishing of the zero mode factors, forcing perturbation theory on top of these mysterious non-perturbative saddles to start from three-loops .

We claim that the complete non-perturbative correction (8.2) to the dressing phase, since it is a formal sum (8.3) of homogeneous solutions, gives also rise to a solution to the homogeneous crossing symmetry equation (8.1), very likely *not* of the simple rational form in x_1, x_2 as the coefficients $\chi_{\text{NP}}^{(L)}$ just encountered.

8.2 Generating solutions to the homogenous crossing symmetry equation

From the large order behaviour (7.6) of the perturbative coefficients (2.8) we can construct a generating functional to obtain solutions to the homogeneous crossing symmetry equations (8.1). In [13] the authors noticed that the perturbative coefficient $c_{p,q}^{(n)}$, with n odd, $n > 1$, generates a contribution to the dressing phase that solves (8.1). Similarly, for $n \gg 1$, we can consider the asymptotic expansion (7.6) and, as we have just seen, for each loop order L , the perturbative coefficient $c_L(p, q)$ yields once again solutions to the homogeneous crossing symmetry equation.

Thus we can consider, similarly to (2.5), the expression

$$\sum_{p=2}^{\infty} \sum_{q=0}^{p-2} \frac{c_{p,q}^{(2z+1)}}{(p-q-1)(p+q)} \frac{1}{x_1^{p-q-1} x_2^{p+q}}. \tag{8.6}$$

When $n = 2z + 1$ is an odd integer, this function reproduces the known perturbative contributions to the dressing phase. Viceversa, when $z \gg 1$, we know from (7.6) that the perturbative coefficients $c_{p,q}^{(2z+1)}$ can be written as an asymptotic expansion in z^{-1} in terms of the non-perturbative sector's coefficients $c_L(p, q)$ (7.21). Thanks to the analysis of the previous section, each one of these terms will produce a solution to (8.1) and equation (8.6) will basically sum up all of these contributions and it will still solve the homogeneous problem since it is a linear problem. Hence equation (8.6) is somehow interpolating between the perturbative and the non-perturbative solutions to (8.1).

Discarding from equation (8.6) an overall factor which is only z -dependent, we consider the generating functional for homogenous solutions to the crossing symmetry equation (8.1) given by

$$\Xi(z; x_1, x_2) = \sum_{p=2}^{\infty} \sum_{q=0}^{p-2} \frac{(-1)^{p+q}}{x_1^{p-q-1} x_2^{p+q}} (z-1/2)_p (z+1/2)_{-p} (z+1/2)_q (z-1/2)_{-q}. \tag{8.7}$$

The sum over q can be easily performed giving

$$\Xi(z; x_1, x_2) = \sum_{p=2}^{\infty} (I_1 + I_2), \tag{8.8}$$

with

$$I_1 = (-1)^p (z-1/2)_p (z+1/2)_{-p} \frac{{}_2F_1\left(1, \frac{1}{2} + z; \frac{3}{2} - z \mid \frac{x_1}{x_2}\right)}{x_1^p x_2^{p-1}}, \tag{8.9}$$

and

$$I_2 = [(z - 1/2)_p (z + 1/2)_{-p}]^2 \frac{{}_2F_1\left(1, p - \frac{1}{2} + z; p + \frac{1}{2} - z \mid \frac{x_1}{x_2}\right)}{x_2^{2p-1}}. \quad (8.10)$$

Note that this analysis is very similar to the one carried out in [21], particularly the discussion in their appendix C. However, in [21] the authors considered only the magnon regime for which x_1 and x_2 take a rather special form at strong coupling, thus allowing to simplify equation (8.7). Furthermore, since the authors were interested in extracting the leading contribution they focused only on the case where z is an even integer. In the present discussion we do not make any assumption on x_1 and x_2 and we are interested in the full series (8.7) and not just in its leading behaviour; this complicates the story considerably.

For I_1 the sum over p is straightforward

$$S_1 = \sum_{p=2}^{\infty} I_1 = \frac{(1+2z) {}_2F_1\left(1, \frac{3}{2} + z; \frac{5}{2} - z \mid \frac{1}{x_1 x_2}\right)}{(2z-3)x_1 x_2} \frac{{}_2F_1\left(1, \frac{1}{2} + z; \frac{3}{2} - z \mid \frac{x_1}{x_2}\right)}{x_2}. \quad (8.11)$$

The first term in S_1 is obviously symmetric in $x_1 \leftrightarrow x_2$, while for $z \in \mathbb{N}$ one can easily show using the inversion formula for the hypergeometric function⁸ that also the second fraction is symmetric. This means that, for $z \in \mathbb{N}$, the contribution of S_1 to the dressing phase (2.4) is actually zero.

The second contribution to Ξ comes from

$$S_2 = \sum_{p=2}^{\infty} I_2 = \sum_{p=2}^{\infty} [(z - 1/2)_p (z + 1/2)_{-p}]^2 \frac{{}_2F_1\left(1, p - \frac{1}{2} + z; p + \frac{1}{2} - z \mid \frac{x_1}{x_2}\right)}{x_2^{2p-1}}. \quad (8.12)$$

This sum is trickier than S_1 because the index of summation p appears in the parameters of the hypergeometric function. We notice that, for $z \in \mathbb{N}$, the difference between the two parameters c and b of the hypergeometric is $p + 1/2 - z - (p - 1/2 + z) = 2z - 1 \in \mathbb{N}$. This allows us to use the reduction formula, see e.g. [56],

$${}_2F_1\left(1, p - \frac{1}{2} + z; p + \frac{1}{2} - z \mid \frac{x_1}{x_2}\right) = \sum_{k=0}^{2z-1} \binom{2z-1}{k} \frac{k!}{(p-z+1/2)_k} \left(\frac{x_1}{x_2-x_1}\right)^k \frac{x_2}{x_2-x_1}.$$

The sum over p can be now performed

$$S_2 = \frac{[(1+2z)\Gamma(3/2-z)]^2}{4x_2^2(x_2-x_1)} \times \sum_{k=0}^{2z-1} \left(\frac{x_1}{x_2-x_1}\right)^k \frac{\Gamma(2z)}{\Gamma(2z-k)} {}_3\tilde{F}_2\left(1, z + \frac{3}{2}, z + \frac{3}{2}; \frac{5}{2} - z, \frac{5}{2} + k - z \mid \frac{1}{x_2^2}\right), \quad (8.13)$$

where ${}_3\tilde{F}_2$ denotes the generalized hypergeometric function regularized.

For $z \in \mathbb{N}$, our generating functional $\Xi(z; x_1, x_2)$ produces only rational functions of x_1, x_2 . In particular, using the explicit formulas (8.11)–(8.13), we can easily check that

⁸We simply used equation (15.8.2) of [61] for the case at hand.

$\Xi(1; x_1, x_2)$ coincides precisely (modulo an overall numerical factor) with the three world sheet loops contribution $\chi^{(3)}(x_1, x_2)$ presented in equation (5.6) of [13].

Similarly, from our studies of large order behaviour (7.6), we expect the following behaviour of the generating functional for $z \gg 1$

$$\Xi(z; x_1, x_2) \sim \chi_{\text{NP}}^{(3)}(x_1, x_2) + \frac{\chi_{\text{NP}}^{(4)}(x_1, x_2)}{(2z-2)} + \frac{\chi_{\text{NP}}^{(5)}(x_1, x_2)}{(2z-2)(2z-3)} + \mathcal{O}(z^{-3}), \quad (8.14)$$

where the rational function $\chi_{\text{NP}}^{(L)}(x_1, x_2)$ are precisely the non-perturbative contributions to the dressing phase (8.5) previously computed.

It would be interesting to obtain an analytic expression for $\Xi(z; x_1, x_2)$ for arbitrary values of z and show that it solves the homogenous crossing symmetry equation.

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A Derivation of the discontinuity of Ω

To obtain the discontinuity of Ω , cf. (3.17), we first apply the reduction technique [56] which allows one to reduce our particular ${}_4F_3$ to a multiple derivative of ${}_2F_1$ with respect to the variable z by means of the following formula

$$\begin{aligned} \Omega &= \sum_{k=0}^{p-2} \sum_{m=0}^q \binom{p-2}{k} \binom{q}{m} \frac{1}{(5/2)_k (3/2)_m} z^m \frac{d^m}{dz^m} z^k \frac{d^k}{dz^k} f \\ &= \sum_{k=0}^{p-2} \sum_{m=0}^q \sum_{s=0}^m \binom{p-2}{k} \binom{q}{m} \binom{m}{s} \frac{1}{(5/2)_k (3/2)_m} \frac{\Gamma(k+1)}{\Gamma(1-m+k+s)} z^{k+s} \frac{d^{k+s} f}{dz^{k+s}}, \end{aligned} \quad (A.1)$$

where we have introduced a concise notation

$$f(z) := {}_2F_1\left(\frac{3}{2} - p, \frac{1}{2} - q, 2, z\right). \quad (A.2)$$

First the sum over m can be straightforwardly taken leaving behind

$$\Omega = \sum_{k=0}^{p-2} \sum_{s=0}^q \frac{3 \times 2^{2(k+s)} \Gamma(-1+p) \Gamma(1+q)}{(3+2k) \Gamma(2+2k) \Gamma(p-k-1) \Gamma(1+q-s) \Gamma(2+2s)} z^{k+s} \frac{d^{k+s} f}{dz^{k+s}}.$$

