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# SPECTRAL ANALYSIS OF ONE-DIMENSIONAL HIGH-CONTRAST ELLIPTIC PROBLEMS WITH PERIODIC COEFFICIENTS*

K. D. CHEREDNICHENKO†, S. COOPER‡, AND S. GUENNEAU§

**Abstract.** We study the behavior of the spectrum of a family of one-dimensional operators with periodic high-contrast coefficients as the period goes to zero, which may represent, e.g., the elastic or electromagnetic response of a two-component composite medium. Compared to the standard operators with moderate contrast, they exhibit a number of new effects due to the underlying nonuniform ellipticity of the family. The effective behavior of such media in the vanishing period limit also differs notably from that of multidimensional models investigated thus far by other authors, due to the fact that neither component of the composite forms a connected set. We then discuss a modified problem, where the equation coefficient is set to a positive constant on an interval that is independent of the period. Formal asymptotic analysis and numerical tests with finite elements suggest the existence of localized eigenfunctions ("defect modes"), whose eigenvalues are situated in the gaps of the limit spectrum for the unperturbed problem.

**Key words.** elliptic differential equations, homogenization, spectrum

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## 1. Introduction.

### 1.1. The general context for the problem in hand.

The description of the effective behavior of high-contrast composites ("high-contrast homogenization") has been of particular interest in the analysis and applied mathematics communities over the last decade. The analytical part of the related literature starts with the work [18], which developed in detail some earlier ideas of [1] concerning the use of "two-scale convergence" for the analysis of the limit behavior of the boundary-value problem

$$-\mathrm{div}(\mathcal{A}^\varepsilon(x/\varepsilon)\nabla u) = f, \qquad f \in L^2(\Omega), \quad u \in H_0^1(\Omega), \quad \mathcal{A}^\varepsilon = \varepsilon^2\chi_0 I + \chi_1 I, \quad \varepsilon > 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\chi_0$, $\chi_1$ are the indicator functions of $[0,1)^n$-periodic sets in $\mathbb{R}^n$ such that $\chi_0 + \chi_1 = 1$.

Several contributions to the high-contrast homogenization followed—in the linear and nonlinear scalar and vector contexts, with various sets of assumptions about the underlying geometry of the composite. With applications mainly in solid mechanics and electromagnetism, high-contrast media have served as a theoretical ground for a number of effects observed in physics experiments, in particular those related to

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†Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK (K.Cherednichenko@bath.ac.uk).

‡Laboratoire de Mécanique et Génie Civil de Montpellier, 34095, Montpellier, France (shane.cooper@fresnel.fr).

§Aix-Marseille Université, CNRS, Centrale Marseille, Institut Fresnel, UMR 7249, 13013 Marseille, France (sebastien.guenneau@fresnel.fr).

photonic band-gap materials and cloaking metamaterials [15]. The range of techniques developed in these contexts and their applications continue their rapid expansion, and the present paper is aimed at addressing some aspects that have thus far been left out of the scope of the related research.

More specifically, we approach the question of the analysis of the spectral behavior of high-contrast composites when the component represented by the function $\chi_1$ (the "matrix" of the composite) is disconnected in $\mathbb{R}^n$. Clearly, this is always so in one dimension ($n = 1$). In the present article we study this particular case.

**1.2. Problem set-up.** We consider solutions $u$ to the following family of elliptic problems on an interval $(a, b) \subset \mathbb{R}$:

$$A^\varepsilon u - \lambda u = f, \quad f \in L^2(a, b), \quad \varepsilon > 0, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

where the operators $A^\varepsilon$ are given by the closed sesquilinear form

$$(A^\varepsilon u, v) = \int_a^b p(x/\varepsilon)\big(\varepsilon^2\chi_0(x/\varepsilon) + \chi_1(x/\varepsilon)\big)u'(x)\overline{v'(x)}dx, \quad u, v \in \mathfrak{H}. \tag{1.2}$$

Here $p = p(y) > 0$ is a 1-periodic function in $\mathbb{R}$ such that $p, p^{-1} \in L^\infty(0, 1)$, the functions $\chi_0$ and $\chi_1$ are the indicator functions of 1-periodic open sets $F_0$ and $F_1$ such that $\overline{F}_0 \cup \overline{F}_1 = \mathbb{R}$, and $\mathfrak{H}$ denotes a closed linear subspace of $H^1(a, b)$ that contains $C_0^\infty(a, b)$. We make no assumptions regarding boundedness of the interval $(a, b)$; in particular, it may coincide with the whole space $\mathbb{R}$.

In applied contexts the problem (1.1) corresponds to, e.g., the study of wave propagation in a layered two-dimensional or three-dimensional composite structure where $f = 0$, $\lambda > 0$. In what follows we study the spectrum $S^\varepsilon$ of the problem (1.1), i.e., the set of values of $\lambda$ for which $A^\varepsilon - \lambda I$ does not have a bounded inverse in $L^2(a, b)$. Throughout the article we employ the notation $\sigma(A)$ for the spectrum of an operator $A$ and the notation $Q$ for the "unit cell" $[0, 1)$ whenever we describe the behavior with respect to the "physical" variables $x, y$. We continue writing $[0, 1)$ for the "Floquet–Bloch dual" cell when we refer to the domain of the quasimomentum $\theta$.

