Accelerating Black Holes

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Abstract. In this presentation, I review recent work [1, 2] with Mike Appels and David Kubizňák on thermodynamics of accelerating black holes. I start by reviewing the geometry of accelerating black holes, focussing on the conical deficit responsible for the 'force' causing the black hole to accelerate. Then I discuss black hole thermodynamics with conical deficits, showing how to include the tension of the deficit as a thermodynamic variable, and introducing a canonically conjugate *thermodynamic length*. Finally, I describe the thermodynamics of the slowly accelerating black hole in anti-de Sitter spacetime.

1. Introduction

Black holes have proved to be perennially fascinating objects to study, both from the theoretical and practical points of view. We have families of exact solutions in General Relativity and beyond, and an ever better understanding of their phenomenology via numerical and observational investigations. However, in all of these situations the description of the black hole is as an isolated object, barely influenced by its sometimes extreme environment, its only possible response being to grow by accretion. The familiar exact solutions in GR describe precisely this – an isolated, topologically spherical, solution to the Einstein equations. The black hole is the ultimate slippery object – to accelerate, we must be able to 'push' or 'pull' on the actual event horizon, yet, by its very nature, anything touching the event horizon must be drawn in, unless it travels locally at the speed of light. Translating this to a physical object touching the event horizon, this means that the energy momentum tensor must locally have the form $T^0_0 \sim T^r_r$. Luckily, we have a candidate object that satisfies this requirement.

A cosmic string [3] is a very thin, quasi-linear object, with an energy momentum dominated by its mass per unit length, and a string tension equal in magnitude to this energy. Cosmic strings can be formed in field theories with non-simply connected vacua, and gravitationally produce no long range local curvature, but generate an overall global conical deficit in the spatial sections orthogonal to the string, thus a string, effectively, is a local conical deficit in the spacetime. It turns out that this conical deficit is precisely what accelerates the black hole.

2. Accelerating black holes

The accelerating black hole metric has been known for many years, though perhaps less widely outside the classical GR community, thus it is worth briefly reviewing its structure. It is described

by the *C-metric* [4, 5, 6], which can be written in Schwarzschild-like coordinates [7] as

$$ds^2 = \frac{1}{(1 + Ar\cos\theta)^2}\left\{ f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\left[\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\phi^2}{K^2}\right]\right\} \tag{1}$$

where ϕ has periodicity 2π, and

$$f(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r}\right) + \frac{r^2}{\ell^2} , \qquad g(\theta) = 1 + 2mA\cos\theta , \tag{2}$$

where m represents the mass scale of the black hole, A the acceleration, K the conical deficit, and ℓ the (negative) cosmological constant, via $\Lambda = -3/\ell^2$. It is worth briefly deconstructing the C-metric to highlight the importance of the various parameters to the geometry and physics of the solution.

The function $f(r)$ determines the horizon structure of the black hole; whether there are one, two, or no horizons, depends on the mass parameter and the competition between the acceleration and cosmological constant. First, to understand the role of K, set $A \to 0$. Then (1) is clearly identifiable as the Schwarzschild-AdS solution, with the proviso that ϕ is now divided by K. Focussing on the axis through the black hole shows that that circles around this axis have a circumference $2\pi(1 - 1/K)$ times their proper radius, instead of the expected 2π of a locally flat space. Thus, the axis contains a conical deficit

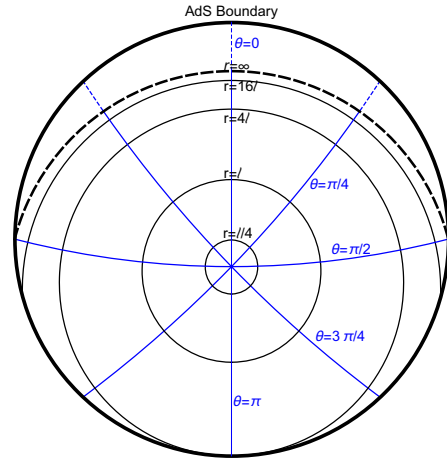$$\delta = 2\pi\left(1 - \frac{1}{K}\right) = 8\pi\mu \tag{3}$$

running through the black hole [8] that can be replaced by a non-singular cosmic string [9].

