BAKRY-ÉMERY CURVATURE AND DIAMETER BOUNDS ON GRAPHS

SHIPING LIU, FLORENTIN MÜNCH, AND NORBERT PEYERIMHOFF

ABSTRACT. We prove finiteness and diameter bounds for graphs having a positive Ricci-curvature bound in the Bakry-Émery sense. Our first result using only curvature and maximal vertex degree is sharp in the case of hypercubes. The second result depends on an additional dimension bound, but is independent of the vertex degree. In particular, the second result is the first Bonnet-Myers type theorem for unbounded graph Laplacians. Moreover, our results improve diameter bounds from [7] and [11] and solve a conjecture from [5].

1. Introduction

The classical Bonnet-Myers theorem states that for a complete, connected Riemannian manifold with Ricci-curvature bounded from below by $K > 0$, the diameter is bounded by the diameter of the sphere with the same dimension and Ricci-curvature K (see [19]). Moreover by Cheng's Rigidy theorem (see [4]), sharpness is obtained if and only if the manifold is a sphere. Bakry and Ledoux [2] successfully established this theorem for abstract Markov generators which are diffusion and satisfy Bakry-Émery-Ricci curvature [1] conditions.

Our aim is to give a simple proof of this theorem in a discrete setting. Indeed, discrete space Markov generators are not diffusion and therefore, the theory of Bakry and Ledoux is not applicable. However, for many discrete curvature notions there is already a Bonnet-Myers-type result established (for sectional curvature on planar graphs, see [6, 10, 14, 22], and for Ollivier-Ricci-curvature, see [20], and for Forman's discrete Ricci curvature, see [8]).

In this article, we focus on Bakry-Émery-Ricci-curvature [1, 21, 16, 13]. Due to the lack of the chain rule in discrete settings, there appear to be great difficulties to adopt results on manifolds to the graph case. In particular it is hard to obtain global results as Li-Yau inequality, volume growth, heat kernel estimates and diameter bounds from discrete curvature dimension conditions. For that reason, there were introduced various non-linear curvature dimension conditions (namely CDE, CDE', $CD\psi$) closely related to the classical linear CD condition [3, 17]. Via these non-linear curvature dimension conditions, the missing chain rule was bypassed and it was possible to prove the global geometric properties mentioned before. Particularly, a discrete Bonnet-Myers type theorem was established under the CDE'-condition in [11]. But unfortunately, the non-linear versions CDE' and $CD\psi$ turn out to be strictly stronger than CD (see [18]). So the initial question is still open: Which global geometric properties can be derived from the CD-condition? Does a positive curvature bound already imply finiteness of the graph? A result related to the second question is given in [7]. They prove that, if the graph is already finite, then the diameter can be estimated in terms of the curvature bound. But this result does not show that positive curvature implies *finiteness* of a possibly infinite graph.

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In this article, we give sharp diameter bounds holding on finite as well as on infinite graphs assuming the linear curvature dimension condition CD with a positive curvaure bound. Thereby, we show finiteness and improve diameter bounds from both [7] and [11] discussed above. Moreover, our results solve Conjecture 8.1 from [5] asking, whether $CD(K,\infty)$ and a bounded vertex degree imply finiteness of the graph and diameter bounds.

A main difference to the case of manifolds, which plays an important role in our arguments, is that we can upper bound the Laplacian by the gradient on graphs, that is, $(\Delta f)^2 \leq C\Gamma f$ for all functions f on the vertices and a constant $C > 0$ depending only on the maximal vertex degree. This property will give us diameter bounds using the vertex degree instead of the dimension parameter in the curvature-dimension condition.

1.1. Organization of the paper and main results.

In Subsection 1.2, we define Bakry-Émery-Ricci-curvature and discuss different distance and diameter notions.

In Section 2, we give two versions of diameter bounds.

The first result (Corollary 2.2) is using $CD(K,\infty)$ and bounded vertex degree $\mathrm{Deg}_{\max}$ and gives the sharp estimate

$$\mathrm{diam}_d(G) \leq \frac{2\,\mathrm{Deg}_{\max}}{K}.$$

This solves Conjecture 8.1 from [5] claiming that for every graph satisfying $CD(K,\infty)$ with $K > 0$, there should exist an upper diameter bound of the graph only depending on K and $\mathrm{Deg}_{\max}$.

