Noname manuscript No.
(will be inserted by the editor)

RANDOM-SETTLEMENT ARBITRATION AND THE GENERALIZED NASH SOLUTION: ONE-SHOT AND INFINITE HORIZON CASES

Nejat Anbarci · Kang Rong · Jaideep Roy

the date of receipt and acceptance should be inserted later

Abstract We study bilateral bargaining á la Nash (1953) but where players face two sources of uncertainty when demands are mutually incompatible. First, there is complete breakdown of negotiations with players receiving zero payoffs, unless with probability p, an arbiter is called upon to resolve the dispute. The arbiter uses the Final-Offer-Arbitration mechanism whereby one of the two incompatible demands is implemented. Second, the arbiter may have a preference bias towards satisfying one of the players that is private information to the arbiter and players commonly believe that the favored party is player 1 with probability q. Following Nash's idea of 'smoothing', we assume that $1-p$ is larger for larger incompatibility of demands. We provide a set of conditions on p such that, as p becomes arbitrarily small, all equilibrium outcomes converge to the Nash solution outcome if $q = 1/2$, that is when the uncertainty regarding the arbiter's bias is maximum. Moreover, with $q \neq 1/2$, convergence is obtained on a special point in the bargaining set that, independent of the nature of the set, picks the generalized Nash solution with as-if bargaining weights q and $1-q$. We then extend these results to infinite-horizon where instead of complete breakdown, players are allowed to re-negotiate.

Keywords Nash bargaining · incompatible demands · arbitration with unknown bias · random settlement · Nash solution

JEL classification C78

We are grateful to the Associate Editor and three anonymous referees for many important suggestions that have helped us improve the presentation and the content of the paper. We also thank the seminar participants at University of Melbourne, Queensland University of Technology, Jadavpur University, Seoul National University, Shanghai University of Finance and Economics, and Southwestern University of Finance and Economics, conference participants at the 34th (2016) Australasian Economic Theory Workshop, 2016 Conference of the Society for the Advancement of Economic Theory, and 2016 World Congress of Game Theory, as well as Kalyan Chatterjee, Youngsub Chun, Nick Feltovich, Emin Karagozoglu, Herve Moulin, Shiran Rachmilevitch, Qiangfeng Tang and Quan Wen, for their useful comments and suggestions. The usual disclaimer applies.

N. Anbarci
Department of Economics, Deakin University, Burwood, Australia
E-mail: nejat.anbarci@deakin.edu.au

K. Rong
School of Economics, Shanghai University of Finance and Economics, Shanghai, China
Tel.: +86-21-65902905
E-mail: rong.kang@mail.shufe.edu.cn

J. Roy
Department of Economics, University of Bath, Bath BA2 7JP, UK
E-mail: J.Roy@bath.ac.uk

1 Introduction

NASH (1950) CONSIDERED TWO-PERSON bargaining problems consisting of a feasible set S and a disagreement point $d \in S$ which the parties receive if no agreement is reached. He showed that there is a unique utility pair in S, now known as the Nash solution, that is isolated by all bargaining solutions satisfying a set of desirable porperties. Using the noncooperative approach, Nash (1953) proposed what is now popularly called the Nash demand game (NDG), where two players simultaneously make demands (after they simultaneously announce their respective disagreement points) and each player receives the payoff he demands if the demands are jointly feasible, but their disagreement payoffs otherwise. Simplicity and reasonability are two celebrated virtues of the NDG that allow to support the Nash solution as a Nash equilibrium outcome. Nevertheless, multiplicity of equilibrium outcomes is a sweeping problem of the NDG as every point on the Pareto frontier is a Nash equilibrium outcome. In addition, it has the 'pure' chilling-effect equilibrium where both players demand their maximum feasible utilities and earn their disagreement payoffs. Nash (1953) resolved this equilibrium-multiplicity problem suggestively, arguing that under a slight perturbation of the game – which has been called 'smoothing' since then and presumably yielding either uncertainty about the feasible set S or requiring outside injection of utilities – there exists a unique equilibrium that is the only necessary limit of the equilibrium points of 'smoothed games'. Abreu and Pearce (2015) formalizes Nash's smoothing and provides a set of conditions for this perturbation environment that yields the Nash solution convergence result.[1]

The seminal contribution of Nash and its recent formalization by Abreu and Pearce not only provide a firm theoretical understanding of non-cooperative bargaining, but are also useful in predicting outcomes in many bargaining environments with uncertainty over the bargaining set or availability of outside resources.[2] However, these 'perturbation' features are not always observed in real world negotiations and important contributions without these assumptions include works by Carlsson (1991) and Dutta (2012). In addition, the literature is relatively underdeveloped when it comes to non-cooperative foundations of the asymmetric 'generalized' Nash solution that, especially since Binmore, Rubinstein and Wolinsky (1986), has become increasingly prominent in the 'money and search' (e.g., Lagos and Wright, 2005), 'legislative bargaining' (e.g., Laruelle and Valenciano, 2007), 'median voter' (e.g., Herings and Predtetchinski, 2010), 'industrial organization' (e.g., Chen, Ding and Liu, 2016) and 'contest' (e.g., Corchón and Dahm, 2010) literatures, among others. We note that Carlsson (1991) is again an exception that addresses convergence to the generalized Nash solutions.

The purpose of the present paper is to understand the significance of the standard and the generalized Nash solutions in an institutionally distinct bargaining environment that connects the NDG with Final-Offer-Arbitration (FOA), a feature that has not received adequate attention in the literature. Our motivation draws upon many real world disputes where private parties are seen to bargain amongst themselves, but upon failing to reach an agreement, an external arbiter is sometimes called upon to resolve the dispute by the use of the FOA mechanism of upholding the demand from one party (and providing the residual to the other). Such real world bargaining situations, as discussed later in this section, are often characterized by two sources of uncertainty that bargainers have to cope with: (i) the act of initiating an arbitration procedure and (ii) the arbiter's bias. We ask how these uncertainties affect bargaining outcomes and, among other things, show that the Nash solution is the only outcome that is obtained in the limit as the probability of initiating an arbitration vanishes but the uncertainty about the arbiter's bias is at its maximum. We also study the 'path' to this convergence and show that when

[1] They then extend the static NDG to an infinite-horizon setting to avoid the "awkward possibility in Nash's original formulation that bargainers who have made incompatible demands will be held perpetually to threats they would both prefer not to carry out" to show that, independent of time preferences of the players, all stationary subgame perfect equilibria of their infinite-horizon model approach the standard Nash solution outcome.

[2] Osborne and Rubinstein (1990) also proposed a variant of NDG where incompatible demands always lead to disagreement, while compatible demands close to the boundary may also lead to disagreement with small probability. Their game also implies that the Nash solution outcome is the limit equilibrium outcome and outside injections of utility are not needed. However, their NDG has this unnatural feature in which players have to waste the surplus with some probability even when their demands are compatible.

bargainers have no prior information about the arbiter's bias and the arbitration initiation probability satisfies some additional properties, equilibrium demands are always incompatible and symmetric around the Nash solution, leading either to breakdown or arbitration.

We consider a standard (one-shot) NDG where each player receives the payoff he demands if the demands are jointly feasible. Otherwise, with probability $1-p$, players receive zero payoffs, while with probability p an arbitration stage is initiated. At the arbitration stage, each player's demand is selected randomly as the outcome with probabilities q and $1-q$, respectively. We call p the arbitration initiation probability (or simply, the *initiation probability*) and its complement $1-p$ the *breakdown probability*. We assume that the arbiter selects one of the two players' demands based upon his preference bias. In real world bargaining environments that motivate this study, it is often the case that the parties in conflict represent different social and economic classes, like landlords vs. tenants, workers vs. firms, man vs. woman, or developed vs. underdeveloped nations. Arbiters may indeed have ideological inclinations towards one party or the other in addition to further information about the dispute that is not always available to the parties in conflict. We interpret q as the common prior held by the two players that the favored party in any unresolved dispute will be player 1. Thus uncertainty about the arbiter's bias is at its maximum when $q = 1/2$. We also make this 'mechanism' share the presentiment of Nash's smoothing: if the players' incompatible demands are farther from each other, players should expect a higher risk of breakdown in negotiations leading to disagreement (i.e., to their receiving zero payoffs, as we will normalize the disagreement payoffs to zero for each player).[3]

The FOA mechanism used by the arbiter is popular in the Industrial Relations literature where biased arbitration has been the subject of serious academic analysis (see for instance Bloom and Cavanagh, 1986) and complains about biased arbiters have been consistently reported.[4] The uncertainty about arbiters' bias is also well documented. For example, as reported by Ashenfelter (1987, p. 343), "[u]nlike the expectations of many, employers have won only about one-third of the final-offer decisions in New Jersey." Likewise, de las Mercedes Adamuz and Ponsatí (2009, p. 280) more recently reported that "[w]hile some arbitrators do act on principle – imposing a fair settlement, independently of the concessions that precede their appointment – there is strong empirical evidence that this is not usually the case."[5]

For a fairly general class of two-person bargaining problems, we provide a set of 'regularity conditions' on the arbitration initiation function p so that as p becomes arbitrarily small, these conditions guarantee convergence of all equilibrium outcomes to the Nash solution outcome when $q = \frac{1}{2}$ and to a particular outcome for each $q \neq \frac{1}{2}$ that can be thought of as the generalized Nash solution (where q and $1-q$ represent the bargaining weights). We then extend the static NDG environment to an infinite-horizon NDG where with probability p any incompatible demands are resolved via arbitration in period t, while with probability $1-p$ the game moves to the next period, $t+1$, in which players make fresh demands. We show that for each discount factor δ, all stationary equilibria of the infinite horizon game tend to the convergent point of the static game if the regularity conditions on p are now satisfied by the function

[3] Our model also shares some features with those on bargaining with observable commitment studied by Crawford (1982), Ellingsen and Miettinen (2008) or Li (2011). In these models, players make simultaneous demands and also have the option to make irreversible commitments to their demands. If only one player commits then the other has to give in, if both commit and the demands are mutually infeasible, the disagreement outcome is implemented, and if neither is committed, then bargaining goes on. In such environments, mixed strategy equilibria can appear and resemble outcomes similar to the one of random settlements studied by us. See also Malueg (2010) for mixed strategy equilibria in NDG in general.

[4] (Biased) random settlements are not only confined to industrial relations but are widespread in sports. In the UEFA European Champions League, in an elimination round a second leg match will go to a penalty shootout - which is after all a random settlement - after a tied overtime score in the second leg, if both teams had identical opposite scores in the original two rounds (e.g., 2-1, 1-2; 1-1, 1-1, etc.). Further, the shootout can be biased towards the home team due to the crowd support.

[5] It has been claimed that the popularity of arbitration as a mechanism to settle disputes has been declining due to concerns about biased arbiters: "International contracts include arbitration clauses more than domestic contracts, but also at a surprisingly low rate" (Eisenberg and Miller, 2007, p. 373). It is a serious concern of arbitration agencies as well. To alleviate biased arbitration, the Federal Mediation and Conciliation Service (FMCS) allows the disputing parties to select an arbitrator by alternately crossing off one name from a panel of seven names (https://www.fmcs.gov/services/arbitration/arbitration-policies-and-procedures/).

$\frac{p}{1-(1-p)\delta}$. Our regularity conditions on the initiation probability are similar, though not identical, to the ones on Nash's smoothing function reported in Abreu and Pearce (2015), suggesting that a 'random settlement' environment like the one we study here can be seen as a close substitute for the bargaining-set uncertainty or the feasibility of outside resources. We also do not require bargainers to end up wasting the surplus when demands are mutually feasible and close to the boundary of the bargaining set, as in Osborne and Rubinstein (1990). This, along with the alternate frameworks of Carlsson (1991) and Dutta (2012), enriches the predictive power and institutional robustness of the Nash solution.

At the heart of the convergence results lies the fact that the chance of initiating the random settlement phase is small as we show that only then do actual bargaining outcomes start to resemble the Nash solution. An example where the arbitration opportunity arose with a very small probability when breakdown risk was very high is as follows. In 2011, the Australian federal government *unexpectedly* chose to intervene in the heated Qantas strike to send it to arbitration. The Minister for Infrastructure and Transport, Anthony Albanese, noted that they "went through 16 hours of hearings to come up with a decision." The Transport Workers Union national secretary Tony Sheldon commented that "The government has stepped in, it's the first to my knowledge in the history of this country" (http://www.smh.com.au/travel/fair-work-ends-qantas-industrial-dispute-20111031-1mqsq.html#ixzz1qS0s0Upn).[6] The results obtained in this paper then have the following policy implication: if one wants to implement efficient bargaining outcomes (such as the standard Nash solution) between two parties, one can achieve this by generating a belief that in the event the parties fail to reach an agreement on their own, there will be a very small probability of mitigation via a FOA mechanism but with a completely unknown bias of the arbiter.

1.1 Related Literature

Variations on NDG and convergence: Alongside Nash (1953) and Abreu and Pearce (2015), there have been a few other papers that studied variants of the NDG setup. Carlsson (1991) provided another noisy variant of NDG, in which (i) players' actions are subject to some 'mistakes' and (ii) if players' demands do not exhaust the available surplus, some of the remainder (including the entire amount) is distributed - thus inefficient outcomes are allowed to be avoided - according to an exogenously fixed 'surplus partition' rule. As a result, the player with a better bargaining weight, who is supposed to obtain a larger share of the remainder, ends up – counter-intuitively – with the lower equilibrium payoff. In the limit as the noise vanishes, the equilibrium outcome converges to one of the generalized Nash solution outcomes.

Breakdown in bargaining: Regarding the risk of breakdown that occurs in our model with probability $1-p$, Zeuthen (1930) was the first to suggest a theory of iterative concessions in which negotiators cared about that risk. Harsanyi (1956) showed that Zeuthen's iterative process converges to the (standard) Nash solution outcome. Later, Aumann and Kurz (1977) - defining a measure of a player's 'boldness' as the maximum probability which makes him willing to take the risk of losing the entire gain against an additional small gain - observed that the Nash solution outcome turns out to be the point at which both players are equally 'bold'. More recently, Howard (1992) proposed a one-shot but a multiple-stage setup providing noncooperative foundations for the Nash solution. Howard's setup was significantly simplified by Rubinstein, Safra and Thomson (1992), which highlighted an endogenous risk of breakdown and the presence of Nature's choice - that are akin to bargaining in real life.