**1.3. Our strategy for the analysis of (1.1).** It has been well understood in the existing literature on the subject (see [2], [18], [20]) that in the analysis of convergence of spectra of families of differential operators with periodic rapidly oscillating coefficients one has to deal with two distinct issues: the lower semicontinuity of the spectra in the sense of Hausdorff convergence of sets and the possibility of spectral pollution, the lack of which is often referred to as "spectral completeness." The former issue, which in the wider spectral analysis context has been looked at from a more general perspective (see, e.g., [4]), is usually dealt with by first proving a variant of the strong resolvent convergence. In the case of periodic operators involving multiple scales, one typically makes use of two-scale convergence (see, e.g., [14], [1], [18]). In the present paper we follow this general approach in proving the related lower semicontinuity statements both for the whole-space problem and for the problem in a bounded interval. This part of the analysis of spectral convergence does not eliminate the need for a study of spectral completeness: unless some assumptions are made concerning the geometry of the periodic composite in question (see, e.g., [18]), one may not get the best possible "lower bound" for the limit spectrum. It has been noticed that in order to capture the behavior with respect to all Bloch components in the limit as $\varepsilon \to 0$, it is preferable to use an advanced, "multicell," version of the standard two-scale convergence; see, e.g., [2], [6, Chapter 5], where this more refined approach is adopted. It is a version of this last, more detailed, procedure that we develop in the present article.

In the proof of spectral completeness, a natural strategy seems to be to analyze the relative strength of different Bloch components in a given (convergent) sequence of eigenfunctions. This idea has been elaborated in [2] in the specific context of "high-frequency" homogenization, with the use of what the authors refer to as the "Bloch measures." A combination of a compactness argument in the related space of measures and a special "slow-variable modulation" construction then yields the simultaneous convergence of the given sequence to a limit eigenfunction and of the associated eigenvalues. In the present work we deal with a situation where such a compactness argument does not suffice, since the limit of an eigenfunction sequence may have a nontrivial part in the orthogonal complement (in the two-scale version of the $L^2$ space) to functions that are constant in the matrix component. (In the set-up studied in [2] this orthogonal complement is zero.) Once we have suitable control of this part, we can prove spectral convergence for some operator families not amenable to the approach of [2], including those considered in [17], [18], and [6, Chapter 4].

The key element in our analysis, which allows us to implement the above idea, is Lemma 3.2 below (see section 3.1). This statement establishes a uniform version of the Poincaré-type inequality between the projection of a given function onto the "poorly behaving" subspace of $H^1$ and the $L^2$-norm of its derivative on the part of the domain where solutions of the eigenvalue problem can be shown to be a priori small as $\varepsilon \to 0$. Different versions of the same idea have appeared in a number of other contexts, serving a similar purpose of somehow compensating for the apparent loss of compactness in the problem, for example, in the form of the Korn inequality in elasticity (see, e.g., [7] and also [19] for its multiscale versions), in the form of the energy method in classical homogenization (see [13]), and, more recently, in the form of a generalized Weyl decomposition for problems with degeneracies (see [11]). For nonlinear variants of the same idea, the reader is referred to the geometric rigidity (see [8]) and $\mathcal{A}$-quasiconvexity (see [9]).

For an easier introduction to the problem, in what follows we start with the analysis of the problem (1.1) in the whole-space case, $(a, b) = \mathbb{R}$; see section 2. While a version of the compactness argument developed in the bounded-interval setting (see section 3) applies here as well (once complemented by a suitable Weyl-sequence argument), we present a different argument, based on some ideas of [6, Chapter 5], where the spectral analysis is carried out in a more challenging setting of the Maxwell system.

Throughout this paper we assume for simplicity that the restriction of $\chi_0$ to the periodicity cell $[0, 1)$ is the indicator function of an open interval $(\alpha, \beta)$, which we also denote by $Q_0$. We use the notation $Q_1$ for the interior of the complement of $Q_0$ to the interval $(0, 1)$.

**2. Limit analysis for the whole space: $(a, b) = \mathbb{R}$.** Let us consider for a moment the case $p \equiv 1$. One well-known procedure for calculating $S^\varepsilon$, the spectrum of the problem (1.1), is the Floquet–Bloch decomposition (see, e.g., [3]) following the rescaling $y = x/\varepsilon$. For $\theta \in [0, 1)$, let $\lambda = \lambda(\theta)$ be the eigenvalues corresponding to $\theta$-quasiperiodic eigenfunctions of the differential expression $((\chi_0 + \varepsilon^{-2}\chi_1)u')'$ on the interval $(0, 1)$. Such eigenvalues are obtained by solving the dispersion equation

$$\frac{1}{2}\left(\frac{1}{\varepsilon} + \varepsilon\right) \sin\left(\varepsilon\sqrt{\lambda}(\alpha - \beta + 1)\right) \sin\left(\sqrt{\lambda}(\alpha - \beta)\right)$$

$$+\cos\Big(\varepsilon\sqrt{\lambda}(\alpha-\beta+1)\Big)\cos\Big(\sqrt{\lambda}(\alpha-\beta)\Big)=\cos(2\pi\theta).$$

Passing to the limit in the above equation as $\varepsilon\to 0$ yields

$$\frac{1}{2}(\alpha-\beta+1)\sqrt{\lambda}\sin\Big(\sqrt{\lambda}(\alpha-\beta)\Big)+\cos\Big(\sqrt{\lambda}(\alpha-\beta)\Big)=\cos(2\pi\theta). \tag{2.1}$$

By varying $\theta$ as indicated, we obtain (for $\alpha=1/4$, $\beta=3/4$) the set shown in Figure 1.