The second result (Theorem 2.4) works in a more general setting, in particular, no boundedness of the vertex degree is needed anymore. But instead, we will assume $CD(K,n)$ with finite n to prove

$$\mathrm{diam}_\rho \leq \pi\sqrt{\frac{n}{K}}$$

where ρ is the resistance metric (see Definition 1.3).

Finally in Subsection 2.3, we compare these diameter bounds to diameter bounds from [7] and [11]. In comparison to [11], we need a weaker curvature assumption and obtain stronger estimates (see Remark 2.5). In comparison to [7], we have an improvement by a factor of 2 under the same curvature assumptions (see Remark 2.7).

1.2. Setup and notations.

A triple $G = (V,w,m)$ is called a *(weighted) graph* if V is a countable set, if $w : V^2 \to [0,\infty)$ is symmetric and zero on the diagonal and if $m : V \to (0,\infty)$. We call V the *vertex set*, w the *edge weight* and m the *vertex measure*. We define the *graph Laplacian* as a map $\mathbb{R}^V \to \mathbb{R}^V$ via $\Delta f(x) := \frac{1}{m(x)}\sum_y w(x,y)(f(y)-f(x))$. In the following, we only consider *locally finite* graphs, i.e., for every $x \in V$ there are only finitely many $y \in V$ with $w(x,y) > 0$. We write $\mathrm{Deg}(x) := \frac{\sum_y w(x,y)}{m(x)}$ and $\mathrm{Deg}_{\max} := \sup_x \mathrm{Deg}(x)$.

Definition 1.1 (Bakry-Émery-curvature)**.** The *Bakry-Émery-operators* are defined via

$$2\Gamma(f,g) := \Delta(fg) - f\Delta g - g\Delta f$$

and

$$2\Gamma_2(f,g) := \Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(g,\Delta f).$$

We write $\Gamma(f) := \Gamma(f,f)$ and $\Gamma_2(f) := \Gamma_2(f,f)$.

A graph G is said to satisfy the *curvature dimension inequality* $CD(K,n)$ for some $K \in \mathbb{R}$ and $n \in (0,\infty]$ if for all f,

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta f)^2 + K\Gamma f.$$

Next, we define combinatorial and resistance metrics and diameters.

Definition 1.2 (Combinatorial metric)**.** Let $G = (V, w, m)$ be a locally finite graph. We define the *combinatorial metric* $d : V^2 \to [0, \infty)$ via

$$d(x, y) := \min\{n : \text{ there exist } x = x_0, \ldots, x_n = y \text{ s.t. } w(x_i, x_{i-1}) > 0 \text{ for all } i = 1 \ldots n\}$$

and the *combinatorial diameter* as $\operatorname{diam}_d(G) := \sup_{x,y\in V} d(x, y)$.

Definition 1.3 (Resistance metric)**.** Let $G = (V, w, m)$ be a locally finite graph. We define the *resistance metric* $\rho : V^2 \to [0, \infty)$ via

$$\rho(x, y) := \sup\{f(y) - f(x) : \|\Gamma f\|_\infty \leq 1\}$$

and the *resistance diameter* as $\operatorname{diam}_\rho(G) := \sup_{x,y\in V} \rho(x, y)$.

In the case of bounded degree, there is a standard estimate between combinatorial and resistance metric.

Lemma 1.4. *(Combinatorial and resistance metric) Let $G = (V, w, m)$ be a locally finite graph with $\mathrm{Deg}_{\max} < \infty$. Then for all $x_0, y_0 \in V$,*

$$d(x_0, y_0) \leq \sqrt{\frac{\mathrm{Deg}_{\max}}{2}}\rho(x_0, y_0).$$

Proof. Let $f : V \to \mathbb{R}$ be defined by $f(x) := d(x, x_0)\sqrt{\frac{2}{\mathrm{Deg}_{\max}}}$. Then for all $x \in V$,

$$\begin{aligned} 0 \leq \Gamma f(x) &= \frac{1}{2m(x)} \sum_y w(x, y)(f(y) - f(x))^2 \\ &\leq \frac{1}{2m(x)} \sum_y w(x, y) \frac{2}{\mathrm{Deg}_{\max}} \\ &= \frac{\mathrm{Deg}(x)}{\mathrm{Deg}_{\max}} \\ &\leq 1. \end{aligned}$$

Hence,

$$\rho(x_0, y_0) \geq f(y_0) - f(x_0) = d(y_0, x_0)\sqrt{\frac{2}{\mathrm{Deg}_{\max}}}.$$

This directly implies the claim. □

2. Bonnet-Myers via the Bakry-Émery curvature-dimension condition

In the first subsection, we obtain sharp diameter bounds for $CD(K, \infty)$. In the second subsection, we obtain diameter bounds for unbounded Laplacians for $CD(K, n)$. In the third subsection we show that our results improve the diameter bounds from [11, Theorem 7.10] and from [7, Corollary 6.4].