[6] Rule 17 of "Rules of Arbitration and Conciliation" of Indian Council of Arbitration also illustrates that arbitration requests of the disputants may not be automatically accepted: "(a) on receipt of an application for arbitration, the Registrar shall have absolute discretion to accept or reject the said application. The Registrar is not bound to give reasons for the exercise of his discretion. Before deciding on the acceptability of an application for arbitration, the Registrar may ask the parties for further information and particulars of their claims. (b) Similarly, if any information or particulars regarding the arbitration agreement furnished by claimant with the application for arbitration are found to be incorrect or false, at any time subsequently, the Registrar shall have a like power to reject the application for arbitration" (http://www.icaindia.co.in/Rules-Arbitration.pdf). So depending on the circumstances, the probability of initiation of arbitration can be small, ill-defined or unspecified.

Bargaining and arbitration: The relation between strategic bargaining and arbitration rules is well studied. It is argued that the chilling equilibrium at the bargaining stage often arises with settlement schemes that use Conventional Arbitration (CA). As in CA, it has been observed that typically arbiters in disputes merely split the difference between the offers of two parties, leading to a 'chilling effect' on negotiations, i.e., making each party reluctant to make concessions - as concessions lead to worse outcomes for that party - and thus making the collective bargaining process lose its entire significance. On the other hand, under Final-offer Arbitration (FOA), each party submits an offer and the arbiter *must* choose one of these two offers as the final outcome (see, for instance, Chatterjee (1981) for a comparative theoretical analysis of FOA and CA). Depending upon the bargainers' beliefs about the arbiter's social preferences of fairness or impartiality, equilibrium demands can be moderated under FOA. Since Stevens (1966) proposed FOA to replace CA, FOA has been used extensively in the U.S. and many other countries. Variations on Steven's ideas were also studied by Crawford (1979) to improve the design of FOAs. Since only one of the demands get selected randomly as the outcome at the settlement stage in our variant of NDG, our settlement mechanism resembles FOA in many ways and remains ex-post efficient as any FOA would. However, in the space of expected settlement outcomes, our mechanism can be interpreted as CA. In fact, all our results go through if we use CA directly on the outcome space as well where the location of the expected outcome is determined by the arbiter's bias. It is important to note in this respect that maximum uncertainty ($q = 1/2$) about the arbiter's bias in the FOA framework leads to unbiased resolution in expected outcomes in the CA sense. Interestingly, however, the threat of complete breakdown (that occurs with probability $1 - p$) allows us to avoid the inevitability of the chilling equilibrium. In addition, our environment endogenizes the initiation probability of using an arbitration mechanism as the arbiter may not even be called for duty if the players' extreme initial positions are hopelessly apart, resembling the case where the breakdown probability $1 - p$ is large.

Arbitration, bargaining and convergence: Anbarci and Boyd III (2011) studied a one-shot, two-stage variant of NDG, where the probability p of initiating a settlement is exogenously given but like us, settlement is randomized. They show that at the highest level of p that allows the players to reach an agreement on the equilibrium path, there is a unique Nash equilibrium in which the outcome coincides with the Kalai-Smorodinsky solution outcome while at lower levels of p, there are multiple equilibria. Further, the 'chilling equilibrium' - one where each player demands the maximum feasible for himself, given S, is a serious problem in that setup too, as in Nash (1953); the chilling equilibrium arises as the unique equilibrium and any agreement equilibrium vanishes when p approaches one.[7]

2 Nash Bargaining with Random Settlement

Preliminaries: Nash's two-person cooperative bargaining problem is a pair (S, d) where $S \subset \mathbb{R}^2$ is the *set of feasible utility allocations* (in short the *feasible set*) and $d \in S$ is the *disagreement point* which is the outcome that results if no agreement is reached by the two parties. For simplicity we normalize $d = (d_1, d_2) = (0, 0)$. Then the bargaining problem will be defined by $S \subset \mathbb{R}^2_+$ alone. We will use the following notation for vector inequalities: $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2$; $x > y$ means $x \geq y$ and there is some i with $x_i > y_i$; $x \gg y$ means $x_i > y_i$ for all i. The set S is assumed to contain some $x \gg (0, 0)$ and to be convex, compact, and comprehensive. *Comprehensiveness* means that if $x \in S$ and $(0, 0) \leq y \leq x$, then $y \in S$. Let $\mathbb{B}$ be the set of all convex, compact, and comprehensive bargaining problems S. A *solution* is a function $f : \mathbb{B} \to \mathbb{R}^2_+$ with $f^S = (f_1^S, f_2^S) \in S$ for all $S \in \mathbb{B}$. Let ∂S denote the *weak Pareto frontier* (or simply, *Pareto frontier*, or *boundary*) of S, where $\partial S = \{x \in S : x' \gg x$ implies $x' \notin S\}$. We will also need $\partial^* S$, the *strong Pareto frontier* of S, where $\partial^* S = \{x \in S : x' > x$ implies $x' \notin S\}$. Clearly, $\partial^* S \subseteq \partial S$. For simplicity, we assume that the slope at any $x \in \partial^* S$ is well-defined. However, we allow the Pareto frontier to have "kinks" at the two ends of the strong Pareto frontier (i.e.,

[7] Anbarci and Feltovich (2012), who experimentally tested the predictions of Anbarci and Boyd III (2011), indeed verified that at very high levels of p, equilibria involving immediate agreement cease to exist and only the chilling-effect equilibria remain.

we allow the Pareto frontier to have a horizontal segment or a vertical segment). Let b_i^S be the maximum or *ideal payoff* that Player i obtains in S, i.e., $b_i^S = \max\{x_i : (x_1, x_2) \in S\}$, for $i = 1, 2$. The *ideal point* of S is given by $b^S = (b_1^S, b_2^S)$. Throughout our analysis, we will assume that bargaining is *non-trivial* in that it is not conflict-free, i.e., $b^S \notin S$.

The Nash Solution: The standard (symmetric) *Nash solution* N in any S is given by $N^S = argmax_{x \in S} x_1 \times x_2$ (Nash, 1950). A *generalized Nash solution (with bargaining weight beta)* $N^{(\beta)}$ in any S is given by $N^{(\beta)S} = argmax_{x \in S}(x_1)^{\beta} \times (x_2)^{1-\beta}$ where $\beta \in (0, 1)$.

The Nash Demand Game (NDG): In an NDG, players 1 and 2 simultaneously make demands x_i with $x_i \in [0, b_i^S]$. If $x = (x_1, x_2) \in S$, Player i receives x_i. Otherwise, both players get 0. NDG has a multiplicity of Nash equilibria since every point on the strong Pareto frontier is a Nash equilibrium outcome as well as the 'pure' chilling-effect equilibrium with payoffs $(0, 0)$. Given (x_1, x_2), denote Player i's payoff function by $L_i = x_i H(x_1, x_2)$ where $H(x_1, x_2) = 1$ for $(x_1, x_2) \in S$ and $H(x_1, x_2) = 0$ otherwise. Nash (1953) smoothed the payoff function L by replacing the indicator function H with a continuous approximation h such that h equals H on S, but then drops off to zero in a continuous way. The smoothed payoff function for Player i is the expected utility $G_i(x_1, x_2) = x_i h(x_1, x_2)$.

Nash Demand Game with Random Settlement (NDG-RS): For any $x_i \in [0, b_i^S]$, let $x_j^*(x_i)$ be the maximum payoff that Player j obtains in S, i.e., $x_j^*(x_i) = \max\{x_j : (x_i, x_j) \in S\}$. Hence, we have $(x_1, x_2^*(x_1)) \in \partial S$ for any $x_1 \in [0, b_1^S]$ and $(x_1^*(x_2), x_2) \in \partial S$ for any $x_2 \in [0, b_2^S]$. In addition, if $(x_1, x_2) \in \partial^* S$, then we must have $x_1 = x_1^*(x_2)$ and $x_2 = x_2^*(x_1)$. We will use "Player i's *demand* x_i" and "Player i's (implicit) *proposal* $(x_i, x_j^*(x_i))$" whenever each one is called for. Let

$$S^I = \{(x_1, x_2) | x_1 \in [0, b_1^S], x_2 \in [0, b_2^S], (x_1, x_2) \notin S\}$$

be the set of combinations of Player 1's demand and Player 2's demand where the two players' demands are incompatible. Note that S and S^I partition the rectangle $[0, b_1^S] \times [0, b_2^S]$. Let p be a probability (function) that is a mapping $p : \partial S \cup S^I \to [0, 1)$ such that $p(x) = 1$ for any $x \in \partial S$ and $0 \leq p(x) < 1$ for any $x \in S^I$. The function p is the *arbitration initiation probability function* (or, *initiation probability function*, for short) and $1 - p$ is thus the *breakdown probability (function)*.

For any given initiation probability function $p(\cdot)$, the Nash Demand Game with Random Settlement (NDG-RS) that we consider is defined as follows:

- Phase 1 ('Agreement' Phase): Players 1 and 2 simultaneously make demands x_i with $x_i \in [0, b_i^S]$. If $x = (x_1, x_2) \in S$, Player i receives x_i. Otherwise, we move to Phase 2;

- Phase 2 ('Breakdown or Random Settlement' Phase): Nature makes a choice: with probability $1 - p(x)$ the game terminates with $(0, 0)$ payoffs while with probability $p(x)$, Nature brings in an arbiter to resolve the dispute. The arbiter implements Player 1's demand x_1 (and thus his effective proposal $(x_1, x_2^*(x_1))$) with probability $q \in (0, 1)$ and Player 2's demand x_2 (and thus his effective proposal $(x_1^*(x_2), x_2)$) with probability $1 - q \in (0, 1)$.

We interpret q and $1 - q$ as the common prior belief held by the players about the arbiter's bias towards Player 1 and Player 2 respectively. We say that an NDG-RS is *Maximum Bias Uncertain* if $q = \frac{1}{2}$. The payoff of Player $i = 1, 2$ at strategy profile $x = (x_1, x_2) \in [0, b_1^S] \times [0, b_2^S]$ is:

$$U_1(x) = \begin{cases} x_1 & \text{if } x \in S \\ p(x)[qx_1 + (1-q)x_1^*(x_2)] & \text{if } x \notin S, \end{cases} \tag{1}$$

and

$$U_2(x) = \begin{cases} x_2 & \text{if } x \in S \\ p(x)[qx_2^*(x_1) + (1-q)x_2] & \text{if } x \notin S. \end{cases} \quad (2)$$

Our focus is on pure-strategy Nash equilibria of an NDG-RS. We say $x \in \mathbb{R}^2$ is a "Phase 1 Nash equilibrium" if $x \in S$ and it is a "Phase 2 Nash equilibrium" if $x \notin S$.

3 Analysis

For any $\epsilon > 0$ and any $(x_1, x_2) \in [0, b_1^S] \times [0, b_2^S]$, let $N_\epsilon(x_1, x_2)$ be a neighborhood of (x_1, x_2) in $[0, b_1^S] \times [0, b_2^S]$, i.e., $N_\epsilon(x_1, x_2)$ consists of all points in $[0, b_1^S] \times [0, b_2^S]$ whose (Euclidean) distances with the point (x_1, x_2) are less than ϵ. Let $N_\epsilon(\partial S)$ be a neighborhood of ∂S in $[0, b_1^S] \times [0, b_2^S]$, i.e., $N_\epsilon(\partial S) = \{(x_1', x_2') \in [0, b_1^S] \times [0, b_2^S] | (x_1', x_2') \in N_\epsilon(x_1, x_2) \text{ for some } (x_1, x_2) \in \partial S\}$.

For any p and any constant $c \in [0, 1]$, let $x_2 = x_2^*(x_1, c|p)$ be the implicit function of the equation $p(x_1, x_2) = c$ (whenever the implicit function of the equation is well defined). That is, $p(x_1, x_2^*(x_1, c|p)) = c$. Thus, for a given c, the function $x_2 = x_2^*(x_1, c|p)$ (as a function of x_1) defines an "iso-probability curve" in the (x_1, x_2) plane on which the probability of initiating random settlement is constant at c (see Figure 1 for an illustration of the iso-probability curve). The slope of the iso-probability curve that passes through the point (x_1, x_2) is $\frac{\partial x_2^*(x_1, c|p)}{\partial x_1}|_{c=p(x_1,x_2)}$, or simply, $\frac{\partial x_2^*(x_1, p(x_1, x_2)|p)}{\partial x_1}$, with slight abuse of notation.

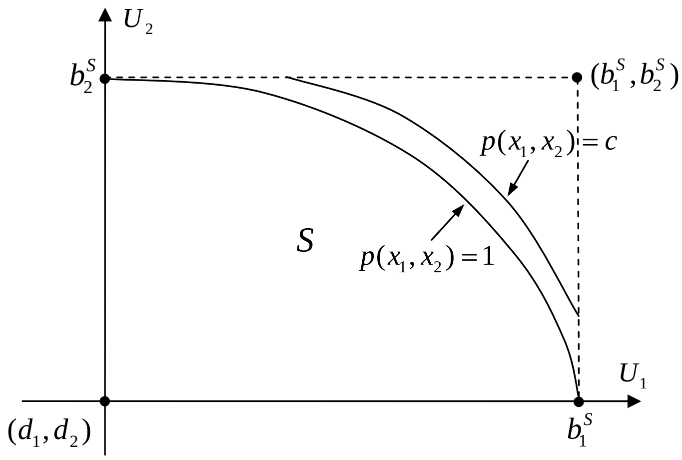

Fig. 1 Iso-probability curves.