And further summation gives

$$\Omega = \frac{3\pi \Gamma(p-1)}{8 \Gamma(q + \frac{3}{2})} \times \sum_{\ell=0}^{p+q-2} \frac{{}_3\tilde{F}_2(\{\frac{3}{2} + \ell, 2 + \ell - p - q, -q\}, \{1 + \ell - q, \frac{5}{2} + \ell - q\}, 1)}{\Gamma(p+q-\ell-1)} z^\ell \frac{d^\ell f}{dz^\ell}, \quad (\text{A.3})$$

where ${}_3\tilde{F}_2$ stands for the regularised hypergeometric function, which is given in fact by the finite sum

$$\frac{{}_3\tilde{F}_2(\{\frac{3}{2} + \ell, 2 + \ell - p - q, -q\}, \{1 + \ell - q, \frac{5}{2} + \ell - q\}, 1)}{\Gamma(p+q-\ell-1)} = \sum_{r=0}^q \frac{\Gamma(\frac{3}{2} + \ell + r) q!}{\Gamma(\frac{3}{2} + \ell) r! (q-r)! \Gamma(1 + \ell - q + r) \Gamma(\frac{5}{2} + \ell - q + r) \Gamma(p+q-1-\ell-r)}. \quad (\text{A.4})$$

The function f has a branch cut on the interval $[1, \infty)$ and the corresponding discontinuity is known to be

$$\text{Disc } f(z) = 2\pi i \frac{(z-1)^{p+q} {}_2F_1(\frac{1}{2} + p, \frac{3}{2} + q, p+q+1, 1-z)}{\Gamma(\frac{3}{2} - p) \Gamma(\frac{1}{2} - q) \Gamma(1 + p + q)}. \quad (\text{A.5})$$

Using the series representation for ${}_2F_1$ we find from this formula

$$z^\ell \frac{d^\ell \text{Disc}(f)}{dz^\ell} = -\frac{8i}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+p+q} z^\ell (z-1)^{n+p+q-\ell} \Gamma(\frac{1}{2} + n + p) \Gamma(\frac{3}{2} + n + q)}{(2p-1)(2q+1) \Gamma(n+1) \Gamma(p+q+n+1-\ell)}. \quad (\text{A.6})$$

Plugging everything together we get

$$\text{Disc}(\Omega) = -3i \frac{\Gamma(p-1)}{\Gamma(q + \frac{3}{2})} \sum_{n=0}^{\infty} \sum_{\ell=0}^{p+q-2} \sum_{r=0}^q \frac{(-1)^{n+p+q} z^\ell (z-1)^{n+p+q-\ell} \Gamma(\frac{1}{2} + n + p) \Gamma(\frac{3}{2} + n + q)}{(2p-1)(2q+1) \Gamma(n+1) \Gamma(p+q+n+1-\ell)} \times \frac{\Gamma(\frac{3}{2} + \ell + r) q!}{\Gamma(\frac{3}{2} + \ell) r! (q-r)! \Gamma(1 + \ell - q + r) \Gamma(\frac{5}{2} + \ell - q + r) \Gamma(p+q-1-\ell-r)}. \quad (\text{A.7})$$

Because of $\Gamma(1 + \ell - q + r)$ in the denominator, the sum over ℓ can be restricted to run from $q - r$ to $p + q - 2$. We therefore make a change of variable $\ell = q - r + s$, so that s runs from 0 to $p + r - 2$. Then we get

$$\text{Disc}(\Omega) = -3i \sum_{n=0}^{\infty} \frac{(-1)^{n+p+q} \Gamma(p-1) \Gamma(\frac{3}{2} + n + q) \Gamma(\frac{1}{2} + n + p)}{\Gamma(1+n) \Gamma(q + \frac{3}{2}) (2p-1)(2q+1)} \sum_{r=0}^q \frac{q!}{r! (q-r)!} \times \sum_{s=0}^{p-2+r} \frac{\Gamma(\frac{3}{2} + q + s) (z-1)^{r-s+n+p} z^{q-r+s}}{\Gamma(1+s) \Gamma(\frac{5}{2} + s) \Gamma(p-1-s) \Gamma(\frac{3}{2} + q - r + s) \Gamma(1+r-s+n+p)}. \quad (\text{A.8})$$

Here in the denominator of the last sum the term $\Gamma(p-1-s)$ cuts the summation range for s at $p-2$. Therefore, we can change the order of summation in r and s and write

$$\begin{aligned} \text{Disc}(\Omega) = & - \sum_{n=0}^{\infty} \frac{3i(-1)^{n+p+q}\Gamma(p-1)}{\Gamma(q+\frac{3}{2})(2p-1)(2q+1)} \sum_{s=0}^{p-2} \frac{q!\Gamma(\frac{3}{2}+q+s)\Gamma(\frac{3}{2}+n+q)\Gamma(\frac{1}{2}+n+p)}{\Gamma(1+n)\Gamma(1+s)\Gamma(\frac{5}{2}+s)\Gamma(p-1-s)} \\ & \times \sum_{r=0}^q \frac{(z-1)^{r-s+n+p}z^{q-r+s}}{r!(q-r)!\Gamma(\frac{3}{2}+q-r+s)\Gamma(1+r-s+n+p)}. \end{aligned} \quad (\text{A.9})$$

The sum over r yields

$$\begin{aligned} \mathcal{R} \equiv & \sum_{r=0}^q \frac{(z-1)^{r-s+n+p}z^{q-r+s}}{r!(q-r)!\Gamma(\frac{3}{2}+q-r+s)\Gamma(1+r-s+n+p)} \\ = & \frac{(z-1)^{n+p-s}z^{q+s} {}_2F_1\left(-q, -\frac{1}{2}-q-s, 1+n+p-s, \frac{z-1}{z}\right)}{\Gamma(1+q)\Gamma(1+n+p-s)\Gamma(\frac{3}{2}+q+s)}. \end{aligned} \quad (\text{A.10})$$

Next with the help of the well-known transformation formula

$${}_2F_1(a, b, c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right) \quad (\text{A.11})$$

we can write

$${}_2F_1\left(-q, -\frac{1}{2}-q-s, 1+n+p-s, \frac{z-1}{z}\right) = z^{-q} {}_2F_1\left(-q, \frac{3}{2}+n+p+q, 1+n+p-s, 1-z\right),$$

so that

$$\mathcal{R} = \frac{(z-1)^{n+p-s}z^s {}_2F_1\left(-q, \frac{3}{2}+n+p+q, 1+n+p-s, 1-z\right)}{\Gamma(1+q)\Gamma(1+n+p-s)\Gamma(\frac{3}{2}+q+s)}. \quad (\text{A.12})$$

Next, the following identity holds

$$\begin{aligned} (z-1)^{n+p-s}z^s {}_2F_1\left(-q, \frac{3}{2}+n+p+q, 1+n+p-s, 1-z\right) = \\ \frac{(-1)^{n+p+q-s}\Gamma(1+n+p-s)}{\Gamma(1+n+p+q-s)} \frac{1}{\sqrt{z}} \frac{d^q}{dz^q} [(1-z)^{q+n+p-s}z^{\frac{1}{2}+q+s}]. \end{aligned} \quad (\text{A.13})$$

Hence, we get

$$\begin{aligned} \text{Disc}(\Omega) = & - \frac{3i}{\sqrt{z}} \frac{d^q}{dz^q} \frac{\Gamma(p-1)}{\Gamma(q+\frac{3}{2})(2p-1)(2q+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n+p)\Gamma(\frac{3}{2}+n+q)}{\Gamma(1+n)} \\ & \times \sum_{s=0}^{p-2} \frac{(-1)^s (1-z)^{q+n+p-s}z^{\frac{1}{2}+q+s}}{\Gamma(1+s)\Gamma(\frac{5}{2}+s)\Gamma(p-1-s)\Gamma(1+n+p+q-s)}. \end{aligned} \quad (\text{A.14})$$

Further, summing over s results into

$$\begin{aligned} \mathcal{S} = & \sum_{s=0}^{p-2} \frac{(-1)^s (1-z)^{q+n+p-s}z^{\frac{1}{2}+q+s}}{\Gamma(1+s)\Gamma(\frac{5}{2}+s)\Gamma(p-1-s)\Gamma(1+n+p+q-s)} \\ = & \frac{4}{3\sqrt{\pi}} \frac{(1-z)^{n+p+q}z^{\frac{1}{2}+q} {}_2F_1\left(2-p, -n-p-q, \frac{5}{2}, \frac{z}{z-1}\right)}{\Gamma(p-1)\Gamma(1+n+p+q)} \\ = & \frac{4}{3\sqrt{\pi}} \frac{(1-z)^{n+q+2}z^{\frac{1}{2}+q} {}_2F_1\left(2-p, \frac{5}{2}+n+p+q, \frac{5}{2}, z\right)}{\Gamma(p-1)\Gamma(1+n+p+q)}, \end{aligned} \quad (\text{A.15})$$

where to obtain the last expression we again used the transformation formula (A.11). Now, taking into account that

$$(1-z)^{n+q+2} z^{\frac{1}{2}+q} {}_2F_1\left(2-p, \frac{5}{2}+n+p+q, \frac{5}{2}, z\right) = \frac{3\sqrt{\pi} z^{q-1}}{4\Gamma(\frac{1}{2}+p)} \frac{d^{p-2}}{dz^{p-2}} [z^{p-\frac{1}{2}}(1-z)^{n+p+q}],$$

the expression for \mathcal{S} acquires the form

$$\mathcal{S} = \frac{z^{q-1}}{\Gamma(\frac{1}{2}+p)\Gamma(p-1)\Gamma(1+n+p+q)} \frac{d^{p-2}}{dz^{p-2}} [z^{p-\frac{1}{2}}(1-z)^{n+p+q}]. \quad (\text{A.16})$$

Thus, for the discontinuity we have

$$\begin{aligned} \text{Disc}(\Omega) = & \quad (\text{A.17}) \\ & - \frac{3i}{\sqrt{z}} \frac{d^q}{dz^q} z^{q-1} \frac{d^{p-2}}{dz^{p-2}} z^{p-\frac{1}{2}} (1-z)^{p+q} \\ & \times \frac{1}{(2p-1)(2q+1)(p+q)!} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{2}+q+n)}{\Gamma(q+\frac{3}{2})} \frac{\Gamma(\frac{1}{2}+p+n)}{\Gamma(\frac{1}{2}+p)} \frac{\Gamma(1+p+q)}{\Gamma(1+p+q+n)} \frac{(1-z)^n}{n!}. \end{aligned}$$

Summing up we finally get the desired formula (3.18).