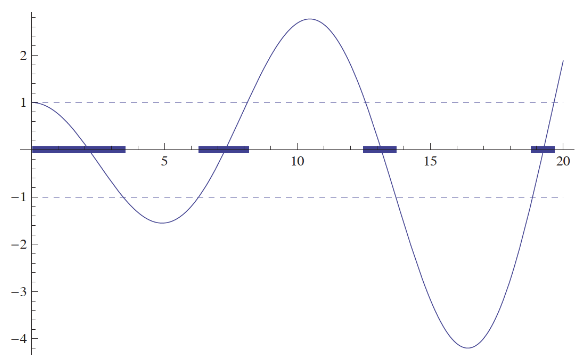


FIG. 1. *The square root of the limit Bloch spectrum for $p\equiv 1$. The oscillating solid line is the graph of the function $f(t)=\cos(t/2)-t\sin(t/2)/4$, where $t$ represents $\sqrt{\lambda}$ in the formula* (2.1) *with $\alpha=1/4$, $\beta=3/4$. The square root of the spectrum is the union of the intervals, indicated by bold lines, that correspond to $t\in\mathbb{R}^+$ such that $-1\le f(t)\le 1$ (the so-called pass bands).*

The following statement is a particular case of our main result, Theorem 2.2 below.

PROPOSITION 2.1. *Let $(a,b)=\mathbb{R}$, $p\equiv 1$. Then the set* $\lim_{\varepsilon\to 0}S^\varepsilon$ *is given by the union of solution sets for* (2.1) *for all $\theta\in[0,1)$.*

In what follows we consider the case of an arbitrary $p$ while adopting the above Floquet–Bloch approach to the description of the set $\lim_{\varepsilon\to 0}S^\varepsilon$ of limit points of $S^\varepsilon$ as $\varepsilon\to 0$. For $\theta\in[0,1)$, we denote by $H^1_\theta(Q)$ the space of functions $u\in H^1(Q)$ that are $\theta$-quasiperiodic, i.e., such that $v(y)=\exp(2\pi\mathrm{i}\theta y)u(y)$, $y\in Q$, for some[1] $u\in H^1_\#(Q)$. We also denote

$$V(\theta):=\left\{v\in H^1_\theta(Q): v'(y)=0 \text{ for } y\in Q_1\right\}. \tag{2.2}$$

Note that $V(\theta)$ is a closed subspace of $H^1(Q)$ and is therefore a Hilbert space when inheriting the standard $H^1$-norm. The sesquilinear form $\mathcal{A}_\theta : V(\theta)\times V(\theta)\to\mathbb{C}$ defined by

$$\mathcal{A}_\theta(u,v):=\int_{Q_0} pu'\overline{v'}$$

[1] As the notation $H^1_0(Q)$ is usually reserved for the space of $H^1(Q)$ functions vanishing on the boundary of $Q$, we denote by $H^1_\#(Q)$ the space $H^1_\theta(Q)$ when $\theta=0$.

is clearly closed and nonnegative on $V(\theta)$. Therefore, it defines a self-adjoint operator $A(\theta)$ (see, e.g., [12]) such that

$$\big(A(\theta)u, v\big) = \mathcal{A}_\theta(u, v) \quad \forall\, v \in V(\theta), \quad u \in \mathrm{dom}(A(\theta)) \subset V(\theta), \tag{2.3}$$

where the domain of $A(\theta)$, denoted by $\mathrm{dom}(A(\theta))$, is a dense subset of $V(\theta)$ with respect to the $L^2$-norm. Henceforth $(\cdot,\cdot)$ denotes the usual inner product in $L^2(Q)$. Under the adopted conditions on the coefficient $p$, the operator $A(\theta)$ is self-adjoint and has a compact inverse (except for the case $\theta = 0$, when it has a compact inverse as an operator on $V(\theta) \ominus \mathbb{C}$). Therefore, the spectrum $\sigma(A(\theta))$ is discrete and unbounded; i.e., it consists of eigenvalues $0 \le \lambda_1(\theta) \le \lambda_2(\theta) \le \dots$ of finite multiplicity with eigenfunctions $v^k(\theta) = v^k(\theta, y)$. The eigenfunctions corresponding to different eigenvalues are automatically orthogonal in $L^2(Q)$. We also carry out the orthogonalization process on those eigenfunctions that correspond to the same eigenvalue, and we normalize each eigenfunction so that $\|v^k(\theta)\|_{L^2(Q)} = 1$ for all $\theta \in [0, 1)$, $k \in \mathbb{N}$.

Our aim within this section is to show that the set $\lim_{\varepsilon\to 0} S^\varepsilon$ coincides with the union of the spectra of the operators $A(\theta)$, $\theta \in [0, 1)$, i.e.,

$$\lim_{\varepsilon\to 0} S^\varepsilon = \bigcup_{\theta\in[0,1)} \sigma\big(A(\theta)\big). \tag{2.4}$$

More precisely, we establish the following theorem.

THEOREM 2.2.