Definition 1 *We say that a sequence of probabilities $\{p_n\}_{n=1}^\infty$ is regular, if it satisfies the following conditions:*
(i) For each n, p_n is differentiable on $(\partial S \cup S^I)\backslash E$ where $E = \{(x_1, x_2) \in \partial S | x_1 = b_1^S \text{ or } x_2 = b_2^S\}$;[8]
(ii) For each n, $-\frac{q_i}{b_i^S} < \frac{\partial p_n(x_1, x_2)}{\partial x_i} \leq 0$ for any $(x_1, x_2) \in \partial^* S$ with $x_i < b_i^S$, where $q_i = q$ for $i = 1$ and $q_i = 1 - q$ for $i = 2$.
(iii) For any $\epsilon > 0$, p_n converges uniformly to 0 on $S^I \backslash N_\epsilon(\partial S)$ as $n \to \infty$;
(iv) For some $i = 1, 2$, there exist some $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and $N_1 > 0$ such that

$$\frac{\partial p_n(x_1, x_2)}{\partial x_i} < -\frac{1}{b_i^S - \epsilon_2} p_n(x_1, x_2)$$

[8] We exclude E from the differentiable domain of p because p can never be differentiable at any point in E. In particular, noticing that the domain of p is $\partial S \cup S^I$, $\frac{\partial p(x_1, x_2)}{\partial x_1}$ is not well defined if $(x_1, x_2) \in \partial S$ and $x_1 = b_1^S$, and $\frac{\partial p(x_1, x_2)}{\partial x_2}$ is not well defined if $(x_1, x_2) \in \partial S$ and $x_2 = b_2^S$.

for all $(x_1, x_2) \in N_{\epsilon_1}(b_1^S, b_2^S)$ *for all* $n > N_1$*;*
(v) For any $i = 1, 2$, $\dfrac{\partial p_n(x_1, x_2)}{\partial x_i} \leq 0$ *on* S^I *for all* n *and there exists an* $\epsilon_3 > 0$ *such that* $\dfrac{\partial p_n(x_1, x_2)}{\partial x_i} < 0$ *on* $S^I \cap N_{\epsilon_3}(\partial S)$ *for all* n*;*
(vi) $\left\{ \dfrac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1} \right\}_{n=1}^{\infty}$ *exists and is uniformly continuous at any* $(x_1, x_2) \in \partial^* S$.[9,10]

We obtain the following result (proved in Appendix 1).

Theorem 1 *Let* $\{p_n\}_{n=1}^{\infty}$ *be a regular sequence of probabilities. Let* $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$ *be a sequence of Nash equilibria where* $(\hat{x}_1^n, \hat{x}_2^n)$ *is a Nash equilibrium of the NDG-RS that uses the breakdown probability* $1 - p_n$ *and that the probability that Player* 1*'s demand is chosen is* $q \in (0, 1)$ *when the game moves to random settlement. Then, as* $n \to \infty$, $(\hat{x}_1^n, \hat{x}_2^n)$ *must converge to the Nash solution outcome if* $q = \frac{1}{2}$ *and to the generalized Nash solution outcome (with as-if bargaining weight* q*) if* $q \neq \frac{1}{2}$.

3.1 Role of Each Condition for Regularity

In this section, we will briefly explain the intuitions behind Theorem 1 and in particular, the role of the various conditions in the definition of the regularity of a sequence of probabilities p. We will then provide a comparison of our regularity conditions with those obtained by Abreu and Pearce (2015).

Condition (ii) requires the initiation probability function p to decrease at a speed that is sufficiently slow as players' demands move slightly away from the Pareto frontier. It is needed to ensure that there is no Phase 1 Nash equilibrium in the game. If this condition is violated, then there may exist multiple Phase 1 Nash equilibria, and they may not shrink to a single point as the initiation probability function p goes to zero. To see this, imagine the extreme case where p is zero at any point on S^I; in this case, all points on the Pareto frontier is a Phase 1 Nash equilibrium outcome.

Condition (iii) implies that for any fixed combination of players' demands, the initiation probability function p will decrease to zero as n goes to infinity. We need Condition (iii) in particular to ensure that as n grows, players' equilibrium proposals will become closer and closer. Otherwise, we cannot guarantee convergence.

Condition (iv) plays the following role. For expositional purposes, suppose that there are no vertical and horizontal segments of the Pareto frontier. Note then that as Player i increases his demand by one unit, the benefit will be pq_i, and the loss will be $-\dfrac{\partial p(x_1, x_2)}{\partial x_i}(q_i x_i + q_j x_i^*(x_j))$, where $q_i = q$ if $i = 1$ and $q_i = 1 - q$ if $i = 2$. Observe that the above marginal loss is approximately equal to $-\dfrac{\partial p(x_1, x_2)}{\partial x_i} q_i b_i$ when (x_1, x_2) is close to (b_1^S, b_2^S). If Condition (iv) holds, then this marginal loss is strictly greater than pq_i, i.e., as (x_1, x_2) approaches (b_1^S, b_2^S), the marginal loss of increasing a player's demand is strictly greater than the marginal benefit from doing so. So, the players' equilibrium proposals can never approach (b_1^S, b_2^S) in the limit. Note that, intuitively, as p goes to zero, players' equilibrium proposals either converge to each other (because making a proposal that is far away from the opponent's proposal will lead to $p = 0$ and is worse than making a proposal that is close to the opponent's proposal) or converge to (b_1^S, b_2^S) (because given that a player's opponent makes an extreme demand, it is a best response for the player

---

[9] When $(x_1, x_2) = (b_1^S, x_2^*(b_1^S))$, $\dfrac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1}$ refers to the left derivative, and when $(x_1, x_2) = (x_1^*(b_2^S), b_2^S)$, $\dfrac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1}$ refers to the right derivative.

[10] Note that a sequence (of functions) $\{f_n(x)\}_{n=1}^{\infty}$ *converges uniformly* to a function f on domain X if for every $\epsilon > 0$ there exists an N such that $|f_n(x) - f(x)| < \epsilon$ for any $n \geq N$ and any $x \in X$. A function sequence $\{f_n(x)\}_{n=1}^{\infty}$ is *uniformly continuous* at some point $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f_n(y) - f_n(x)| < \epsilon$ for all n for any $y \in X$ for which $|y - x| < \delta$.

to also make an extreme demand if there are no vertical and horizontal segments in the Pareto frontier). So, Condition (iv) rules out the latter case.[11]

Condition (v) is required in addition to ensure that the iso-probability curve of p exists in some neighborhood of the Pareto frontier. In particular, if Condition (v) holds, then $\frac{\partial x_2^*(x_1, p_n(x_1,x_2)|p_n)}{\partial x_1}$ is well defined for any combination of demands (x_1, x_2) that are incompatible but is sufficiently close to the Pareto frontier as in this case,

$$\frac{\partial x_2^*(x_1, p_n(x_1,x_2)|p_n)}{\partial x_1} = -\frac{\frac{\partial p_n(x_1,x_2)}{\partial x_1}}{\frac{\partial p_n(x_1,x_2)}{\partial x_2}}.$$

Condition (vi) says that the initiation probability function p is "smooth" in the neighborhood of the Pareto frontier in the sense that as players' (incompatible) demands converge to the Pareto frontier, the slope of the iso-probability curve of p also converges to the slope of the Pareto frontier.[12] The condition guarantees that if players' equilibrium proposals converge to each other, then they must converge to the generalized Nash solution outcome. To see this point, again, we use the marginal benefit/loss argument. If Player i increases his demand by Δx_i, then the benefit is $pq_i \Delta x_i$, and the loss is $\Delta p(q_i \hat{x}_i + (1-q_i)x_i^*(\hat{x}_j))$, which is approximately $\Delta p \hat{x}_i$ in the limit if the two players' equilibrium proposals converge to each other; but in equilibrium the marginal benefit equals the marginal loss, which implies that $pq_i \Delta x_i = \Delta p \hat{x}_i$ for $i = 1, 2$. So, we have:

$$\frac{\Delta p/\Delta x_1}{\Delta p/\Delta x_2} = \frac{q_1 \hat{x}_2}{q_2 \hat{x}_1}. \tag{3}$$

Note that $\frac{\Delta p/\Delta x_1}{\Delta p/\Delta x_2}$ is exactly the slope of the iso-probability curve (i.e., $\frac{\partial x_2^*(\hat{x}_1, p(\hat{x}_1,\hat{x}_2)|p)}{\partial x_1}$). With Condition (vi), $\frac{\Delta p/\Delta x_1}{\Delta p/\Delta x_2}$ converges to the slope of the Pareto frontier as $(\hat{x}_1, \hat{x}_2)$ converges to some point (x_1, x_2) on the Pareto frontier. So, equation (3) implies:

$$\frac{\Delta x_2}{\Delta x_1}\bigg|_{\partial S} = \frac{q_1 x_2}{q_2 x_1} = \frac{q x_2}{(1-q)x_1}.$$

This in turn, however, implies that the limit outcome must be the generalized Nash solution outcome with bargaining weight q.

3.1.1 Comparison with Abreu and Pearce

The institutional setup for resolving incompatible demands is different from ours in Abreu and Pearce (2015) where breakdown is prevented with probability p by meeting the demands through outside injection of resources. They also provide regularity conditions on the 'perturbation' function p under which the equilibrium outcomes of the perturbed NDG converge to the Nash solution outcome as the perturbation goes to zero. Their conditions and ours share common features and are very similar in spirit. In particular, both our Condition (iv) and Abreu and Pearce (2015)'s Condition (i) require that as a player's demand moves further away from the feasible set, p decreases very rapidly. This is also consistent with Nash's argument: "...as less and less smoothing is used, h will decrease more and more rapidly on moving away from B..." (Nash (1953, p. 133)). Note that these conditions are necessary for the Nash

[11] See more on this in Remark 4 below.

[12] Notice that if $(x_1, x_2) \in \partial^* S$, then $x_2^*(x_1, p_n(x_1,x_2)|p_n) = x_2^*(x_1, 1|p_n) = x_2^*(x_1)$. So, we have $\frac{\partial x_2^*(x_1, p_n(x_1,x_2)|p_n)}{\partial x_1} = \frac{\partial x_2^*(x_1)}{\partial x_1}$ for any $(x_1, x_2) \in \partial^* S$.

solution convergence because players' demands may be trapped at the chilling outcome (b_1^S, b_2^S) as p goes to zero and the above conditions rule this situation out. In addition, both our Condition (vi) and Condition (ii) in Abreu and Pearce (2015) require that as players' (incompatible) demands move close to the Pareto frontier, the slope of the iso-probability curve be close to that of the Pareto frontier. As explained in Section 3.1, Condition (vi) is a key one which guarantees the Nash solution convergence. It is also worth to note that our Condition (ii)—which requires that p does not decrease at a too-fast speed at the boundary of S as players demands slightly move away from the boundary—is 'seemingly' absent in Abreu and Pearce (2015). However, in their paper, the domain of their perturbation function includes the entire feasible set, S, and the perturbation function is assumed to be (continuously) differentiable in its entire domain. Since the perturbation function is held constant at 1 for all $x \in S$, the continuous differentiability of the perturbation function implies that the derivative of the perturbation function with respect to any player's demand must equal *zero* at the boundary of S. Hence, Condition (ii) is trivially satisfied in Abreu and Pearce (2015). Note that Condition (ii) and Condition (iv) (along with Condition (v)), roughly imply that as the distance between two players' proposals increases, p first decreases at a slow speed, but then decreases at a sufficiently fast speed as players' proposals are close to the extreme demands - as is also the case in Abreu and Pearce (2015).

We end this section with the following two remarks.

Remark 1 As $n \to \infty$, it is not immediately clear whether the value of p at the equilibrium demands in the NDG-RS will converge to zero or one. This is because as n increases, there exist two effects. The first is that p will decrease for any *given* combination of players' incompatible demands. The second is that the two players' equilibrium proposals become closer, which will increase p. It can be shown that the second effect completely dominates the first effect and that the value of p at the equilibrium demands will converge to one. This also implies that the players' expected equilibrium payoffs will converge to the Nash solution payoffs as n goes to infinity. In Appendix 2, we show the following: *Let* $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$ *be a sequence of Nash equilibria where* $(\hat{x}_1^n, \hat{x}_2^n)$ *is a Nash equilibrium of the NDG-RS that uses the breakdown probability* $1 - p_n$ *where* $\{p_n\}_{n=1}^{\infty}$ *is regular and that the probability that Player 1's demand is chosen is* $q \in (0, 1)$ *when the game moves to random settlement. If* $(\hat{x}_1^n, \hat{x}_2^n)$ *converges to some point on* $\partial^* S$ *as* $n \to \infty$*, then it must be true that* $p_n(\hat{x}_1^n, \hat{x}_2^n) \to 1$ *as* $n \to \infty$*.*

Remark 2 We now compare our setup and Nash's smoothing setup graphically (see Figure 2). In our NDG-RS, when players make incompatible demands, the game turns into a lottery where either the disagreement point is chosen or a point inside S (in particular, the middle point of the two players' incompatible proposals) is chosen.[13] However, in Nash's smoothing, when players make incompatible demands, either the disagreement point or a point outside S is chosen as the outcome.

3.2 Examples of Endogenous Breakdown Probabilities

In this section, we will first give an intuitive example in which the sequence of probabilities p is regular and thus the Nash solution convergence result holds. Then we will give one example in which the sequence is not regular (and consequently there is no convergence result), even though both examples are based on the common concept of the "Minimal Agreement Point" which will be defined next. For simplicity, we only consider the case $q = 1/2$ throughout this section.

3.2.1 Minimal Agreement Point and Convergence

Consider a particular regular sequence of probabilities based on the "Minimal Agreement Point" (MAP) defined as follows. Let $A_{(x_1,x_2)} = \{(y_1, y_2) \in S : (y_1, y_2) \geq (x_1, x_2)\}$ be the set of all points that (weakly)

[13] The lottery generated by our game in fact has three states, the agreement point and the two incompatible proposals. The latter can be merged into one in expectations (denoted in the figure by the point 'Ours') as the players are expected utility maximizers.

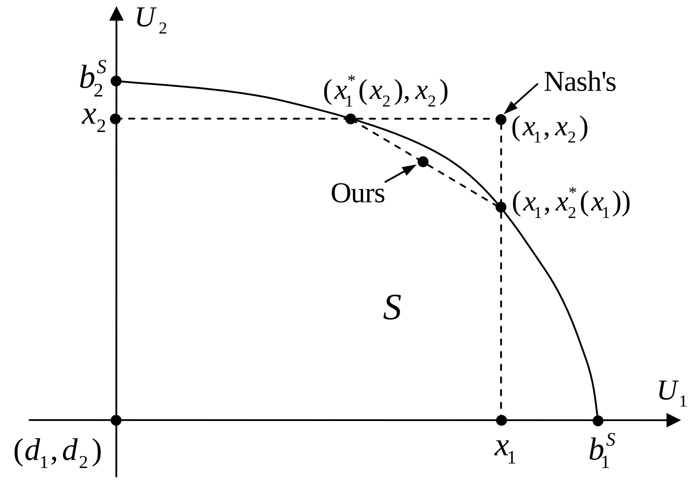

Fig. 2 A comparison with Nash's smoothing.