Now consider the acceleration parameter, A. This in part counterbalances the effect of the cosmological constant in $f(r)$, and if m is nonzero, gives a distortion of the two-spheres at constant r. To isolate the effect of A, setting $m = 0$ and $K = 1$ in (1) and keeping $A\ell < 1$ means that there is no acceleration horizon, and a simple coordinate transformation

Figure 1. AdS spacetime in slowly accelerating coordinates.

$$1 + \frac{R^2}{\ell^2} = \frac{1 + (1 - A^2\ell^2)r^2/\ell^2}{(1 - A^2\ell^2)\Omega^2} \qquad ; \qquad R\sin\Theta = \frac{r\sin\theta}{\Omega} \tag{4}$$

gives pure AdS spacetime [10]. The C-metric coordinates in (1) represent an off-centre perspective of AdS – they are the coordinates of an observer hovering at fixed distance from the centre of AdS, who is actually accelerating (see figure 1), but not sufficiently strongly to form an acceleration horizon.

Putting together, A gives the acceleration of the black hole that counterbalances the effect of the cosmological constant in the Schwarzschild potential. The acceleration also gives an

imbalance between the North and South axes that now have different conical deficits:

- $\theta \to \theta_+ = 0 \quad ds^2_{\theta,\phi} \propto d\theta^2 + \dfrac{(1 + 2mA)^2}{K^2} \theta^2 d\phi^2$

- $\theta \to \theta_- = \pi \quad ds^2_{\theta,\phi} \propto d\theta^2 + \dfrac{(1 - 2mA)^2}{K^2} (\pi - \theta)^2 d\phi^2$

$$\tag{5}$$

leading to the interpretation of the acceleration being induced by the imbalance of tension between the North and South axes:

$$8\pi\mu_\pm = \delta_\pm = 2\pi \left(1 - \frac{g(\theta_\pm)}{K} \right) = 1 - \frac{1}{K} \left(1 \pm 2mA + e^2 A^2 \right) \tag{6}$$

Conventionally, we choose to make the N axis regular by setting $K = 1 + 2mA$, leaving the deficit on the S axis corresponding to a cosmic string of tension $\mu_S = mA/K$ [11] (see figure 2. We will also demand that we have a *slowly accelerating* black hole, i.e. the acceleration is sufficiently gentle that we do not get an acceleration horizon (almost equivalent to $A\ell < 1$) [12]. The geometry therefore looks like a black hole in AdS displaced from the centre, the force required to hold it being supplied by the cosmic string or conical deficit running from the horizon to the boundary. Interpreting this tension as the force, and A as the acceleration, Newton's Law suggests that the mass[1] of the black hole should then be identified as $M = m/K$, so that $\mu = F = MA$ – we will see presently that this is exactly what the first law of thermodynamics requires.

Figure 2. A representation of the accelerating black hole.

3. Thermodynamics with conical defects

Before discussing the accelerating black hole, set $A = 0$ and look at the effect on the Schwarzschild-AdS black hole of the conical deficit (see [14] for early work with fixed tension deificits). The horizon is defined as $f(r) = 0$, so we examine incremental changes in the black hole

$$f(r_+ + \delta r_+) = f'(r_+)\delta r_+ + \frac{\partial f}{\partial m}\delta m + \frac{\partial f}{\partial \ell}\delta\ell = 0 \tag{7}$$

where we are allowing not only for a change in the black hole mass, but a possible change in the cosmological constant, as might occur for example when vacuum energy is due to a scalar field that is allowed to vary (see [15] for a discussion of black hole thermodynamics in this situation).

