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# Complex hyperbolic triangle groups with 2-fold symmetry

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**Abstract.** In this paper we will consider the 2-fold symmetric complex hyperbolic triangle groups generated by three complex reflections through angle $2\pi/p$ with $p \geqslant 2$. We will mainly concentrate on the groups where some elements are elliptic of finite order. Then we will classify all such groups which are candidates for being discrete. There are only 4 types.

## 1. Introduction

A complex hyperbolic triangle is a triple $(C_1, C_2, C_3)$ of complex geodesics in $\mathbf{H}^2_{\mathbb{C}}$. If each pair of complex geodesics intersects in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ and the angles between $C_{k-1}$ and $C_k$ for $k = 1, 2, 3$ (the indices are taken mod 3) are $\pi/p_1, \pi/p_2, \pi/p_3$, where $p_1, p_2, p_3 \in \mathbb{N} \cup \{\infty\}$, we call the triangle $(C_1, C_2, C_3)$ a $(p_1, p_2, p_3)$-*triangle*. The intersection points of pairs of complex geodesics are called the *vertices* of the complex hyperbolic triangle. A group $\Gamma$ is called a $(p_1, p_2, p_3)$-*triangle group*, if $\Gamma$ is generated by three complex reflections $R_1, R_2, R_3$ fixing sides $C_1, C_2, C_3$ of $(p_1, p_2, p_3)$-*triangle*. Note that a complex reflection may have order greater than 2. In what follows we suppose that $R_1$, $R_2$ and $R_3$ all have order $p \in \mathbb{Z}$ with $p \geqslant 2$.

Any two real hyperbolic triangle groups with the same intersection angles are conjugate in $\mathrm{Isom}^+(\mathbf{H}^2)$, which is the orientation preserving isometry group of real hyperbolic plane, see section 10.6 in [1]. If we consider the groups in $\mathrm{PU}(2,1) = \mathrm{Aut}(\mathbf{H}^2_{\mathbb{C}})$, we will get the nontrivial deformations. The deformation theory of complex hyperbolic triangle groups was begun

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in [3] in which they investigated $\Gamma$ of type $(\infty, \infty, \infty)$ with $p = 2$ (complex hyperbolic ideal triangle groups). Since then, there have been many developments referring to other types, such as [15, 6, 13] among which they mainly gave the necessary conditions of $\Gamma$ to be discrete. Especially Parker and Paupert in [10] and [11] investigated the equilateral triangle group generated by three complex reflections with finite order. These include *Deraux's lattice, Livné's lattices, Mostow's lattices.* Our starting point is a result given by Thompson [14] where he investigated the non-equilateral triangle groups generated by three complex involutions (that is the order of the reflections is $p = 2$). He obtained his result using a computer search. Using [11] we see that Thompson's results apply to groups with $p > 2$ as well. In what follows we will give the specific case about the triangles group with 2-fold symmetry and we give a rigorous proof.

We will restrict to the complex hyperbolic triangle groups generated by three complex reflections with finite order $p \geq 2$. Suppose that the polar vector of a complex geodesic $C_1$ is $\mathbf{v_1}$ (see Section 2 for a more precise explanation). We consider the complex reflection $R_1$ in the complex geodesic $C_1$. This map sends $\mathbf{v_1}$ to $e^{i\phi}\mathbf{v_1}$ and acts as the identity on the orthogonal complement of $\mathbf{v}_1$, that is on vectors that project to $C_1$. We will always restrict to the case where $\phi = 2\pi/p$ and so $R_1$ has order $p \geqslant 2$. Then $R_1$ is given by the following formula:

$$R_1(\mathbf{z}) = \mathbf{z} + (e^{i\phi} - 1)\frac{\langle \mathbf{z}, \mathbf{v_1}\rangle}{\langle \mathbf{v_1}, \mathbf{v_1}\rangle}\mathbf{v_1}. \tag{1.1}$$

In order to convert $R_1$ into a matrix with determinant 1, we need to multiply the expression in (1.1) by $e^{-i\phi/3}$. The ambiguity involved in this choice is precisely the ambiguity involved in lifting an isometry in $\mathrm{PU}(2,1)$ to a matrix in $\mathrm{SU}(2,1)$.

Here we recall the terminology for *braid relations* between group elements (see Section 2.2 of Mostow [8]). Let $G$ be a group and $a$, $b \in G$. Then $a$ and $b$ satisfy a braid relation of length $l \in \mathbb{Z}_+$ if

$$(ab)^{l/2} = (ba)^{l/2},$$

where powers means that the corresponding alternating product of $a$ and $b$ should have $l$ factors. For example, $(ab)^{3/2} = aba$, $(ba)^2 = baba$. We denote the *braid length* $l$ by $br(a, b)$ to be the minimum length of a braid relation satisfied by $a$ and $b$.

We define the $(l_1, l_2, l_3; l_4)$-triangle groups to be the triangle groups with the following braid relations:

$$br(R_2, R_3) = l_1, \qquad br(R_1, R_3) = l_2,$$
$$br(R_1, R_2) = l_3, \qquad br(R_1, R_3^{-1}R_2R_3) = l_4,$$

where each $R_j$ is of order $p$.

In this paper we aim to list the candidates of discrete triangle groups generated by $R_1, R_2, R_3$ with $l_1 = l_2$ and $l_3 = l_4$ as stated in Theorem 2.4.

## 2. The parameter space, traces and main result

Firstly we recall some fundamentals about complex hyperbolic 2-space. Please refer to [4, 9] for more details about the complex hyperbolic space. Let $\mathbb{C}^{2,1}$ denote the vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3$$

of signature $(2, 1)$, where $\mathbf{z} = [z_1, z_2, z_3]^t$ and $\mathbf{w} = [w_1, w_2, w_3]^t$. The Hermitian form divides $\mathbb{C}^{2,1}$ into three parts $V_-$, $V_0$ and $V_+$, which are

$$V_- = \{\mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$
$$V_0 = \{\mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle = 0\},$$
$$V_+ = \{\mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle > 0\}.$$

We denote by $\mathbb{CP}^2$ the complex projectivisation of $\mathbb{C}^{2,1}$ and by

$$\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{CP}^2$$

the natural projectivisation map. The *complex hyperbolic* 2*-space* $\mathbf{H}^2_{\mathbb{C}}$ is defined as $\mathbb{P}(V_-)$. It is called the *standard projective model* of the complex hyperbolic space. Correspondingly the boundary of $\mathbf{H}^2_{\mathbb{C}}$ is $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0 \setminus \{0\})$. One can also consider the *unit ball model* whose boundary is the sphere $\mathbb{S}^3$ by taking $z_3 = 1$, which can be simply written as

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}.$$

The complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is a Kähler manifold of constant holomorphic sectional curvature $-1$. The holomorphic automorphism group of $\mathbf{H}^2_{\mathbb{C}}$ is the projectivisation $\mathrm{PU}(2,1)$ of the group $\mathrm{U}(2,1)$ of complex linear transformations on $\mathbb{C}^{2,1}$, which preserve the Hermitian form. Especially $\mathrm{SU}(2,1)$ is the subgroup of $\mathrm{U}(2,1)$ with the determinant of each element being 1.

Let $x, y \in \mathbf{H}^2_{\mathbb{C}}$ be points corresponding to vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{2,1} \setminus \{0\}$. Then the *Bergman metric* $\rho$ on $\mathbf{H}^2_{\mathbb{C}}$ is given by

$$\cosh^2\left(\frac{\rho(x,y)}{2}\right) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle},$$

where $\mathbf{x}$, $\mathbf{y} \in V_-$ are the lifts of $x$, $y$ respectively. It is easy to check that this definition is independent of the choice of lifts.

Given two points $x$ and $y$ in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$, with lifts $\mathbf{x}$ and $\mathbf{y}$ to $\mathbb{C}^{2,1}$ respectively, the complex span of $\mathbf{x}$ and $\mathbf{y}$ projects to a *complex line* in $\mathbb{CP}^2$ passing through $x$ and $y$. The intersection of a complex line with $\mathbf{H}^2_{\mathbb{C}}$

will be called a *complex geodesic* $C$ (which is homeomorphic to an open 2-dimensional disk), which can be uniquely determined by a positive vector $\mathbf{v} \in V_+$, i.e. $C = \mathbb{P}(\{\mathbf{z} \in \mathbb{C}^{2,1} \setminus \{0\} \mid \langle \mathbf{z}, \mathbf{v} \rangle = 0\})$. We call $\mathbf{v}$ a *polar vector* to $C$. As stated in Section 1, we will consider $(l_1, l_2, l_3; l_4)$-triangle groups $\Gamma$ generated by three complex reflections, see (1.1), through angle $\phi$ in three complex geodesics.