Pareto dominate (x_1, x_2) in S. For any $X \subseteq S$, let $\alpha(X)$ be the area of X. For simplicity, we normalize $\alpha(S) = 1$. Let $(x_1, x_2) \notin S$. Then, $m(x_1, x_2) = (x_1^*(x_2), x_2^*(x_1))$ (or in short $m(x) = (x_1^*, x_2^*)$) is called the *Minimal Agreement Point* since Player 1's demand already acknowledges that Player 2 should get $x_2^*(x_1)$ at least (i.e., even if Player 1's demand prevails) and Player 2's demand already acknowledges that Player 1 should get at least $x_1^*(x_2)$ (i.e., even if Player 2's demand prevails). Thus, $m(x)$ is a very intuitive reference point for generating breakdown probability functions $1 - p(x)$.

Note that $A_{m(x_1,x_2)}$ (or simply, $A_{m(x)}$) is the set of all points that are (weakly) Pareto superior to MAP. Figure 3 illustrates $m(x)$ and $A_{m(x)}$.

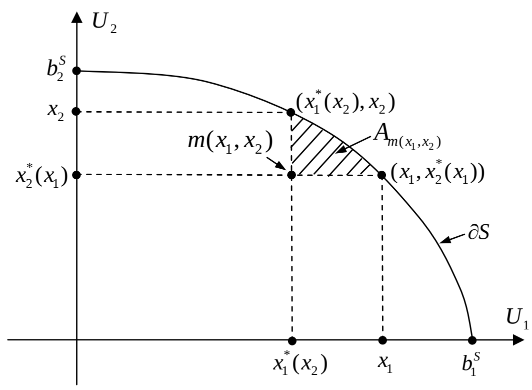

Fig. 3 Pareto superior sets with respect to the Minimal Agreement Point $m(x_1, x_2)$.

Let the *MAP-based breakdown probability* be $1-p(x) = \alpha(A_{m(x)})$ and the *n-th MAP-based breakdown probability* be $1 - p_n(x) = 1 - [1 - \alpha(A_{m(x)})]^n$. So, the *n-th MAP-based initiation probability* is $p_n(x) = [1 - \alpha(A_{m(x)})]^n$. It can be verified that the sequence of the n-th MAP-based initiation probabilities is regular.

Proposition **1** *As $n \to \infty$, the two players' equilibrium proposals $(\hat{x}_1^n, \hat{x}_2^n)$ in any Nash equilibrium of the NDG-RS that uses the n-th MAP-based breakdown probability $1 - p_n$ and has maximum bias uncertainty (viz. $q = 1/2$) converge to the Nash solution outcome. Moreover, $(\hat{x}_1^n, x_2^*(\hat{x}_1^n))$ and $(x_1^*(\hat{x}_2^n), \hat{x}_2^n)$, must surround the Nash solution outcome on the Pareto frontier (i.e., $(\hat{x}_1^n, x_2^*(\hat{x}_1^n))$ must lie on the lower right of the Nash solution outcome on the Pareto frontier or coincide with the Nash solution outcome, and $(x_1^*(\hat{x}_2^n), \hat{x}_2^n)$ must lie on the upper left of the Nash solution outcome on the Pareto frontier or coincide with the Nash solution outcome).*

The proof of Proposition 1 is moved to Appendix 1. For now, consider the following feasible set that yields a linear Pareto frontier:

$$S = \{(x_1, x_2)|0 \leq x_1 \leq \sqrt{2}, 0 \leq x_2 \leq \sqrt{2} \text{ and } 0 \leq x_1 + x_2 \leq \sqrt{2}\}.$$

Noting that the area of S is 1, let $1 - p_n(x) = 1 - [1 - \alpha(A_{m(x)})]^n$. Let $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$ be a sequence of Nash equilibria where $(\hat{x}_1^n, \hat{x}_2^n)$ is a Nash equilibrium of the NDG-RS that uses the breakdown probability $1 - p_n$ and is maximum bias uncertain. Then it can be verified that

$$\hat{x}_1^n = \hat{x}_2^n = \frac{\sqrt{2}}{2} + \frac{\sqrt{2n^2+2} - \sqrt{2}n}{2}.$$

It is obvious that as $n \to \infty$, both players' equilibrium proposals converge to $(\sqrt{2}/2, \sqrt{2}/2)$ (i.e., the Nash solution outcome). The convergence path is depicted in Figure 4 below.

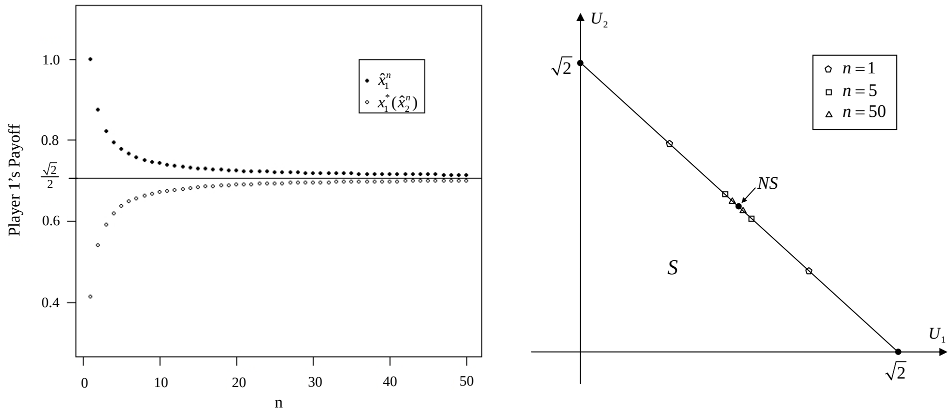

Fig. 4 Convergence of Nash equilibrium to the Nash solution.

3.2.2 Minimal Agreement Point and Non-convergence

Convergence to the Nash solution outcome is not automatic for any type of breakdown probability that looks reasonable. To make this point consider a breakdown probability $1 - p$ which is based on the measure of *all outcomes not weakly Pareto-dominated by MAP,* (x_1^*, x_2^*), instead of the one that is used in the previous subsection which is based on the measure of *all outcomes weakly Pareto superior to MAP,* (x_1^*, x_2^*). Note that the latter set is embedded in the former set. We will show that the Nash solution convergence never happens in this case; in particular, any efficient demand pair turns out to be a Nash equilibrium (as in the original NDG without smoothing), while the only viable Phase 2 Nash equilibrium turns out to be the chilling-effect equilibrium.

Let $B_{(x_1,x_2)} = \{(y_1, y_2) \in S : (y_1, y_2) \leq (x_1, x_2)\}$ be the set of all points that are (weakly) Pareto dominated by (x_1, x_2) in S. Then, $B_{m(x_1,x_2)}$ (or simply, $B_{m(x)}$) is the set of all points that are (weakly) Pareto inferior to the *minimal agreement point, MAP* (see Figure 5). Let the *alternative MAP-based breakdown probability* be $1 - p(x) = 1 - \alpha(B_{m(x)})$. So, the *alternative MAP-based initiation probability* is $p(x) = \alpha(B_{m(x)})$.

Remark 3 Note that the MAP-based breakdown probability $1 - p(x) = \alpha(A_{m(x)})$ is continuous in the 'divergence' between the players' demands while the alternative one $1 - p(x) = 1 - \alpha(B_{m(x)})$ is not. To see this, consider two demands x_1 and x_2 such that $(x_1, x_2) \in \partial^* S$. Then, the probability of obtaining disagreement payoffs is 0 (since players' demands are compatible). Suppose now that x_1 increases to $x_1' = x_1 + \varepsilon$, where $\varepsilon > 0$ but is arbitrarily small. Under $1 - p(x) = \alpha(A_{m(x)})$, the probability of obtaining disagreement payoffs remains arbitrarily close to zero while under the alternative one $1 - p(x) = 1 - \alpha(B_{m(x)})$, it involves a 'jump'.

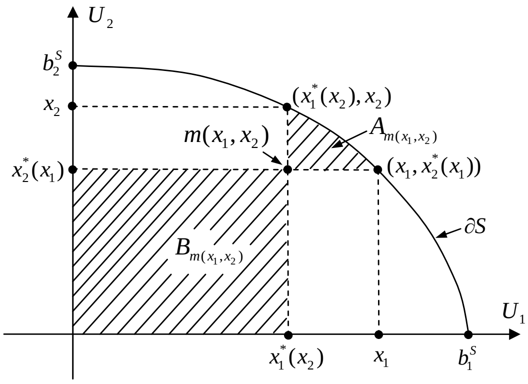

Fig. 5 Pareto inferior sets with respect to the minimal agreement point $m(x_1, x_2)$.

Remark 3 implies that the alternative MAP-based initiation probability does not satisfy condition (ii). This implies that the n-th alternative MAP-based initiation probability (i.e., $p_n = \alpha(B_{m(x)})^n$) also does not. Thus, the sequence of the n-th alternative MAP-based initiation probability cannot be regular, and the Nash solution convergence result is not guaranteed if we use the n-th alternative MAP-based initiation probability.

Our next result, proved in Appendix 1, illustrates that there exist two types of Nash equilibria in the NDG-RS that uses the alternative MAP-based breakdown probability and has maximum bias uncertainty. First, any $(x_1, x_2) \in \partial^* S$ is an equilibrium and thus there is a continuum of agreement equilibria. Second, the 'chilling-effect' outcome (b_1^S, b_2^S) is the *only possible* Phase 2 Nash equilibrium, although in this equilibrium, the breakdown probability $1 - p$ is one since $\alpha(S) = 1$.

PROPOSITION **2** *In the NDG-RS that uses the alternative MAP-based breakdown probability and has maximum bias uncertainty (viz. $q = 1/2$), any $(x_1, x_2) \in \partial^* S$ is a Phase 1 Nash equilibrium and the chilling-effect equilibrium (b_1^S, b_2^S) is the only possible Phase 2 Nash equilibrium. In addition, (b_1^S, b_2^S) is a Phase 2 Nash equilibrium if and only if $x_1^*(b_2^S) = 0$ and $x_2^*(b_1^S) = 0$.*

As alluded to in Remark 3 above, $1 - p(x) = \alpha(A_{m(x)})$ differs from $1 - p(x) = 1 - \alpha(B_{m(x)})$ in that the former is continuous while the latter is discontinuous in the divergence of individual demands. This difference leads to the fact that any point on the Pareto frontier is a Nash equilibrium outcome with $1 - p(x) = 1 - \alpha(B_{m(x)})$, but not with $1 - p(x) = \alpha(A_{m(x)})$. Another difference between them is that with $1 - p(x) = 1 - \alpha(B_{m(x)})$, if a player makes an extreme demand, then $1 - p$ is one no matter how generous the opponent's proposal will be (assuming that the Pareto frontier has no vertical or horizontal segments), while with $1 - p(x) = \alpha(A_{m(x)})$, if a player makes an extreme demand, $1 - p$ will decrease as the opponent's proposal becomes more generous. This second difference implies that with $1 - p(x) = 1 - \alpha(B_{m(x)})$, both players making extreme demands is a Nash equilibrium in the case where the Pareto frontier has no vertical or horizontal segments.

The comparison of $1 - p(x) = \alpha(A_{m(x)})$ vs. $1 - p(x) = 1 - \alpha(B_{m(x)})$ reveals a number of interesting features of our NDG-RS schemes. First that if a breakdown probability is $\alpha(A_{m(x)})$, then negotiation moves to Phase 2 that involves potential agreements on two outcomes that surround the Nash solution outcome. In terms of expectation, this yields an inefficient outcome. However, as this breakdown probability approaches one, these two outcomes converge to the Nash solution outcome and both players indeed demand what the Nash solution outcome would yield them. However, this convergence is trivially violated when one moves to the alternative MAP-based breakdown probability $1 - \alpha(B_{m(x)})$ in two ways: any efficient agreement remains robust as an equilibrium outcome as does the only Phase 2 Nash equilibrium outcome (i.e., the chilling-effect outcome) albeit only when the probability of moving to the random settlement stage is also zero. Thus, the interesting contrast is that while the alternative MAP-based breakdown probability $1-\alpha(B_{m(x)})$ fails in obtaining the Nash solution outcome as a unique equilibrium outcome, with the MAP-based breakdown probability $\alpha(A_{m(x)})$, all equilibria are Phase 2 Nash equilibria.[14]

Remark 4 Condition (iv) of Definition 1 is a sufficient condition for the non-existence of chilling-effect equilibrium. Hence any violation of it will not guarantee the existence of an equilibrium with chilling-effect. However, consider the following natural variant of the above MAP-based continuation probability p used in section 3.2.1 that violates Condition (iv) in a special way, but satisfies all other conditions of Definition 1: $p(x_1, x_2) = 0$ if $x_i = b_i^S$ for at least some $i = 1, 2$. Such a 'mechanism' can emerge out of an institutional norm where it is understood that the institution is willing to help bargainers by allowing for some chance for arbitration provided every bargainer shows at least some sign of giving something positive to her bargaining partner; otherwise, that is, if even one individual shows no mercy, the institution will punish that individual by disallowing any form of continuation. Here it is easy to see that, provided there are no vertical or horizontal segments on the Pareto frontier, there is an equilibrium where each player $i = 1, 2$ demands b_i^S and this is the only equilibrium where at least for one i, the demand is b_i^S. Thus, Condition (iv) of Definition 1 (and Condition (i) as in Abreu and Pearce (2015)) is an important element of regularity to get away from the chilling-effect equilibrium present originally in Nash's NDG.

The example in Section 3.2.2 violates both Condition (ii) and Condition (iv) (as it also has the feature that $p(x_1, x_2) = 0$ if $x_i = b_i^S$ for at least some i). In the example, violating Condition (ii) leads to the fact that any point on the Pareto frontier is an equilibrium, and violating Condition (iv) leads to the fact that (b_1^S, b_2^S) is the only possible Phase 2 equilibrium. So, making the chilling-effect outcome the only possible Phase 2 equilibrium here is due to the violation of Condition (iv), rather than the violation of Condition (ii).