Defining temperature in the usual way via surface gravity, $T = f'(r_+)/4\pi$, we see that the first term in (7) will be related to a change in entropy, however, now the entropy, given by the area of the horizon [16], also depends on K:

$$S = \frac{\pi r_+^2}{K} \to f'(r_+)\delta r_+ = \frac{2K}{r_+} \left(T\delta S + \frac{r_+^2 f'(r_+)}{4} \frac{\delta K}{K^2} \right) \tag{8}$$

To identify the physics in changing K, note that tension was related to K, (3), from which we deduce

$$\delta\mu = \frac{\delta K}{4K^2} \tag{9}$$

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[1] Note that unlike the rapidly accelerating black hole, this is a genuine ADM-style mass, and not a "rearrangement of dipoles" as discussed in [13] where a boost mass was introduced.

Finally, writing the thermodynamic pressure [17, 18, 19, 20]

$$P = -\Lambda = \frac{3}{8\pi\ell^2} \qquad V = \frac{4\pi r_+^3}{3K} \tag{10}$$

allows us to identify a first law from (7)

$$\delta(\frac{m}{K}) = T\delta S + 2(m - r_+)\delta\mu + V\delta P \tag{11}$$

Leading to the definition of mass for the black hole $M = m/K$, as suggested by Newton's Law. Indeed, taking this definition of M also gives a Smarr formula [21] for the black hole:

$$M = 2TS - 2PV \tag{12}$$

Let us look more closely at this first law. The term multiplying the variation in tension has the form of a length, and we define this to be the *thermodynamic length*[2] associated to the tension charge of the string:

$$\lambda = r_+ - m \tag{13}$$

Note, we do not include the factor of 2, but instead have the length of the string from each pole – this will be particularly relevant when we look at the accelerating black hole that has different deficits at each pole. This reinforces the interpretation of M as the enthalpy [17] of the black hole: if the black hole grows, it swallows some string, but has also displaced the same amount of energy from the environment.
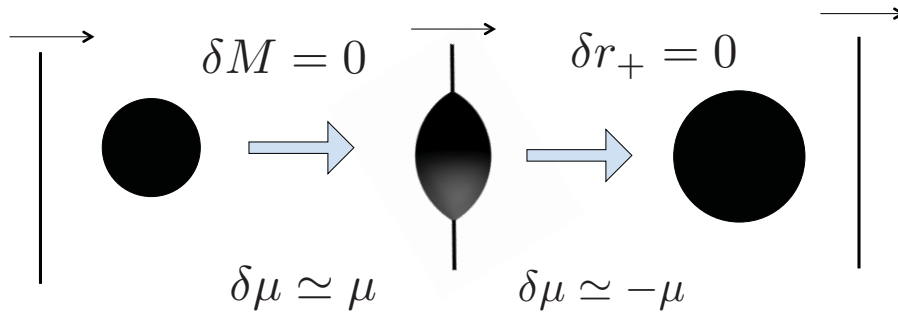
Figure 3. A black hole capturing a piece of cosmic string.

Does this definition and the first law make sense? In order to explore this, consider the situation of a cosmic string interacting with a black hole, first proposed in [23], and for simplicity let $\Lambda = 0$, for which the first law reduces to

$$\delta M = T\delta S - 2\lambda\delta\mu \tag{14}$$

Initially, the string is outside the black hole, but moving towards it (see figure 3). The black hole then captures the string and thus acquires a positive conical deficit, however because the string is still moving, it eventually pulls off the black hole, leaving a purely spherical black hole behind. The first step of the process has a positive $\delta\mu$ and the second an equal but negative

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[2] Note, a similar expression was obtained by Kastor and Traschen [22] in the context of an angular gravitational tension associated to the rotational symmetry of the black hole.

tension change, $-\delta\mu$. In the first stage, we would expect the total energy of the black hole to be constant, thus $\delta M = 0$, the physical intuition being that a cosmic string passing through a spherical shell of matter would not change the total energy throughout the process, For the black hole however, string cannot emerge from within the event horizon, and we have the interpretation of a segment of string being captured by the black hole, with the black hole increasing its mass accordingly.