2.1. Diameter bounds and $CD(K, \infty)$.

The key to prove diameter bounds from $CD(K, \infty)$ is the semigroup characterization of $CD(K, \infty)$ which is equivalent to

$$\Gamma P_t f \leq e^{-2Kt} P_t \Gamma f.$$

Here, P_t denotes the heat semigroup operator. For details, see e.g. [9, 15].

Theorem 2.1 (Distance bounds under $CD(K,\infty)$)**.** *Let (V,w,m) be a connected graph satisfying $CD(K,\infty)$ and $\mathrm{Deg}_{\max} < \infty$. Then for all $x_0, y_0 \in V$,*

$$\rho(x_0,y_0) \leq \frac{\sqrt{2\,\mathrm{Deg}(x_0)} + \sqrt{2\,\mathrm{Deg}(y_0)}}{K}.$$

Proof. By [15, Theorem 3.1] and $\mathrm{Deg}_{\max} < \infty$, we have that $CD(K,\infty)$ is equivalent to

$$\Gamma P_t f \leq e^{-2Kt} P_t \Gamma f \tag{2.1}$$

for all bounded functions $f : V \to \mathbb{R}$. Due to Cauchy-Schwarz, $(\Delta g(x))^2 \leq 2\mathrm{Deg}(x)(\Gamma g)(x)$ for all $x \in V$ and all $g : V \to \mathbb{R}$. We fix $x_0, y_0 \in V$ and $\varepsilon > 0$. Then by definition of ρ, there is a function $f : V \to \mathbb{R}$ s.t. $f(y_0) - f(x_0) > \rho(x_0,y_0) - \varepsilon$ and $\Gamma f \leq 1$. W.l.o.g., we can assume that f is bounded. Putting everything together yields for all $x \in V$,

$$|\partial_t P_t f(x)|^2 = |\Delta P_t f(x)|^2 \leq 2\mathrm{Deg}(x)\Gamma P_t f(x) \leq 2\mathrm{Deg}(x) e^{-2Kt} P_t \Gamma f(x) \leq 2\,\mathrm{Deg}(x) e^{-2Kt}.$$

By taking the square root and integrating from $t = 0$ to ∞, we obtain

$$|P_T f(x) - f(x)| \leq \int_0^\infty |\partial_t P_t f(x)|\, dt \leq \int_0^\infty \sqrt{2\,\mathrm{Deg}(x)} e^{-Kt} dt = \frac{\sqrt{2\,\mathrm{Deg}(x)}}{K}$$

for all $T > 0$ and $x \in V$. Now, the triangle inequality yields

$$\begin{aligned} \rho(x_0,y_0) - \varepsilon &\leq |f(x_0) - f(y_0)| \\ &\leq |P_t f(x_0) - f(x_0)| + |P_t f(x_0) - P_t f(y_0)| + |P_t f(y_0) - f(y_0)| \\ &\leq \frac{\sqrt{2\,\mathrm{Deg}(x_0)} + \sqrt{2\,\mathrm{Deg}(y_0)}}{K} + |P_t f(x_0) - P_t f(y_0)| \\ &\overset{t\to\infty}{\longrightarrow} \frac{\sqrt{2\,\mathrm{Deg}(x_0)} + \sqrt{2\,\mathrm{Deg}(y_0)}}{K} \end{aligned}$$

where $|P_t f(x_0) - P_t f(y_0)| \to 0$, since the graph is connected and since $\Gamma P_t f \to 0$ as $t \to \infty$ because of (2.1). Taking the limit $\varepsilon \to 0$ finishes the proof. □

We now use the distance bound to obtain a bound on the combinatorial diameter.