Apart from Nash's original smoothing and its foundations by Abreu and Pearce (2015), the chilling-effect has been ruled out in the existing literature by others as well. For example, Dutta (2012) adds a second stage to the standard NDG where bargainers, facing incompatible demands, can decide whether to stick to their original demands or give in to the demands of their partners by incurring a cost. Very high initial demands give competitors the option to generate a large negative surplus that then gets shared between the bargainers, thereby putting oneself into a position where yielding to the competitor's original demand becomes a dominant strategy in the second stage. This keeps players away from extreme demands.

4 Infinite-horizon

We now consider an *infinite-horizon NDG-RS* where in each period, if players' demands (x_1, x_2) are compatible, then the game ends immediately with each player obtaining what he demands. If players' demands (x_1, x_2) are incompatible, then with initiation probability $p(x_1, x_2)$, the game moves to random settlement stage (in which $(x_1, x_2^*(x_1))$ is chosen with probability q and $(x_1^*(x_2), x_2)$ is chosen with probability $1 - q$), while with probability $1 - p(x_1, x_2)$, the game moves to the next period in which players make new demands in the next period, and this continues. We focus on stationary (subgame-perfect) equilibria in which players' demands in any period t do not depend on the history of the game before t. Let (x_1, x_2) be a stationary equilibrium for the infinite-horizon NDG-RS, where players make the demands (x_1, x_2) in all periods. If $(x_1, x_2) \in S$, Player i's utility in such an equilibrium is $U_i(x_1, x_2) = x_i$. If $(x_1, x_2) \notin S$, then

$$U_i(x_1, x_2) = p(x_1, x_2)[q_i x_i + (1 - q_i)x_i^*(x_j)] + (1 - p(x_1, x_2))\delta U_i(x_1, x_2),$$

where δ is the discount factor. This yields

$$U_i(x_1, x_2) = \frac{p(x_1, x_2)}{1 - (1 - p(x_1, x_2))\delta}[q_i x_i + (1 - q_i)x_i^*(x_j)].$$

[14] In the second part of Appendix 2 we provide another example of a seemingly desirable breakdown probability, based on Euclidean distance between 'offers,' that does not yield convergence to the Nash solution outcome.

Note that if Player 1 deviates to a demand x_1' in *all* periods, then Player 1's utility will be $U_1(x_1', x_2) = \frac{p(x_1',x_2)}{1-(1-p(x_1',x_2))\delta}[q_1 x_1' + (1-q_1)x_1^*(x_2)]$ if $(x_1', x_2) \notin S$ and $U_1(x_1', x_2) = x_1'$ if $(x_1', x_2) \in S$. We can define $U_2(x_1, x_2')$ similarly. Since (x_1, x_2) is an equilibrium, it implies that $U_1(x_1, x_2) \geq U_1(x_1', x_2)$ for any $x_1' \in [0, b_1^S]$ and $U_2(x_1, x_2) \geq U_2(x_1, x_2')$ for any $x_2' \in [0, b_2^S]$.[15] This implies that if (x_1, x_2) is a stationary equilibrium of the infinite horizon game, then it is also a Nash equilibrium in the static game where the initiation probability function p is replaced by another initiation probability function $g = \frac{p}{1-(1-p)\delta}$. So, we obtain the following result.

Theorem 2 *Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of probabilities and let $g_n = \frac{p_n}{1-(1-p_n)\delta}$. Suppose that $\{g_n\}_{n=1}^{\infty}$ is regular. Let $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$ be a sequence of stationary equilibria where $(\hat{x}_1^n, \hat{x}_2^n)$ is a stationary equilibrium of the infinite-horizon NDG-RS that uses the breakdown probability $1-p_n$ and where Player 1's demand is chosen with probability $q \in (0,1)$ when the game moves to the random settlement stage in any period. Then, as $n \to \infty$, $(\hat{x}_1^n, \hat{x}_2^n)$ must converge to the Nash solution outcome if $q = \frac{1}{2}$ and to the generalized Nash solution outcome (with as-if bargaining weight q) if $q \neq \frac{1}{2}$.*

5 Conclusion

There are many real world situations where negotiation between two parties - like employers and workers, divorce partners or nation-states - are undertaken under the possibility that if negotiations break down then an external arbiter is called upon to resolve the crisis. In such situations, the bargaining parties may face uncertainty regarding whether such arbitration will take place, and if it does, whether the arbiter would be biased towards one of them. We have incorporated these features in an otherwise standard NDG that probabilistically allows players to resolve their incompatible demands via a random settlement scheme where a biased arbiter uses the FOA mechanism to select one of the two demands. The 'randomness' of the settlement mechanism in the eyes of the bargainers is driven by incomplete information about the arbiter's bias. The probability of initiating the random settlement scheme is endogenous as larger incompatibility of individual demands reduces this initiation probability.

We have provided conditions on this initiation probability function and the stochastic structure of the random settlement mechanism to guarantee that every Nash equilibrium of our game converges to the standard or the generalized Nash solution outcome as the chances of arbitration vanishes. We show that the standard Nash solution is related to maximum entropy concerning the uncertainty over the arbiter's bias. We have then extended the static environment to an infinite-horizon (dynamic) setup and proved that these conditions are also useful to obtain convergence of stationary equilibria to the Nash solution in such a dynamic model where players are given the option to renegotiate. From a policy perspective, our results suggest that giving the bargainers a tiny chance of resolving disagreements through arbiters whose bias are totally unknown can yield desirable outcomes.

6 Appendix 1: Proofs

Proof of Theorem 1

Our proof consists of three steps.

Step 1. In this step, we will show that if $(\hat{x}_1, \hat{x}_2)$ is a Nash equilibrium of an NDG-RS that has settlement bias $q \in (0,1)$ (i.e., Player 1's demand is chosen with probability q when the game moves to random settlement) and uses the breakdown probability $1-p$ where p satisfies Condition (ii) (i.e., $-\dfrac{q_i}{b_i^S} < \dfrac{\partial p(x_1,x_2)}{\partial x_i} \leq 0$ for any $(x_1, x_2) \in \partial^* S$ with $x_i < b_i^S$, where $q_i = q$ for $i = 1$ and $q_i = 1-q$ for $i = 2$), then $(\hat{x}_1, \hat{x}_2)$ must be a Phase 2 Nash equilibrium, that is $(\hat{x}_1, \hat{x}_2) \notin S$. Moreover, for any i, if $\hat{x}_i < b_i^S$, then $p(\hat{x}_1, \hat{x}_2)q_i = -\dfrac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_i}(q_i \hat{x}_i + (1-q_i)x_i^*(\hat{x}_j))$, and if $\hat{x}_i = b_i^S$, then $p(\hat{x}_1, \hat{x}_2)q_i \geq -\dfrac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_i}(q_i \hat{x}_i + (1-q_i)x_i^*(\hat{x}_j))$.

[15] Note that these inequalities are the necessary (but not sufficient) conditions for (x_1, x_2) to be an equilibrium in the infinite-horizon NDG-RS.

We now prove the above statements. There are two possible types of Nash equilibria: Phase 1 Nash equilibria and Phase 2 Nash equilibria. So, we have the following two cases.

(i) *Phase 1 Nash equilibria, i.e., the Nash equilibria where players' demands are compatible with each other*: For this case, let $(\hat{x}_1, \hat{x}_2) \in S$ be a Phase 1 Nash equilibrium. Then, we must have $(\hat{x}_1, \hat{x}_2) \in \partial^* S$ (if not, then $(\hat{x}_1, \hat{x}_2)$ cannot be a Nash equilibrium because one of the two players will have an incentive to deviate to a slightly higher demand, and the two players' proposals are still compatible when the player makes the deviation). Suppose $\hat{x}_1 < b_1^S$. Notice that Player 1's payoff is $U_1(x_1, x_2) = p(x_1, x_2)(qx_1 + (1-q)x_1^*(x_2))$ if the two players' demands $(x_1, x_2) \in \partial S \cup S^I$. Taking the derivative of Player 1's payoff with respect to x_1 at $(\hat{x}_1, \hat{x}_2)$ on the domain $\partial S \cup S^I$, we have[16] $\frac{\partial U_1(\hat{x}_1, \hat{x}_2)}{\partial x_1} = \frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1}(q\hat{x}_1 + (1-q)x_1^*(\hat{x}_2)) + p(\hat{x}_1, \hat{x}_2)q = \frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1}\hat{x}_1 + p(\hat{x}_1, \hat{x}_2)q = \frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1}\hat{x}_1 + q \geq \frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1} b_1^S + q > -q + q = 0$, where the second equality follows from the fact that $x_1^*(\hat{x}_2) = \hat{x}_1$ (which is due to the fact that $(\hat{x}_1, \hat{x}_2) \in \partial^* S$), and the second inequality follows from Condition (ii). The above inequality then implies that Player 1 is strictly better off by making a slightly higher demand. If $\hat{x}_1 = b_1^S$, then we must have $\hat{x}_2 < b_2^S$ because $(\hat{x}_1, \hat{x}_2) \in \partial^*(S)$ and $(b_1^S, b_2^S) \notin \partial^*(S)$. In this case, it can be shown that Player 2 has an incentive to deviate to a slightly higher demand (the proof is similar to the case where $\hat{x}_1 < b_1^S$). So, we have proved that $(\hat{x}_1, \hat{x}_2)$ cannot be a Nash equilibrium. Thus, there is no Phase 1 Nash equilibrium.

(ii) *Phase 2 Nash equilibria, i.e., the Nash equilibria where players' demands are not compatible with each other*: Let $(\hat{x}_1, \hat{x}_2) \notin S$ be a Phase 2 Nash equilibrium. Then, we must have $\hat{x}_1 > x_1^*(\hat{x}_2)$ and $\hat{x}_2 > x_2^*(\hat{x}_1)$. For Player i, we have either $\hat{x}_i < b_i^S$ or $\hat{x}_i = b_i^S$. We consider Player 1 first. We have the following two cases.

Case 1: $\hat{x}_1 < b_1^S$. Since $(\hat{x}_1, \hat{x}_2)$ is a Nash equilibrium, we must have:

$$\hat{x}_1 = \mathrm{argmax}_{x_1^*(\hat{x}_2) < x_1 < b_1^S} U_1(x_1, \hat{x}_2) = \mathrm{argmax}_{x_1^*(\hat{x}_2) < x_1 < b_1^S} p(x_1, \hat{x}_2)(qx_1 + (1-q)x_1^*(\hat{x}_2)) \tag{4}$$

The first order condition of the maximization problem in (4) is that $\frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1}(q\hat{x}_1 + (1-q)x_1^*(\hat{x}_2)) + p(\hat{x}_1, \hat{x}_2)q = 0$. That is, $p(\hat{x}_1, \hat{x}_2)q = -\frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1}(q\hat{x}_1 + (1-q)x_1^*(\hat{x}_2))$.

Case 2: $\hat{x}_1 = b_1^S$. In this case, it must be true that Player 1's marginal utility at $(\hat{x}_1, \hat{x}_2)$ is non-negative. So, we must have $p(\hat{x}_1, \hat{x}_2)q \geq -\frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1}(q\hat{x}_1 + (1-q)x_1^*(\hat{x}_2))$.

Similarly, for Player 2, if $\hat{x}_2 < b_2^S$, then $p(\hat{x}_1, \hat{x}_2)(1-q) = -\frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_2}((1-q)\hat{x}_2 + qx_2^*(\hat{x}_1))$. If $\hat{x}_2 = b_2^S$, then $p(\hat{x}_1, \hat{x}_2)(1-q) \geq -\frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_2}((1-q)\hat{x}_2 + qx_2^*(\hat{x}_1))$.

Finally, if in addition we have that p it is a twice differentiable function and is (weakly) decreasing as well as strictly concave in x_i (for $i = 1, 2$), then the necessary conditions above are also sufficient. Therefore a Nash equilibrium must exist.

Step 2. In this step, we show that if $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$ is a sequence of Nash equilibria where $(\hat{x}_1^n, \hat{x}_2^n)$ is a Nash equilibrium of the NDG-RS that has settlement bias q and uses the breakdown probability $1 - p_n$ where $\{p_n\}_{n=1}^{\infty}$ is a regular sequence of probabilities, then the two players' equilibrium proposals, $(\hat{x}_1^n, x_2^*(\hat{x}_1^n))$ and $(x_1^*(\hat{x}_2^n), \hat{x}_2^n)$, will converge to each other as n goes to infinity. In particular, we will show that $\lim_{n\to\infty}(\hat{x}_1^n - x_1^*(\hat{x}_2^n)) = 0$ and $\lim_{n\to\infty}(\hat{x}_2^n - x_2^*(\hat{x}_1^n)) = 0$.

Suppose that $\lim_{n\to\infty}(\hat{x}_1^n - x_1^*(\hat{x}_2^n)) = 0$ does not hold. Then, using the fact that $\hat{x}_1^n - x_1^*(\hat{x}_1^n) > 0$ for all n (noticing that for any n, $(\hat{x}_1^n, \hat{x}_2^n)$ must be a Phase 2 Nash equilibrium according to Step 1), there must exist a subsequence of $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$, say $\{(\hat{x}_1^{n_k}, \hat{x}_2^{n_k})\}_{k=1}^{\infty}$, such that $\hat{x}_1^{n_k} - x_1^*(\hat{x}_2^{n_k}) > \epsilon$ for all k and for some $\epsilon > 0$. Notice that the sequence $\{\hat{x}_1^{n_k} - x_1^*(\hat{x}_2^{n_k})\}_{k=1}^{\infty}$ lies in a compact set (because $|\hat{x}_1^{n_k} - x_1^*(\hat{x}_2^{n_k})| \leq b_1^S$ for any k), so it must have a convergent subsequence, say $\{\hat{x}_1^{n_{k_l}} - x_1^*(\hat{x}_2^{n_{k_l}})\}_{l=1}^{\infty}$. In addition, we must have $\lim_{l\to\infty}(\hat{x}_1^{n_{k_l}} - x_1^*(\hat{x}_2^{n_{k_l}})) = \epsilon'$ for some $\epsilon' \geq \epsilon$.