B From the Borel image $\hat{\phi}_{p,q}$ to its representation $\hat{\Phi}_{p,q}$

B.1 First proof

The main ingredient of the formula (3.14) is its non-polynomial part represented by the hypergeometric function $\Omega(z)$, cf. (3.17), where we have introduced a variable $z = x^2$. To proceed, we will use representation (A.3), where we analytically continue the function f in the complex plane for the values $|\text{Arg}(-z)| < \pi$. The corresponding formula is well known and reads

$$\begin{aligned} f(z) = \frac{i}{\Gamma(\frac{1}{2}-q)} & \left[\frac{(-1)^{q+1} z^{q-\frac{1}{2}}}{\Gamma(p+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-p)_{n+p-q-1} (\frac{1}{2}-p)_{n+p-q-1}}{n!(n+p-q-1)!} z^{-n} (\log(-z) + h(p, q, n)) \right. \\ & \left. + (-1)^p z^{p-\frac{3}{2}} \sum_{n=0}^{p-q-2} \frac{(\frac{3}{2}-p)_n \Gamma(p-q-1-n)}{n! \Gamma(\frac{1}{2}+p-n)} z^{-n} \right], \quad (\text{B.1}) \end{aligned}$$

where

$$h(p, q, n) = \psi(p-q+n) + \psi(n+1) - \psi\left(\frac{1}{2}-q+n\right) - \psi\left(\frac{3}{2}+q-n\right). \quad (\text{B.2})$$

Obviously, the function $f(z)$ has a cut on the real axis. Taking into account that p and q are positive integers it is elementary to find the real part for $f(z)$ for z positive. Using the fact that $\log(-z) = \log|z| + i\pi$, we find that

$$\Re f(z) = \frac{(-1)^{p+q}}{\Gamma(\frac{3}{2}-p)\Gamma(\frac{1}{2}-q)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n-q)\Gamma(-\frac{1}{2}+n-q)}{n!(n+p-q-1)!} z^{q-n-\frac{1}{2}}, \quad z > 0. \quad (\text{B.3})$$

As a next step we compute

$$z^\ell \frac{d^\ell (\Re f)}{dz^\ell} = \pi \frac{(-1)^p}{\Gamma(\frac{3}{2} - p)\Gamma(\frac{1}{2} - q)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n - q - \frac{1}{2})}{n! \Gamma(n + p - q) \Gamma(\frac{1}{2} - \ell - n + q)} z^{q-n-\frac{1}{2}}. \quad (\text{B.4})$$

Substituting this result into the real part of (A.3) and replacing the regularised hypergeometric function via its normal counterpart, we obtain

$$\Re \Omega = \frac{(-1)^p}{\Gamma(\frac{3}{2} - p)\Gamma(\frac{1}{2} - q)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n - q - \frac{1}{2})}{n! \Gamma(n + p - q)} S(n) z^{q-n-\frac{1}{2}},$$

where we need to compute the following sum

$$S(n) = \varkappa \sum_{\ell=0}^{p+q-2} \frac{{}_3F_2\left(\left\{\frac{3}{2} + \ell, 2 + \ell - p - q, -q\right\}, \left\{1 + \ell - q, \frac{5}{2} + \ell - q\right\}, 1\right)}{\Gamma(1 + \ell - q)\Gamma(\frac{5}{2} + \ell - q)\Gamma(\frac{1}{2} - \ell - n + q)\Gamma(p + q - 1 - \ell)}, \quad (\text{B.5})$$

where the coefficient \varkappa is chosen for convenience to be

$$\varkappa = \frac{3\pi^2}{8} \frac{\Gamma(p-1)}{\Gamma(q + \frac{3}{2})}. \quad (\text{B.6})$$

Here for the hypergeometric function we can substitute its definition

$${}_3F_2\left(\left\{\frac{3}{2} + \ell, 2 + \ell - p - q, -q\right\}, \left\{1 + \ell - q, \frac{5}{2} + \ell - q\right\}, 1\right) = \sum_{r=0}^q \frac{(-1)^r}{r!} \frac{\Gamma(r-q)}{\Gamma(-q)} \frac{\Gamma(\frac{3}{2} + \ell + r)}{\Gamma(\frac{3}{2} + \ell)} \frac{\Gamma(1 + \ell - q)}{\Gamma(1 + \ell - q + r)} \frac{\Gamma(\frac{5}{2} + \ell - q)}{\Gamma(\frac{5}{2} + \ell - q + r)} \frac{\Gamma(p + q - 1 - \ell)}{\Gamma(p + q - 1 - \ell - r)}. \quad (\text{B.7})$$

After the change of order of summation eq. (B.5) acquires the form

$$S(n) = \varkappa \sum_{r=0}^q (-1)^r \frac{\Gamma(r-q)}{r! \Gamma(-q)} \times \sum_{\ell=q-r}^{p+q-2-r} \frac{\Gamma(\frac{3}{2} + \ell + r)}{\Gamma(\frac{3}{2} + \ell)\Gamma(\frac{1}{2} - \ell - n + q)\Gamma(1 + \ell - q + r)\Gamma(\frac{5}{2} + \ell - q + r)\Gamma(p + q - 1 - \ell - r)}, \quad (\text{B.8})$$

where the restrictions on the summation variable ℓ are clear from the arguments of the Γ -functions entering the denominators of the second sum. In the second sum we change variable $\ell \rightarrow \ell = q - r + s$, summing now over s we obtain

$$S(n) = \frac{\varkappa}{\pi} (-1)^n \sum_{s=0}^{p-2} \frac{(-1)^s \Gamma(\frac{3}{2} + q + s)}{s! \Gamma(\frac{5}{2} + s) \Gamma(p-1-s)} \sum_{r=0}^q \frac{\Gamma(\frac{1}{2} + n + s - r)}{\Gamma(\frac{3}{2} + q + s - r)} \frac{q! (-1)^r}{r! (q-r)!}.$$

The internal sum is given by

$$\sum_{r=0}^q \frac{\Gamma(\frac{1}{2} + n + s - r)}{\Gamma(\frac{3}{2} + q + s - r)} \frac{q! (-1)^r}{r! (q-r)!} = \frac{\Gamma(\frac{1}{2} + n + s)}{\Gamma(\frac{3}{2} + q + s)} \frac{\Gamma(\frac{1}{2} - n - s)}{\Gamma(\frac{1}{2} + q - n - s)} \frac{\Gamma(2q + 1 - n)}{\Gamma(q + 1 - n)}.$$

Finally, to perform the last sum over r we have to carefully distinguish two cases: $q \neq 0$ and $q = 0$. We treat these cases in turn.

1. Case $q \neq 0$. We have

$$S(n) = \frac{3\pi^2}{8} \frac{1}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)} \begin{cases} \frac{\Gamma(p+q-n)\Gamma(1+2q-n)}{\Gamma(\frac{1}{2}+q-n)\Gamma(1+q-n)\Gamma(2+q-n)}, & 0 \leq n \leq q; \\ 0, & q+1 \leq n \leq p+q-1; \\ (-1)^{p+q} \frac{\Gamma(n-1-q)\Gamma(n-q)}{\Gamma(n-2q)\Gamma(1+n-p-q)\Gamma(\frac{1}{2}-n+q)}, & n \geq p+q. \end{cases} \quad (\text{B.9})$$

Now we are ready to compute the real part of Ω

$$\begin{aligned} \Re\Omega &= \frac{3}{2} \frac{(-1)^q}{(1-2p)(1+2q)} \\ &\times \left[z^{-\frac{1}{2}} \sum_{n=0}^q \frac{(-1)^n \Gamma(n-q-\frac{1}{2})}{n! \Gamma(n+p-q)} \frac{\Gamma(p+q-n)\Gamma(1+2q-n)}{\Gamma(\frac{1}{2}+q-n)\Gamma(1+q-n)\Gamma(2+q-n)} z^{q-n} \right. \\ &\quad \left. + (-1)^{p+q} z^{-\frac{1}{2}} \sum_{n=p+q}^{\infty} \frac{(-1)^n \Gamma(n-q-\frac{1}{2})}{n! \Gamma(n+p-q)} \frac{\Gamma(n-1-q)\Gamma(n-q)}{\Gamma(n-2q)\Gamma(1+n-p-q)\Gamma(\frac{1}{2}-n+q)} z^{q-n} \right]. \end{aligned} \quad (\text{B.10})$$

Changing the summation indices appropriately, we find the final answer

$$\begin{aligned} \Re\Omega(z) &= \\ &\frac{1}{(2p-1)(2q+1)} \left[3z^{-\frac{1}{2}} {}_4F_3 \left(\left\{ 1-p, p, -q, 1+q \right\}, \left\{ \frac{1}{2}, \frac{3}{2}, 2 \right\}, z \right) \right. \\ &\quad \left. - z^{-\frac{1}{2}-p} \frac{3 \times 2^{3-4p} (-1)^{p+q} \Gamma(2p-2)}{\Gamma(p-q)\Gamma(p+q+1)} {}_4F_3 \left(\left\{ p-1, p-\frac{1}{2}, p, p+\frac{1}{2} \right\}, \left\{ 2p, p-q, p+q+1 \right\}; \frac{1}{z} \right) \right]. \end{aligned} \quad (\text{B.11})$$

Now we recall that function $\Re \hat{\phi}_{p,q}$ reads as

$$\begin{aligned} \Re \hat{\phi}_{p,q}(x) &= \frac{4}{3} (p-q-1)(p+q)x^2 \\ &\times \left[3 {}_4F_3 \left(\left\{ 1-p, p, -q, 1+q \right\}, \left\{ \frac{1}{2}, \frac{3}{2}, 2 \right\}, x^2 \right) - (2p-1)(2q+1)x \Re\Omega(x^2) \right]. \end{aligned} \quad (\text{B.12})$$

Substituting here (B.11) we find that the polynomial part cancels out completely and we are left with the desired result (3.22).