Throughout this paper, we assume that $R_1$, $R_2$, $R_3$ are three complex reflections in complex geodesics $C_1$, $C_2$, $C_3$ respectively. We parameterize the triangle groups generated by $R_1$, $R_2$, $R_3$ by three complex numbers $\rho$, $\sigma$ and $\tau$. Up to the action of $\mathrm{PU}(2,1)$, we can parameterize the collection of three pairwise distinct complex lines in $\mathbf{H}^2_{\mathbb{C}}$ by four real parameters, see Proposition 1 of [12]. The parameters we choose are $|\rho|$, $|\sigma|$, $|\tau|$ and $\arg(\rho\sigma\tau)$. In particular, we can freely choose the argument of two out of the three parameters.

Write $u = e^{i\phi/3} = e^{2\pi i/3p}$. The group $\Gamma$ has generators given by

$$R_1 = \begin{pmatrix} u^2 & \rho & -u\bar{\tau} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad R_2 = \begin{pmatrix} \bar{u} & 0 & 0 \\ -u\bar{\rho} & u^2 & \sigma \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad R_3 = \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ \tau & -u\bar{\sigma} & u^2 \end{pmatrix} \tag{2.1}$$

which preserve the Hermitian form

$$H = \begin{pmatrix} \alpha & \beta_1 & \overline{\beta}_3 \\ \overline{\beta}_1 & \alpha & \beta_2 \\ \beta_3 & \overline{\beta}_2 & \alpha \end{pmatrix}, \tag{2.2}$$

where $\alpha = \sqrt{2 - u^3 - \bar{u}^3}$, $\beta_1 = -i\bar{u}^{1/2}\rho$, $\beta_2 = -i\bar{u}^{1/2}\sigma$, $\beta_3 = -i\bar{u}^{1/2}\tau$ (note that here we take $\bar{u}^{1/2} = e^{-\pi i/3p}$).

This Hermitian form has signature $(2,1)$ if and only if $\det(H) < 0$. That is,

$$\begin{aligned} 0 &< \alpha|\beta_1|^2 + \alpha|\beta_2|^2 + \alpha|\beta_3|^2 - \alpha^3 - \beta_1\beta_2\beta_3 - \bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 \\ &= \alpha^2|\rho|^2 + \alpha^2|\sigma|^2 + \alpha^2|\tau|^2 - \alpha^3 - i\bar{u}^{3/2}\rho\sigma\tau + iu^{3/2}\bar{\rho}\bar{\sigma}\bar{\tau}. \end{aligned}$$

In terms of these parameters

$$\begin{aligned} \mathrm{tr}(R_1R_2) &= u(2 - |\rho|^2) + \bar{u}^2, \\ \mathrm{tr}(R_2R_3) &= u(2 - |\sigma|^2) + \bar{u}^2, \\ \mathrm{tr}(R_1R_3) &= u(2 - |\tau|^2) + \bar{u}^2, \\ \mathrm{tr}(R_1R_3^{-1}R_2R_3) &= u(2 - |\sigma\tau - \bar{\rho}|^2) + \bar{u}^2. \end{aligned} \tag{2.3}$$

**Lemma 2.1.** [11, Corollary 2.5] *If $|\rho| = 2\cos\zeta$, then the three eigenvalues of $R_1R_2$ will be $\bar{u}^2$, $-ue^{2i\zeta}$, $-ue^{-2i\zeta}$.*

**Proof.** Each point on $C_1$ is a $\bar{u} = e^{-i\phi/3}$ eigenvector of $R_1$ and each point on $C_2$ is a $\bar{u} = e^{-i\phi/3}$ eigenvector of $R_2$, see (1.1). Therefore if $\mathbf{z} \in C_1 \cap C_2$, then we will get that

$$R_1R_2(\mathbf{z}) = e^{-i\phi/3}R_1(\mathbf{z}) = e^{-2i\phi/3}\mathbf{z}$$

Hence $\mathbf{z}$ is a $\bar{u}^2 = e^{-2i\phi 3}$ eigenvector of $R_1R_2$. Hence the sum of the other two eigenvalues of $R_1R_2$ is $u(2 - |\rho|^2)$. By the assumption $|\rho| = 2\cos\zeta$, we know that $R_1R_2$ is not loxodromic, see Section 6.2 in [4]. Therefore each eigenvalue of $R_1R_2$ is of modulus one. Then we can get that the three eigenvalues of $R_1R_2$ will be $\bar{u}^2$, $-ue^{2i\zeta}$, $-ue^{-2i\zeta}$ from the form of $\mathrm{tr}(R_1R_2)$ in (2.3). □

**Remark 2.2.** We suppose that $m \in \mathbb{N}$, $m \geqslant 2$. If $|\rho| = 2\cos(\pi/m)$ then $br(R_1, R_2) = m$, see Section 2.2 in [8] for details or more precisely [2, Proposition 2.3]. (In fact this is true if $|\rho| = 2\cos(k\pi/m)$ where $k$ is coprime to $m$.) In the following Theorem 2.4, we suppose

$$|\rho| = |\sigma\tau - \bar{\rho}| = 2\cos(\pi/m), \qquad |\sigma| = |\tau| = 2\cos(\pi/n),$$

which is the case of interest in [2].

If $R_1$, $R_2$ are complex involutions ($p = 2$), then the order of $R_1R_2$ will be of $m$.

Assume that

$$br(R_1,\, R_2) = br(R_1,\, R_3^{-1}R_2R_3), \qquad br(R_2,\, R_3) = br(R_1,\, R_3).$$

From Remark 2.2 and (2.3), our hypothesis on braiding implies that

$$|\rho| = |\sigma\tau - \bar{\rho}|, \qquad |\sigma| = |\tau|.$$

Since we are free to choose the argument of two of the three parameters, we impose the condition that $\sigma$ and $\tau$ should be real and non-negative, which means that $\mathrm{Im}(\rho) = \mathrm{Im}(\sigma\tau - \bar{\rho})$. So the condition $|\rho| = |\sigma\tau - \bar{\rho}|$ becomes either $\sigma\tau = \rho + \bar{\rho}$ or $\sigma\tau = 0$. In the latter case the group is reducible, so we do not consider it. Hence we suppose $\mathrm{Re}(\rho) > 0$ and $\sigma = \tau = \sqrt{\rho + \bar{\rho}}$.

Suppose that $|\rho| = 2\cos(\pi/m)$ and $\sigma = \tau = 2\cos(\pi/n)$, where $m, n \in \mathbb{N}$ and $m, n \geqslant 3$. Then the matrices in (2.1) become:

$$R_1 = \begin{pmatrix} u^2 & \rho & -u\sqrt{\rho+\bar{\rho}} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}, \tag{2.4}$$

$$R_2 = \begin{pmatrix} \bar{u} & 0 & 0 \\ -u\bar{\rho} & u^2 & \sqrt{\rho+\bar{\rho}} \\ 0 & 0 & \bar{u} \end{pmatrix}, \tag{2.5}$$

$$R_3 = \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ \sqrt{\rho+\bar{\rho}} & -u\sqrt{\rho+\bar{\rho}} & u^2 \end{pmatrix}. \tag{2.6}$$

Furthermore, the Hermitian form $H$ (2.2) has signature $(2,1)$ if and only if

$$\begin{aligned} 0 &< \alpha|\rho|^2 + 2\alpha(\rho+\bar{\rho}) - \alpha^3 - i\bar{u}^{3/2}(\rho^2+|\rho|^2) + iu^{3/2}(\bar{\rho}^2+|\rho|^2) \\ &= 2\alpha(\rho+\bar{\rho}) - \alpha^3 - i\bar{u}^{3/2}\rho^2 + iu^{3/2}\bar{\rho}^2. \end{aligned} \tag{2.7}$$

**Proposition 2.3.** *Let*

$$S = \begin{pmatrix} \rho & u(1-\rho-\bar{\rho}) & u^2\sqrt{\rho+\bar{\rho}} \\ \bar{u} & 0 & 0 \\ 0 & \bar{u}\sqrt{\rho+\bar{\rho}} & -1 \end{pmatrix}.$$