Corollary 2.2 (Diameter bounds under $CD(K,\infty)$)**.** *Let (V,w,m) be a connected graph satisfying $CD(K,\infty)$. Then,*

$$\mathrm{diam}_d(G) \leq \frac{2\,\mathrm{Deg}_{\max}}{K}.$$

Proof. Due to Lemma 1.4 and the above theorem, we have for all x_0, y_0,

$$d(x_0,y_0) \leq \sqrt{\frac{\mathrm{Deg}_{\max}}{2}} \rho(x_0,y_0) \leq \sqrt{\frac{\mathrm{Deg}_{\max}}{2}} \frac{\sqrt{2\,\mathrm{Deg}(x_0)} + \sqrt{2\,\mathrm{Deg}(y_0)}}{K} \leq \frac{2\,\mathrm{Deg}_{\max}}{K}.$$

Thus, $\mathrm{diam}_d(G) \leq \frac{2\,\mathrm{Deg}_{\max}}{K}$ as claimed. □

Indeed, this diameter bound is sharp for the n-dimensional hypercube which has diameter n, curvature bound $K = 2$ and vertex degree $\mathrm{Deg}_{\max} = n$.

An interesting question is whether an analog of Cheng's rigidy theorem (see [4]) holds true. In particular, we ask whether hypercubes are the only graphs for which the above diameter bound is sharp.

2.2. **Diameter bounds and $CD(K,n)$.** We can also give diameter bounds for unbounded Laplacians. We need two ingredients to do so. First, we have to replace the combinatorial metric by a resistance metric. Second, we have to assume a finite dimension bound. Furthermore, we will need completeness of the graph and non-degenerate vertex measure to obtain the semigroup characterization of $CD(K,n)$ (see [9, Theorem 3.3], [12, Theorem 1.1]). For definitions of completeness of graphs and non-degenerate vertex measure, see Sections 1 and 2.1 of [12] or [9, Definition 2.9, Definition 2.13].

We start with an easy but useful consequence of Gong and Lin's semigroup characterization of $CD(K,\infty)$, see [9, Theorem 3.3].

Lemma 2.3 (Semigroup property of $CD(K,n)$)**.** *Let $G=(V,w,m)$ be a complete graph with non-degenerate vertex measure. Suppose G satisfies $CD(K,n)$. Then for all bounded $f:V\to\mathbb{R}$ with bounded Γf,*

$$\Gamma P_t f \le e^{-2Kt}P_t\Gamma f - \frac{1-e^{-2Kt}}{Kn}(\Delta P_t f)^2. \tag{2.2}$$

Proof. We first assume that f is compactly supported. By [9, Theorem 3.3], we have

$$\Gamma P_t f \le e^{-2Kt}P_t\Gamma f - \frac{2}{n}\int_0^t e^{-2Ks}P_s(\Delta P_{t-s}f)^2 ds. \tag{2.3}$$

Jensen's inequality yields $P_s g^2 \ge (P_s g)^2$ for all g and thus by $g:=\Delta P_{t-s}f$,

$$\frac{2}{n}\int_0^t e^{-2Ks}P_s(\Delta P_{t-s}f)^2 ds \ge \frac{2}{n}\int_0^t e^{-2Ks}(P_s\Delta P_{t-s}f)^2 ds = \frac{1-e^{-2Kt}}{Kn}(\Delta P_t f)^2. \tag{2.4}$$

Putting (2.3) and (2.4) together yields the claim for compactly supported f. We now prove the claim for all bounded f with bounded $\Gamma(f)$ by completeness and a density argument. Completeness implies that there are compactly supported $(\eta_k)_{k\in\mathbb{N}}$ s.t. $\eta_k\to 1$ from below and $\Gamma\eta_k\le 1$. Due to compact support, (2.2) holds for $\eta_k f$. Obviously since $\eta_k\to 1$ from below, we have $\Gamma P_t(\eta_k f)\to\Gamma P_t f$ and $\Delta P_t(\eta_k f)\to\Delta P_t f$, pointwise for $k\to\infty$. It remains to show

$$P_t\Gamma(\eta_k f)\to P_t\Gamma f,$$

pointwise for $k\to\infty$. We observe

$$\begin{aligned}[(\eta_k f)(y)-(\eta_k f)(x)]^2 &= [\eta_k(y)(f(y)-f(x))+f(x)(\eta_k(y)-\eta_k(x))]^2\\ &\le 2\,[\eta_k(y)(f(y)-f(x))]^2+2\,[f(x)(\eta_k(y)-\eta_k(x))]^2\\ &\le 2\,\|\eta_k\|_\infty^2\,(f(y)-f(x))^2+2\|f\|_\infty^2(\eta_k(y)-\eta_k(x))^2\end{aligned}$$

and thus,

$$\Gamma(\eta_k f)\le 2\,\|\eta_k\|_\infty^2\,\Gamma f+2\|f\|_\infty^2\Gamma\eta_k.$$