1. *For a given $\theta \in [0, 1)$ let $\lambda \in \sigma(A(\theta))$. There exist $\lambda_\varepsilon \in S^\varepsilon$ such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$.*
2. *Let $\lambda_\varepsilon \in S^\varepsilon$ be such that $\lambda_\varepsilon \to \lambda \in \mathbb{R}$. Then there exist $\theta \in [0, 1)$ and $u \in V(\theta) \setminus \{0\}$ such that*

$$\int_\alpha^\beta p u' \overline{\varphi'} = \lambda \int_0^1 u \overline{\varphi} \qquad \forall \varphi \in V(\theta). \tag{2.5}$$

In order to demonstrate property 1 we use an appropriate modification of the strong two-scale resolvent convergence, introduced in [18]. By showing, for each $N \in \mathbb{N}$, that a subsequence of $A^\varepsilon$ strongly two-scale resolvent converges to an "intermediate" operator $A_N$ on the space of $L^2_{\mathrm{loc}}$-functions that are $NQ$-periodic, we establish property 1 for $\theta = j/N$, $0 \le j \le N-1$. The details of this argument, which rely on a procedure that we refer to as "$NQ$-periodic homogenization," are given in Appendix A.

An essential ingredient in extending property 1 to hold for $\theta \in [0, 1)$ and in proving Property 2 is the following continuity property of the set $V(\theta)$ with respect to $\theta$.

LEMMA 2.3. *The family $V(\theta)$ is continuous in $\theta \in [0, 1)$ in the following sense. For fixed $\theta \in [0, 1)$, let $\theta_\varepsilon \in [0, 1)$ be such that $\theta_\varepsilon \to \theta$ as $\varepsilon \to 0$. Then for any $\varphi \in V(\theta)$ there exist $\varphi_\varepsilon \in V(\theta_\varepsilon)$ such that $\varphi_\varepsilon \to \varphi$ strongly in $H^1(Q)$ as $\varepsilon \to 0$.*

*Proof.* For $\theta \in [0, 1)$ the space $V(\theta)$ consists of functions that are $\theta$-quasiperiodic and constant in each connected component of $Q_1$; that is, for any $\varphi \in V(\theta)$ one has $\varphi(y) = \eta(\theta, y)c + v(y)$, $y \in Q$, where $c \in \mathbb{C}$, $v \in H^1_0(\alpha, \beta)$, and

$$\eta(\theta, y) := \begin{cases} 1, & y \in [0, \alpha), \\ (\exp(2\pi \mathrm{i}\theta) - 1)\,(\beta - \alpha)^{-1}(y - \alpha) + 1, & y \in [\alpha, \beta], \\ \exp(2\pi \mathrm{i}\theta), & y \in (\beta, 1). \end{cases}$$

For each value of $\varepsilon$ we now define $\varphi_\varepsilon$ by the formula $\varphi_\varepsilon(y) = \eta(\theta_\varepsilon, y)c + v(y)$, $y \in Q$. Notice that by construction $\varphi_\varepsilon \in V(\theta_\varepsilon)$, and, since $\eta$ is uniformly continuous with respect to $\theta$, one has $\varphi_\varepsilon \to \varphi$ strongly in $H^1(Q)$. ∎

Next we prove Theorem 2.2.

*Proof. Property* 1: We shall outline the proof here and refer the reader to Appendix A for the full details. Note that property 1 holds for rational $\theta$, the set $\lim_{\varepsilon\to 0} S^\varepsilon$ is closed, and the rationals are dense in $[0,1)$. Therefore, it suffices to show that the eigenvalues $\lambda(\theta)$ of $A(\theta)$ are continuous with respect to $\theta$. Indeed, in Appendix B we show this to follow from Lemma 2.3, i.e., from the continuity of $V(\theta)$.

*Property* 2: For each $\varepsilon > 0$, since $\lambda_\varepsilon \in S^\varepsilon$, by the Floquet–Bloch decomposition there exists $u_\varepsilon \in H^1_{\theta_\varepsilon}(\varepsilon Q) \setminus \{0\}$ such that

$$\int_{\varepsilon Q} p(x/\varepsilon)\big(\varepsilon^2\chi_0(x/\varepsilon) + \chi_1(x/\varepsilon)\big)u_\varepsilon'(x)\overline{\varphi'(x)}\mathrm{d}x = \lambda_\varepsilon \int_{\varepsilon Q} u_\varepsilon(x)\overline{\varphi(x)}\mathrm{d}x \tag{2.6}$$

for all $\varphi \in H^1_{\theta_\varepsilon}(\varepsilon Q)$. Rescaling the formulation (2.6) with $y = x/\varepsilon$ yields the existence of $v_\varepsilon \in H^1_{\theta_\varepsilon}(Q)$, $\|v_\varepsilon\|_{L^2(Q)} = 1$, such that

$$\varepsilon^{-2}\int_{Q_1} p(y)v_\varepsilon'(y)\overline{\varphi'(y)}\mathrm{d}y + \int_{Q_0} p(y)v_\varepsilon'(y)\overline{\varphi'(y)}\mathrm{d}y = \lambda_\varepsilon \int_Q v_\varepsilon(y)\overline{\varphi(y)}\mathrm{d}y \tag{2.7}$$

for all $\varphi \in H^1_{\theta_\varepsilon}(Q)$.

The sequence $\theta_\varepsilon$ is bounded, and therefore there exists some $\theta \in [0,1]$ such that, up to a subsequence which we do not relabel, $\theta_\varepsilon \to \theta$. Without loss of generality, if $\theta = 1$ we set $\theta = 0$ so that $\theta \in [0,1)$. By substituting $\varphi = v_\varepsilon$ into (2.7), the sequence $v_\varepsilon$ satisfies the bounds

$$\big\|\chi_1 v_\varepsilon'\big\|_{L^2(Q)} \le C\varepsilon, \qquad \big\|\chi_0 v_\varepsilon'\big\|_{L^2(Q)} \le C, \tag{2.8}$$

with a constant $C > 0$ independent of $\varepsilon$.