*Then*

(*a*) $S^2 = R_1R_2R_3$,

(*b*)

$$\begin{aligned} SR_1S^{-1} &= R_1R_2R_1^{-1}, \\ SR_2S^{-1} &= R_1R_3R_1R_3^{-1}R_1^{-1}, \\ SR_3S^{-1} &= R_1R_3R_1^{-1}. \end{aligned}$$

*In particular,*

$$S(R_2R_3)S^{-1} = R_1R_3, \qquad S(R_1R_3^{-1}R_2R_3)S^{-1} = R_1R_2.$$

*Moreover, $S$ is the only matrix in $SU(2,1)$ satisfying $(a)$ and $(b)$.*

**Proof.** Suppose that $S$ satisfies $(b)$. The basis vectors $\mathbf{v}_1 = [1,0,0]^t$, $\mathbf{v}_2 = [0,1,0]^t$ and $\mathbf{v}_3 = [0,0,1]^t$ are the polar vectors to the fixed complex geodesics of $R_1$, $R_2$, $R_3$ respectively. Since $SR_1S^{-1} = R_1R_2R_1^{-1}$, we see that $S$ sends $\mathbf{v}_1$ to a vector that is polar to the fixed complex geodesic of $R_1R_2R_1^{-1}$, which is a non-zero multiple of $R_1\mathbf{v}_2$. Similarly for the other complex reflections. Therefore

$$S\mathbf{v}_1 = \lambda R_1\mathbf{v}_2, \qquad S\mathbf{v}_2 = \mu R_1R_3\mathbf{v}_1, \qquad S\mathbf{v}_3 = \nu R_1\mathbf{v}_3.$$

Hence any matrix $S$ satisfying $(b)$ has the form:

$$S = \begin{pmatrix} \lambda\rho & \mu u(1-\rho-\bar{\rho}) & -\nu u\sqrt{\rho+\bar{\rho}} \\ \lambda\bar{u} & 0 & 0 \\ 0 & \mu\bar{u}\sqrt{\rho+\bar{\rho}} & \nu\bar{u} \end{pmatrix},$$

where $\lambda$, $\mu$, $\nu \in \mathbb{C} - \{0\}$. Now squaring $S$ and comparing its entries with the entries of $R_1R_2R_3$, we see that if such a matrix $S$ also satisfies $(a)$, then we must have:

$$\lambda^2 = 1, \qquad \lambda\mu = 1, \qquad \mu\nu = -u, \qquad \lambda\nu = -u, \qquad \nu^2 = u^2.$$

Also, since $S \in SU(2,1)$ we have $1 = \det(S) = -\lambda\mu\nu\bar{u}$. The only solution to these equations is $\lambda = \mu = 1$ and $\nu = -u$. Hence $S$ has the form we claimed.

Finally, it is easy to check directly that the matrix $S$ in the statement of the proposition lies in $\mathrm{SU}(2,1)$ and satisfies $(a)$ and $(b)$. □

In the following we will classify all discrete triangle groups generated by $R_1, R_2, R_3$ with the 2-fold symmetry given by $S$ satisfying the conditions $(a)$ and $(b)$ in Proposition 2.3.

**Theorem 2.4.** *Let $R_1, R_2, R_3$ be three complex reflections of order $p$ (with $p \geqslant 2$) in $SU(2,1)$ so that $R_i$ keeps a complex geodesic $C_i$ ($i = 1, 2, 3$) invariant. Assume that there is $S \in SU(2,1)$ such that*

$$\begin{aligned} SR_1S^{-1} &= R_1R_2R_1^{-1}, \\ SR_2S^{-1} &= R_1R_3R_1R_3^{-1}R_1^{-1}, \\ SR_3S^{-1} &= R_1R_3R_1^{-1}, \\ S^2 &= R_1R_2R_3. \end{aligned}$$

*Let $\rho$ and $\sigma$ be as in (2.3). Suppose $|\rho| = 2\cos(\pi/m)$ and $|\sigma| = 2\cos(\pi/n)$, which implies that $br(R_1,R_3) = n$, $br(R_1,R_2) = m$ (where $m, n \in \mathbb{N}$ and $m,n \geqslant 3$). Suppose also that $R_1R_2R_3$ is of finite order. Then the the possible values for $(n,m)$ are $(3,4)$, $(3,5)$, $(4,3)$, $(5,4)$, $(8,6)$ and $(k,k)$ ($k \in \mathbb{N}$ and $k \geqslant 3$).*

*Moreover, in each case the group preserves a Hermitian form $H$. When $(n,m)$ is one of $(3,5)$ or $(k,k)$ for $k \geqslant 5$ the form $H$ has signature $(2,1)$ for all $p \geqslant 2$. For the other values of $(n,m)$ the form $H$ only has signature $(2,1)$ for the following values of $p$:*

| | | |
|---|---|---|
| $(3,4)$, $p \geqslant 5$; | $(4,3)$, $p \geqslant 4$; | $(5,4)$, $p \geqslant 3$; |
| $(8,6)$, $p \geqslant 3$; | $(3,3)$, $p \geqslant 4$; | $(4,4)$, $p \geqslant 3$. |

Note that the solutions correspond to the following parameter values, or their complex conjugates:

| $(n,m)$ | $\rho$ | $s = \rho - 1$ | $\sigma = \tau$ |
|---|---|---|---|
| $(3,4)$ | $(1+i\sqrt{7})/2$ | $e^{2\pi i/7} + e^{4\pi i/7} + e^{-6\pi i/7}$ | $1$ |
| $(3,5)$ | $2e^{2\pi i/5}\cos(\pi/5)$ | $e^{2\pi i/5} + e^{7\pi i/15} + e^{-13\pi i/15}$ | $1$ |
| $(4,3)$ | $1$ | $0$ | $\sqrt{2}$ |
| $(5,4)$ | $(1+i\sqrt{3})(\sqrt{5}-i\sqrt{3})/4$ | $e^{-2\pi i/3} + e^{2\pi i/15} + e^{8\pi i/15}$ | $(1+\sqrt{5})/2$ |
| $(8,6)$ | $(1+i)(1-i/\sqrt{2})$ | $e^{\pi i/2} + e^{\pi i/12} + e^{-7\pi i/12}$ | $\sqrt{2+\sqrt{2}}$ |
| $(k,k)$ | $2e^{i\pi/k}\cos(\pi/k)$ | $e^{2\pi i/k}$ | $2\cos(\pi/k)$ |

## 3. The proof

Firstly a direct computation will show that the symmetry $S$ conjugates $R_1R_2$ to $R_1R_3^{-1}R_2R_3$ and conjugates $R_2R_3$ to $R_1R_3$. It means that

$$br(R_1,\, R_2) = br(R_1,\, R_3^{-1}R_2R_3), \qquad br(R_2,\, R_3) = br(R_1,\, R_3)$$

by recalling Remark 2.2 and (2.3). By the parameterization of the triangle groups in Section 2 and the assumption in Theorem 2.4, one could get the matrix representation of $H$, $R_1$, $R_2$, $R_3$ as (2.2), (2.4), (2.5), (2.6), where

$$|\rho| = 2\cos(\pi/m), \qquad \sigma = \tau = 2\cos(\pi/n).$$

Throughout the proof we let $\zeta = \pi/m$ and $\eta = \pi/n$. Then by Proposition 2.3, we can get the unique matrix from of $S \in \mathrm{SU}(2,1)$.

Because of $S^2 = R_1R_2R_3$, we can restrict ourselves to $S$, which is elliptic of finite order. Equivalently, there exist $a$ and $b$ that are rational multiples of $\pi$ for which:

$$\mathrm{tr}(S) = -1 + \rho = e^{ia} + e^{ib} + e^{-i(a+b)}. \tag{3.1}$$

Observe that there is some ambiguity in the choice of $a$ and $b$. First, we can permute the three terms in this expression, and so permute $\{a,\, b,\, -a-b\}$; secondly we can change the sign of all three terms and, finally, since $\mathrm{tr}(S)$ is only defined up to multiplying by a cube root of unity, we can add the same integer multiple of $2\pi/3$ to both $a$ and $b$. We will use these operations to simplify things in our calculations below.