2. Case $q = 0$. In this situation we have

$$S_{q=0}(n) = \frac{3\pi^2}{8} \frac{1}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2})} \begin{cases} \frac{\Gamma(p-n)}{\Gamma(\frac{1}{2}-n)\Gamma(2-n)}, & n = 0, 1; \\ 0, & 2 \leq n \leq p-1; \\ (-1)^p \frac{\Gamma(n-1)}{\Gamma(1+n-p)\Gamma(\frac{1}{2}-n)}, & n \geq p. \end{cases} \quad (\text{B.13})$$

We therefore find

$$\begin{aligned} \Re\Omega &= \frac{3}{2} \frac{1}{1-2p} \left[z^{-\frac{1}{2}} \sum_{n=0}^1 \frac{(-1)^n \Gamma(n-\frac{1}{2})}{n! \Gamma(n+p)} \frac{\Gamma(p-n)}{\Gamma(\frac{1}{2}-n)\Gamma(2-n)} z^{-n} \right. \\ &\quad \left. + (-1)^p z^{-\frac{1}{2}} \sum_{n=p}^{\infty} \frac{(-1)^n \Gamma(n-\frac{1}{2})}{n! \Gamma(n+p)} \frac{\Gamma(n-1)}{\Gamma(1+n-p)\Gamma(\frac{1}{2}-n)} z^{-n} \right], \end{aligned}$$

which gives

$$\Re\Omega = \frac{1}{2p-1} \left[3z^{-\frac{1}{2}} - \frac{3}{4p(p-1)} z^{-\frac{3}{2}} - z^{-\frac{1}{2}-p} \frac{3 \times 2^{3-4p} (-1)^p \Gamma(2p-2)}{\Gamma(p)\Gamma(p+1)} {}_3F_2 \left(\left\{ p-1, p-\frac{1}{2}, p+\frac{1}{2} \right\}, \{2p, p+1\}; \frac{1}{z} \right) \right]. \quad (\text{B.14})$$

Then we specify (B.12) for $q = 0$ and get

$$\begin{aligned} \Re \hat{\phi}_{p,q}(x) = & \quad (\text{B.15}) \\ & \frac{4}{3} p(p-1)x^2 \\ & \times \left[3 - x \left[3x^{-1} - \frac{3}{4p(p-1)} x^{-3} - x^{-1-2p} \frac{3 \times 2^{3-4p} (-1)^p \Gamma(2p-2)}{\Gamma(p)\Gamma(p+1)} {}_3F_2 \left(\left\{ p-1, p-\frac{1}{2}, p+\frac{1}{2} \right\}, \{2p, p+1\}; \frac{1}{x^2} \right) \right] \right], \end{aligned}$$

which finally boils down to

$$\begin{aligned} \Re \hat{\phi}_{p,q}(x) = & \quad (\text{B.16}) \\ & 1 + (-1)^p 2^{5-4p} p(p-1) \frac{\Gamma(2p-2)}{\Gamma(p)\Gamma(p+1)} \\ & \times x^{2-2p} {}_3F_2 \left(\left\{ p-1, p-\frac{1}{2}, p+\frac{1}{2} \right\}, \{2p, p+1\}; \frac{1}{x^2} \right). \end{aligned}$$

Thus, we have proved that in all the cases $\Re \hat{\phi}_{p,q}$ is equivalent to eq. (3.21).

B.2 Second proof

Another approach is based on the Mellin-Barnes integral representation [61] for the hypergeometric function (3.17):

$$\frac{\prod_{k=1}^4 \Gamma(a_k)}{\prod_{k=1}^3 \Gamma(b_k)} {}_4F_3(\mathbf{a}, \mathbf{b}, z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^4 \Gamma(a_i + s)}{\prod_{i=1}^3 \Gamma(b_i + s)} \Gamma(-s) (-z)^s ds, \quad (\text{B.17})$$

where the integration contour separates the poles of $\Gamma(a_k + s)$, $k = 1, \dots, 4$, from those of $\Gamma(-s)$. The right-hand side of (B.17) provides the analytic continuation of the left-hand side from the open unit disk to the sector $|\text{Arg}(1-z)| < \pi$. In our case parameters are: $b_1 = \frac{3}{2}$, $b_2 = 2$, $b_3 = \frac{5}{2}$ and $a_{i+1} = a_i + m_i$, $i = 1, 2, 3$, with $a_1 = \frac{3}{2} - p$, $m_1 = p - q - 1$, $m_2 = 2q + 1$, $m_3 = p - q - 1$. Since all $m_i \in \mathbb{N}$, the function of integration in (B.17) has the following poles:

- first order poles $s = -a_1 - k$, $k = 0, 1, 2, \dots, m_1 - 1$;
- second order poles $s = -a_2 - k$, $k = 0, 1, 2, \dots, m_2 - 1$;
- third order poles $s = -a_3 - k$, $k = 0, 1, 2, \dots, m_3 - 1$;
- fourth order poles $s = -a_4 - k$, $k = 0, 1, 2, \dots, \infty$.

The method is thus build on the further calculation of the residues of the integrand at each pole. The full expression for the residues includes a huge amount of summands. Therefore, we present only the needed real part which comes from the terms of the form $(-z)^{1/2+n} \log(-z)$, $n \in \mathbb{N}$.

The 1st order poles. In this case the residue is given by

$$\text{Res}_1 = (-1)^k (-z)^{-\frac{3}{2}+p-k} \frac{\Gamma(\frac{3}{2} + k - p) \Gamma(2p - k - 1) \Gamma(p - q - 1 - k) \Gamma(p + q - k)}{k! \Gamma(p - k) \Gamma(p - k + \frac{1}{2}) \Gamma(p - k + 1)} \quad (\text{B.18})$$

and is purely imaginary. Thus, $\Re \text{Res}_1 = 0$.

The 2nd order poles. At the k -th pole, $0 \leq k \leq 2q$, we have the following expression for the real part of the residue:

$$\Re \text{Res}_2 = (i\pi) (-1)^{p-q+1} (-z)^{-\frac{1}{2}-k+q} \frac{\Gamma(\frac{1}{2} + k - q) \Gamma(p + q - k) \Gamma(2q - k + 1)}{k! (p - q + k - 1)! \Gamma(q - k + 1) \Gamma(\frac{3}{2} + q - k) \Gamma(q - k + 2)}. \quad (\text{B.19})$$

Note, that the residue is non-zero only for $0 \leq k \leq q$. After changing factorials by Gamma functions, transforming $k \rightarrow q - s$, $0 \leq s \leq q$, and using $\Gamma(1 - z) = \pi / (\Gamma(z) \sin(\pi z))$, one gets

$$\Re \text{Res}_2 = 2(-1)^{p-q+1} \pi z^{-\frac{1}{2}} \frac{(1-p)_s (p)_s (-q)_s (1+q)_s}{(1/2)_s (3/2)_s (2)_s} \frac{z^s}{\Gamma(1+s)}. \quad (\text{B.20})$$

Summing over all the second order poles ($0 \leq s \leq q$), we arrive at

$$\Re \text{Res}_2 = 2(-1)^{p-q+1} \pi z^{-\frac{1}{2}} {}_4F_3 \left(\{1 - p, p, -q, 1 + q\}, \left\{ \frac{1}{2}, \frac{3}{2}, 2 \right\}, z \right). \quad (\text{B.21})$$

Putting all the coefficients from (B.17) and (3.14), one gets the polynomial part in $\Re \hat{\phi}_{p,q}(x)$ with a minus sign. This means that the first term in (3.14) is cancelled by this residue term.

The 3rd order poles. Here $0 \leq k \leq p - q - 2$ and the the real part of the residue at the k -th pole is given by

$$\begin{aligned} \Re \text{Res}_3 &= \frac{(i\pi) (-1)^{p+q-1} (-z)^{-\frac{3}{2}-k-q}}{\Gamma(-k-q) \Gamma(1-k-q)} \frac{\Gamma(-1-k+p-q) \Gamma(\frac{3}{2} + k + q)}{k! (k+p+q)! (1+k+2q)! \Gamma(\frac{1}{2} - k - q)} \quad (\text{B.22}) \\ &\times \left(\log(z) + \psi(1+k) - \psi(-k-q) - \psi\left(\frac{1}{2} - k - q\right) - \psi(1-k-q) \right. \\ &\quad \left. + \psi(-1-k+p-q) - \psi\left(\frac{3}{2} + k + q\right) + \psi(1+k+p+q) + \psi(2+k+2q) \right). \end{aligned}$$

Note that for $0 \leq k \leq p - q - 2$ the above expression goes to zero thanks to the term $1/(\Gamma(-k-q)\Gamma(1-k-q))$ that behaves as $1/\infty^2$, leading to $\Re \text{Res}_3 = 0$.