Since $(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})$ must be a Phase 2 Nash equilibrium for any l, it must be true that in the NDG-RS that uses the breakdown probability $p_{n_{k_l}}$ and has settlement bias q, making the demand $\hat{x}_1^{n_{k_l}}$ is not worse than making the demand $x_1^*(\hat{x}_2^{n_{k_l}})$ for Player 1 (the latter demand will make the two players' demands just compatible). That is, $(q\hat{x}_1^{n_{k_l}} + (1-q)x_1^*(\hat{x}_2^{n_{k_l}}))p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}}) \geq x_1^*(\hat{x}_2^{n_{k_l}})$. The fact that $\lim_{l\to\infty}(\hat{x}_1^{n_{k_l}} - x_1^*(\hat{x}_2^{n_{k_l}})) = \epsilon' > 0$ and the fact that p_n converges to 0 uniformly on any domain where an arbitrarily small neighborhood of ∂S is excluded (i.e., Condition (iii)) imply that

[16] In the remainder of the paper, we use $\frac{\partial f(\hat{x}_1, \hat{x}_2)}{\partial x_i}$ to denote $\frac{\partial f(x_1, x_2)}{\partial x_i}|_{(x_1,x_2)=(\hat{x}_1,\hat{x}_2)}$.

$p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})$ converges to zero as $l \to \infty$. This implies that $x_1^*(\hat{x}_2^{n_{k_l}}) \to 0$ as $l \to \infty$. So, $\hat{x}_2^{n_{k_l}} \to b_2^S$ as $l \to \infty$. Similarly, we have $x_2^*(\hat{x}_1^{n_{k_l}}) \to 0$ as $l \to \infty$, and thus $\hat{x}_1^{n_{k_l}} \to b_1^S$ as $l \to \infty$.

Suppose that Condition (iv) holds for $i = 1$. Let $\hat{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$. Since $\hat{x}_1^{n_{k_l}} \to b_1^S$ and $\hat{x}_2^{n_{k_l}} \to b_2^S$, we have $|\hat{x}_1^{n_{k_l}} - b_1^S| < \hat{\epsilon}$ and $|\hat{x}_2^{n_{k_l}} - b_2^S| < \hat{\epsilon}$ for all $l > L$ for some $L > N_1$. This implies that $(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}}) \in N_{\hat{\epsilon}}(b_1^S, b_2^S) \subseteq N_{\epsilon_1}(b_1^S, b_2^S)$ for all $l > L$. It also implies that $q\hat{x}_1^{n_{k_l}} + (1-q)x_1^*(\hat{x}_2^{n_{k_l}}) \geq q\hat{x}_1^{n_{k_l}} \geq q(b_1^S - \hat{\epsilon})$ for all $l > L$. So, for any $l > L$, we have $-\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1}(q\hat{x}_1^{n_{k_l}} + (1-q)x_1^*(\hat{x}_2^{n_{k_l}})) \geq -\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1}q(b_1^S - \hat{\epsilon}) \geq -\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1}q(b_1^S - \epsilon_2) > \frac{1}{b_1^S - \epsilon_2}p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})q(b_1^S - \epsilon_2) = p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})q$, where the first inequality follows from the fact that $q\hat{x}_1^{n_{k_l}} + (1-q)x_1^*(\hat{x}_2^{n_{k_l}}) \geq q(b_1^S - \hat{\epsilon})$ and $-\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1} \geq 0$ (according to Condition (v)), the second inequality follows from the fact that $-\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1} \geq 0$ and $b_1^S - \hat{\epsilon} \geq b_1^S - \epsilon_2$, and the last inequality follows from the fact that $-\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1} > \frac{1}{b_1^S - \epsilon_2}p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})$ for any $(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}}) \in N_{\epsilon_1}(b_1^S, b_2^S)$ (according to Condition (iv)).

Since $(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})$ is a Phase 2 Nash equilibrium, according to Step 1, we must have $p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})q \geq -\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1}(q\hat{x}_1^{n_{k_l}} + (1-q)x_1^*(\hat{x}_2^{n_{k_l}}))$ (and $p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})(1-q) \geq -\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_2}((1-q)\hat{x}_2^{n_{k_l}} + qx_2^*(\hat{x}_1^{n_{k_l}})))$ for any l. However, we have shown that $-\frac{\partial p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})}{\partial x_1}(q\hat{x}_1^{n_{k_l}} + (1-q)x_1^*(\hat{x}_2^{n_{k_l}})) > p_{n_{k_l}}(\hat{x}_1^{n_{k_l}}, \hat{x}_2^{n_{k_l}})q$ when l is sufficiently large, a contradiction.

So, $\lim_{n\to\infty}(\hat{x}_1^n - x_1^*(\hat{x}_2^n)) = 0$ must hold. Similarly, we can show that $\lim_{n\to\infty}(\hat{x}_2^n - x_2^*(\hat{x}_1^n)) = 0$ must hold. Hence, we have proved that the two players' equilibrium "proposals" must converge to each other.

Step 3. In this step, we show that if $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^\infty$ is a sequence of Nash equilibria where $(\hat{x}_1^n, \hat{x}_2^n)$ is a Nash equilibrium of the NDG-RS that has settlement bias q and uses the breakdown probability $1 - p_n$ where $\{p_n\}_{n=1}^\infty$ is a regular sequence of probabilities, then the sequence $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^\infty$ must converge to the generalized Nash solution outcome with bargaining weight q as $n \to \infty$, i.e., $\lim_{n\to\infty}\hat{x}_1^n = x_1^{N^q}$ and $\lim_{n\to\infty}\hat{x}_2^n = x_2^{N^q}$ where $(x_1^{N^q}, x_2^{N^q})$ is the generalized Nash solution outcome with bargaining weight q.

Suppose that $\lim_{n\to\infty}\hat{x}_1^n = x_1^{N^q}$ does not hold. Then there exists a subsequence of $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^\infty$, say $\{(\hat{x}_1^{n_i}, \hat{x}_2^{n_i})\}_{i=1}^\infty$, such that $|\hat{x}_1^{n_i} - x_1^{N^q}| > \epsilon$ for all i and for some $\epsilon > 0$. Notice that the sequence $\{\hat{x}_1^{n_i}\}_{i=1}^\infty$ lies in a compact set, so it must have a convergent subsequence, say $\{\hat{x}_1^{n_{i_j}}\}_{j=1}^\infty$. Let x_1' be the limit of this subsequence, then we must have $|x_1' - x_1^{N^q}| \geq \epsilon$. Since $\lim_{n\to\infty}(\hat{x}_2^n - x_2^*(\hat{x}_1^n)) = 0$ and $x_2^*(x_1)$ is a continuous function, we have $\lim_{j\to\infty}\hat{x}_2^{n_{i_j}} = \lim_{j\to\infty}x_2^*(\hat{x}_1^{n_{i_j}}) = x_2^*(x_1')$. Observing that $x_1' = b_1^S$ and $x_2^*(x_1') = b_2^S$ cannot hold simultaneously because $(x_1', x_2^*(x_1')) \in \partial^* S$ and $(b_1^S, b_2^S) \notin \partial^* S$, we thus have the following three cases.

Case 1. $x_1' = b_1^S$ and $x_2^*(x_1') < b_2^S$. Since $\lim_{j\to\infty}\hat{x}_2^{n_{i_j}} = x_2^*(x_1') < b_2^S$, we have $\hat{x}_2^{n_{i_j}} < b_2^S$ when j is sufficiently large. So, we have $p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})(1-q) = -\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}((1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}}))$ when j is sufficiently large. On the other hand, we have $p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})q \geq -\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}(q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}}))$ (notice that this is true regardless of whether $\hat{x}_1^{n_{i_j}} = b_1^S$ or $\hat{x}_1^{n_{i_j}} < b_1^S$). So, $\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}(q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})) \geq \frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}((1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}}))\frac{q}{1-q}$ when j is sufficiently large. So, $\frac{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}}{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}} \leq \frac{(1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}})}{q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})}\frac{q}{1-q}$ when j is sufficiently large (observing that $\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2} < 0$ when j is sufficiently large because $(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}}) \in S^I \cap N_{\epsilon_3}(\partial S)$ when j is sufficiently large, according to Condition (v)). We now consider the following two subcases.

The first subcase is that $x_2^*(x_1') = x_2^*(b_1^S) = 0$. In this case, we have $\lim_{j\to\infty} \frac{(1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}})}{q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})} \frac{q}{1-q} = \frac{(1-q)\times 0 + q\times 0}{qb_1^S + (1-q)b_1^S} \frac{q}{1-q} = 0$. On the other hand, we have

$$\lim_{j\to\infty} \frac{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}}{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}} = \lim_{j\to\infty} \left(-\frac{\partial x_2^*(\hat{x}_1^{n_{i_j}}, p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})|p_{n_{i_j}})}{\partial x_1}\right) = -\frac{\partial x_2^*(b_1^S-)}{\partial x_1} > 0.$$

The first equality in the above equation follows from the fact that by differentiating $p_n(x_1, x_2^*(x_1, c|p_n)) = c$ (for $c < 1$) with respect to x_1 we have $\frac{\frac{\partial p_n(x_1, x_2^*(x_1, c|p_n))}{\partial x_1}}{\frac{\partial p_n(x_1, x_2^*(x_1, c|p_n))}{\partial x_2}} = -\frac{\partial x_2^*(x_1, c|p_n)}{\partial x_1}$, which implies that $\frac{\frac{\partial p_n(x_1, x_2)}{\partial x_1}}{\frac{\partial p_n(x_1, x_2)}{\partial x_2}} = -\frac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1}$ by setting $c = p_n(x_1, x_2)$. The second equality in the above equation follows from the facts that $\lim_{j\to\infty} \hat{x}_1^{n_{i_j}} = x_1' = b_1^S$ and $\lim_{j\to\infty} \hat{x}_2^{n_{i_j}} = \lim_{j\to\infty} x_2^*(\hat{x}_1^{n_{i_j}}) = x_2^*(x_1') = x_2^*(b_1^S)$ and the facts that $\{\frac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1}\}_{n=1}^\infty$ is uniformly continuous at any point on $\partial^* S$ and $\frac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1} = \frac{\partial x_2^*(x_1)}{\partial x_1}$ for any $(x_1, x_2) \in \partial^* S$ and any p_n. So, we have shown that $\lim_{j\to\infty} \frac{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}}{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}} > \lim_{j\to\infty} \frac{(1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}})}{q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})} \frac{q}{1-q}$.
This is a contradiction with the fact that $\frac{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}}{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}} \le \frac{(1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}})}{q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})} \frac{q}{1-q}$ for sufficiently large j.

The second subcase is that $x_2^*(x_1') = x_2^*(b_1^S) > 0$. In this case, there must exist a vertical segment from $(b_1^S, 0)$ to $(b_1^S, x_2^*(b_1^S))$ on the Pareto frontier. In addition, we have $\lim_{j\to\infty} \frac{(1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}})}{q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})} \frac{q}{1-q} = \frac{(1-q)x_2^*(b_1^S) + qx_2^*(b_1^S)}{qb_1^S + (1-q)b_1^S} \frac{q}{1-q} = \frac{x_2^*(b_1^S)}{b_1^S} \frac{q}{1-q}$. On the other hand, we have $\lim_{j\to\infty} \frac{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}}{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}} = \lim_{j\to\infty}\left(-\frac{\partial x_2^*(\hat{x}_1^{n_{i_j}}, p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})|p_{n_{i_j}})}{\partial x_1}\right) = -\frac{\partial x_2^*(b_1^S-)}{\partial x_1}$. So, we have $-\frac{\partial x_2^*(b_1^S-)}{\partial x_1} \le \frac{x_2^*(b_1^S)}{b_1^S} \frac{q}{1-q}$. This implies that $(b_1^S, x_2^*(b_1^S))$ must be the generalized Nash solution outcome with bargaining weight q. This is a contradiction with the fact that $x_1' \neq x_1^{N^q}$.

Case 2. $x_1' < b_1^S$ and $x_2^*(x_1') = b_2^S$. This case cannot hold. The analysis is similar to Case 1 and is omitted.

Case 3. $x_1' < b_1^S$ and $x_2^*(x_1') < b_2^S$. The facts that $\lim_{j\to\infty} \hat{x}_1^{n_{i_j}} = x_1' < b_1^S$ and that $\lim_{j\to\infty} \hat{x}_2^{n_{i_j}} = x_2^*(x_1') < b_2^S$ imply that $\hat{x}_1^{n_{i_j}} < b_1^S$ and $\hat{x}_2^{n_{i_j}} < b_2^S$ when j is sufficiently large. So, we have $p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}}) = \frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}(q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}}))\frac{1}{q} = \frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}((1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}}))\frac{1}{1-q}$ when j is sufficiently large. So, $\frac{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}}{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}} = \frac{(1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}})}{q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})} \frac{q}{1-q}$ when j is sufficiently large. However, noticing that $\lim_{j\to\infty} \frac{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_1}}{\frac{\partial p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})}{\partial x_2}} = \lim_{j\to\infty}\left(-\frac{\partial x_2^*(\hat{x}_1^{n_{i_j}}, p_{n_{i_j}}(\hat{x}_1^{n_{i_j}}, \hat{x}_2^{n_{i_j}})|p_{n_{i_j}})}{\partial x_1}\right) = -\frac{\partial x_2^*(x_1')}{\partial x_1}$ and

$\lim_{j\to\infty} \dfrac{(1-q)\hat{x}_2^{n_{i_j}} + qx_2^*(\hat{x}_1^{n_{i_j}})}{q\hat{x}_1^{n_{i_j}} + (1-q)x_1^*(\hat{x}_2^{n_{i_j}})} = \lim_{j\to\infty} \dfrac{(1-q)x_2^*(x_1') + qx_2^*(x_1')}{qx_1' + (1-q)x_1'} = \dfrac{x_2^*(x_1')}{x_1'}\dfrac{q}{1-q}$, we must have $-\dfrac{\partial x_2^*(x_1')}{\partial x_1} = \dfrac{x_2^*(x_1')}{x_1'}\dfrac{q}{1-q}$. This implies that $x_1' = x_1^{N^q}$, which is a contradiction with the assumption that $x_1' \neq x_1^{N^q}$.

So, we have shown that $\lim_{n\to\infty} \hat{x}_1^n = x_1^{N^q}$ must hold. This implies that $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$ must converge to the generalized Nash solution outcome with bargaining weight q.

