Corrigendum: "Auctions with a buy price: The case of reference-dependent preferences"

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In this short note, we correct a mistake in the main result of Shunda (2009). He studies an auction model with reserve prices and buy prices, and with reference-dependent preferences as in Rosenkranz and Schmitz (2007) in which the reference point is a convex combination of the reserve price and the buy price, and claims that (page 653, last paragraph):

> A seller would set [...] a buy price that some bidder type would accept with positive probability in equilibrium. This result clearly contrasts with the finding that a risk-neutral seller facing risk-neutral bidders would set a buy price so high that no bidder would accept it with positive probability in equilibrium.

We shall show that this result is not correct. In particular, we shall show that although it is true that it is optimal for the auctioneer to use buy prices, the optimal buy price is so high that it is never exercised by the bidders in equilibrium. This finding implies that the model of Shunda rationalizes why sellers may want to use buy prices, but it does not explain why bidders exercise them.

We also find another related mistake in Shunda's analysis. The maximization problem that the seller solves is not well-defined for some parameter values, as the auctioneer can achieve unbounded expected revenue with an appropriate choice of reserve prices and buy prices.

Formally, we shall show that Theorem 1 in Shunda (2009) should be amended and it should say instead:[1]

Theorem 1. *If $\epsilon < \frac{1}{\sqrt{1-\lambda}}$, a risk-neutral seller maximizes her expected revenue by setting some $v^* = \overline{v}$ and thus by posting a buy price $B(\overline{v}, r)$, i.e. a buy price that none of the bidder types less than $\overline{v}$ would exercise with positive probability in equilibrium.*

If $\epsilon \geq \frac{1}{\sqrt{1-\lambda}}$, the auctioneer can achieve arbitrarily large expected payoffs: when $\epsilon < \frac{1}{1-\lambda}$, making r arbitrarily large and $B^* = B(\overline{v}, r)$; and when $\epsilon \geq \frac{1}{1-\lambda}$, making B arbitrarily large and $r = 0$.

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[1] We have tried to follow the notation and statement of the original theorem in Shunda (2009). However, we find that the statement may be slightly confusing. We state in Theorem 2 an equivalent more transparent statement that uses the notation we introduce below.

To understand Shunda's mistake note that he points out that the reason why he obtains that the auctioneer does not find it optimal to fix a buy price so high that it is not exercised by any bidder is that (page 654, line 14):

> [...] one effect of increasing the auction's buy price is to attract low valuation bidders to the auction who would not have participated otherwise, and this has an adverse effect on the seller's revenue.

We show that although this statement may be correct for some given reserve price, the auctioneer can encompass the increase in the buy price with an increase in the reserve price so that the set of low valuation bidders that participate in the auction does not change and the auctioneer's expected revenue increases.

In our analysis we depart from Shunda's notation with respect to the use of B^* and v^*. He finds that there is a monotone relationship between the buy price B^* and the threshold v^* for any given reserve price r. Thus, instead of solving for the optimal buy price B^*, he solves for the optimal threshold v^* for any reserve price. We find this approach slightly confusing and indeed may explain Shunda's mistake. We find more natural and tractable to solve directly for the optimal buy price B^* and reserve price r.

It is tedious but otherwise standard (see the details in the appendix) to show that the auctioneer's expected revenue is equal to:

$$\int_{\underline{\underline{v}}(r,B^*)}^{v^*(r,B^*)} \left(\beta(y,r,B^*) - \frac{1-F(y)}{f(y)(1+\epsilon)}\right) dF(y)^n + \beta(v^*(r,B^*),r,B^*)(1-F(v^*(r,B^*))^n). \quad (1)$$