This implies that $\Gamma(\eta_k f)$ is uniformly bounded in k and since $\eta_k f\to f$ pointwise, we obtain $P_t\Gamma(\eta_k f)\to P_t\Gamma f$ as desired. ☐

With this semigroup property in hands, we now can prove diameter bounds. We will use similar methods as in the proof of Theorem 2.1.

Theorem 2.4 (Diameter bounds under $CD(K,n)$)**.** *Let $G=(V,w,m)$ be a connected, complete graph with non-degenerate vertex measure. Suppose G satisfies $CD(K,n)$ for some $K>0$ and $n<\infty$. Then,*

$$\operatorname{diam}_\rho(G)\le\pi\sqrt{\frac{n}{K}}.$$

Proof. Suppose the opposite. Then, there are $x, y \in V$ s.t. $\rho(x,y) > \pi\sqrt{\frac{n}{K}}$, and there is a function $f : V \to \mathbb{R}$ s.t. $\Gamma f \leq 1$ and $f(y) - f(x) > \pi\sqrt{\frac{n}{K}}$. W.l.o.g., we can assume that f is bounded. By the semigroup property of $CD(K,n)$ from Lemma 2.3 and by ignoring the non-negative term $\Gamma P_t f$, we have

$$\frac{1-e^{-2Kt}}{Kn}(\Delta P_t f)^2 \leq e^{-2Kt} P_t \Gamma f.$$

Taking square root and applying $\Gamma f \leq 1$ yields

$$|\partial_t P_t f| = |\Delta P_t f| \leq \sqrt{Kn}\sqrt{\frac{e^{-2Kt}}{1-e^{-2Kt}}} = \sqrt{Kn}\sqrt{\frac{1}{e^{2Kt}-1}}.$$

Integrating from $t = 0$ to ∞ yields for all T,

$$|P_T f - f| \leq \sqrt{Kn}\int_0^\infty \sqrt{\frac{1}{e^{2Kt}-1}}dt = \sqrt{Kn}\frac{\arctan\sqrt{e^{2Kt}-1}}{K}\Bigg|_{t=0}^{\infty} = \frac{\pi}{2}\sqrt{\frac{n}{K}}$$

Due to $CD(K,\infty)$ which implies $\|\Gamma P_t f\|_\infty \overset{t\to\infty}{\longrightarrow} 0$ and since G is connected, we infer $|P_t f(x) - P_t f(y)| \overset{t\to\infty}{\longrightarrow} 0$. We now apply the triangle inequality and obtain

$$\begin{aligned}
\pi\sqrt{\frac{n}{K}} &< |f(y) - f(x)| \\
&\leq |P_t f(y) - f(y)| + |P_t f(y) - P_t f(x)| + |P_t f(x) - f(x)| \\
&\leq \pi\sqrt{\frac{n}{K}} + |P_t f(y) - P_t f(x)| \\
&\overset{t\to\infty}{\longrightarrow} \pi\sqrt{\frac{n}{K}}.
\end{aligned}$$

This is a contradiction and thus, $\operatorname{diam}_\rho(G) \leq \pi\sqrt{\frac{n}{K}}$ as claimed. ☐

In contrast to Corollary 2.2, we cannot have sharpness in Theorem 2.4 since in the proof, we have thrown away $\Gamma P_t f$ which is strictly positive for $t > 0$.