Due to the weak compactness of bounded sets in $H^1(Q)$, the bounds (2.8), along with $\|v_\varepsilon\|_{L^2(Q)} = 1$, imply that, up to extracting a subsequence, $v_\varepsilon$ converges weakly in $H^1(Q)$, and therefore strongly in $L^2(Q)$, to some $v_0 \in H^1(Q)$, $\|v_0\|_{L^2(Q)} = 1$. Clearly, for $w_\varepsilon(y) := \exp(-2\pi\mathrm{i}\theta_\varepsilon y)v_\varepsilon(y)$, $y \in Q$, one has $w_\varepsilon \in H^1_\#(Q)$, and the uniform convergence of $\exp(2\pi\mathrm{i}\theta_\varepsilon\cdot)$ to $\exp(2\pi\mathrm{i}\theta\cdot)$ as $\varepsilon \to 0$ implies that $w_\varepsilon$ converges weakly in $H^1_\#(Q)$ to $w_0$ given by the formula $w_0(y) = \exp(-2\pi\mathrm{i}\theta y)v_0(y)$, $y \in Q$, such that $v_0 \in H^1_\theta(Q)$. Furthermore, (2.8) implies that $\chi_1 v_\varepsilon' \to 0$ strongly in $L^2(Q)$; hence $v_0 \in V(\theta)$.

In order to show that $v_0$ satisfies the limit identity (2.5), for a fixed $\varphi_0 \in V(\theta)$ let $\varphi_\varepsilon \in V(\theta_\varepsilon)$ be given by Lemma 2.3. Substituting $\varphi_\varepsilon$ into (2.7), we obtain

$$\int_\alpha^\beta p v_\varepsilon'\overline{\varphi_\varepsilon'} = \lambda_\varepsilon \int_0^1 v_\varepsilon\overline{\varphi_\varepsilon}. \tag{2.9}$$

By virtue of the facts that $\varphi_\varepsilon \to \varphi_0$ strongly in $H^1(Q)$ and $v_\varepsilon \rightharpoonup v_0$ weakly in $H^1(Q)$, passing to the limit $\varepsilon \to 0$ in (2.9) immediately implies (2.5) for $u = v_0$. ∎

*Remark* 2.1. The above "limit spectrum" $\lim_{\varepsilon\to 0} S^\varepsilon$ is strictly larger than the set obtained by the two-scale analysis of the operator $A^\varepsilon$ of the paper [18]. In particular, the spectrum of the one-dimensional version of the homogenized operator obtained in [18] coincides with $\{\lambda_k(0)\}_{k=1}^\infty$, using our notation.

Our analysis above shows that the set $\lim_{\varepsilon\to 0} S^\varepsilon$ has, in fact, a band-gap structure, with infinitely many gaps opening in the interval $[0,\infty)$, as $\varepsilon \to 0$. This fact suggests possible applications of the above composite structures to the design of optical or acoustic band-gap materials, which we discuss in section 5. The above effect also raises a mathematical question regarding the analysis of the limit behavior of the operators $A^\varepsilon$ in the case when $(a,b)$ is a bounded interval, which we study in the next section.

**3. Spectral behavior on a bounded interval.** It is known that the classical, "moderate-contrast," analogue of the problem (1.1)–(1.2) leads to limit spectra of different kinds for problems on bounded and unbounded intervals $(a,b)$: the limit set in the case of the problem in the whole space is purely absolutely continuous, while in the case $-\infty < a < b < \infty$ it is purely discrete; i.e., it consists of eigenvalues with finite multiplicities; see, e.g., [3]. A similar situation occurs in multidimensional high-contrast problems where the inclusion $F_0 \cap Q$ has a nonzero distance to the boundary of $Q$; see [18], where, in addition, some eigenvalues of infinite multiplicity are present. As we shall see below, this is not the case for the problem (1.1)–(1.2), when the spectrum of the operator $A^\varepsilon$ defined by (1.2) contains the right-hand side of (2.4), i.e., the spectrum of the operator defined by the form (1.2) with $(a,b)$ replaced by $\mathbb{R}$ and $\mathfrak{H}$ replaced by $H^1(\mathbb{R})$. This makes the limit spectrum in question considerably richer than that described in [18].

In this section we employ, for convenience, the following notation: $\Omega := (a,b)$, $\Omega_1^\varepsilon := \Omega \cap (\varepsilon F_1)$, $\Omega_0^\varepsilon := \Omega \cap (\varepsilon F_0)$, $\widetilde{\Omega}^\varepsilon := \varepsilon^{-1}\Omega$, $\widetilde{\Omega}_1^\varepsilon := \varepsilon^{-1}\Omega_1^\varepsilon$, and $\widetilde{\Omega}_0^\varepsilon := \varepsilon^{-1}\Omega_0^\varepsilon$.