As in Shunda (2009), $\underline{\underline{v}}$, v^* and β on (r,B^*) denote respectively the minimum type that participates in the auction, the minimum type that exercise the buy price, and $\beta(y,r,B^*)$ the maximum willingness to pay of a bidder with type y when the reserve price is r and the buy price B^*. This is:

$$\beta(v,r,B^*) \equiv \frac{v + \epsilon(\lambda r + (1-\lambda)B^*)}{1+\epsilon},$$

$$\underline{\underline{v}}(r,B^*) = \max\{\underline{v}, \min\{\overline{v}, (1+\epsilon(1-\lambda))r - \epsilon(1-\lambda)B^*\}\},$$

and $v^*(r,B^*) \in [\underline{\underline{v}}(r,B^*), \overline{v}]$ is uniquely defined by:

$$\int_{\underline{\underline{v}}(r,B^*)}^{v^*} F(y)^{n-1} dy = \left(\frac{1-F(v^*)^n}{n(1-F(v^*))}\right)(\beta(v^*,r,B^*)) - B^*)(1+\epsilon),$$

if

$$\int_{\underline{\underline{v}}(r,B^*)}^{\overline{v}} F(y)^{n-1} dy < (\beta(\overline{v},r,B^*)) - B^*)(1+\epsilon),$$

and $v^*(r,B^*) \equiv \overline{v}$ otherwise.

Note that unlike Shunda, we have made explicit the dependence of $\underline{\underline{v}}$, v^* and β on (r,B^*) and the boundary conditions that must be verified by $\underline{\underline{v}}$ and v^*.

The auctioneer chooses (r,B^*) that maximize Equation (1) subject to the constraint that $B^* \leq \gamma(r)$ where $\gamma(r)$ is a function implicitly defined by the unique solution to:

$$\int_{\underline{\underline{v}}(r,\gamma)}^{\overline{v}} F(y)^{n-1} dy = (\beta(\overline{v},r,\gamma) - \gamma)(1+\epsilon),$$

if $\epsilon < \frac{1}{1-\lambda}$, and $\gamma(r) = +\infty$ otherwise. Note that such γ is s continuous function.

The constraint $B^* \leq \gamma(r)$ corresponds to the assumption in Shunda (2009), first full paragraph in page 655.[2] To see why, note that the condition that defines $\gamma(r)$ is equivalent to say that a given bidder with type $\overline{v}$ is indifferent between participating in the auction or exercising the buy price when all the other bidders' types above $\underline{\underline{v}}(r, \gamma(r))$ go to the auction. It is easy to see that a lower type strictly prefers the auction to the buy price.

We can write our version of Theorem 1 with this new notation as follows:

Theorem 2. *If $\epsilon < \frac{1}{\sqrt{1-\lambda}}$, for any reserve price and buy price (r, B^*) such that an open set of bidders' type chooses the buy price in equilibrium, i.e. $B^* < \gamma(r)$, we can always find a reserve price r' and buy price that is not exercised with positive probability in equilibrium, i.e. $\gamma(r')$ that gives strictly greater expected revenue.*

If $\epsilon \geq \frac{1}{\sqrt{1-\lambda}}$, the auctioneer can achieve arbitrarily large expected payoffs: when $\epsilon < \frac{1}{1-\lambda}$, making r arbitrarily large and $B^* = \gamma(r)$; and when $\epsilon \geq \frac{1}{1-\lambda}$, making B arbitrarily large and $r = 0$.

Proof. Consider first the case $\epsilon < \frac{1}{\sqrt{1-\lambda}}$. The fact that β is increasing in B^* and r and that[3]

$$\int_{\underline{\underline{v}}(r,B^*)}^{\hat{v}} \left(\beta(y, r, B^*) - \frac{1-F(y)}{f(y)(1+\epsilon)} \right) dF(y)^n + \beta(\hat{v}, r, B^*)(1 - F(\hat{v})^n), \tag{2}$$

is increasing in v^* means that it is sufficient to show that for any (r, B^*) such that $B^* < \gamma(r)$, there exists a $(r', \gamma(r')) > (r, B^*)$ such that $\underline{\underline{v}}(r', \gamma(r')) = \underline{\underline{v}}(r, B^*)$. This can be proved noting that that the graph of γ intersects with the set O of points $(\tilde{r}, \tilde{B}^*)$ such that $\underline{\underline{v}}(\tilde{r}, \tilde{B}^*)$ to the right and above of (r, B^*). Consider for simplicity the case $\underline{\underline{v}}(\tilde{r}, \tilde{B}^*) > \underline{v}$ (the case $\underline{\underline{v}}(\tilde{r}, \tilde{B}^*) = \underline{v}$ is similar). By the implicit function theorem $\gamma'(r) \leq \frac{\lambda}{\frac{1}{\epsilon}-(1-\lambda)}$, whereas O to the right of (r, B^*) is a straight line with slope $\frac{\frac{1}{\epsilon}+(1-\lambda)}{(1-\lambda)}$. The former slope is less than the latter when $\epsilon < \frac{1}{\sqrt{1-\lambda}}$, i.e. $\frac{1}{\epsilon} > \sqrt{1-\lambda}$:

$$\frac{\lambda}{\frac{1}{\epsilon} - (1-\lambda)} < \frac{\lambda}{\sqrt{1-\lambda} - (1-\lambda)} = \frac{\sqrt{1-\lambda} + (1-\lambda)}{1-\lambda} < \frac{\frac{1}{\epsilon} + (1-\lambda)}{1-\lambda},$$

and hence both functions cross to the right and above of (r, B^*) as desired.

Consider next the case $\epsilon \in \left[\frac{1}{\sqrt{1-\lambda}}, \frac{1}{1-\lambda}\right)$. We shall show that r arbitrarily large and $B^* = \gamma(r)$ gives unboundedly large expected revenue to the auctioneer. Since β grows unboundedly with B^* and r and $v^*(r, \gamma(r)) = \overline{v}$, a direct inspection of Equation (1) shows that a sufficient condition is that $\underline{\underline{v}}(r, \gamma(r)) = \underline{v}$ for any r. Since $\underline{\underline{v}}$ is decreasing in its first argument, we shall show that for any r, $\gamma(r)$ is greater than the infimum of the set of points (r, B^*) for which $\underline{\underline{v}}(r, B^*) = \underline{v}$. This infimum is a continuous function equal to 0 for any $r \in \left[0, \frac{\underline{v}}{1+\epsilon(1-\lambda)}\right]$ and a straight line with slope $\frac{\frac{1}{\epsilon}+(1-\lambda)}{(1-\lambda)}$, otherwise. To complete the argument just note that $\gamma(0) > 0$ and by the implicit function theorem $\gamma'(r) = \frac{\lambda}{\frac{1}{\epsilon}-(1-\lambda)}$

[2]An equivalent and perhaps more natural assumption is that the reference price ρ is equal to $\lambda r + (1-\lambda)\min\{\overline{v}, B^*\}$.

[3]Just check by differentiation that the function is convex in $\hat{v}$ with slope equal to zero at $\hat{v} = \overline{v}$.

if $\underline{v}(r, B^*) = \underline{v}$, which is greater than the former slope when $\epsilon \in \left[\frac{1}{\sqrt{1-\lambda}}, \frac{1}{1-\lambda}\right)$, i.e. $\frac{1}{\epsilon} \in \left(1-\lambda, \sqrt{1-\lambda}\right]$:

$$\frac{\lambda}{\frac{1}{\epsilon} - (1-\lambda)} \geq \frac{\lambda}{\sqrt{1-\lambda} - (1-\lambda)} = \frac{\sqrt{1-\lambda} + (1-\lambda)}{1-\lambda} \geq \frac{\frac{1}{\epsilon} + (1-\lambda)}{1-\lambda}.$$

Finally, in the case $\epsilon \geq \frac{1}{1-\lambda}$, note that we can argue as above that $r = 0$ and B^* arbitrarily large gives unbounded expected revenue, just note that in the case $\epsilon \geq \frac{1}{1-\lambda}$, $\gamma(0) = +\infty$ and hence the constraint $B^* \leq \gamma(0)$ is never binding.

