On the expected diameter of planar Brownian motion

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Abstract

Known results show that the diameter d_1 of the trace of planar Brownian motion run for unit time satisfies $1.595 \leq \mathbb{E}d_1 \leq 2.507$. This note improves these bounds to $1.601 \leq \mathbb{E}d_1 \leq 2.355$. Simulations suggest that $\mathbb{E}d_1 \approx 1.99$.

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Let $(b_t, t \in [0,1])$ be standard planar Brownian motion, and consider the set $b[0,1] = \{b_t : t \in [0,1]\}$. The Brownian convex hull $\mathcal{H}_1 := \operatorname{hull} b[0,1]$ has been well-studied from Lévy [5, §52.6, pp. 254–256] onwards; the expectations of the perimeter length ℓ_1 and area a_1 of $\mathcal{H}_1$ are given by the exact formulae $\mathbb{E}\ell_1 = \sqrt{8\pi}$ (due to Letac and Tákacs [4,6]) and $\mathbb{E}a_1 = \pi/2$ (due to El Bachir [1]).

Another characteristic is the *diameter*

$$d_1 := \operatorname{diam} \mathcal{H}_1 = \operatorname{diam} b[0,1] = \sup_{x,y \in b[0,1]} \|x - y\|,$$

for which, in contrast, no explicit formula is known. The exact formulae for $\mathbb{E}\ell_1$ and $\mathbb{E}a_1$ rest on geometric integral formulae of Cauchy; since no such formula is available for d_1, it may not be possible to obtain an explicit formula for $\mathbb{E}d_1$. However, one may get bounds.

By convexity, we have the almost-sure inequalities $2 \leq \ell_1/d_1 \leq \pi$, the extrema being the line segment and shapes of constant width (such as the disc). In other words,

$$\frac{\ell_1}{\pi} \leq d_1 \leq \frac{\ell_1}{2}.$$

The formula of Letac and Takács [4,6] says that $\mathbb{E}\ell_1 = \sqrt{8\pi}$, so we get:

Proposition 1. $\sqrt{8/\pi} \leq \mathbb{E}d_1 \leq \sqrt{2\pi}$.

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Note that $\sqrt{8/\pi} \approx 1.5958$ and $\sqrt{2\pi} \approx 2.5066$. In this note we improve both of these bounds.

For the lower bound, we note that $b[0,1]$ is compact and thus, as a corollary of Lemma 6 below, we have the formula

$$d_1 = \sup_{0 \le \theta \le \pi} r(\theta), \tag{1}$$

where r is the parametrized range function given by

$$r(\theta) = \sup_{0 \le s \le 1} (b_s \cdot \mathbf{e}_\theta) - \inf_{0 \le s \le 1} (b_s \cdot \mathbf{e}_\theta),$$

with $\mathbf{e}_\theta$ being the unit vector $(\cos\theta, \sin\theta)$. Feller [2] established that

$$\mathbb{E}r(\theta) = \sqrt{8/\pi} \quad \text{and} \quad \mathbb{E}(r(\theta)^2) = 4\log 2, \tag{2}$$

and the density of $r(\theta)$ is given explicitly as

$$f(r) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \exp\{-k^2 r^2/2\}, \ (r \ge 0). \tag{3}$$

Combining (1) with (2) gives immediately $\mathbb{E}d_1 \ge \mathbb{E}r(0) = \sqrt{8/\pi}$, which is just the lower bound in Proposition 1. For a better result, a consequence of (1) is that $d_1 \ge \max\{r(0), r(\pi/2)\}$. Observing that $r(0)$ and $r(\pi/2)$ are independent, we get:

Lemma 2. $\mathbb{E}d_1 \ge \mathbb{E}\max\{X_1, X_2\}$, *where* X_1 *and* X_2 *are independent copies of* $X := r(0)$.