**3.1. The main convergence result.** The following theorem holds.

THEOREM 3.1. *Consider an operator $A^\varepsilon$ from the class described in section* 1.2. *The limit set $\lim_{\varepsilon\to 0} S^\varepsilon$ is given by the union $\sigma_{\rm Bloch} \cup \sigma_{\rm boundary}$, where*

$$\sigma_{\rm Bloch} := \bigcup_{\theta\in[0,1)} \sigma\bigl(A(\theta)\bigr)$$

*(cf.* (2.4)*), and*

$$\sigma_{\rm boundary} := \Bigl\{\lambda \in \mathbb{R} : \lambda = \lim_{\varepsilon\to 0} \lambda_\varepsilon,\ A^\varepsilon u_\varepsilon = \lambda_\varepsilon u_\varepsilon,\ u_\varepsilon \in \mathfrak{H},\ \|u_\varepsilon\|_{L^2(a,b)} = 1,$$

$$\|u_\varepsilon\|_{L^2(a+\delta_\varepsilon, b-\delta_\varepsilon)} \to 0 \text{ for some } \delta_\varepsilon \to 0 \text{ as } \varepsilon \to 0\Bigr\}. \tag{3.1}$$

An essential element to proving Theorem 3.1 is validating the following statement.

LEMMA 3.2. *There exists a constant $C > 0$, which depends on $\alpha$ and $\beta$ only, such that for all $\theta \in [0,1)$ one has*

$$\|w\|_{H^1(Q)} \le C\|w'\|_{L^2(Q_1)} \qquad \forall w \in V^\perp(\theta). \tag{3.2}$$

*Here $V^\perp(\theta)$ denotes the orthogonal complement of $V(\theta)$ (see* (2.2)*) in the space $H^1_\theta(Q)$.*

*Proof.* It suffices to show that the statement of the lemma holds for the equivalent $H^1$-norm

$$|||u||| := \left(\left|\int_{Q_1} u\right|^2 + \|u'\|^2_{L^2(Q)}\right)^{1/2}.$$

Indeed, assuming that the statement of the lemma holds for the norm $|||\cdot|||$, with a constant $C = \widetilde{C}$ in the analogue of (3.2) for $|||\cdot|||$, and $\hat{c} := \sup_{|||w|||=1} \|w\|_{H^1(Q)}$, then for all $w \in H^1_\theta(Q)$ we obtain

$$\inf_{v\in V(\theta)} \|w - v\|_{H^1(Q)} \le \hat{c} \inf_{v\in V(\theta)} |||w - v||| \le \hat{c}\widetilde{C}\|w'\|_{L^2(Q_1)},$$

e.g., the claim of the lemma with $C = \hat{c}\widetilde{C}$. In what follows we keep the notation $V^\perp(\theta)$ for the orthogonal complement to $V(\theta)$ with respect to the inner product induced by $|||\cdot|||$. We first note some properties of functions that belong to the space $V^\perp(\theta)$, which follow immediately from the characterization of the space $V(\theta)$ given in the proof of Lemma 2.3.

PROPOSITION 3.3. *Let $w \in V^\perp(\theta)$; then the following hold.*

(i) *The equation $w''(y) = 0$ holds for $y \in Q_0$. In particular, the function $w$ is linear on the $Q_0$-component of the unit cell:*

$$w(y) = \big(w(\beta) - w(\alpha)\big)(\beta - \alpha)^{-1}(y - \alpha) + w(\alpha), \quad y \in Q_0.$$

(ii) *The formula*

$$\int_{Q_1} w = \frac{\big(w(\beta) - w(\alpha)\big)\big(1 - \exp(-2\pi \mathrm{i}\theta)\big)}{(\beta - \alpha)\big(\alpha + (1 - \beta)\exp(-2\pi \mathrm{i}\theta)\big)}$$

*holds.*

We now return to the proof of Lemma 3.2. We consider three different cases, depending on the location of the quasimomentum $\theta$ within the Floquet–Bloch cell $[0, 1)$.

Case I: $\theta = 0$. Using Proposition 3.3, we find that

$$|||w|||^2 = \int_{Q_1} |w'|^2 + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha}.$$

Since $w(1) = w(0)$, we obtain the estimate

$$\begin{aligned} |||w|||^2 &\le \|w'\|^2_{L^2(Q_1)} + \frac{2}{\beta - \alpha}\left(|w(\beta) - w(1)|^2 + |w(0) - w(\alpha)|^2\right) \\ &= \|w'\|^2_{L^2(Q_1)} + \frac{2}{\beta - \alpha}\left(\left|\int_\beta^1 w'\right|^2 + \left|\int_0^\alpha w'\right|^2\right) \\ &\le \left(1 + \frac{2|Q_1|}{\beta - \alpha}\right)\|w'\|^2_{L^2(Q_1)}. \end{aligned}$$

Case II: $\theta \in (0, \delta) \cup (1 - \delta, 1)$, where $0 < \delta < 1/2$ is to be chosen appropriately. Again, Proposition 3.3 implies that

$$\begin{aligned} |||w|||^2 &= \left|\int_{Q_1} w\right|^2 + \|w'\|^2_{L^2(Q_1)} + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha} \\ &= \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2}\right)|w(\beta) - w(\alpha)|^2 + \|w'\|^2_{L^2(Q_1)}, \end{aligned}$$

where $d_\theta := (\beta-\alpha)(1-\exp(-2\pi i\theta))^{-1}(\alpha+(1-\beta)\exp(-2\pi i\theta))$. From the fact that $w(1)=\exp(2\pi i\theta)w(0)$ we infer

$$\begin{aligned}|w(\beta)-w(\alpha)|^2 &\le 3\left(|w(1)-w(\beta)|^2+|w(\alpha)-w(0)|^2+|w(1)-w(0)|^2\right)\\ &\le 3\,|Q_1|\,\|w'\|^2_{L^2(Q_1)}+3\,|\exp(2\pi i\theta)-1|^2\,|w(0)|^2,\end{aligned}$$

and therefore

$$\begin{aligned}|||w|||^2 \le &\left(1+3|Q_1|\left(\frac{1}{\beta-\alpha}+\frac{1}{|d_\theta|^2}\right)\right)\|w'\|^2_{L^2(Q_1)}\\ &+3\left(\frac{1}{\beta-\alpha}+\frac{1}{|d_\theta|^2}\right)|\exp(2\pi i\theta)-1|^2\,|w(0)|^2.\end{aligned}$$