2.3. **Comparison with other discrete diameter bounds.** In [11, Theorem 7.10], Horn, Lin, Liu and Yau have proven

$$CDE'(K,n) \quad \Longrightarrow \quad \operatorname{diam}_d(G) \leq 2\pi\sqrt{\frac{6\operatorname{Deg}_{\max} n}{K}}.$$

Indeed, due to [18, Corollary 3.3] and Theorem 2.4, this result can be improved to

$$CDE'(K,n) \quad \Longrightarrow \quad CD(K,n) \Longrightarrow \quad \operatorname{diam}_\rho(G) \leq \pi\sqrt{\frac{n}{K}}.$$

In case of $\operatorname{Deg}_{\max} < \infty$, we have

$$\operatorname{diam}_d(G) \leq \sqrt{\frac{\operatorname{Deg}_{\max}}{2}}\operatorname{diam}_\rho(G) \leq \pi\sqrt{\frac{\operatorname{Deg}_{\max} n}{2K}},$$

where the first estimate is due to Lemma 1.4.

Remark 2.5. Summarizing, we can say that our approach improves [11, Theorem 7.10] by a factor of $4\sqrt{3}$ and by having weaker curvature assumptions.

Let us also compare our results to the results on Markov-chains in [7]. Translated into the graph setting, we have the following result in [7].

Theorem 2.6. *(see* [7, Corollary 6.4]*) Let* $G = (V, w, m)$ *be a graph with* $\mathrm{Deg}_{\max} < \infty$ *and* $m(V) := \sum_x m(x) < \infty$. *Suppose* G *satisfies* $CD(K, \infty)$ *for some* $K > 0$. *Then for all* $x, y \in V$,

$$\rho(x,y) \leq 2\sqrt{2}\frac{\left(\sqrt{\mathrm{Deg}(x)} + \sqrt{\mathrm{Deg}(y)}\right)}{K}.$$

Proof. Indeed, the theorem is just a reformulation of [7, Corollary 6.4]. Let $G = (V, w, m)$ be a graph with $m(V) < \infty$ and $\mathrm{Deg}_{\max} < \infty$. We define the corresponding Markov kernel as

$$K(x,y) := \begin{cases} \frac{w(x,y)}{m(x)\,\mathrm{Deg}_{\max}} & : x \neq y \\ 1 - \frac{\mathrm{Deg}(x)}{\mathrm{Deg}_{\max}} & : x = y \end{cases}$$

and the corresponding measure $\mu(x) := m(x)/m(V)$ for all $x \in V$ according to [7]. Then, $Lf(x) := \sum_y (f(y) - f(x))K(x,y) = \frac{\Delta f(x)}{\mathrm{Deg}_{\max}}$.

Since Δ satisfies $CD(K, \infty)$, we have that L satisfies $CD(\frac{K}{\mathrm{Deg}_{\max}}, \infty)$. As in [7], we set

$$J(x) := 1 - K(x,x) = \frac{\mathrm{Deg}(x)}{\mathrm{Deg}_{\max}}.$$

For all $x, y \in V$, we have

$$d_\Gamma(x,y) := \sup\left\{ f(y) - f(x) : \frac{1}{2}L(f^2) - fLf \leq 1 \right\} = \sqrt{\mathrm{Deg}_{\max}}\rho(x,y).$$

Now, [7, Corollary 6.4] yields for all $x, y \in V$,

$$\begin{aligned} \rho(x,y)\frac{K}{\sqrt{\mathrm{Deg}_{\max}}} = d_\Gamma(x,y)\frac{K}{\mathrm{Deg}_{\max}} &\leq 2\sqrt{2}\left(\sqrt{J(x)} + \sqrt{J(y)}\right) \\ &= \frac{2\sqrt{2}\left(\sqrt{\mathrm{Deg}(x)} + \sqrt{\mathrm{Deg}(y)}\right)}{\sqrt{\mathrm{Deg}_{\max}}}. \end{aligned}$$

Multiplying with $\frac{\sqrt{\mathrm{Deg}_{\max}}}{K}$ finishes the proof. ☐

Remark 2.7. We observe that Theorem 2.1 improves Fathi's and Shu's result [7, Corollary 6.4], see Theorem 2.6, by a factor of 2 and by the fact, that we allow $m(V) = \infty$ which corresponds to an infinite reversible invariant measure in the Markov-chain setting.

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S. Liu, School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui Province, China

E-mail address: spliu@ustc.edu.cn

F. Münch, Institut für Mathematik, Universität Potsdam, 14476 Potsdam, Germany

E-mail address: chmuench@uni-potsdam.de

N. Peyerimhoff, Department of Mathematical Sciences, Durham University, Durham DH1 3LE, United Kingdom

E-mail address: norbert.peyerimhoff@durham.ac.uk