In the first stage (taking $\mu \ll 1$), fixing M implies

$$T\delta S = 2\lambda\mu \tag{15}$$

consistent with $\delta m = 4m\delta\mu$, and $S = 4\pi m^2/K$. Because the energy of the black hole has been fixed, the event horizon moves outwards to compensate for the conical deficit, resulting in an increase of entropy indicating this is an irreversible thermodynamic process. In the second stage, the string pulls off the black hole, so $\delta\mu = -\mu$, and since entropy cannot decrease, M must increase:

$$\delta M = T\delta S + 2m\mu = \delta m + 4m\mu \tag{16}$$

in this case, it seems reasonable to suppose that the local geodesic congruence defining the event horizon cannot contract, giving $\delta m = 0$, and $\delta M = 4m\mu$. Thus, the net result of this adiabatic process is that the black hole has 'swallowed' the segment of cosmic string that fell inside its event horizon.

4. Thermodynamics of accelerating black holes

To find the first law for accelerating black holes, the idea is to follow the same steps: the horizon is defined by $f(r) = 0$, and consider small increments in horizon radius:

$$\delta f(r_+) = f'_+ \delta r_+ - 2\frac{\delta m}{r_+}(1 - A^2 r_+^2) - 2A\delta A r_+(r_+ - 2m) - 2\frac{r_+^2}{\ell^3}\delta\ell = 0 \tag{17}$$

Using the usual Euclidean method we can relate temperature to f'_+,

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi r_+^2}\left[m(1 - A^2 r_+^2) + \frac{r_+^3}{\ell^2(1 - A^2 r_+^2)}\right] \tag{18}$$

but now the area of the event horizon gives a more involved relation for δr_+:

$$\delta S = \delta\left(\frac{\pi r_+^2}{K(1 - A^2 r_+^2)}\right) = \frac{2\pi r_+ \delta r_+}{K(1 - A^2 r_+^2)^2} + \frac{2\pi r_+^4 A\delta A}{K(1 - A^2 r_+^2)^2} - \frac{\pi r_+^2}{(1 - A^2 r_+^2)}\frac{\delta K}{K^2} \tag{19}$$

The expressions for the tensions (6) allow us to replace $\delta K/K^2 = 2(\delta\mu_+ + \delta\mu_-)$ and $m\delta A/K = -\delta\mu_+ + \delta\mu_- - A\delta M$, having identified $M = m/K$ as before. Putting together, this gives the First Law as (see also [24])

$$\delta M = V\delta P + T\delta S - \lambda_+\delta\mu_+ - \lambda_-\delta\mu_- . \tag{20}$$

where we now have a thermodynamic length associated to the piece of string attached at each pole:

$$\lambda_\pm = \frac{r_+}{1 \pm Ar_+} - KM \tag{21}$$

This obviously agrees with (13) for the string threading the black hole, where r_+ has now been replaced by $r_+/\Omega(r_+, \theta_\pm)$ at each pole. We also have a Smarr relation,

$$M = 2TS - 2PV , \tag{22}$$

as before. The string does not contribute to the overall mass of the black hole as the string is gravitationally 'neutral' – it has no overall Riemann curvature, the spacetime is locally asymptotically flat – the effect of the string being simply a global deficit angle.
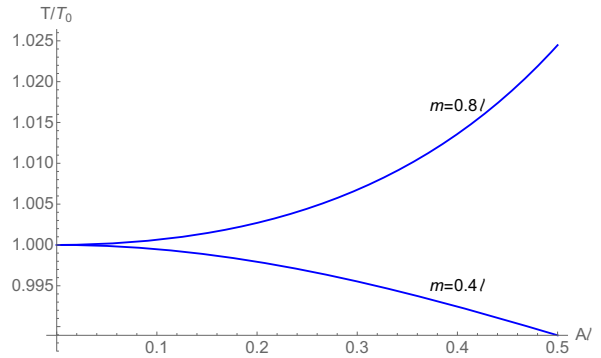
5. Exploring thermodynamics