Notice that $|d_\theta|^2=(\beta-\alpha)^2(2-2\cos(2\pi\theta))^{-1}(\alpha^2+(1-\beta)^2+2\alpha(1-\beta)\cos(2\pi\theta))$; hence $|d_\theta|$ vanishes at $\theta=1/2$ for the special case $\alpha=1-\beta$. In view of this observation and in order to have a bound below on $|d_\theta|$, we require that $\delta<1/4$. Further, by continuity of the embedding of $H^1(Q)$ in $C(\overline{Q})$, there exists a constant $\hat{c}$, which is independent of $\theta$, such that

$$\big|w(0)\big| \le \hat{c}|||w|||,$$

and thus

$$\begin{aligned}|||w|||^2 \le &\left(1+3|Q_1|\left(\frac{1}{\beta-\alpha}+\frac{1}{|d_\theta|^2}\right)\right)\|w'\|^2_{L^2(Q_1)}\\ &+3\left(\frac{1}{\beta-\alpha}+\frac{1}{|d_\theta|^2}\right)|\exp(2\pi i\theta)-1|^2\,\hat{c}^2|||w|||^2. \qquad (3.3)\end{aligned}$$

We now choose $\delta<1/4$ so that $3\left((\beta-\alpha)^{-1}+|d_\theta|^{-2}\right)|\exp(2\pi i\theta)-1|^2\,\hat{c}^2<1/2$, and hence $|d_\theta|^{-2}$ is bounded above by a constant independent of $\theta$ in the intervals considered. The inequality (3.3) now immediately implies the required estimate.

Case III: $\theta\in[\delta,1-\delta]$. For given $x\in(\beta,1], y\in[0,\alpha)$ the identities

$$w(x)=\int_\beta^x w'+w(\beta), \qquad w(y)=-\int_y^\alpha w'+w(\alpha)$$

imply, in view of Proposition 3.3(ii),

$$w(x)-w(y)=w(\beta)-w(\alpha)+\left(\int_\beta^x+\int_y^\alpha\right)w'=d_\theta\int_{Q_1}w+\left(\int_\beta^x+\int_y^\alpha\right)w'.$$

In particular, substituting $x=1$, $y=0$ and using the fact that $w(1)=\exp(2\pi i\theta)w(0)$, we obtain

$$w(1)=\frac{d_\theta}{1-\exp(-2\pi i\theta)}\int_{Q_1}w+\frac{1}{1-\exp(-2\pi i\theta)}\int_{Q_1}w',$$

whence

$$w(x)=-\int_x^1 w'+w(1)=-\int_x^1 w'+\frac{1}{1-\exp(-2\pi i\theta)}\left(d_\theta\int_{Q_1}w+\int_{Q_1}w'\right).$$

Integrating the last identity over $(\beta, 1]$ yields

$$\int_\beta^1 w = -\int_\beta^1 \left(\int_x^1 w'\right) \mathrm{d}x + \frac{1-\beta}{1-\exp(-2\pi \mathrm{i}\theta)}\left(d_\theta \int_{Q_1} w + \int_{Q_1} w'\right). \tag{3.4}$$

Similarly, we write

$$w(y) = \int_0^y w' + \frac{\exp(-2\pi \mathrm{i}\theta)}{1-\exp(-2\pi \mathrm{i}\theta)}\left(d_\theta \int_{Q_1} w + \int_{Q_1} w'\right),$$

which upon integration over $(0, \alpha)$ yields

$$\int_0^\alpha w = \int_0^\alpha \left(\int_0^y w'\right) \mathrm{d}y + \frac{\alpha \exp(-2\pi \mathrm{i}\theta)}{1-\exp(-2\pi \mathrm{i}\theta)}\left(d_\theta \int_{Q_1} w + \int_{Q_1} w'\right). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\left(1 - \frac{(1-\beta+\alpha\exp(-2\pi \mathrm{i}\theta))\, d_\theta}{1-\exp(-2\pi \mathrm{i}\theta)}\right)\int_{Q_1} w = \int_0^\alpha \left(\int_0^y w'\right) \mathrm{d}y$$

$$-\int_\beta^1 \left(\int_x^1 w'\right) \mathrm{d}x + \frac{(1-\beta+\alpha\exp(-2\pi \mathrm{i}\theta))}{1-\exp(-2\pi \mathrm{i}\theta)}\int_{Q_1} w'.$$

Squaring both sides and using the Cauchy–Schwarz inequality yields

$$\left|1 - \frac{(1-\beta+\alpha\exp(-2\pi \mathrm{i}\theta))\, d_\theta}{1-\exp(-2\pi \mathrm{i}\theta)}\right|^2 \left|\int_{Q_1} w\right|^2$$

$$\le 2\left(4 + \frac{\left|1-\beta+\alpha\exp(-2\pi \mathrm{i}\theta)\right|^2}{|1-\exp(-2\pi \mathrm{i}\theta)|^2}\right)\|w'\|^2_{L^2(Q_1)}.$$

A direct calculation shows that the coefficient on the left-hand side of the last inequality is separated from zero in the range of $\theta$ considered. Similarly, the coefficient on the right-hand side is bounded.