The 4th order poles. In this case the expression for the real part of the residue $\Re \text{Res}_4$ is huge. Nevertheless, again thanks to the $\Gamma(1-k-p)\Gamma(2-k-p)$ factor in the denominator most of the terms vanish. The rest is given by

$$\begin{aligned} \Re \text{Res}_4 &= (i\pi)(-z)^{-\frac{1}{2}-k-p} \frac{\Gamma(\frac{1}{2}+k+p)}{\Gamma(\frac{3}{2}-k-p)} \frac{1}{2k!(-1+k+2p)!(-1+k+p-q)!(k+p+q)!} \\ &\times \frac{(\psi(1-k-p) + \psi(2-k-p))^2 - (\psi'(1-k-p) + \psi'(2-k-p))}{\Gamma(1-k-p)\Gamma(2-k-p)}. \end{aligned} \quad (\text{B.23})$$

This can be simplified with the help of the following identities: $\psi(1-z) = \pi \cot(\pi z) + \psi(z)$, $\psi'(1-z) = \pi^2/\sin^2(\pi z) - \psi'(z)$ and $\Gamma(1-z) = \pi/(\Gamma(z)\sin(\pi z))$:

$$\Re \text{Res}_4 = z^{-\frac{1}{2}-p} \frac{\Gamma(k+p-1)\Gamma(k+p-\frac{1}{2})\Gamma(k+p+\frac{1}{2})\Gamma(k+p)}{\Gamma(k+2p)\Gamma(k+p-q)\Gamma(k+p+q+1)} \frac{z^{-k}}{k!}. \quad (\text{B.24})$$

Summing over all the fourth order poles ($0 \leq k < \infty$), taking into account all the coefficients from (B.17) and (3.14), one comes to the desired expression for $\Re \hat{\Phi}_{p,q}$ which proves (3.22) for $q \neq 0$.

$q = 0$. Strictly speaking, the direct substitution $q = 0$ into the results obtained above leads to the wrong answer. In this case one has to perform the same procedure from the very beginning. This happens because (3.17) reduces to ${}_3F_2(\{\frac{3}{2}-p, \frac{1}{2}, p+\frac{1}{2}\}, \{2, \frac{5}{2}\}, z)$ and now we have a different system of poles. The final result gives an additional factor 1 which completely corresponds to (3.22).

C Derivation of Q

In section 5 we introduced the following function

$$Q(z) = \sum_{k=0}^{\infty} \frac{(-h)^{k+1}}{k!} z^{p-\frac{1}{2}} \frac{d^{p-2}}{dz^{p-2}} z^{q-1} \frac{d^q}{dz^q} z^{\frac{1}{2}(k+1)}. \quad (\text{C.1})$$

Performing straightforward differentiations and then summation over k we arrive at the following formula

$$Q(z) = -\frac{\pi h z {}_1F_2(\{\frac{3}{2}\}, \{\frac{5}{2}-p, \frac{3}{2}-q\}, \frac{h^2 z}{4})}{2\Gamma(\frac{5}{2}-p)\Gamma(\frac{3}{2}-q)} + \frac{h^2 z^{3/2} {}_2F_3(\{1, 2\}, \{\frac{3}{2}, 3-p, 2-q\}, \frac{h^2 z}{4})}{\Gamma(3-p)\Gamma(2-q)}. \quad (\text{C.2})$$

Further one can show that for $p \geq q+2$, $q \geq 0$, the following identity takes place

$$\frac{h^2 z^{3/2} {}_2F_3(\{1, 2\}, \{\frac{3}{2}, 3-p, 2-q\}, \frac{h^2 z}{4})}{\Gamma(3-p)\Gamma(2-q)} = -\frac{4\sqrt{\pi}(\frac{h^2 z}{4})^{p-\frac{1}{2}} \Gamma(p) {}_1F_2(\{p\}, \{p-\frac{1}{2}, p-q\}, \frac{h^2 z}{4})}{h \Gamma(p-\frac{1}{2})\Gamma(p-q)}.$$

Thus, $Q(z)$ is essentially written as the sum of two ${}_1F_2$ functions that both have the same characteristic feature. Namely, if the upper parameter is $\rho+q$, where $q \geq 0$, then among the lower parameters there is ρ . In this situation we can apply the following reduction

formula, where ${}_1F_2$ gets replaced by a finite sum of ${}_0F_1$, the latter being expressed via the modified Bessel function of the first kind I_ν ,

$$\begin{aligned}
 Q(z) = & -\frac{\pi}{h} \sum_{k=0}^q 2^{\frac{1}{2}-k-p} (h\sqrt{z})^{\frac{1}{2}+k+p} \frac{\Gamma(1+q)}{\Gamma(1+k)\Gamma(1+q-k)\Gamma(\frac{3}{2}+k-q)} I_{\frac{3}{2}+k-p}(h\sqrt{z}) \\
 & + \frac{\sqrt{\pi}}{h} \sum_{k=0}^q 2^{\frac{3}{2}-k-p} (h\sqrt{z})^{\frac{1}{2}+k+p} \frac{\Gamma(1+q)\Gamma(p)}{\Gamma(1+k)\Gamma(1+q-k)\Gamma(p+k-q)} I_{-\frac{3}{2}+k+p}(h\sqrt{z}).
 \end{aligned}
 \tag{C.3}$$

Note that due to our restrictions on the range of p and q the index ν of the first Bessel function $I_{\frac{3}{2}+k-p}$ is always negative, while the index of the second one, $I_{-\frac{3}{2}+k+p}$, is always positive. Moreover, the index always takes half-integer values which means that I_ν can be written via elementary functions. Indeed, let us introduce the following auxiliary functions

$$\begin{aligned}
 \mathcal{J}_n^e(x) &= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\Gamma(1+n+2m)}{(2m)!\Gamma(1+n-2m)} (2x)^{-2m}, \\
 \mathcal{J}_n^o(x) &= \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma(2+n+2m)}{(2m+1)!\Gamma(n-2m)} (2x)^{-2m-1}, \\
 \mathcal{J}_n(x) &= \sum_{m=0}^n \frac{\Gamma(n+m+1)}{m!\Gamma(n+1-m)} (2x)^{-m} = e^x \sqrt{\frac{2x}{\pi}} K_{\frac{1}{2}+n}(x),
 \end{aligned}
 \tag{C.4}$$

where $[n] \equiv \text{Floor}(n)$ and $K_\nu(x)$ is the Macdonald function. Then for an integer $n \geq 0$ we have

$$\begin{aligned}
 I_{\frac{1}{2}+n}(x) &= e^{-x} (-1)^{n+1} \frac{\mathcal{J}_n^e(x) + \mathcal{J}_n^o(x)}{\sqrt{2\pi x}} + e^x \frac{\mathcal{J}_n^e(x) - \mathcal{J}_n^o(x)}{\sqrt{2\pi x}}, \\
 I_{-\frac{1}{2}-n}(x) &= e^{-x} (-1)^{n+1} \frac{\mathcal{J}_n^e(x) + \mathcal{J}_n^o(x) - 2\mathcal{J}_n(x)}{\sqrt{2\pi x}} + e^x \frac{\mathcal{J}_n^e(x) - \mathcal{J}_n^o(x)}{\sqrt{2\pi x}}.
 \end{aligned}
 \tag{C.5}$$

First, we note that each individual I_ν involves the growing exponent, e^x , as $x \rightarrow \infty$. On the other hand, as is obvious from (5.5), these terms cannot appear in the final answer for $Q(z)$. Thus, upon summing up they all must cancel. Second, concerning the terms with the damping exponent e^{-x} , our numerical analysis shows that the terms involving the functions $\mathcal{J}_n^e(x)$ and $\mathcal{J}_n^o(x)$ all cancel in the sum, so that the only contribution left comes from $\mathcal{J}_n(x)$. In this way we find

$$Q(z) = \frac{\pi}{h} \sum_{k=0}^q (h\sqrt{z})^{\frac{1}{2}+k+p} \frac{2^{\frac{3}{2}-k-p}\Gamma(1+q)}{\Gamma(1+k)\Gamma(1+q-k)\Gamma(\frac{3}{2}+k-q)} \frac{e^{-h\sqrt{z}}(-1)^{p+k+1} \mathcal{J}_{p-k-2}(h\sqrt{z})}{\sqrt{2\pi h\sqrt{z}}},$$

where upon substituting the series representation for $\mathcal{J}_{p-k-2}(h\sqrt{z})$ and replacing $h \rightarrow h_n$, we obtain the desired formula (5.9). Note also that in terms of K_ν , the above formula reads as

$$Q(z) = \frac{1}{h} \sum_{k=0}^q (-1)^{p+k+1} 2^{\frac{3}{2}-k-p} (h\sqrt{z})^{\frac{1}{2}+p+k} \frac{K_{p-k-\frac{3}{2}}(h\sqrt{z})\Gamma(1+q)}{\Gamma(1+k)\Gamma(1+q-k)\Gamma(\frac{3}{2}+k-q)}.$$

D Details for the construction of the asymptotic expansion for $\Delta S_{p,q}(g)$

Here we resolve several technical issues concerning construction of the asymptotic expansion of the discontinuity $\Delta S_{p,q}(g)$ and also present an alternative method to derive the same asymptotic expansion.