Having derived a sensible first law for the accelerating black hole, now let us briefly explore some consequences of the expressions (for a full discussion see [1, 2]).
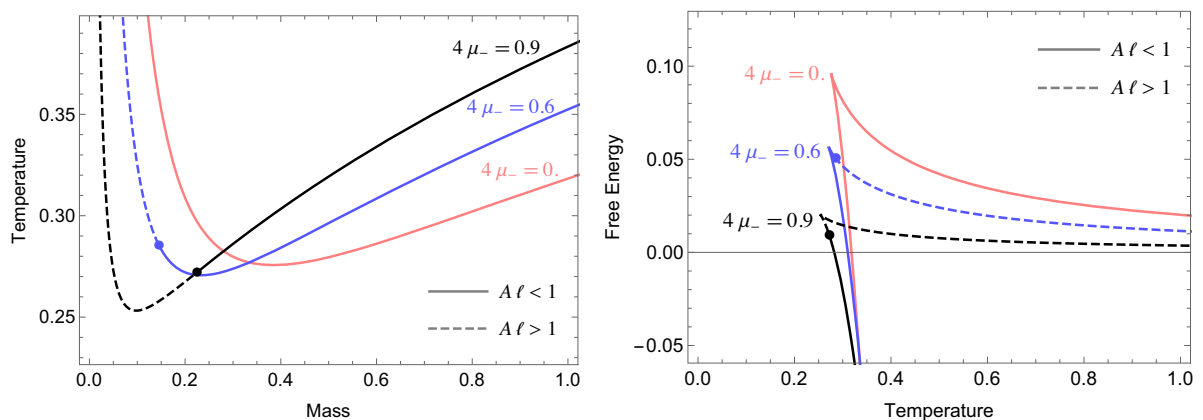
First consider the temperature of the accelerating black hole (18). Typically, we associate a temperature to the surface gravity of an horizon; apart from the black hole, horizons arise in the context of cosmology with a positive cosmological constant, and acceleration. Here we have explicitly taken our acceleration sufficiently low that we never get an acceleration horizon, but nonetheless, the effect of acceleration counterbalances the cosmological constant in the gravitational potential: $1/\ell^2 \to 1/\ell^2 - A^2 = 1/\ell_{\mathrm{eff}}^2$. We could perhaps expect that the temperature of the black hole is somehow a combination of the inherent Hawking temperature and the Rindler temperature, but in this case we would expect a systematic shift with A, and this is not what

Figure 4. The variation of temperature of a fixed mass black hole with acceleration.

happens. Instead, for smaller mass black holes the temperature decreases with acceleration, but for larger mass black holes it increases (see figure 4). To some extent this can be understood as a result of the varying tensions that are required to accelerate a fixed mass black hole differently.

Another phenomenon we can explore is that of the Hawking-Page phase transition [25]. For small black holes in AdS, the vacuum curvature is subdominant to the local tidal curvature of the black hole, thus thermodynamics is similar to vacuum with black holes being thermodynamically unstable. For black holes larger than the AdS radius however, the local tidal curvature becomes subdominant to the vacuum curvature, and the black hole has positive specific heat. Thus in AdS, there is a minimum temperature for the black hole, and below this temperature only a radiation bath can be a solution to the Einstein equations at finite T. At higher temperatures, one can have either a small or large black hole (or radiation bath). The Hawking-Page transition occurs at $T > 1/\pi\ell$, where the large black hole is thermodynamically preferred to the radiation bath, as can be seen by plotting the Gibbs free energy as a function of temperature. For the

Figure 5. A plot of temperature as a function of mass (left), and the Gibbs free energy as a function of temperature (right).

accelerating black hole, we fix the tension of the string, and plot the temperature as a function of (thermodynamic) mass in figure 5. This plot explicitly demonstrates that acceleration makes a black hole of given mass *more* thermodynamically stable in the sense of positive specific heat. A plot of the corresponding Gibbs free energy is also shown.