We denote $\mathrm{tr}(S)$ by $s$, then get that

$$\begin{aligned} |s|^2 = 1 + |\rho|^2 - 2\,\mathrm{Re}\,(\rho) &= |e^{ia} + e^{ib} + e^{-i(a+b)}|^2, && (3.2)\\ \mathrm{Re}\,(s) = -1 + \mathrm{Re}\,(\rho) &= \cos(a) + \cos(b) + \cos(a+b), && (3.3)\end{aligned}$$

Recall that

$$\begin{aligned} |\rho|^2 &= 4\cos^2\zeta = 2\cos(2\zeta) + 2,\\ \mathrm{Re}\,(\rho) &= \frac{\sigma\tau}{2} = 2\cos^2\eta = \cos(2\eta) + 1.\end{aligned}$$

The above two equations can be simplified to

$$\begin{aligned} 1 &= \cos(2\zeta) - \cos(2\eta) - \cos(a-b) - \cos(a+2b) - \cos(2a+b), && (3.4)\\ 0 &= \cos(2\eta) - \cos(a) - \cos(b) - \cos(a+b). && (3.5)\end{aligned}$$

In what follows we will repeatedly use the following result given by A. Monaghan, which generalizes the result of Conway and Jones for vanishing sums of cosines of rational multiples of $\pi$.

**Proposition 3.1.** [7, Theorem 2.4.3.1] *Suppose that we have at most five distinct rational numbers of $\pi$, for which some rational linear combination of* *their cosines is rational but no proper subset has this property. If $\phi \in (0, \pi)$ and all other angles are normalized to lie in $(0, \frac{\pi}{2})$, then the appropriate linear combination is proportional to one of the following:*

(a) $0 = \cos(\phi) + \cos(\phi + \frac{2\pi}{3}) + \cos(\phi + \frac{4\pi}{3})$,

(b) $0 = \cos(\phi) + \cos(\phi \pm \frac{2\pi}{5}) - \cos(\phi \pm \frac{2\pi}{15}) + \cos(\phi \pm \frac{7\pi}{15})$,

(c) $0 = \cos(\phi) - \cos(\phi \pm \frac{\pi}{5}) + \cos(\phi \pm \frac{\pi}{15}) - \cos(\phi \pm \frac{4\pi}{15})$,

(d) $\frac{1}{2} = \cos(\frac{\pi}{3})$,

(e) $\frac{1}{2} = \cos(\frac{\pi}{5}) - \cos(\frac{2\pi}{5})$,

(f) $\frac{1}{2} = \cos(\frac{\pi}{5}) - \cos(\frac{\pi}{15}) + \cos(\frac{4\pi}{15})$,

(g) $\frac{1}{2} = -\cos(\frac{2\pi}{5}) + \cos(\frac{2\pi}{15}) - \cos(\frac{7\pi}{15})$,

(h) $\frac{1}{2} = -\cos(\frac{\pi}{15}) + \cos(\frac{2\pi}{15}) + \cos(\frac{4\pi}{15}) - \cos(\frac{7\pi}{15})$,

(i) $\frac{1}{2} = \cos(\frac{\pi}{7}) - \cos(\frac{2\pi}{7}) + \cos(\frac{3\pi}{7})$,

(j) $\frac{1}{2} = \cos(\frac{\pi}{7}) - \cos(\frac{2\pi}{7}) + \cos(\frac{2\pi}{21}) - \cos(\frac{5\pi}{21})$,

(k) $\frac{1}{2} = \cos(\frac{\pi}{7}) + \cos(\frac{3\pi}{7}) - \cos(\frac{\pi}{21}) + \cos(\frac{8\pi}{21})$,

(l) $\frac{1}{2} = -\cos(\frac{2\pi}{7}) + \cos(\frac{3\pi}{7}) + \cos(\frac{4\pi}{21}) + \cos(\frac{10\pi}{21})$,

(m) $\frac{1}{2} = \cos(\frac{\pi}{7}) - \cos(\frac{\pi}{21}) + \cos(\frac{2\pi}{21}) - \cos(\frac{5\pi}{21}) + \cos(\frac{8\pi}{21})$,

(n) $\frac{1}{2} = -\cos(\frac{2\pi}{7}) + \cos(\frac{2\pi}{21}) + \cos(\frac{4\pi}{21}) - \cos(\frac{5\pi}{21}) + \cos(\frac{10\pi}{21})$,

(o) $\frac{1}{2} = \cos(\frac{3\pi}{7}) - \cos(\frac{\pi}{21}) + \cos(\frac{4\pi}{21}) + \cos(\frac{8\pi}{21}) + \cos(\frac{10\pi}{21})$.

Since the right hand side of equation (3.4) is 1 (rather than 0 or 1/2), Monaghan's theorem implies that it must be a sum of (at least) two similar sums involving fewer cosines. We begin by showing that at least one of the cosines must itself be rational.

**Proposition 3.2.** *Suppose that $\zeta = \pi/m$, $\eta = \pi/n$ and $a$, $b$ are rational multiples of $\pi$ so that equations* (3.4) *and* (3.5) *hold. Then one of the cosines in equation* (3.4) *must be rational.*

**Proof.** Suppose that none of the cosines are rational. Then (3.4) splits into two rational sums, one of length two and the other of length three, neither of which has a rational subsum. By inspection from Proposition 3.1 we see that these two sums must have the value 0, $\pm 1/2$. Since they sum to 1, they must both be 1/2. Therefore, the sum of length 2 must be (e) and the sum of length 3 must be one of (f), (g) or (i).

(1) $1/2 = \cos(2\zeta) - \cos(2\eta) = -\cos(a-b) - \cos(a+2b) - \cos(2a+b)$. Since $\zeta = \pi/m$ and $\eta = \pi/n$ the sum (e) implies $2\zeta = \pi/5$ and $2\eta = 2\pi/5$.

For the second equation, there are certain symmetry operations on $a$ and $b$ described in the paragraph after equation (3.1) above. Up to these operations, we now list the possible values of $a$ and $b$. In the first column we indicate which of the identities (a) to (o) in Proposition 3.1 we mainly used.

| | $a-b$ | $a+2b$ | $2a+b$ | $a$ | $b$ |
|---|---|---|---|---|---|
| (f) | $\pi/15$ | $11\pi/15$ | $4\pi/5$ | $13\pi/45$ | $2\pi/9$ |
| (g) | $2\pi/5$ | $7\pi/15$ | $13\pi/15$ | $19\pi/45$ | $\pi/45$ |
| (i) | $2\pi/7$ | $4\pi/7$ | $6\pi/7$ | $-2\pi/7$ | $-4\pi/7$ |

Using $2\eta = 2\pi/5$, we see that none of the values in this table satisfy (3.5). Therefore we get no solutions.

(2) $1/2 = \cos(2\zeta) - \cos(a-b) = -\cos(2\eta) - \cos(a+2b) - \cos(2a+b)$. The first equation gives $2\zeta = \pi/5$ as in case (1) and so $a - b = 2\pi/5$. Since $a-b = (2a+b) - (a+2b)$ the difference of two of the angles in the second equation must be $2\pi/5$. By inspection, we see the only solution is $a+2b = 7\pi/15$ and $2a+b = 13\pi/15$. This means $2\eta = 2\pi/5$ and we are back in case (1).

(3) $1/2 = -\cos(2\eta) - \cos(a-b) = \cos(2\zeta) - \cos(a+2b) - \cos(2a+b)$. The first equation gives $2\eta = 2\pi/5$ as in (1) and so $a - b = 4\pi/5$. Substituting in the second equation, we see $a+2b = -\pi/15$ and $2a+b = 11\pi/15$. Thus $2\zeta = \pi/5$ and we are back in case (1) again.

(4) $1/2 = -\cos(a-b) - \cos(a+2b) = \cos(2\zeta) - \cos(2\eta) - \cos(2a+b)$. Up to symmetries of $a$, $b$ and $-a-b$, the first sum implies that $a - b = 2\pi/5$ and $a + 2b = 4\pi/5$. Hence $2a + b = 6\pi/5$ and so the second sum must be (f). Thus $\cos(2\zeta) = \cos(2\pi/m) = \cos(4\pi/15)$ or $\cos(2\eta) = \cos(2\pi/n) = -\cos(4\pi/15)$, so either $m$ or $n$ is not an integer. Therefore there are no solutions. □

As a consequence of this result, we can consider separate cases where each of the cosines in (3.4) is rational. If either $\cos(2\zeta)$ or $\cos(2\eta)$ is rational it must be 0 or $\pm 1/2$ since $\zeta = \pi/m$ and $\eta = \pi/n$ where $m$ and $n$ are at least 3. If one of the other three cosines is rational we can use the allowable symmetries of $a$ and $b$, we to assume that $\cos(a-b)$ is rational. We treat each of these cases separately below. First we eliminate a simple situation which gives us many solutions and will recur in the different cases.