Proof of Proposition 1

For the first part of Proposition 1, it is sufficient to show that the sequence of the n-th MAP-based probabilities is regular.

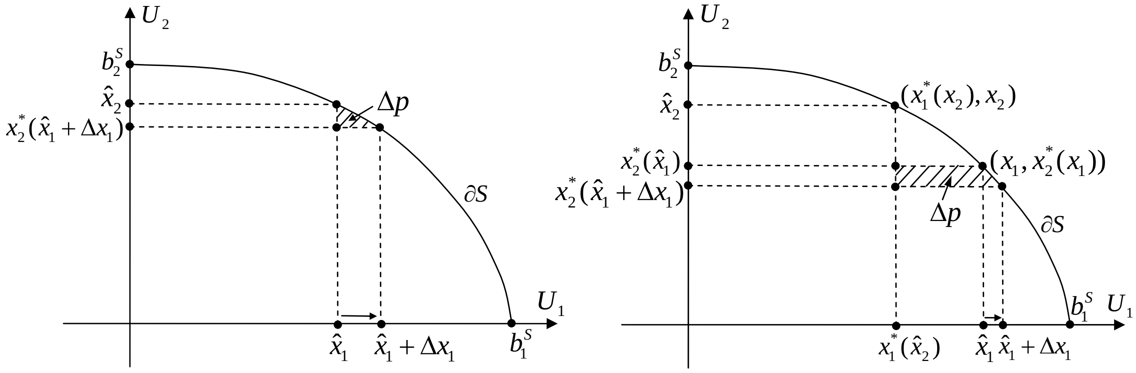

Fig. 6 Impact of an increase in Player 1's demand.

Let $\hat{p}(x) = 1 - \alpha(A_{m(x)})$. So, the n-th MAP-based initiation probability $p_n = \hat{p}^n$. Notice that $\dfrac{\partial \hat{p}^n(x_1,x_2)}{\partial x_i} = n\hat{p}^{n-1}(x_1,x_2)\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_i} = 0$ for any $(x_1,x_2) \in \partial^* S$ with $x_i < b_i^S$ because $\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_i} = 0$ for any $(x_1,x_2) \in \partial^* S$ with $x_i < b_i^S$.[17] So, p_n satisfies Condition (ii) for any n. Condition (iii) is satisfied because as $n \to \infty$, p_n converges to zero at any $(x_1,x_2) \in S^I \backslash N_\epsilon(\partial S)$ for any $\epsilon > 0$ and $\{p_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of functions. For Condition (iv), notice that when the initiation probability is $\hat{p}$, if the two players' demands (x_1,x_2) are incompatible and Player 1 increases his demand by Δx_1, then the initiation probability p will decrease by $\Delta p \approx \frac{1}{2}(\Delta x_1)(|\dfrac{\partial x_2^*(x_1)}{\partial x_1}|\Delta x_1) + (|\dfrac{\partial x_2^*(x_1)}{\partial x_1}|\Delta x_1)(x_1 - x_1^*(x_2))$ (see the right figure of Figure 6). So, $\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1} = \dfrac{\partial x_2^*(x_1)}{\partial x_1}(x_1 - x_1^*(x_2))$. If (x_1,x_2) is close to (b_1^S, b_2^S), then $\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1} \approx \dfrac{\partial x_2^*(x_1)}{\partial x_1}(b_1^S - x_1^*(b_2^S))$. Notice that $\dfrac{\partial p_n(x_1,x_2)}{\partial x_1} = n\hat{p}^{n-1}\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1} < -\dfrac{1}{b_1-\epsilon_2}\hat{p}^n$ if and only if $n\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1} < -\dfrac{1}{b_1-\epsilon_2}\hat{p}$. The latter inequality holds if (x_1,x_2) is in a sufficiently small neighborhood of (b_1^S, b_2^S) because $\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1} \approx \dfrac{\partial x_2^*(x_1)}{\partial x_1}(b_1^S - x_1^*(b_2^S))$ is bounded away from zero when (x_1,x_2) is sufficiently close to (b_1^S,b_2^S) (which implies that $n\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1}$ goes to negative infinity as $n \to \infty$). So, Condition (iv) is satisfied. Condition (v) is satisfied obviously. For Condition (vi), note that

$$\frac{\partial x_2^*(x_1, p_n(x_1,x_2)|p_n)}{\partial x_1} = -\frac{\dfrac{\partial p_n(x_1,x_2)}{\partial x_1}}{\dfrac{\partial p_n(x_1,x_2)}{\partial x_2}}$$

$$= -\frac{n\hat{p}^{n-1}\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1}}{n\hat{p}^{n-1}\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_2}} = -\frac{\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_1}}{\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_2}} = -\frac{\dfrac{\partial x_2^*(x_1)}{\partial x_1}(x_1 - x_1^*(x_2))}{\dfrac{\partial x_1^*(x_2)}{\partial x_2}(x_2 - x_2^*(x_1))}.$$

[17] In order to see the latter point, notice that when Player i increases his demand from a level (say x_i) that is just compatible with his opponent's demand to $x_i + \Delta x_i$, the probability of moving to settlement stage will decrease by an amount that is on the order of $(\Delta x_i)^2$ (see the left figure of Figure 6). So, we have $\dfrac{\partial \hat{p}(x_1,x_2)}{\partial x_i} = 0$ for any $(x_1,x_2) \in \partial^*(S)$ with $x_i < b_i^S$.

As (x_1, x_2) converges to some point on the strong Pareto frontier, the last term in the above equation will converge to the slope of the Pareto frontier at that point (because $\frac{x_1 - x_1^*(x_2)}{x_2 - x_2^*(x_1)}$ will converge to $\frac{\partial x_1^*(x_2)}{\partial x_2}$). That is, $\frac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1}$ is continuous at any point on the strong Pareto frontier. Since $\frac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1}$ is the same for all n, this implies that $\frac{\partial x_2^*(x_1, p_n(x_1, x_2)|p_n)}{\partial x_1}$ is uniformly continuous at any point on the strong Pareto frontier.

We now prove the second part of Proposition 1. The Nash solution outcome (x_1^N, x_2^N) must be such that $(x_1^N, x_2^N) \in \partial^* S$ and $\frac{x_2^N}{x_1^N} = -\frac{\partial x_2^*(x_1^N)}{\partial x_1}$. This implies that if $(x_1', x_2') \in \partial S$ is such that $\frac{x_2'}{x_1'} > -\frac{\partial x_2^*(x_1')}{\partial x_1}$, then (x_1', x_2') must lie on the upper left of the Nash solution outcome on the Pareto frontier or coincide with the Nash solution outcome. If $(x_1', x_2') \in \partial S$ is such that $\frac{x_2'}{x_1'} < -\frac{\partial x_2^*(x_1')}{\partial x_1}$, then (x_1', x_2') must lie on the lower right of the Nash solution outcome on the Pareto frontier or coincide with the Nash solution outcome.

Let $(\hat{x}_1^n, \hat{x}_2^n)$ be a Phase 2 Nash equilibrium of the NDG-RS that uses the n-th MAP-based breakdown probability $1-p_n$, where $p_n = \hat{p}^n$ with $\hat{p}(x) = 1 - \alpha(A_{m(x)})$. If $\hat{x}_1^n = b_1^S$, then it is obvious that $(\hat{x}_1^n, x_2^*(\hat{x}_1^n))$ must lie on the lower right of the Nash solution outcome on the Pareto frontier or coincide with the Nash solution outcome. If $\hat{x}_1^n < b_1^S$, then according to Step 1 of the proof of Theorem 1, we have $\hat{p}^n(\hat{x}_1^n, \hat{x}_2^n) = -\frac{\partial \hat{p}^n(\hat{x}_1^n, \hat{x}_2^n)}{\partial x_1}(\hat{x}_1^n + x_1^*(\hat{x}_2^n)) = -n\hat{p}^{n-1}\frac{\partial \hat{p}(\hat{x}_1^n, \hat{x}_2^n)}{\partial x_1}(\hat{x}_1^n + x_1^*(\hat{x}_2^n)) = -n\hat{p}^{n-1}\frac{\partial x_2^*(\hat{x}_1^n)}{\partial x_1}(\hat{x}_1^n - x_1^*(\hat{x}_2^n))(\hat{x}_1^n + x_1^*(\hat{x}_2^n))$ (where the last equality is because $\frac{\partial \hat{p}(x_1, x_2)}{\partial x_1} = \frac{\partial x_2^*(x_1)}{\partial x_1}(x_1 - x_1^*(x_2))$ if (x_1, x_2) are incompatible; see also the proof for the first part of Proposition 1). For Player 2, we have $\hat{p}^n(\hat{x}_1^n, \hat{x}_2^n) \geq -\frac{\partial \hat{p}^n(\hat{x}_1^n, \hat{x}_2^n)}{\partial x_2}(\hat{x}_2^n + x_2^*(\hat{x}_1^n)) = -n\hat{p}^{n-1}\frac{\partial \hat{p}(\hat{x}_1^n, \hat{x}_2^n)}{\partial x_2}(\hat{x}_2^n + x_2^*(\hat{x}_1^n)) = -n\hat{p}^{n-1}\frac{\partial x_1^*(\hat{x}_2^n)}{\partial x_2}(\hat{x}_2^n - x_2^*(\hat{x}_1^n))(\hat{x}_2^n + x_2^*(\hat{x}_1^n))$, regardless of whether $\hat{x}_2 < b_2^S$ or $\hat{x}_2 = b_2^S$. So, we have $-\frac{\partial x_2^*(\hat{x}_1^n)}{\partial x_1}(\hat{x}_1^n - x_1^*(\hat{x}_2^n))(\hat{x}_1^n + x_1^*(\hat{x}_2^n)) \geq -\frac{\partial x_1^*(\hat{x}_2^n)}{\partial x_2}(\hat{x}_2^n - x_2^*(\hat{x}_1^n))(\hat{x}_2^n + x_2^*(\hat{x}_1^n))$. That is, $\frac{\hat{x}_2^n + x_2^*(\hat{x}_1^n)}{\hat{x}_1^n + x_1^*(\hat{x}_2^n)} \leq \frac{\frac{\partial x_2^*(\hat{x}_1^n)}{\partial x_1}(\frac{\partial x_1^*(\hat{x}_2^n)}{\partial x_2})^{-1}}{\frac{\hat{x}_2^n - x_2^*(\hat{x}_1^n)}{\hat{x}_1^n - x_1^*(\hat{x}_2^n)}}$. Noting that $\frac{\hat{x}_2^n - x_2^*(\hat{x}_1^n)}{\hat{x}_1^n - x_1^*(\hat{x}_2^n)} \geq -(\frac{\partial x_1^*(\hat{x}_2^n)}{\partial x_2})^{-1}$ (which is due to the fact that the two players' demands $\hat{x}_1^n$ and $\hat{x}_2^n$ are incompatible and the fact that the Pareto frontier is concave because S is convex) and $\frac{\hat{x}_2^n + x_2^*(\hat{x}_1^n)}{\hat{x}_1^n + x_1^*(\hat{x}_2^n)} > \frac{x_2^*(\hat{x}_1^n)}{\hat{x}_1^n}$, we must have $\frac{x_2^*(\hat{x}_1^n)}{\hat{x}_1^n} < -\frac{\partial x_2^*(\hat{x}_1^n)}{\partial x_1}$. So, $(\hat{x}_1^n, x_2^*(\hat{x}_1^n))$ must be on the lower right of the Nash solution outcome on the Pareto frontier or coincide with the Nash solution outcome.

Similarly, we can show that Player 2's equilibrium proposal $(x_1^*(\hat{x}_2^n), \hat{x}_2^n)$ must lie on the upper left of the Nash solution outcome on the Pareto frontier or coincide with the Nash solution outcome. □

Proof of Proposition 2

(i) Phase 1 Nash equilibria: Let $(\hat{x}_1, \hat{x}_2)$ be a Phase 1 Nash equilibrium. Then, we must have $(\hat{x}_1, \hat{x}_2) \in S$. In addition, we must have $(\hat{x}_1, \hat{x}_2) \in \partial^* S$ (if not, then $(\hat{x}_1, \hat{x}_2)$ cannot be a Nash equilibrium because one of the two players will have an incentive to deviate to a higher demand, and the two players' proposals are still compatible when the player makes the deviation). We will next show that any $(x_1, x_2) \in \partial^* S$ is a Nash equilibrium. Let $(\hat{x}_1, \hat{x}_2)$ be a given point on $\partial^* S$. Let Player 2's demand $\hat{x}_2$ be given. If Player 1 makes the demand $\hat{x}_1$, then the two players' demands are just compatible and Player 1's payoff is thus $\hat{x}_1$. Suppse Player 1 makes the demand x_1' with $x_1' > \hat{x}_1$; Player 1 is never better off by decreasing his demand, so the only possible deviation for Player 1 is to increase his demand. Then Player 1's payoff is $\frac{\hat{x}_1 + x_1'}{2}p$, where p is the initiation probability when Player 1's demand is x_1' and Player 2's demand is $\hat{x}_2$. Notice that $p = \alpha(B_{m(x_1', \hat{x}_2)}) = \alpha(B(x_1^*(\hat{x}_2), x_2^*(x_1'))) = \alpha(B(\hat{x}_1, x_2^*(x_1'))) = \hat{x}_1 x_2^*(x_1')$, where the third equality follows from the fact that $(\hat{x}_1, \hat{x}_2) \in \partial^* S$. So, $\frac{\hat{x}_1 + x_1'}{2}p = \frac{\hat{x}_1 + x_1'}{2}\hat{x}_1 x_2^*(x_1') = \hat{x}_1 \frac{\hat{x}_1 x_2^*(x_1') + x_1' x_2^*(x_1')}{2} \leq \hat{x}_1 \frac{2\alpha(S)}{2} = \hat{x}_1$ (the inequality is strict if $\hat{x}_1 > 0$). So, Player 1 is better off by making the demand $\hat{x}_1$ (which is just compatible with Player 2's demand) than deviating to any higher demand. Similarly, Player 2 is better off by making the demand $\hat{x}_2$ (which is just compatible with Player 1's demand) than deviating to any higher demand. So, we have proved that any $(x_1, x_2) \in \partial^* S$ must be a Nash equilibrium.