It seems hard to explicitly compute $\mathbb{E}\max\{X_1, X_2\}$ in Lemma 2, because although the density given at (3) is known explicitly, it is not very tractable. Instead we obtain a lower bound. Since

$$\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$$

we get

$$\mathbb{E}\max\{X_1, X_2\} = \mathbb{E}X + \frac{1}{2}\mathbb{E}|X_1 - X_2|. \tag{4}$$

Thus with Lemma 2, the lower bound in Proposition 1 is improved given any non-trivial lower bound for $\mathbb{E}|X_1 - X_2|$. Using the fact that for any $c \in \mathbb{R}$, if m is a median of X, $\mathbb{E}|X - c| \ge \mathbb{E}|X - m|$, we see that

$$\mathbb{E}|X_1 - X_2| \ge \mathbb{E}|X - m|.$$

Again, the intractability of the density at (3) makes it hard to exploit this. Instead, we provide the following as a crude lower bound on $\mathbb{E}|X_1 - X_2|$.

Lemma 3. *For any* $a, h > 0$,

$$\mathbb{E}|X_1 - X_2| \ge 2h\,\mathbb{P}(X \le a)\,\mathbb{P}(X \ge a + h).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}|X_1 - X_2| &\ge \mathbb{E}\left[|X_1 - X_2|\mathbf{1}\{X_1 \le a, X_2 \ge a + h\}\right] \\
&\quad + \mathbb{E}\left[|X_1 - X_2|\mathbf{1}\{X_2 \le a, X_1 \ge a + h\}\right] \\
&\ge h\,\mathbb{P}(X_1 \le a)\,\mathbb{P}(X_2 \ge a + h) + h\,\mathbb{P}(X_2 \le a)\,\mathbb{P}(X_1 \ge a + h) \\
&= 2h\,\mathbb{P}(X \le a)\,\mathbb{P}(X \ge a + h),
\end{aligned}$$

which proves the statement. □

This lower bound yields the following result.

Proposition 4. *For $a, h > 0$ define*

$$g(a,h) := h\left(\frac{4}{\pi}\exp\left\{-\frac{\pi^2}{2a^2}\right\} - \frac{4}{3\pi}\exp\left\{-\frac{9\pi^2}{2a^2}\right\}\right)\left(1 - \frac{4}{\pi}\exp\left\{-\frac{\pi^2}{8(a+h)^2}\right\}\right).$$

Then $\mathbb{E}d_1 \geq \sqrt{8/\pi} + g(1.492, 0.337) \approx 1.6014$.

Proof. Consider

$$Z := \sup_{0\leq s\leq 1} |b_s \cdot \mathbf{e}_0|.$$

Then it is known (see [3]) that for $x > 0$,

$$\frac{4}{\pi}\exp\left\{-\frac{\pi^2}{8x^2}\right\} - \frac{4}{3\pi}\exp\left\{-\frac{9\pi^2}{8x^2}\right\} \leq \mathbb{P}(Z < x) \leq \frac{4}{\pi}\exp\left\{-\frac{\pi^2}{8x^2}\right\}. \tag{5}$$

Moreover, we have

$$Z \leq X \leq 2Z.$$

Since $X \leq 2Z$, we have

$$\mathbb{P}(X \leq a) \geq \mathbb{P}(Z \leq a/2) \geq \frac{4}{\pi}\exp\left\{-\frac{\pi^2}{2a^2}\right\} - \frac{4}{3\pi}\exp\left\{-\frac{9\pi^2}{2a^2}\right\},$$

by the lower bound in (5). On the other hand,

$$\mathbb{P}(X \geq a+h) \geq \mathbb{P}(Z \geq a+h) \geq 1 - \frac{4}{\pi}\exp\left\{-\frac{\pi^2}{8(a+h)^2}\right\},$$

by the upper bound in (5). Combining these two bounds and applying Lemma 3 we get $\mathbb{E}|X_1 - X_2| \geq 2g(a,h)$. So from (4) and the fact that $\mathbb{E}X = \sqrt{8/\pi}$ by (2) we get $\mathbb{E}d_1 \geq \sqrt{8/\pi} + g(a,h)$. Numerical evaluation using MAPLE suggests that $(a,h) = (1.492, 0.337)$ is close to optimal, and this choice gives the statement in the proposition. □

We also improve the upper bound in Proposition 1.

Proposition 5. $\mathbb{E}d_1 \leq \sqrt{8\log 2} \approx 2.3548$.

Proof. First, we claim that

$$d_1^2 \leq r(0)^2 + r(\pi/2)^2. \tag{6}$$

It follows from (6) and (2) that

$$\mathbb{E}(d_1^2) \leq \mathbb{E}(X_1^2 + X_2^2) = 2\mathbb{E}(X^2) = 8\log 2.$$

The result now follows by Jensen's inequality.