D.1 Solution of the difference equation for c_ℓ

The coefficients c_ℓ which arise in the asymptotic expansion of the function ${}_2F_3$ can be determined recurrently by using Riney's method. The corresponding recurrence formula reads [61]

$$c_0 = 1, \quad c_\ell = -\frac{1}{4\ell} \sum_{k=0}^{\ell-1} c_k e_{\ell,k}, \quad (\text{D.1})$$

where

$$e_{\ell,k} = \sum_{j=1}^4 \frac{\Gamma(1-\nu-2b_j+\ell+2)}{\Gamma(1-\nu-2b_j+k)} \frac{\prod_{i=1}^2 (a_i - b_j)}{\prod_{i=1, i \neq j}^4 (b_i - b_j)}, \quad b_4 \equiv 1. \quad (\text{D.2})$$

Here the coefficients a_1, \dots, b_3 are given by (6.4) and $\nu = a_1 + a_2 - b_1 - b_2 - b_2 + \frac{1}{2} = -2 - p - q - 2\epsilon$. For our purposes, however, this recurrence formula is not enough as we need to determine these coefficients in the closed form, i.e. without referring to any recurrence procedure.

We start our analysis with some observations. First, computing $e_{\ell,k}$ explicitly we note that they do not depend on ϵ and, as a consequence, c_ℓ are also ϵ independent. Second, the coefficients c_ℓ satisfy certain difference equations. To understand this issue, consider the differential equation for the function ${}_2F_3$:

$$(\vartheta(\vartheta + b_1 - 1)(\vartheta + b_2 - 1)(\vartheta + b_3 - 1) - t(\vartheta + a_1)(\vartheta + a_2)) {}_2F_3(\{a_1, a_2, a_3\}, \{b_1, b_2\}, t) = 0,$$

where $\vartheta = t \frac{d}{dt}$. To derive the difference equations for c_ℓ , it is enough to substitute in this equation the part of the asymptotic expansion for ${}_2F_3$ which contains either damping or growing exponent

$$\mathcal{F}_3^- = t^{\frac{\nu}{2}} e^{i\pi\nu - 2\sqrt{t}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell c_\ell}{2^\ell t^{\ell/2}}, \quad \mathcal{F}_3^+ = t^{\frac{\nu}{2}} e^{2\sqrt{t}} \sum_{\ell=0}^{\infty} \frac{c_\ell}{2^\ell t^{\ell/2}}. \quad (\text{D.3})$$

Both lead to the same difference equations, so it is enough to consider only one of them. Substituting for instance \mathcal{F}_3^- , we get an expression which contains ϵ -independent term and the term proportional to ϵ . Since the original equation is valued for arbitrary ϵ these terms must separately vanish. Collecting terms proportional to $t^{\ell/2}$ in the first, ϵ -independent,

term we get the following difference equation:

$$\begin{aligned}
 & (-1 + \ell + p - q)(1 + \ell - p + q)(1 + \ell + p + q)^2 c_{\ell-1} \\
 & - \left(1 + 4\ell^3 - 2p^3 + 2p^2(1 + q) - 2q(1 + q)(2 + q) + 2p(2 + q)^2 \right. \\
 & \quad \left. + 6\ell^2(2 + p + q) + 2\ell(5 + 3q + p(7 + 4q)) \right) c_{\ell} \quad (D.4) \\
 & + \left(11 + 5\ell^2 + 7p - (p - q)^2 + 3q + \ell(15 + 4(p + q)) \right) c_{\ell+1} - 2(2 + \ell)c_{\ell+2} = 0.
 \end{aligned}$$

From the second term proportional to ϵ we find a simpler difference equation, namely,

$$\begin{aligned}
 & (-1 + \ell + p - q)(1 + \ell - p + q)(1 + \ell + p + q)c_{\ell-1} \quad (D.5) \\
 & + \left(-3\ell^2 - \ell(5 + 2p + 2q) + (p - q)^2 - 3p + q - 1 \right) c_{\ell} + 2(1 + \ell)c_{\ell+1} = 0.
 \end{aligned}$$

In fact, the second equation (D.5) implies the first. Shifting in eq. (D.5) the variable $\ell \rightarrow \ell + 1$, solving for $c_{\ell+2}$ and plugging this solution into (D.4), we observe that the last equation factorises and it contains the left hand side of eq. (D.5) as a factor. Thus, fulfilment of (D.5) implies the fulfilment of (D.4).

Now we explain how to find a closed formula for the coefficients c_{ℓ} . It is not difficult to see that these coefficients must arise in the large s -expansion of the following ratio of the gamma functions

$$\frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)\Gamma(s + p + 1)\Gamma(s + q + 2)} \sim \sum_{\ell=0}^{\infty} \frac{2^{2s+5/2+p+q}}{(2\pi)^{1/2}\Gamma(2s + 3 + p + q + j)} c_{\ell}. \quad (D.6)$$

Indeed, according to the discussion in chapter 2.2.2 by [62], the numbers c_{ℓ} can be computed from the following recursion formula

$$c_{\ell} = -\frac{1}{4\ell} \sum_{k=0}^{\ell-1} c_k E_{\ell,k}, \quad (D.7)$$

where

$$E_{\ell,k} = \sum_{j=1}^3 \frac{\Gamma(5 + p + q - 2b_j)}{\Gamma(3 + p + q - 2b_j)} \frac{(1/2 - b_j)}{\prod_{i=1}^3 (b_i - b_j)}. \quad (D.8)$$

Here $b_1 = 1 + p$, $b_2 = 2 + q$, $b_3 = 1$, and the prime signifies omission of the term with $i = j$. Computing recurrently the first few coefficients

$$\begin{aligned}
 c_0 &= 1, \\
 c_1 &= \frac{1}{2} (1 - p^2 - q(1 + q) + p(3 + 2q)), \\
 c_2 &= \frac{1}{8} \left(9 + p^4 + (q - 1)q^2(3 + q) - 2p^3(3 + 2q) \right. \\
 & \quad \left. - 2p(3 + 2q)(-4 + q + q^2) + p^2(5 + 2q(7 + 3q)) \right), \\
 & \dots,
 \end{aligned} \quad (D.9)$$

we verify that they form a sequence satisfying the difference equation (D.5). It is however unknown how to produce an expansion of the left hand side of eq. (D.6) in a way which would allow one to read off the closed formula for an arbitrary coefficient c_ℓ . What is however known in the closed form is the following asymptotic expansion

$$\frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)\Gamma(s + p + 1)\Gamma(s + q + 2)} \sim \sum_{j=0}^{\infty} \frac{v(j)}{\Gamma(s + 5/2 + p + q + j)\Gamma(s + 1)} \quad (\text{D.10})$$

with the coefficients

$$v(j) = \frac{(1/2 + p)_j(3/2 + q)_j}{j!}, \quad (\text{D.11})$$

see formula (2.2.39) in [62]. At this point it is natural to use the large s asymptotic expansion of the inverse product of two gamma functions, cf. eq. (2.2.34) in [62],

$$\frac{1}{\Gamma(s + 5/2 + p + q + j)\Gamma(s + 1)} \sim \frac{2^{2s+5/2+p+q+j}}{(2\pi)^{1/2}} \sum_{k=0}^{\infty} \frac{\sigma(k, j)}{\Gamma(2s + 3 + p + q + j + k)} \quad (\text{D.12})$$

where

$$\begin{aligned} \sigma(k, j) &= \frac{(-2)^{-k}}{k!} \prod_{r=1}^k \left(\left(\frac{3}{2} + p + q + j \right)^2 - \left(r - \frac{1}{2} \right)^2 \right) \\ &= \frac{(-1 - j - p - q)_k (2 + j + p + q)_k}{2^k k!}. \end{aligned} \quad (\text{D.13})$$

Thus, we arrive at the following double sum representation

$$\frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)\Gamma(s + p + 1)\Gamma(s + q + 2)} \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{2s+5/2+p+q}}{(2\pi)^{1/2}} \frac{2^j v(j) \sigma(k, j)}{\Gamma(2s + 3 + p + q + j + k)}. \quad (\text{D.14})$$

This expansion is to be compared with (D.6). To this end we make a change of the summation variables $j + k = \ell$ and get

$$\frac{\Gamma(s + \frac{1}{2})}{\Gamma(s + 1)\Gamma(s + p + 1)\Gamma(s + q + 2)} \sim \sum_{\ell=0}^{\infty} \frac{2^{2s+5/2+p+q}}{(2\pi)^{1/2}\Gamma(2s+3+p+q+\ell)} \sum_{j=0}^{\ell} 2^j v(j) \sigma(\ell - j, j). \quad (\text{D.15})$$

In this way we get an explicit formula for the coefficients c_ℓ :

$$c_\ell = \sum_{j=0}^{\ell} 2^j v(j) \sigma(\ell - j, j). \quad (\text{D.16})$$

Substituting here the corresponding coefficients and performing the summation over j we arrive at the following result

$$\begin{aligned} c_\ell &= (-1)^\ell \frac{\Gamma(2 + \ell + p + q)}{\Gamma(\frac{1}{2} + p)\Gamma(\frac{3}{2} + q)} \sum_{j=0}^{\ell} \frac{(-1)^j 2^{2j-1} \Gamma(\frac{1}{2} + j + p)\Gamma(\frac{3}{2} + j + q)}{\Gamma(1 + \ell - j)\Gamma(1 + j)\Gamma(2 + 2j - \ell + p + q)} \\ &= \frac{\Gamma(2 + \ell + p + q) {}_3F_2(\{-\ell, \frac{1}{2} + p, \frac{3}{2} + q\}, \{1 - \frac{\ell}{2} + \frac{p}{2} + \frac{q}{2}, \frac{3}{2} - \frac{\ell}{2} + \frac{p}{2} + \frac{q}{2}\}, 1)}{(-2)^\ell \Gamma(\ell + 1)\Gamma(2 - \ell + p + q)}. \end{aligned} \quad (\text{D.17})$$

One can now directly verify that the coefficients c_ℓ given by the formula above satisfy the difference equation (D.5) and coincide with those found through the recursion formula.