**Lemma 3.3.** *Suppose that* $\cos(2\zeta) = \cos(2\eta)$, *or equivalently* $m = n$, *then putting* $s = e^{\pm 2\pi i/m}$ *gives a solution to equations* (3.4) *and* (3.5) *for all* $m \geqslant 3$.

**Proof.** Substituting $\cos(2\zeta) = \cos(2\eta)$ into (3.4) gives:

$$0 = 1 + \cos(a-b) + \cos(a+2b) + \cos(2a+b)$$

$$= 2\cos^2\big((a-b)/2\big) + 2\cos\big((a-b)/2\big)\cos\big(3(a+b)/2\big)$$
$$= 4\cos\big((a-b)/2\big)\cos\big((a+2b)/2\big)\cos\big((2a+b)/2\big).$$

Therefore one of $(a-b)$, $(a+2b)$ or $(2a+b)$ is an odd multiple of $\pi$. Without loss of generality, we suppose that $a+2b=(2k+1)\pi$. Then we get $-a-b=b-(2k+1)\pi$ which yields $s=e^{ia}$, where $a$ is a rational multiple of $\pi$. Because $\mathrm{Re}\,(s)=-1+\mathrm{Re}\,(\rho)=-1+\frac{|\sigma|^2}{2}=\cos(2\eta)$, we see that $\cos(a)=\cos(2\pi/m)=\cos(2\pi/n)$.

Now we consider the signature of the Hermitian form

$$\mathrm{Det}(H) = ie^{-\frac{4\theta+3\phi}{2}i}(-1+e^{(2\theta+\phi)i})(e^{i\theta}+e^{i\phi})^2.$$

TABLE 3.1. Signature of Hermitian form

| $s$ | (2, 1) | degenerate | (3, 0) |
|---|---|---|---|
| $e^{\frac{2\pi i}{3}}$ $(m=n=3)$ | $p\geqslant 4$ | $p=3$ | $p=2$ |
| $e^{\frac{2\pi i}{4}}$ $(m=n=4)$ | $p\geqslant 3$ | $p=2$ | none |
| $e^{\frac{2\pi i}{k}}$ $(m=n=k\geqslant 5)$ | $p\geqslant 2$ | none | none |

In this case, we get the solution $n=m$. □

We now consider the cases where $\cos(2\zeta)$, $\cos(2\eta)$ or $\cos(a-b)$ are rational. We will use the following result proved by Parker when he was analyzing the triangle groups with 3-fold symmetry [10]. In [10] the last two cases were missed out, but this was corrected in [2].

**Proposition 3.4.** [10, Proposition 3.2] *Let $\theta$, $a$ and $b$ be rational multiples of $\pi$. Write $s=e^{ia}+e^{ib}+e^{-i(a+b)}$. Then the only possible solutions to the equation*

$$\cos(2\theta)-\cos(a-b)-\cos(a+2b)-\cos(2a+b)=\frac{1}{2}$$

*give rise to the following values of $\theta$ and $s$, up to changing the sign of $\theta$ and up to conjugating $s$ and multiplying it by a power of $\omega=e^{2\pi i/3}$:*

(i) $2\theta=2\pi/3$ *and* $s=-e^{-i\psi/3}$ *for some angle $\psi$ that is a rational multiple of $\pi$;*

(ii) $2\theta=\psi$ *and* $s=e^{2i\psi/3}+e^{-i\psi/3}=e^{i\psi/6}2\cos\frac{\psi}{2}$ *for some angle $\psi$ that is a rational multiple of $\pi$;*

(iii) $2\theta=\pi/3$ *and* $s=e^{i\pi/3}+e^{-i\pi/6}2\cos\frac{\pi}{4}$*;*

(iv) $2\theta=\pi/5$ *and* $s=e^{i\pi/3}+e^{-i\pi/6}2\cos\frac{\pi}{5}$*;*

(v) $2\theta = 3\pi/5$ *and* $s = e^{i\pi/3} + e^{-i\pi/6} 2\cos\frac{2\pi}{5}$;

(vi) $2\theta = \pi/2$ *and* $s = e^{2\pi i/7} + e^{4\pi i/7} + e^{-6\pi i/7}$;

(vii) $2\theta = \pi/2$ *and* $s = e^{2\pi i/9} + e^{-i\pi/9} 2\cos\frac{2\pi}{5}$;

(viii) $2\theta = \pi/2$ *and* $s = e^{2\pi i/9} + e^{-i\pi/9} 2\cos\frac{4\pi}{5}$;

(ix) $2\theta = \pi/7$ *and* $s = e^{2\pi i/9} + e^{-i\pi/9} 2\cos\frac{2\pi}{7}$;

(x) $2\theta = 5\pi/7$ *and* $s = e^{2\pi i/9} + e^{-i\pi/9} 2\cos\frac{4\pi}{7}$;

(xi) $2\theta = 3\pi/7$ *and* $s = e^{2\pi i/9} + e^{-i\pi/9} 2\cos\frac{6\pi}{7}$;

(xii) $2\theta = 2\pi/5$ *and* $s = 1 + 2\cos\frac{2\pi}{5}$;

(xiii) $2\theta = 4\pi/5$ *and* $s = 1 + 2\cos\frac{4\pi}{5}$.

Note that for the groups Parker was considering $s = e^{ia} + e^{ib} + e^{-ia-ib}$ was the trace of $R_1 J$, whereas in our case it is the trace of $S$. In the cases where $\cos(2\zeta) = 1/2$ or $\cos(2\eta) = -1/2$ then equation (3.4) reduces to the equation from Proposition 3.4, and we can use that result to find solutions.

**Lemma 3.5.** *Suppose that* $\cos(2\eta)$ *is rational. Then the only solutions to* (3.4) *and* (3.5) *are* $\cos(2\zeta) = \cos(2\pi/m)$ *and* $\cos(2\eta) = \cos(2\pi/n)$ *where* $(n, m)$ *is one of* $(3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, $(4,4)$ *or* $(6,6)$.

**Proof.** Since $\cos(2\eta)$ is rational and not equal to $\pm 1$ it can only be 0 or $\pm 1/2$. We treat each case separately.

(1) $\cos(2\eta) = -\frac{1}{2}$, which gives $n = 3$. Note that

$$\frac{1}{2} = \cos(2\eta) + 1 = \operatorname{Re}(\rho) = \operatorname{Re}(s) + 1$$

and so $\operatorname{Re}(s) = -1/2$. We rewrite (3.4) to give the equation from Proposition 3.4 with $\theta = \zeta$.

By direct calculation, we just need to consider cases (i), (ii) and (vi) because of $\operatorname{Re}(s) = -1/2$.

(i) $s = -e^{-i\psi/3}$ and so $|s| = 1$. This yields that $|\rho| = 2\cos(\pi/m) = 1$, and so $m = 3$. By considering $\operatorname{Re}(s) = -\cos(\theta/3)$, we know that $\theta = \pm\pi + 6k\pi$ $(k \in \mathbb{Z})$ which means that $s = -e^{\mp i\pi/3}$. From (2.7), we get that

$$\operatorname{Det}(H) = \mp\sqrt{3}\cos(\phi/2) + \sin(\phi/2) - 2\sin(3\phi/2).$$

We list the corresponding signature of Hermitian form for different $s$ in Table 3.2. In this case, we get that $n = m = 3$.