It remains to prove the claim (6). Note that the diameter is an increasing function, that is, if $A \subseteq B$ then $\operatorname{diam} A \leq \operatorname{diam} B$. Note also, that by the definition of $r(\theta)$, $b[0,1] \subseteq \mathbf{z} + [0, r(0)] \times [0, r(\pi/2)] =: R_{\mathbf{z}}$ for some $\mathbf{z} \in \mathbb{R}^2$. Since the diameter of the set $R_{\mathbf{z}}$ is attained at the diagonal,

$$\operatorname{diam} R_{\mathbf{z}} = \sqrt{r(0)^2 + r(\pi/2)^2},$$

for all $\mathbf{z} \in \mathbb{R}^2$, and we have $\operatorname{diam} b[0,1] \leq \operatorname{diam} R_{\mathbf{z}}$, the result follows. □

We make one further remark about second moments. In the proof of Proposition 5, we saw that $\mathbb{E}(d_1^2) \leq 8 \log 2 \approx 5.5452$. A bound in the other direction can be obtained from the fact that $d_1^2 \geq \ell_1^2/\pi^2$, and we have (see [7, §4.1]) that

$$\mathbb{E}(\ell_1^2) = 4\pi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_0^\infty \mathrm{d}u \cos\theta \frac{\cosh(u\theta)}{\sinh(u\pi/2)} \tanh\left(\frac{(2\theta+\pi)u}{4}\right) \approx 26.1677,$$

which gives $\mathbb{E}(d_1^2) \geq 2.651$.

Finally, for completeness, we state and prove the lemma which was used to obtain equation (1).

Lemma 6. *Let $A \subset \mathbb{R}^d$ be a nonempty compact set, and let $r_A(\theta) = \sup_{x \in A}(x \cdot \mathbf{e}_\theta) - \inf_{x \in A}(x \cdot \mathbf{e}_\theta)$. Then*

$$\operatorname{diam} A = \sup_{0 \leq \theta \leq \pi} r_A(\theta).$$

Proof. Since A is compact, for each θ there exist $x, y \in A$ such that

$$\begin{aligned} r_A(\theta) &= x \cdot \mathbf{e}_\theta - y \cdot \mathbf{e}_\theta \\ &= (x-y) \cdot \mathbf{e}_\theta \leq \|x-y\|. \end{aligned}$$

So $\sup_{0 \leq \theta \leq \pi} r_A(\theta) \leq \sup_{x,y \in A} \|x-y\| = \operatorname{diam} A$.

It remains to show that $\sup_{0 \leq \theta \leq \pi} r_A(\theta) \geq \operatorname{diam} A$. This is clearly true if A consists of a single point, so suppose that A contains at least two points. Suppose that the diameter of A is achieved by $x, y \in A$ and let $z = y - x$ be such that $\hat{z} := z/\|z\| = \mathbf{e}_{\theta_0}$ for $\theta_0 \in [0, \pi]$. Then

$$\begin{aligned} \sup_{0 \leq \theta \leq \pi} r_A(\theta) &\geq r_A(\theta_0) \geq y \cdot \mathbf{e}_{\theta_0} - x \cdot \mathbf{e}_{\theta_0} \\ &= z \cdot \hat{z} = \|z\| = \operatorname{diam} A, \end{aligned}$$

as required. □

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References

[1] M. El Bachir, *L'enveloppe convexe du mouvement brownien*, Ph.D. thesis, Université Toulouse III—Paul Sabatier, 1983.

[2] W. Feller, The asymptotic distribution of the range of sums of independent random variables, *Ann. Math. Statist.* **22** (1951) 427–432.

[3] N.C. Jain and W.E. Pruitt, The other law of the iterated logarithm, *Ann. Probab.* **3** (1975) 1046–1049.

[4] G. Letac, Advanced problem 6230, *Amer. Math. Monthly* **85** (1978) 686.

[5] P. Lévy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.

[6] L. Takács, Expected perimeter length, *Amer. Math. Monthly* **87** (1980) 142.

[7] A.R. Wade and C. Xu, Convex hulls of random walks and their scaling limits, *Stochastic Process. Appl.* **125** (2015) 4300–4320.