(ii) *Phase 2 Nash equilibria*: Let $(\hat{x}_1, \hat{x}_2)$ be a Phase 2 Nash equilibrium. Then we must have $(\hat{x}_1, \hat{x}_2) \notin S$. This implies that $\hat{x}_1 > x_1^*(\hat{x}_2)$ and $\hat{x}_2 > x_2^*(\hat{x}_1)$.

If Player 2's demand $\hat{x}_2$ is such that $x_1^*(\hat{x}_2) > 0$, then according to the proof of (i), Player 1 is strictly better off by making the demand $x_1^*(\hat{x}_2)$, rather than $\hat{x}_1$. So, $(\hat{x}_1, \hat{x}_2)$ cannot be a Nash equilibrium.

If Player 2's demand $\hat{x}_2$ is such that $x_1^*(\hat{x}_2) = 0$ (which must imply that $\hat{x}_2 = b_2^S$), then for Player 1, any $\hat{x}_1 \in [0, b_1^S]$ is a best response (because the initiation probability p will be zero regardless of Player 1's demand). However, only if $\hat{x}_1$ is such that $x_2^*(\hat{x}_1) = 0$ (which must imply that $\hat{x}_1 = b_1^S$), will $\hat{x}_2 = b_2^S$ be a best responses of Player 2. So, (b_1^S, b_2^S) is the only possible Phase 2 Nash equilibrium, and it is a Phase 2 Nash equilibrium if and only if $x_1^*(b_2^S) = 0$ and $x_2^*(b_1^S) = 0$ (note that, if $x_1^*(b_2^S) > 0$ - which occurs when there is a horizontal segment in the Pareto frontier - or $x_2^*(b_1^S) > 0$ - which occurs when there is a vertical segment in the Pareto frontier - then there is no Phase 2 Nash equilibrium). In addition, there is no other Phase 2 Nash equilibria. □

7 Appendix 2

Proof of Remark 1

Since $\{p_n\}_{n=1}^{\infty}$ is regular, $(\hat{x}_1^n, \hat{x}_2^n)$ can only be a Phase 2 Nash equilibrium for any n (according to Step 1 in the proof of Theorem 1). This implies that $(q\hat{x}_1^n + (1-q)x_1^*(\hat{x}_2^n))p_n(\hat{x}_1^n, \hat{x}_2^n) \geq x_1^*(\hat{x}_2^n)$ and $((1-q)\hat{x}_2^n + qx_2^*(\hat{x}_1^n))p_n(\hat{x}_1^n, \hat{x}_2^n) \geq x_2^*(\hat{x}_1^n)$ for all n (because for Player i, making the demand $\hat{x}_i$ cannot be strictly worse off than making the demand that is just compatible with the opponent's demand). Let $(x_1', x_2') \in \partial^* S$ be the limit point of $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$, i.e., $\lim_{n\to\infty} \hat{x}_1^n = x_1'$ and $\lim_{n\to\infty} \hat{x}_2^n = x_2'$. Since $(x_1', x_2') \in \partial^* S$, we must have either $x_1' > 0$ or $x_2' > 0$. We thus have the following two cases.

(i) $x_1' > 0$. In this case, $(q\hat{x}_1^n + (1-q)x_1^*(\hat{x}_2^n))p_n(\hat{x}_1^n, \hat{x}_2^n) \geq x_1^*(\hat{x}_2^n)$ implies that $p_n(\hat{x}_1^n, \hat{x}_2^n) \geq \dfrac{x_1^*(\hat{x}_2^n)}{q\hat{x}_1^n + (1-q)x_1^*(\hat{x}_2^n)}$ when n is sufficiently large (using the fact that $q\hat{x}_1^n + (1-q)x_1^*(\hat{x}_2^n) > 0$ when n is sufficiently large because $\lim_{n\to\infty} \hat{x}_1^n = x_1' > 0$). We have $\lim_{n\to\infty} \dfrac{x_1^*(\hat{x}_2^n)}{q\hat{x}_1^n + (1-q)x_1^*(\hat{x}_2^n)} = 1$ because $\lim_{n\to\infty} \hat{x}_1^n = x_1'$ and $\lim_{n\to\infty} x_1^*(\hat{x}_2^n) = x_1^*(x_2') = x_1'$ (where the last equality holds because $(x_1', x_2') \in \partial^* S$). In addition, $p_n(\hat{x}_1^n, \hat{x}_2^n) \leq 1$ for all n. So, we have $\lim_{n\to\infty} p_n(\hat{x}_1^n, \hat{x}_2^n) = 1$.

(ii) $x_2' > 0$. The proof is similar to case (i) and is omitted.

We thus have proved that $\lim_{n\to\infty} p_n(\hat{x}_1^n, \hat{x}_2^n) = 1$ if $\{(\hat{x}_1^n, \hat{x}_2^n)\}_{n=1}^{\infty}$ converges to some point on $\partial^* S$ where $(\hat{x}_1^n, \hat{x}_2^n)$ is a Nash equilibrium in the *NDG-RS* that uses the breakdown probability $1 - p_n$ where $\{p_n\}_{n=1}^{\infty}$ is regular and that the probability that Player 1's demand is chosen is $q \in (0,1)$ when the game moves to random settlement.

Euclidean Divergence and Non-convergence

We next consider an intuitively-appealing p function that uses the Euclidean distance between the two players' demands to determine the probability of moving to settlement stage. Even though such a p is continuous at the boundary of S, the resulting NDG-RS fails to have the Nash solution outcome as the limit equilibrium outcome.

When players make their demands, one can convert these demands into proposals by using the Pareto frontier of S. Given two such resulting proposals, a breakdown probability is based on Euclidean divergence if the breakdown probability decreases as the Euclidean distance between the two proposals falls. Noting that the maximum possible distance between any two proposals in S is $\sqrt{(b_1^S)^2 + (b_2^S)^2}$, an Euclidean divergence breakdown probability $1 - p_d$ is given by

$$1 - p_d(x_1, x_2) = \frac{\sqrt{(x_1 - x_1^*(x_2))^2 + (x_2 - x_2^*(x_1))^2}}{\sqrt{(b_1^S)^2 + (b_2^S)^2}}$$

for any $(x_1, x_2) \in S^I$ (and $1 - p_d(x_1, x_2) = 0$ for any $(x_1, x_2) \in \partial S$). It turns out that this otherwise intuitively appealing breakdown probability does not lead to Nash solution convergence. To see this, notice that $(x_1, x_2) \in \partial^* S$ is a Phase 1 Nash equilibrium if and only if no player has an incentive to deviate to some higher demand. This means that for Player 1, if he deviates to some higher demand, say x_1', then his payoff $p_d(x_1', x_2)\dfrac{x_1' + x_1}{2}$, must not be greater than x_1 (the payoff when Player 1 chooses not to deviate). That is,

$$\left(1 - \frac{\sqrt{(x_1' - x_1^*(x_2))^2 + (x_2 - x_2^*(x_1'))^2}}{\sqrt{(b_1^S)^2 + (b_2^S)^2}}\right)\frac{x_1' + x_1}{2} \leq x_1,$$

i.e.,

$$\left(\frac{\sqrt{(x_1' - x_1)^2 + (x_2^*(x_1) - x_2^*(x_1'))^2}}{\sqrt{(b_1^S)^2 + (b_2^S)^2}}\right)\frac{x_1' + x_1}{2} \geq \frac{x_1' - x_1}{2}$$

since $(x_1, x_2) \in \partial^* S$. That is, we have

$$x_1' + x_1 \geq \frac{\sqrt{(b_1^S)^2 + (b_2^S)^2}}{\sqrt{1 + (\dfrac{x_2^*(x_1') - x_2^*(x_1)}{x_1' - x_1})^2}}$$

for any $x_1' > x_1$. Similarly, Player 2 has no incentive to deviate to some higher demand $x_2' > x_2$ if and only if

$$x_2' + x_2 \geq \frac{\sqrt{(b_1^S)^2 + (b_2^S)^2}}{\sqrt{1 + (\dfrac{x_1^*(x_2') - x_1^*(x_2)}{x_2' - x_2})^2}}.$$

Suppose that $b_1^S = b_2^S = 1$ and that $S = \{(x_1, x_2) | 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1$ and $0 \leq x_1 + x_2 \leq k\}$ for some $k \in [1, \frac{3}{2})$. Then, it can be verified that any $(x_1, x_2) \in \partial^* S$ with $\dfrac{1}{2} \leq x_1 \leq 1$ and $\dfrac{1}{2} \leq x_2 \leq 1$ is a Phase 1 Nash equilibrium because if $x_1 \geq \dfrac{1}{2}$ and $x_2 \geq \dfrac{1}{2}$, then for any $x_1' > x_1$, we have

$$x_1' + x_1 > \frac{\sqrt{2}}{\sqrt{1 + \left(\dfrac{x_2^*(x_1') - x_2^*(x_1)}{x_1' - x_1}\right)^2}} = 1$$

and for any $x_2' > x_2$, we have

$$x_2' + x_2 > \frac{\sqrt{2}}{\sqrt{1 + \left(\dfrac{x_1^*(x_2') - x_1^*(x_2)}{x_2' - x_2}\right)^2}} = 1.$$

Let $p_n = p_d^n$. Note that as n increases, p_n decreases for any given incompatible demands (i.e., for any $(x_1, x_2) \in S^I$). This implies that all Phase 1 Nash equilibria when the initiation probability is p_d will remain as Phase 1 Nash equilibria when the initiation probability is p_n for any $n > 1$, because as n increases, a player will have less incentive to deviate from a demand that is compatible with his opponent's demand to some higher demand. So, if we use $\{p_n\}_{n=1}^{\infty}$ as the sequence of probabilities, then the Nash solution convergence result cannot hold.

References

1. Abreu, Dilip and David Pearce (2015), "A dynamic reinterpretation of Nash bargaining with endogenous threats." *Econometrica*, 83, 1641-1655.
2. Anbarci, Nejat and John Boyd III (2011), "Nash demand game and the Kalai-Smorodinsky solution." *Games Economic Behavior*, 71, 14-22.
3. Anbarci, Nejat and Nick Feltovich (2012), "Bargaining with random implementation: an experimental study." *Games Economic Behavior*, 76, 495-514.
4. Ashenfelter, Orley (1987), "Arbitrator behavior." American Economic Review, Papers and Proceedings, 77, 342-346.
5. Aumann, Robert and Mordecai Kurz (1977), "Power and taxes." *Econometrica*, 45, 1137-1161.
6. Binmore, Ken G., Ariel Rubinstein and Asher Wolinsky (1986), "The Nash bargaining solution in economic modelling." *RAND Journal of Economics,* 17, 176-188.
7. Bloom, David E., and Christopher L. Cavanagh (1986), "An analysis of the selection of arbitrators." *American Economic Review*, 76, 408–22.
8. Carlsson, Hans (1991), "A bargaining model where parties make errors." *Econometrica*, 59, 1487-1496.
9. Chatterjee, Kalyan (1981), "Comparison of arbitration procedures: models with complete and incomplete information." *IEEE Transactions on Systems, Man and Cybernetics*, 11, 101-109.
10. Chen, Zhiqi, Hong Ding and Zhiyang Liu (2016), "Downstream competition and the effects of buyer power." *Review of Industrial Organization*, 49, 1-23.
11. Corchón, Luis and Matthias Dahm (2010), "Foundations for contest success functions." *Economic Theory*, 43, 81-98.
12. Crawford, Vincent P. (1979), "On compulsory-arbitration schemes." *Journal of Political Economy*, 87, 131-159.
13. Crawford, Vincent P. (1982), "A theory of disagreement in bargaining." *Econometrica*, 50, 607-637.
14. de las Mercedes Adamuz, Maria and Clara Ponsatí (2009), "Arbitration systems and negotiations." *Review of Economic Design*, 13, 279-303.
15. Dutta, Rohan (2012), "Bargaining with revoking costs." *Games and Economic Behavior*, 74, 144-153.
16. Eisenberg, Theodore and Geoffrey P. Miller (2007), "The flight from arbitration: an empirical study of ex ante arbitration clauses in the contracts of publicly held companies." Cornell Law Faculty Publications. Paper 348. Available at http://scholarship.law.cornell.edu/facpub/348.
17. Ellingsen, Tore and Topi Miettinen (2008), "Commitment and conflict in bilateral bargaining." *American Economic Review*, 98, 1629-1635.

18. Harsanyi, John (1956). “Approaches to the bargaining problem before and after the theory of games: a critical discussion of Zeuthen’s, Hicks’, and Nash’s theories.” *Econometrica*, 24, 144-157.
19. Herings, P. Jean-Jacques and Arkadi Predtetchinski (2010), “One-dimensional bargaining with Markov recognition probabilities.” *Journal of Economic Theory*, 145, 189-215.
20. Howard, John (1992), “A social choice rule and its implementation in perfect equilibrium.” *Journal of Economic Theory*, 56, 142-159.
21. Lagos, Ricardo and Randall Wright (2005), “A unified framework for monetary theory and policy analysis.” *Journal of Political Economy*, 113, 463-484.
22. Laruelle, Annick and Federico Valenciano (2007), “Bargaining in committees as an extension of Nash’s bargaining theory.” *Journal of Economic Theory*, 132, 291-305.
23. Li, Duozhe (2011), “Commitment and compromise in bargaining.” *Journal of Economic Behavior & Organization*, 77, 203-211.
24. Malueg, David (2010), “Mixed-strategy equilibria in the Nash Demand Game.” *Economic Theory*, 44, 243-270.
25. Nash, John F. (1950), “The bargaining problem.” *Econometrica*, 18, 155-162.
26. Nash, John F. (1953), “Two-person cooperative games.” *Econometrica*, 21, 128-140.
27. Osborne, Martin J., and Ariel Rubinstein (1990), *Bargaining and Markets*. Academic Press, London.
28. Rubinstein, Ariel, Zvi Safra and William Thomson (1992), “On the interpretation of the Nash bargaining solution and its extension to non-expected utility preferences.” *Econometrica*, 60, 1171-1186.
29. Stevens, Carl M. (1966), “Is compulsory arbitration compatible with bargaining?” *Industrial Relations*, 5, 38-52.
30. Zeuthen, Frederik (1930), *Problems of Monopoly and Economic Welfare*. Routledge and Kegan Paul, London.