TABLE 3.2. Signature of Hermitian form

| $s$ | (2, 1) | degenerate | (3, 0) |
|---|---|---|---|
| $-e^{-i\pi/3}$ | $p \geqslant 4$ | $p = 3$ | $p = 2$ |
| $-e^{i\pi/3}$ | none | $p = 6$ | $p \neq 6$ |

(ii) $s = e^{2i\psi/3} + e^{-i\psi/3} = e^{i\psi/6} 2\cos(\psi/2)$ where $\psi = 2\theta$. By solving

$$\begin{aligned} -1/2 &= \mathrm{Re}\,(s) = \cos(4\theta/3) + \cos(2\theta/3) \\ &= 2\cos^2(2\theta/3) + \cos(2\theta/3) - 1 \end{aligned}$$

we obtain $\cos(2\theta/3) = (-1 \pm \sqrt{5})/4$. That is, $2\theta/3 = \pm 2\pi/5 + 2k\pi$ or $\pm 4\pi/5 + 2k\pi$. Hence $\theta = \pm 3\pi/5 + 3k\pi$ or $\pm 6\pi/5 + 3k\pi$. The only solution to $|s| = 2|\cos(\theta)| = 2\cos(\pi/m)$ is $m = 5$ (coming from $2\theta/3 = 4\pi/5 - 2\pi$).
Therefore $s = e^{2\pi i/5} + e^{4\pi i/5}$ or $s = e^{-2\pi i/5} + e^{-4\pi i/5}$. In these cases we find, respectively, that:

$$\mathrm{Det}(H) = -\sqrt{5 + 2\sqrt{5}} \cos\frac{\phi}{2} - (2 + \sqrt{5} + 4\cos\phi)\sin\frac{\phi}{2},$$
$$\mathrm{Det}(H) = \sqrt{5 + 2\sqrt{5}} \cos\frac{\phi}{2} - (2 + \sqrt{5} + 4\cos\phi)\sin\frac{\phi}{2}.$$

TABLE 3.3. Signature of Hermitian form

| $s$ | (2, 1) | degenerate | (3, 0) |
|---|---|---|---|
| $e^{-\frac{2\pi i}{5}} + e^{-\frac{4\pi i}{5}}$ | $p \leqslant 7$ | none | $p \geqslant 8$ |
| $e^{\frac{2\pi i}{5}} + e^{\frac{4\pi i}{5}}$ | $p \geqslant 2$ | none | none |

In this case, we get that $n = 3$, $m = 5$.

(vi) Since $s = e^{2\pi i/7} + e^{4\pi i/7} + e^{-6\pi i/7} = (-1 + \sqrt{7}i)/2$, it follows that $\mathrm{Re}\,(s) = -1/2$ and $|s| = \sqrt{2}$ which indicates that $m = 4$. A simple calculation yields that

$$\mathrm{Det}(H) = \frac{1}{2}(1 - 8\cos\phi)\sin\frac{\phi}{2}$$

from which it follows that the signature of the Hermitian form will be of $(2, 1)$ for $p \geqslant 5$, otherwise it will be positive. In this case, we get that $n = 3$, $m = 4$.

Therefore we obtain the solutions $(n, m) = (3, 3)$, $(3, 4)$ and $(3, 5)$.

(2) $\cos(2\eta) = 0$. Now we have $|\sigma|^2 = 2$ which yields $\mathrm{Re}\,(s) = 0$. Therefore one can get the following two equations

$$\begin{cases} \cos(2\zeta) - \cos(a-b) - \cos(a+2b) - \cos(2a+b) = 1, \\ \cos a + \cos b + \cos(a+b) = 0. \end{cases} \tag{3.6}$$

Since the first of these has 1 on the right hand side, it must split as the sum of (at least) two minimal subsums. Treating these case by case we see that the only possibilities are $\cos(2\zeta) = 0$, which yields $m = n = 4$ and $\cos(2\zeta) = -1/2$, which gives $n = 4$ and $m = 3$. The former case is a particular instance of Lemma 3.3. In the latter case we rewrite (3.2) as

$$|s|^2 = 1 + |\rho|^2 - 2\,\mathrm{Re}\,(\rho) = 1 + 2\cos(2\zeta) - 2\cos(2\eta) = 0.$$

Therefore, the only solution is $s = 0$, or equivalently $\rho = 1$. This implies that

$$\mathrm{Det}(H) = -2\sin\frac{3\phi}{2} = -2\sin\frac{3\pi}{p},$$

and the signature of the Hermitian form will be positive if $p = 2$, degenerate if $p = 3$, negative (of signature $(2,1)$) if $p \geqslant 4$. Therefore in this case we get that $n = 4$, $m = 3$. Hence the only solutions we get in this case are $(n,m) = (4,3)$ and $(4,4)$.

(3) $\cos(2\eta) = 1/2$. Now we have $|\sigma|^2 = 3$ from which it follows that $\mathrm{Re}\,(s) = 1/2$. We rewrite the two equations

$$\begin{cases} \cos(2\zeta) - \cos(a-b) - \cos(a+2b) - \cos(2a+b) = \dfrac{3}{2}, \\ \cos a + \cos b + \cos(a+b) = \dfrac{1}{2}. \end{cases} \tag{3.7}$$

If the second equation is irreducible, then it must be one of Proposition 3.1 parts (f), (g) or enum:posi:1:i. We see in each case that the angles involved do not sum to 0 (making each cosine positive, the sum is $\pi$ times the ratio of two odd integers for each choice of sign). If the second equation splits as the sum of two rational subsums then, without loss of generality, $\cos(a)$ is rational. Hence it is in the set $\{0, \pm 1/2. \pm 1\}$. Simple trigonometry shows that

$$\begin{aligned} 2\cos(a/2)\cos(a/2+b) &= \cos(b) + \cos(a+b) \\ &= 1/2 - \cos(a), \\ \cos(a+2b) + 1 &= 2\cos^2(a/2+b) \\ &= \big(1/2 - \cos(a)\big)^2 / \big(1 + \cos(a)\big), \\ \cos(a-b) + \cos(2a+b) &= 2\cos(3a/2)\cos(a/2+b) \end{aligned}$$

$$= -2\big(1/2 - \cos(a)\big)^2.$$

Substituting these identities in the first equation, we see that $\cos(2\zeta)$ is a rational function of $\cos(a)$, and so is rational. Substituting the different values of $\cos(a)$ gives a solution with $\zeta = \pi/m$ only when $\cos(a) = \pm 1/2$. In both cases, $\cos(2\zeta) = 1/2$ and so $m = 6$. Thus we obtain the solution $(n, m) = (6, 6)$. □

**Lemma 3.6.** *Suppose that* $\cos(2\zeta)$ *is rational. Then the only solutions to* (3.4) *and* (3.5) *are* $\cos(2\zeta) = \cos(2\pi/m)$ *and* $\cos(2\eta) = \cos(2\pi/n)$, *where* $(n, m)$ *is one of* $(3, 3)$, $(4, 3)$, $(4, 4)$, $(5, 4)$, $(6, 6)$ *or* $(8, 6)$.

**Proof.** Since $\cos(2\zeta)$ is rational and not equal to $\pm 1$ it can only be 0 or $\pm 1/2$. We treat each case separately.

(1) $\cos(2\zeta) = 1/2$, which gives $m = 6$. In this case, we know

$$|s|^2 = 1 + 2\cos(2\zeta) - 2\cos(2\eta) = 2 - 2\cos(2\eta).$$

In this case, we rewrite equation (3.4) to give the equation from Proposition 3.4 with $2\theta = \pi - 2\eta$. Checking one by one, we will find that there is no value of $s$ in Proposition 3.4 satisfying (3.5) except the cases (i) and (ii). For (i) we have $2\eta = \pi - 2\theta = \pi/3$ and so $n = 6$ (we have analyzed this case previously). For(ii), we have $\psi = \pi - 2\eta$ and $s = e^{2i\pi/3 - 4i\eta/3} + e^{-i\pi/3 + 2i\eta/3}$. Substituting in equation (3.5) gives

$$\begin{aligned} 0 &= \cos(2\eta) - \mathrm{Re}(s) \\ &= -\cos(\pi - 2\eta) - \cos(2\pi/3 - 4\eta/3) - \cos(\pi/3 - 2\eta/3) \\ &= -\cos(2\pi/3 - 4\eta/3)\big(1 + 2\cos(\pi/3 - 2\eta/3)\big). \end{aligned}$$

The only solution with $\eta = \pi/n$ is when $2\pi/3 - 4\eta/3 = \pi/2$. That is, $n = 8$. By calculating $\mathrm{Det}(H) = -2\cos(\phi)(1 + 2\sin\phi)$, we see that $H$ is of signature $(3, 0)$ for $p = 2$ and is of signature $(2, 1)$ for any $p \geqslant 3$. In this case, we get $(n, m) = (6, 6)$ or $(8, 6)$.