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Appendix

In the last line of page 654 of Shunda (2009), he shows that the auctioneer's expected revenue when she posts a reserve price r and a Buy price $B^* \geq r$ is equal in equilibrium to:

$$rnF(\underline{v})^{n-1}(F(v^*)-F(\underline{v}))+\int_{\underline{v}}^{v^*} \beta(y)n(n-1)F(y)^{n-2}f(y)(F(v^*)-F(y))dy+B^*(1-F(v^*)^n),$$

where,

$$\beta(v) = \frac{v+\epsilon\rho}{1+\epsilon}, \text{ for } \rho = \lambda r + (1-\lambda)B^*$$

and $\underline{v}$ is such that $\beta(\underline{v}) = r$, and v^* is the unique value in $[\underline{v}, \overline{v}]$ that verifies (this is Equation (4) in Shunda (2009) evaluated at $v_i = v^*$ and $B(v_i, r) = B^*$):

$$\int_{\underline{v}}^{v^*} F(y)^{n-1}dy = \left(\frac{1-F(v^*)^n}{n(1-F(v^*))}\right)(\beta(v^*) - B^*)(1+\epsilon),$$

if any, and otherwise $v^* = \overline{v}$. In the limit when v^* tends to $\overline{v}$, this equation becomes:

$$\int_{\underline{v}}^{\overline{v}} F(y)^{n-1}dy = (\beta(\overline{v}) - B^*)(1+\epsilon).$$

This equation determines a relationship between B^* and r. This relationship determines the set of points for which $v^* = \overline{v}$.

To show that $v^* < \overline{v}$ is suboptimal, we shall consider some r and B^* such that this inequality is verified and show that the auctioneer can strictly improve by increasing B^* and r in an appropriate manner. In this case, we can substitute r and B^* above to get that the auctioneer's expected revenue is equal to:

$$\begin{aligned}\beta(\underline{v})nF(\underline{v})^{n-1}(F(v^*) - F(\underline{v})) \\ + \int_{\underline{v}}^{v^*} \beta(y)n(n-1)F(y)^{n-2}f(y)(F(v^*) - F(y))dy \\ + \beta(v^*)(1-F(v^*)^n) - \frac{n(1-F(v^*))}{1+\epsilon}\int_{\underline{v}}^{v^*} F(y)^{n-1}dy. \quad (3)\end{aligned}$$

The first integral is equal to:

$$\int_{\underline{\underline{v}}}^{v^*} \beta(y)d\left(F(y)^n + n(F(v^*) - F(y))F(y)^{n-1}\right) =$$

$$\int_{\underline{\underline{v}}}^{v^*} \beta(y)dF(y)^n + \int_{\underline{\underline{v}}}^{v^*} \beta(y)d\left(n(F(v^*) - F(y))F(y)^{n-1}\right) =$$

$$\int_{\underline{\underline{v}}}^{v^*} \beta(y)dF(y)^n - \beta(\underline{\underline{v}})n(F(v^*) - F(\underline{\underline{v}}))F(\underline{\underline{v}})^{n-1} - \int_{\underline{\underline{v}}}^{v^*} \left(n(F(v^*) - F(y))F(y)^{n-1}\right) d\beta(y). \tag{4}$$

Also note that the last integral above is equal to:

$$-\int_{\underline{\underline{v}}}^{v^*} \frac{n(F(v^*) - F(y))F(y)^{n-1}}{1+\epsilon} dy =$$

$$-\int_{\underline{\underline{v}}}^{v^*} \frac{n(1 - F(y))F(y)^{n-1}}{1+\epsilon} dy + \int_{\underline{\underline{v}}}^{v^*} \frac{n(1 - F(v^*))F(y)^{n-1}}{1+\epsilon} dy =$$

$$-\int_{\underline{\underline{v}}}^{v^*} \frac{1 - F(y)}{f(y)(1+\epsilon)} dF(y)^n + \frac{n(1 - F(v^*))}{1+\epsilon} \int_{\underline{\underline{v}}}^{v^*} F(y)^{n-1} dy \tag{5}$$

As a consequence, the auctioneer's expected revenue is equal to:

$$\int_{\underline{\underline{v}}}^{v^*} \left(\beta(y) - \frac{1 - F(y)}{f(y)(1+\epsilon)}\right) dF(y)^n + \beta(v^*)(1 - F(v^*)^n).$$

If we increase B^* and r so that $\underline{\underline{v}}$ remains constant, i.e. if the increment in B^* is equal to $\frac{1+\epsilon(1-\lambda)}{\epsilon(1-\lambda)}$ the increment in r, ρ increases, and hence $\beta(y)$, and v^*

References

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