D.2 Simplifying the expression for $\Delta S_{p,q}(g)$

We start with the expression (6.22) for $\Delta S_{p,q}(g)$, isolate the sum over ℓ and substitute there the coefficients c_ℓ in the form of the sum (D.17). We get the following double sum

$$\begin{aligned} \mathcal{W}(k, m) &= \sum_{\ell=0}^{L-3} \sum_{j=0}^{\ell} \frac{\Gamma(2+\ell+p+q)}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)} \frac{(-1)^j 2^{2j-1} \Gamma(\frac{1}{2}+j+p)\Gamma(\frac{3}{2}+j+q)}{\Gamma(1+\ell-j)\Gamma(1+j)\Gamma(2+2j-\ell+p+q)} \\ &\quad \times \frac{1}{\Gamma(1+\ell+p+q)\Gamma(4+p+q-L+\ell)\Gamma(L-2-\ell+k-m-q)}. \end{aligned} \quad (\text{D.18})$$

Now we isolate from $\Delta S_{p,q}(g)$ the sum over ℓ and substitute there c_ℓ in the form of the sum (D.17). We get the following double sum

$$\begin{aligned} \mathcal{W}(k, m) &= \sum_{\ell=0}^{L-3} \sum_{j=0}^{\ell} \frac{\Gamma(2+\ell+p+q)}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)} \frac{(-1)^j 2^{2j-1} \Gamma(\frac{1}{2}+j+p)\Gamma(\frac{3}{2}+j+q)}{\Gamma(1+\ell-j)\Gamma(1+j)\Gamma(2+2j-\ell+p+q)} \\ &\quad \times \frac{1}{\Gamma(1+\ell+p+q)\Gamma(4+p+q-L+\ell)\Gamma(L-2-\ell+k-m-q)}. \end{aligned} \quad (\text{D.19})$$

As a next step we interchange the order of summation and get

$$\begin{aligned} \mathcal{W}(k, m) &= \sum_{j=0}^{L-3} \sum_{\ell=j}^{L-3} \frac{\Gamma(2+\ell+p+q)}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)} \frac{(-1)^j 2^{2j-1} \Gamma(\frac{1}{2}+j+p)\Gamma(\frac{3}{2}+j+q)}{\Gamma(1+\ell-j)\Gamma(1+j)\Gamma(2+2j-\ell+p+q)} \\ &\quad \times \frac{1}{\Gamma(1+\ell+p+q)\Gamma(4+p+q-L+\ell)\Gamma(L-2-\ell+k-m-q)}. \end{aligned} \quad (\text{D.20})$$

Taking the sum over ℓ we obtain

$$\begin{aligned} \mathcal{W}(k, m) &= \sum_{j=0}^{L-3} \frac{(-1)^j 2^{2j-1} \Gamma(\frac{1}{2}+j+p)\Gamma(\frac{3}{2}+j+q)}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)\Gamma(1+j)} \\ &\quad \times \left[\frac{{}_3F_2(\{-1-j-p-q, 3+j-k-L+m+q, 2+j+p+q\}, \{1+j+p+q, 4+j+p+q-L\}, 1/2)}{\Gamma(-2-j+k+L-m-q)\Gamma(1+j+p+q)\Gamma(4+j+p+q-L)} \right. \\ &\quad \left. - 2^{j-L+3} \frac{\Gamma(L+p+q)}{\Gamma(2+p+q)\Gamma(L+p+q-1)} \right. \\ &\quad \left. \times \frac{{}_4F_3(\{1, -2j+L-3-p-q, 1-k+m+q, L+p+q\}, \{-1-j+L, 2+p+q, L-1+p+q\}, 1/2)}{\Gamma(-1-j+L)\Gamma(k-m-q)\Gamma(4+2j-L+p+q)} \right]. \end{aligned} \quad (\text{D.21})$$

Note that for the allowed values of k and m the function ${}_4F_3$ is always finite, but the gamma function $\Gamma(k-m-q)$ which divides it is always infinite because $k \leq q$ and $m \geq 0$. Thus, the term of \mathcal{W} containing ${}_4F_3$ does not contribute to the discontinuity $\Delta S_{p,q}$ and we can safely omit it. We therefore consider

$$\begin{aligned} \mathcal{W}(s, m) &= \sum_{k=0}^{L-3} \frac{(-1)^k 2^k \Gamma(\frac{1}{2}+k+p)\Gamma(\frac{3}{2}+k+q)}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)\Gamma(1+k)} \\ &\quad \times \frac{{}_3F_2(\{-1-k-p-q, 3-L-s+k+m+q, 2+k+p+q\}, \{1+k+p+q, 4+k+p+q-L\}, 1/2)}{\Gamma(-2+L+s-k-m-q)\Gamma(1+k+p+q)\Gamma(4+k+p+q-L)}, \end{aligned} \quad (\text{D.22})$$

where for further clarity we replace the index k by s and j by k . Further, the hypergeometric function

$$\mathcal{V} = {}_3F_2(\{-1-k-p-q, 3-L-s+k+m+q, 2+k+p+q\}, \{1+k+p+q, 4+k+p+q-L\}, 1/2)$$

featuring in the last formula can be reduced, namely,

$$\begin{aligned} \mathcal{V} &= 2 {}_2F_1(-1-k-p-q, 3+k-L+m+q-s, 4+k+p+q-L, 1/2) \\ &\quad - 2 {}_2F_1(-k-p-q, 3+k-L+m+q-s, 4+k+p+q-L, 1/2). \end{aligned} \quad (\text{D.23})$$

To each of these two ${}_2F_1$'s we apply an identity

$${}_2F_1(a, b, c, z) = (1-z)^{-a} {}_2F_1(a, c-b, c, z/(z-1))$$

and get

$$\begin{aligned} \mathcal{V} &= 2^{-k-p-q} [{}_2F_1(-1-k-p-q, 1-m+p+s, 4+k+p+q-L, -1) \\ &\quad - {}_2F_1(-k-p-q, 1-m+p+s, 4+k+p+q-L, -1)] \\ &= 2^{-k-p-q} \frac{1-m+p+s}{4+k+p+q-L} {}_2F_1(-k-p-q, 2-m+p+s, 5+k+p+q-L, -1). \end{aligned} \quad (\text{D.24})$$

Therefore,

$$\begin{aligned} \mathcal{W}(s, m) &= \sum_{k=0}^{L-3} \frac{(-1)^k 2^{-p-q} \Gamma(\frac{1}{2} + k + p) \Gamma(\frac{3}{2} + k + q)}{\Gamma(\frac{1}{2} + p) \Gamma(\frac{3}{2} + q) \Gamma(1 + k)} (p - m + s + 1) \\ &\quad \times \frac{{}_2F_1(-k-p-q, 2-m+p+s, 5+k+p+q-L, -1)}{\Gamma(-2+L+s-k-m-q) \Gamma(1+k+p+q) \Gamma(5+k+p+q-L)}. \end{aligned} \quad (\text{D.25})$$

For the discontinuity we therefore find

$$\begin{aligned} \Delta S_{p,q}(g) &= ig(p-q-1)(p+q) \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} \sum_{s=0}^q \sum_{m=0}^{p-s-2} \mathcal{W}(s, m) \\ &\quad \times \sqrt{\pi} \frac{(-1)^{p+s} 2^{3-s-m+q} q! \Gamma(p+m-s-1) \Gamma(p-m+s+1) \Gamma(-2+s+p+L-m)}{s! m! (q-s)! \Gamma(p-m-s-1) \Gamma(\frac{3}{2} + s - q)} \end{aligned}$$

or explicitly

$$\begin{aligned} \Delta S_{p,q}(g) &= \\ &ig(p-q-1)(p+q) \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} \sum_{k=0}^{L-3} \frac{\Gamma(\frac{1}{2} + k + p) \Gamma(\frac{3}{2} + k + q)}{\Gamma(\frac{1}{2} + p) \Gamma(\frac{3}{2} + q) \Gamma(1 + k)} \\ &\quad \times \sum_{s=0}^q \sum_{m=0}^{p-s-2} \sqrt{\pi} \frac{(-1)^{p+s+k} 2^{3-s-m-p} q! \Gamma(p+m-s-1) \Gamma(p-m+s+2) \Gamma(-2+s+p+L-m)}{s! m! (q-s)! \Gamma(p-m-s-1) \Gamma(\frac{3}{2} + s - q)} \\ &\quad \times \frac{{}_2F_1(-k-p-q, p-m+s+2, 5+k+p+q-L, -1)}{\Gamma(-2+L+s-k-m-q) \Gamma(1+k+p+q) \Gamma(5+k+p+q-L)}. \end{aligned} \quad (\text{D.26})$$

We note as an interesting fact that the hypergeometric function entering in the last expression is expressible via the following Jacobi polynomial

$$\begin{aligned} {}_2F_1(-k-p-q, p-m+s+2, 5+k+p+q-L, -1) &= \\ &\frac{(k+p+q)! (-2)^{k+p+q}}{(5+k-L+p+q, k+p+q)} J_{k+p+q}^{(-2-k+m-2p-q-s, -2+k-L+p+q)}(0). \end{aligned} \quad (\text{D.27})$$

Expanding ${}_2F_1$ into the hypergeometric series, one comes to another representation

$$\Delta S_{p,q}(g) = 4ig(p-q-1)(p+q) \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} C_L(p,q), \quad (\text{D.28})$$

where the coefficients $C_L(p,q)$ are

$$\begin{aligned} C_L(p,q) &= \sum_{k=0}^{L-3} \frac{\Gamma(\frac{1}{2}+k+p)\Gamma(\frac{3}{2}+k+q)}{\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{2}+q)\Gamma(1+k)} \\ &\times \sum_{\tau=0}^{p+q-2} (-1)^{k+\tau} 2^{-1-\tau} \sqrt{\pi} \frac{\Gamma(1+q)\Gamma(4+\tau)\Gamma(\tau+L)}{\Gamma(p+q+k+1)\Gamma(p+q-\tau-1)\Gamma(L+\tau-k-p-q)} \\ &\times {}_2\tilde{F}_1(-k-p-q, 4+\tau; 5+k-L+p+q; -1) \\ &\times {}_3\tilde{F}_2\left(\left\{\frac{1}{2}(4-2p+\tau), \frac{1}{2}(5-2p+\tau), 2-p-q+\tau\right\}, \left\{3-p+\tau, \frac{7}{2}-p-q+\tau\right\}, 1\right), \end{aligned} \quad (\text{D.29})$$

where “ \sim ” denotes that the hypergeometric function is regularised.