For simplicity, I focussed primarily in this presentation on uncharged non-rotating black holes to demonstrate the procedure for determining the thermodynamic charges and potentials. One can add charge to the black hole, which now allows for a much richer thermodynamic phase structure. Briefly, the charges and potentials in the presence of electric charge become [2]):

$$M = \frac{m}{K(1 + e^2 A^2)} \qquad Q = \frac{e}{K}$$

$$\Phi = \frac{e}{r_+} - \frac{meA^2}{1 + e^2 A^2} \quad \lambda_\pm = \frac{r_+}{1 \pm Ar_+} - \frac{m(1 - e^2 A^2)}{(1 + a^2 A^2)^2} \mp \frac{e^2 A}{1 + e^2 A^2} \tag{23}$$

with the entropy, thermodynamic pressure and volume unchanged in form.

With charge, there is a lower bound on the black hole mass from the extremal limit, thus at fixed Q, black holes have positive specific heat near this extremal limit, and at large mass, but in between can have negative specific heat if the charge is sufficiently low that the event horizon lies below the AdS scale. As with the uncharged black hole, acceleration makes the black hole more thermodynamically stable, as seen in figure 6.
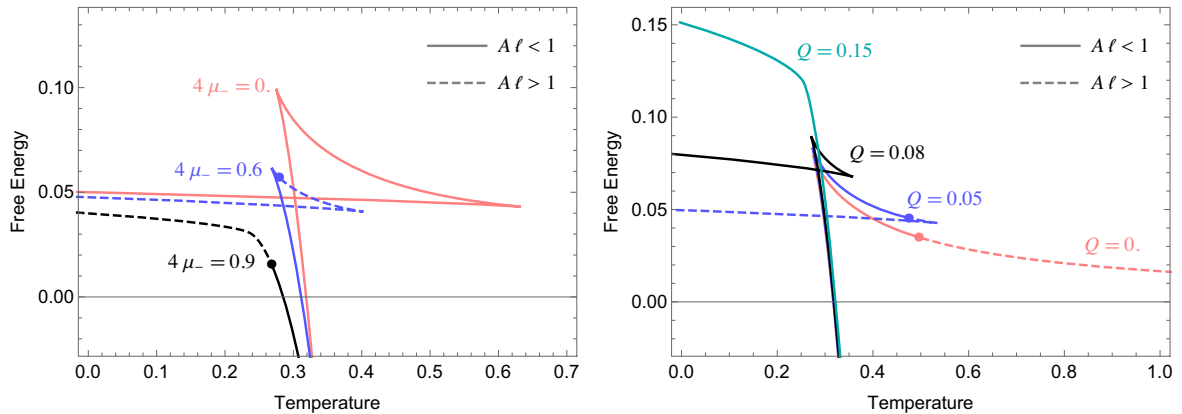
Figure 6. A plot of the free energy as a function of temperature for varying tension with $Q = 0.05\ell$ on the left, and varying charge with $4\mu_- = 0.3$ on the right.

6. Conclusion

To sum up: we have shown how to allow for varying tension in the thermodynamics of black holes. In accelerating black holes, if one does not allow the tension of the deficits to vary, then the various thermodynamic charges are not free to vary independently, and there is no guarantee that any thus derived thermodynamic quantities will be the correct ones, it is therefore mandatory to consider μ as a variable in order to correctly explore thermodynamics. Work on the thermodynamics of rotating accelerating black holes is reported on in the poster session [26].

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