(2) $\cos(2\zeta) = 0$, which gives $m = 4$. Then we get that $|\rho|^2 = 2$ and $|s|^2 = 3 - |\sigma|^2$. Also (3.4) can be replaced by

$$-\cos(2\eta) - \cos(a - b) - \cos(a + 2b) - \cos(2a + b) = 1.$$

We have already analyzed the case where $\cos(2\eta) = 0$ or $-1/2$, which lead to the solution $(n, m) = (3, 4)$ or $(4, 4)$. If $\cos(2\eta) = 1/2$, then $|\sigma|^2 = 3$ induces $s = 0$, which contradicts $\mathrm{Re}\,(s) = -1 + |\sigma|^2/2 = 1/2$. Then it suffices for us to consider the following possible values due to $\eta = \pi/n$,

| | $2\eta$ | $a-b$ | $a+2b$ | $2a+b$ | $a$ | $b$ |
|---|---|---|---|---|---|---|
| (g) | $2\pi/5$ | $2\pi/3$ | $7\pi/15$ | $17\pi/15$ | $3\pi/5$ | $-\pi/15$ |
| (e) | $2\pi/5$ | $2\pi/5$ | $4\pi/5$ | $6\pi/5$ | $8\pi/5$ | $2\pi/15$ |

From this table, we know that $n = 5$ and the pair values $a = 3\pi/5$ and $b = -\pi/15$ do not satisfy the equation (3.5). However the second line $a = 8\pi/5$ and $b = 2\pi/15$ satisfy the equation (3.5) by applying the equation (g) in Proposition 3.1. Then we calculate the signature of the Hermitian form $H$ using (2.7). We see that $H$ is of signature $(3, 0)$ for $p = 2$ and is of signature $(2, 1)$ for any $p \geqslant 3$. In this case, we get that $m = 4$ and $n = 5$.

(3) $\cos(2\zeta) = -\frac{1}{2}$. It follows that $m = 3$ and

$$-\cos(2\eta) - \cos(a-b) - \cos(a+2b) - \cos(2a+b) = \frac{3}{2}.$$

Also, $\cos(2\zeta) = -1/2$ implies $|\rho| = 1$ and so $\cos(2\eta) + 1 = \mathrm{Re}\,(\rho) \leqslant 1$. This means that $\cos(2\eta) \leq 0$ and so either $\cos(2\eta) = \cos(2\pi/n) = -1/2$ or 0. We have analyzed both of these cases already. These give solutions $(n, m) = (3, 3)$ or $(4, 3)$. □

Now we begin to consider the remaining case in which $\cos(a-b)$ is rational.

**Lemma 3.7.** *Suppose that* $\cos(a-b) = -1$, *then* $\cos(2\zeta) - \cos(2\eta) = 0$, *and the possible solutions are given in Lemma* 3.3 *in which* $n = m$.

**Proof.** It follows from $\cos(a-b) = -1$ that $b = a + (2k+1)\pi$. Hence we have $\cos(a+2b) = \cos(3a)$ and $\cos(2a+b) = -\cos(3a)$. Therefore, equation (3.4) reduces to $\cos(2\zeta) - \cos(2\eta) = 0$, which we have already treated in Lemma 3.3. □

**Lemma 3.8.** *Suppose that* $\cos(a-b) = -1/2$, $\cos(2\zeta)$ *and* $\cos(2\eta)$ *are not rational,* $\cos(2\zeta) - \cos(2\eta) \neq 0$. *Then we get no solutions for* $n$, $m$ *such that* (3.4) *and* (3.5) *hold.*

**Proof.** It follows from $\cos(a-b) = -1/2$ that $b = a \pm 2\pi/3 + 2k\pi$. Hence we have $\cos(a+2b) = \cos(3a \mp 2\pi/3)$ and $\cos(2a+b) = \cos(3a \pm 2\pi/3)$. Therefore equation (3.4) becomes

$$\begin{aligned} 1/2 &= 1 + \cos(a-b) \\ &= \cos(2\zeta) - \cos(2\eta) - \cos(a+2b) - \cos(2a+b) \\ &= \cos(2\zeta) - \cos(2\eta) + \cos(3a). \end{aligned}$$

Since we have supposed that $\cos(2\zeta)$ and $\cos(2\eta)$ are not rational, the only way this equation can split into to rational subsums is for $\cos(3a)$ to be rational. Investigating the different possibilities, we see that (3.5) then implies $\cos(2\eta)$ is rational.

Now suppose the equation does not split into two rational sums of cosines. We list the possible values of $2\zeta,\ 2\eta,\ a,\ b$ in Table 3.4, up to the allowable symmetries of $a$ and $b$.

However, we note that there are no values of $2\eta,\ a,\ b$ in the list satisfying (3.5). Therefore there are no solutions for $n, m$. □

TABLE 3.4.

| | $2\zeta$ | $2\eta$ | $3a$ | $a-b$ | $a$ | $b$ |
|---|---|---|---|---|---|---|
| (e) | $\pi/5$ | $2\pi/5$ | $\pi/2$ | $2\pi/3$ | $\pi/6$ | $-\pi/2$ |
| (e) | $2\pi/5$ | $\pi/5$ | $0$ | $2\pi/3$ | $0$ | $-2\pi/3$ |
| (f) | $\pi/5$ | $\pi/15$ | $4\pi/15$ | $2\pi/3$ | $4\pi/45$ | $-26\pi/45$ |
| (g) | $2\pi/15$ | $2\pi/5$ | $8\pi/15$ | $2\pi/3$ | $8\pi/45$ | $-22\pi/45$ |
| (i) | $\pi/7$ | $2\pi/7$ | $3\pi/7$ | $2\pi/3$ | $\pi/7$ | $-11\pi/21$ |

**Lemma 3.9.** *Suppose that* $\cos(a-b) = 0,\ 1/2$ *or* $1$, $\cos(2\zeta)$ *and* $\cos(2\eta)$ *are not rational,* $\cos(2\zeta) - \cos(2\eta) \neq 0$*. Then there are no solutions for* $n$*,* $m$ *satisfying both* (3.4) *and* (3.5)*.*

**Proof.** We immediately get that

$$\cos(2\zeta) - \cos(2\eta) - \cos(a+2b) - \cos(2a+b) = 1 \text{ or } \frac{3}{2} \text{ or } 2. \tag{3.8}$$

Since the right hand side is not 0, $\pm 1/2$, we see that this sum must split into shorter rational sums of cosines.We break down into the following three cases.

(1) Case $\cos(2\zeta) - \cos(2\eta) = \pm 1/2$.

(i) Suppose $\cos(2\zeta) - \cos(2\eta) = 1/2$. Note that $\zeta = \pi/m$ and $\eta = \pi/n$, where $m, n \in \mathbb{N}$. Therefore we know that $(n, m)$ is $(5, 10)$ and

$$\cos(a-b) + \cos(a+2b) + \cos(2a+b) = -\frac{1}{2}.$$

We have supposed that $\cos(a-b)$ is rational, then using elementary trigonometry arguments, we see that

$$2\cos\big((a-b)/2\big)\cos\big(3(a+b)/2\big) = -\frac{1}{2} - \cos(a-b).$$

Squaring both sides and rearranging gives

$$\cos(3a+3b) = \frac{\cos^2(a-b) - 3/4}{\cos(a-b)+1}.$$

We have assumed that either $\cos(a-b)=0$ or $\cos(a-b)=1/2$ or $\cos(a-b)=1$, which means that $\cos(3a+3b)=-3/4$ or $-1/3$ or $-1/8$. It gives a contradiction here.

(ii) $\cos(2\zeta)-\cos(2\eta)=-1/2$. It follows that

$$\cos(a+2b)-\cos(2a+b)=-\frac{3}{2} \text{ or } -2 \text{ or } -\frac{5}{2}.$$

This sum must again split and so both cosines are rational. Therefore the possible values for $\cos(a+2b)$ are just $-1$ or $-1/2$ which are equivalent to the case where $\cos(a-b)$ is this value, see Lemma 3.7 and Lemma 3.8. However we assumed $\cos(2\zeta)-\cos(2\eta)\neq 0$, therefore there are no solutions for $n$, $m$ satisfying both (3.4) and (3.5).

(2) Assume that $\cos(2\zeta)-\cos(x)$ (or $\cos(2\eta)+\cos(y)$) is $1/2$ or $-1/2$, where $x, y \in \{a+2b, 2a+b\}$.