In appendix D.3 we provide an alternative but simpler expression for $C_L(p,q)$.

D.3 Alternative derivation of the asymptotic expansion for $\Delta S_{p,q}(g)$

An alternative method to compute the discontinuity in (5.5), is by using the Gauss series expansion for the hypergeometric function

$${}_2F_1\left(\frac{1}{2}+p, \frac{3}{2}+q, p+q+1, 1-z\right) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}+p)_k (\frac{3}{2}+q)_k}{(p+q+1)_k \Gamma(k+1)} (1-z)^k. \quad (\text{D.30})$$

We need just to compute the following integral

$$f = \int_1^{\infty} dz (-h_n \sqrt{z}) e^{-h_n \sqrt{z}} \frac{d^q}{dz^q} \left(z^{q-1} \frac{d^{p-2}}{dz^{p-2}} [z^{p-\frac{1}{2}} (1-z)^{p+q+k}] \right). \quad (\text{D.31})$$

First we use the binomial expansion to expand $(1-z)^{p+q+k}$ and rewrite

$$f = \sum_{s=0}^{p+q+k} \frac{(-1)^s \Gamma(p+q+1+k)}{\Gamma(s+1)\Gamma(p+q+1+k-s)} \int_1^{\infty} dz (-h_n \sqrt{z}) e^{-h_n \sqrt{z}} \frac{d^q}{dz^q} \left(z^{q-1} \frac{d^{p-2}}{dz^{p-2}} z^{p+s-\frac{1}{2}} \right).$$

Now we compute the derivatives by using $d^m z^n = \Gamma(n+1)/\Gamma(n+1-m) z^{n-m}$, so the integral takes the form

$$\begin{aligned} f &= \sum_{s=0}^{p+q+k} (-1)^s \frac{\Gamma(p+q+1+k)}{\Gamma(s+1)\Gamma(p+q+1+k-s)} \\ &\times \frac{\Gamma(p+s+\frac{1}{2})\Gamma(q+s+\frac{3}{2})}{\Gamma(s+\frac{3}{2})\Gamma(s+\frac{5}{2})} \int_1^{\infty} dz (-h_n) e^{-h_n \sqrt{z}} z^{s+1}. \end{aligned} \quad (\text{D.32})$$

We pass back to the original variable $x^2 = z$ and note that

$$\begin{aligned} 2 \int_1^{\infty} dx (-h_n) e^{-h_n x} x^{2s+3} &= 2h_n \frac{d^{2s+3}}{dh_n^{2s+3}} \int_1^{\infty} dx e^{-h_n x} \\ &= 2h_n \frac{d^{2s+3}}{dh_n^{2s+3}} \left(\frac{e^{-h_n}}{h_n} \right) = (-2) \sum_{l=0}^{2s+3} \frac{\Gamma(2s+4)}{\Gamma(2s+4-l)} \frac{e^{-h_n}}{h_n^l}. \end{aligned}$$

Here the summation range of l can be extended all the way to infinity thanks to $\Gamma(2s+4-l)$ in the denominator. The sum over n in $\Delta S_{p,q}$ is now trivial and gives

$$\sum_{n=1}^{\infty} \frac{e^{-h_n}}{h_n^l} = \frac{\text{Li}_l(e^{-4\pi g})}{(4\pi g)^l}. \quad (\text{D.33})$$

By putting everything together we obtain

$$\begin{aligned} \Delta S_{p,q}(g) &= (4ig)(p-q-1)(p+q) \sum_{l=0}^{\infty} \frac{\text{Li}_l(e^{-4\pi g})}{(4\pi g)^l} \sum_{k=0}^{\infty} \frac{\Gamma(p+\frac{1}{2}+k)\Gamma(q+\frac{3}{2}+k)}{\Gamma(p+\frac{1}{2})\Gamma(q+\frac{3}{2})\Gamma(k+1)} \\ &\quad \times \sum_{s=0}^{p+q+k} \frac{(-1)^s}{\Gamma(s+1)\Gamma(p+q+1+k-s)} \frac{\Gamma(p+s+\frac{1}{2})\Gamma(q+s+\frac{3}{2})}{\Gamma(s+\frac{3}{2})\Gamma(s+\frac{5}{2})} \frac{\Gamma(2s+4)}{\Gamma(2s+4-l)}. \end{aligned} \quad (\text{D.34})$$

The sum over s can be extended all the way to infinity thanks to $\Gamma(p+q+1+k-s)$ in the denominator and this can be performed:

$$\begin{aligned} \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(p+q+1+k-s)} \frac{\Gamma(p+s+\frac{1}{2})\Gamma(q+s+\frac{3}{2})}{\Gamma(s+\frac{3}{2})\Gamma(s+\frac{5}{2})} \frac{\Gamma(2s+4)}{\Gamma(2s+4-l)} = \quad (\text{D.35}) \\ 2^l \frac{\Gamma(p+\frac{1}{2})\Gamma(q+\frac{3}{2})}{\Gamma(p+q+k+1)} {}_4\tilde{F}_3 \left(\left\{ 2, p+\frac{1}{2}, q+\frac{3}{2}, -p-q-k \right\}, \left\{ \frac{3}{2}, 2-\frac{l}{2}, \frac{5}{2}-\frac{l}{2} \right\}; 1 \right), \end{aligned}$$

where ${}_4\tilde{F}_3$ is the regularized generalized hypergeometric function.

It can be shown, cf. [56], that ${}_4\tilde{F}_3(\{2, p+\frac{1}{2}, q+\frac{3}{2}, -p-q-k\}, \{\frac{3}{2}, 2-\frac{l}{2}, \frac{5}{2}-\frac{l}{2}\}; 1)$ vanishes for $k+2 > l$. This implies that in (D.34) the sum over l actually starts from $l=2$, while k runs from 0 to $l-2$. We can therefore shift $l=L-1$ and finally arrive at

$$\begin{aligned} \Delta S_{p,q}(g) &= (4ig)(p-q-1)(p+q) \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} 2^{L-1} \quad (\text{D.36}) \\ &\quad \times \sum_{k=0}^{L-3} \frac{\Gamma(p+\frac{1}{2}+k)\Gamma(q+\frac{3}{2}+k)}{\Gamma(k+1)\Gamma(p+q+k+1)} {}_4\tilde{F}_3 \left(\left\{ 2, p+\frac{1}{2}, q+\frac{3}{2}, -p-q-k \right\}, \left\{ \frac{3}{2}, \frac{5-L}{2}, \frac{6-L}{2} \right\}; 1 \right). \end{aligned}$$

Thus, the discontinuity takes the form

$$\Delta S_{p,q}(g) = (4ig)(p-q-1)(p+q) \sum_{L=3}^{\infty} \frac{\text{Li}_{L-1}(e^{-4\pi g})}{(4\pi g)^{L-1}} c_L(p, q), \quad (\text{D.37})$$

where the coefficients $c_L(p, q)$ are given by

$$\begin{aligned} c_L(p, q) &= 2^{L-1} \sum_{k=0}^{L-3} \frac{\Gamma(p+\frac{1}{2}+k)\Gamma(q+\frac{3}{2}+k)}{\Gamma(k+1)\Gamma(p+q+k+1)} \quad (\text{D.38}) \\ &\quad \times {}_4\tilde{F}_3 \left(\left\{ 2, p+\frac{1}{2}, q+\frac{3}{2}, -p-q-k \right\}, \left\{ \frac{3}{2}, \frac{5-L}{2}, \frac{6-L}{2} \right\}; 1 \right), \end{aligned}$$

or equivalently by expanding the hypergeometric function

$$c_L(p, q) = \sum_{k=0}^{L-3} \frac{\Gamma(p + \frac{1}{2} + k)\Gamma(q + \frac{3}{2} + k)}{\Gamma(p + \frac{1}{2})\Gamma(q + \frac{3}{2})\Gamma(k + 1)} \tag{D.39}$$

$$\times \sum_{n=0}^{k+p+q} \frac{(-1)^n 2^{2n+3} (n+1)}{\sqrt{\pi}} \frac{\Gamma(p + \frac{1}{2} + n)\Gamma(q + \frac{3}{2} + n)}{\Gamma(p + q + 1 + k - n)\Gamma(n + \frac{3}{2})\Gamma(2n + 5 - L)}.$$

Note that these coefficients $c_L(p, q)$ entering the expansion (D.37) seem very different from the previously computed $C_L(p, q)$ given by (D.29), nonetheless we have numerically checked that the two expressions coincide k by k once we fix values for p, q and L .

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