Recalling the equation (3.8), $\cos(2\zeta)-\cos(x)=\pm 1/2$ means that $\cos(2\eta)+\cos(y)$ is one of the values $\{-5/2, -2, -3/2, -1, -1/2\}$. We just need to consider the case $\cos(2\eta)+\cos(y)=-1/2$, because other values of $\cos(2\eta)+\cos(y)$ mean that $\cos(2\eta)$ will be rational. Without loss of generality, we suppose that $x=a+2b$, $y=2a+b$ and list the values of $2\zeta$, $a+2b$, $2\eta$, $2a+b$ and corresponding $a-b$:

| | $2\zeta$ | $a+2b$ | $2\eta$ | $2a+b$ | $a-b$ |
|---|---|---|---|---|---|
| (e) | $\pi/5$ | $2\pi/5$ | $2\pi/5$ | $4\pi/5$ | $2\pi/5$ |
| (e) | $\pi/5$ | $-2\pi/5$ | $2\pi/5$ | $4\pi/5$ | $6\pi/5$ |

There are no values of $a-b$ such that $\cos(a-b)=0$ or $\cos(a-b)=1/2$ or $\cos(a-b)=1$. Therefore there are no solutions for $n$, $m$ satisfying both (3.4) and (3.5) in this case.

(3) Suppose that $\cos(x)$ is rational, where $x\in\{a+2b, 2a+b\}$. By suitable changes of $a$ and $b$, the cases $\cos(a+2b)$ or $\cos(2a+b)$ is $-1/2$ or $-1$ are equivalent to the cases in Lemma 3.7 and Lemma 3.8. Therefore there are no solutions for $n$, $m$ because we supposed $\cos(2\zeta)-\cos(2\eta)\neq 0$.

Then we consider the condition for $\cos(x)$ to be 0, $1/2$ or 1 and suppose that $x=a+2b$. We get that

$$\cos(2\zeta)-\cos(2\eta)-\cos(2a+b)\in\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\},$$

which can be reduced to $\cos(2\zeta)-\cos(2a+b)$ or $\cos(2\eta)+\cos(2a+b)$ is rational which has been considered above.

Now we can get that there are no solutions for $n$, $m$ satisfying both (3.4) and (3.5) under the conditions in Lemma 3.9. □

We sum up all the possible values for $n$, $m$ from above process,

Lemma 3.3 $n = m \geqslant 3$;

Lemma 3.5 $(n, m) \in (3,3), (3,4), (3,5), (4,3), (4,4)$ or $(6,6)$;

Lemma 3.6 $(n, m) \in (3,3), (4,3), (4,4), (5,4), (6,6)$ or $(8,6)$;

which we desired. Also we could see the range of $p$ for each possible value to hold from the above analysis.

**Remark 3.10.** Note that the new candidates for $(n, m)$ to be $(5,4)$, $(4,3)$ and $(8,6)$ do not appear on Thompson's list in [14]. However referring to [2], in what follows we will see that the triangle groups for $(n, m)$ to be $(5,4)$ corresponds to Thompson groups $\mathbf{S}_2$ and the triangle groups for $(m, n)$ to be $(4,3)$ is of actually Mostow groups with braiding $(2,3,4;4)$. The pair $(n, m) = (8,6)$ was also found by Deraux when he was making a similar computer search to Thompson (private communication).

Case 1: $(n, m) = (5,4)$. Suppose that $M_1$, $M_2$, $M_3$ are three complex reflections of order $p$, which satisfy

$$\begin{aligned} &br(M_1, M_2) = 4, \\ &br(M_1, M_3) = br(M_2, M_3) = 3, \\ &br(M_1, M_3^{-1} M_2 M_3) = 5. \end{aligned}$$

Actually, $M_1$, $M_2$, $M_3$ will be Thompson group $\mathbf{S}_2$. Write $R_1 = M_2^{-1} M_1 M_2$, $R_2 = M_1 M_2 M_1^{-1}$, $R_3 = M_3$. We claim that

$$br(R_1, R_2) = br(R_1, R_3^{-1} R_2 R_3) = 4, \qquad br(R_1, R_3) = br(R_2, R_3) = 5.$$

First, observe, we also have $br(M_2^{-1} M_1 M_2, M_3) = br(M_1^{-1} M_2 M_1, M_3) = 5$. Thus

$$\begin{aligned} br(R_1, R_3) &= br(M_2^{-1} M_1 M_2, M_3) = 5, \\ br(R_2, R_3) &= br(M_1 M_2 M_1^{-1}, M_3) = 5. \end{aligned}$$

Using $br(M_1, M_2) = 4$, we have

$$R_1 R_2 = (M_2^{-1} M_1 M_2)(M_1 M_2 M_1^{-1}) = M_2^{-1} (M_2 M_1 M_2 M_1) M_1^{-1} = M_1 M_2.$$

Hence $br(R_1, R_2) = br(M_1, M_2) = 4$. We denote $M_1$, $M_1^{-1}$ by $1$, $\bar{1}$ simply and so on. Now we consider

$$\begin{aligned} R_1 R_3^{-1} R_2 R_3 &= M_2^{-1} M_1 M_2 M_3^{-1} M_1 M_2 M_1^{-1} M_3 \\ &= \bar{2}12\bar{3}12\bar{1}3 \\ &= (123123)(\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \cdot \bar{2}12\bar{3}12\bar{1}3 \cdot 123123)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \\ &= (123123)(\bar{3}\bar{2}\bar{1}\bar{3} \cdot 12\bar{1} \cdot \bar{3}12\bar{1}3 \cdot 123123)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \\ &= (123123)(\bar{3}\bar{2}(\bar{1}\bar{3}1)2(\bar{1}\bar{3}1)2(\bar{1}31)23123)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \end{aligned}$$

$$= (123123)(\bar{3}\bar{1}31\bar{2}\bar{1}\bar{3}13123)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1}$$
$$= (123123)(1\bar{3}\bar{2}323)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1}$$
$$= (123123)(12)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1}.$$

Since $R_1R_3^{-1}R_2R_3$ is conjugate to $M_1M_2$ we see that

$$br(R_1, R_3^{-1}R_2R_3) = br(M_1, M_2) = 4$$

as claimed. In particular, this shows that this case is equivalent to Thompson groups $\mathbf{S}_2$.

Case 2: $(n, m) = (4, 3)$. In this case, it is easy to check that $R_2$, $R_3$ (also $R_3$, $R_1$) braid with length 4, $R_1$, $R_2$ (also $R_1$, $R_3^{-1}R_2R_3$) braid with length 3, $R_1$, $R_2R_3R_2^{-1}$ (also $R_3$, $R_1R_2R_1^{-1}$) braid with length 2 (i.e. they commute) and $R_1R_2R_3$ is regular elliptic of order 3. Note that $\mathrm{Det}(H) < 0$ when $p \geqslant 4$.

As the same fashion in [5], we define $\iota$ by the reflection of group that acts on the generating set $(R_1, R_2, R_3)$ as follows,

$$\iota(R_1) = R_1, \qquad \iota(R_2) = R_1R_2R_1^{-1}, \qquad \iota(R_3) = R_3.$$

Under the action of $\iota$, the $(4, 4, 3; 3)$-triangle groups will be sent to the triangle groups with braiding $(2, 3, 4; 4)$

$$\Big\langle \iota(R_1), \iota(R_2)\,\iota(R_3) \ : \ \iota(R_2R_3) = \iota(R_3R_2), (\iota(R_1R_2))^{\frac{3}{2}} = (\iota(R_2R_1))^{\frac{3}{2}},$$
$$(\iota(R_1R_3))^2 = (\iota(R_3R_1))^2,$$
$$(\iota(R_1R_2R_3R_2^{-1}))^2 = (\iota(R_2R_3R_2^{-1}R_1))^2\Big\rangle.$$

Recall the Mostow groups $\Gamma(p, t)$ mentioned in [8, 10]. For Mostow groups, there exists a complex hyperbolic isometry $J$ of order 3 so that $R_{j+1} = JR_jJ^{-1}$ and $R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}$. We could rewrite them as triangle groups with braiding $(2, 3, 4; 4)$ as follows

$$\langle R_1, R_2, J(R_1R_2)^{-1} : R_2J(R_1R_2)^{-1} = J(R_1R_2)^{-1}R_2, (R_1R_2)^{\frac{3}{2}} = (R_2R_1)^{\frac{3}{2}},$$
$$(R_1J(R_1R_2)^{-1})^2 = (J(R_1R_2)^{-1}R_1)^2,$$
$$(R_1R_2J(R_1R_2)^{-1}R_2^{-1})^2 = (R_2J(R_1R_2)^{-1}R_2^{-1}R_1)^2\rangle.$$

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