Finally, we argue that

$$|w(\beta) - w(\alpha)| \le |w(\beta) - w(1)| + |w(0) - w(\alpha)| + |w(0)| + |w(1)|$$

$$= \left|\int_\beta^1 w'\right| + \left|\int_0^\alpha w'\right| + |w(0)| + |w(1)|,$$

and

$$(1-\beta)w(1) = \int_\beta^1 \left(\int_x^1 w'\right)\mathrm{d}x + \int_\beta^1 w, \qquad \alpha w(0) = -\int_0^\alpha \left(\int_0^x w'\right)\mathrm{d}x + \int_0^\alpha w.$$

The required inequality follows since by Proposition 3.3(i) one has

$$|||w|||^2 = \left|\int_{Q_1} w\right|^2 + \frac{\left|w(\beta) - w(\alpha)\right|^2}{\beta - \alpha} + \|w'\|^2_{L^2(Q_1)}.$$

This completes the proof of Lemma 3.2. ∎

We now prove Theorem 3.1.

*Proof.* The inclusion $\bigcup_{\theta\in[0,1)}\sigma\bigl(A(\theta)\bigr)\subset\lim_{\varepsilon\to0}S^\varepsilon$ is proved in the same way as in the case of the whole-space problem; see the proof of Theorem 2.2. In what follows we therefore discuss the converse inclusion; i.e., for $\lambda_\varepsilon\in S^\varepsilon$, $\lambda_\varepsilon\to\lambda$ such that $\lambda\notin\sigma_{\rm boundary}$ one has $\lambda\in\bigcup_{\theta\in[0,1)}\sigma\bigl(A(\theta)\bigr)$. Let us consider such a sequence $\lambda_\varepsilon$. Notice first that for each $\lambda_\varepsilon\in S^\varepsilon$, there exist $u_\varepsilon\in\mathfrak{H}$, $\|u_\varepsilon\|_{L^2(a,b)}=1$, such that for any sequence $\delta_\varepsilon\overset{\varepsilon\to0}{\longrightarrow}0$, there exists a constant $c>0$ uniformly bounding the $L^2$-norms of $u_\varepsilon$ away from the boundary of $\Omega$,

$$\|u_\varepsilon\|_{L^2(a+\delta_\varepsilon,b-\delta_\varepsilon)}\geq c\quad\forall\varepsilon,$$

and the identity

$$\int_{\Omega_1^\varepsilon}p\left(\tfrac{x}{\varepsilon}\right)u_\varepsilon'(x)\overline{\varphi'(x)}{\rm d}x+\varepsilon^2\int_{\Omega_0^\varepsilon}p\left(\tfrac{x}{\varepsilon}\right)u_\varepsilon'(x)\overline{\varphi'(x)}{\rm d}x=\lambda_\varepsilon\int_\Omega u_\varepsilon(x)\overline{\varphi(x)}{\rm d}x$$

holds for all $\varphi\in\mathfrak{H}$. Indeed, if this were not the case, then $\lambda$ would be an element of $\sigma_{\rm boundary}$; see (3.1). Rescaling the above statement with $y=x/\varepsilon$ yields the existence of functions $v_\varepsilon\in H^1(\widetilde{\Omega}^\varepsilon)$, $\|v_\varepsilon\|_{L^2(\widetilde{\Omega}^\varepsilon)}=1$, such that

$$\|v_\varepsilon\|_{L^2(\varepsilon^{-1}a+\varepsilon^{-1}\delta_\varepsilon,\varepsilon^{-1}b-\varepsilon^{-1}\delta_\varepsilon)}\geq c\quad\forall\varepsilon,\tag{3.6}$$

and

$$\varepsilon^{-2}\int_{\widetilde{\Omega}_1^\varepsilon}p(y)v_\varepsilon'(y)\overline{\varphi'(y)}{\rm d}y+\int_{\widetilde{\Omega}_0^\varepsilon}p(y)v_\varepsilon'(y)\overline{\varphi'(y)}{\rm d}y=\lambda_\varepsilon\int_{\widetilde{\Omega}^\varepsilon}v_\varepsilon(y)\overline{\varphi(y)}{\rm d}y\tag{3.7}$$

for all $\varphi\in\widetilde{\mathfrak{H}}:=\{v\mid v(y)=u(\varepsilon y)\text{ for some }u\in\mathfrak{H}\}$. Choosing $\varphi=v_\varepsilon$ in (3.7) yields the estimates (a priori bounds)

$$\|v_\varepsilon'\|_{L^2(\widetilde{\Omega}_1^\varepsilon)}\leq\varepsilon C_{\rm B},\qquad\|v_\varepsilon'\|_{L^2(\widetilde{\Omega}_0^\varepsilon)}\leq C_{\rm B},\tag{3.8}$$

where $C_{\rm B}>0$ is independent of $\varepsilon$.

*Step* 1. Without loss of generality, we assume that $\overline{\widetilde{\Omega}^\varepsilon}\subset N_\varepsilon Q$ for some $N_\varepsilon\in\mathbb{N}$, such that $\varepsilon N_\varepsilon\to1$ as $\varepsilon\to0$. We also take a sequence $\delta_\varepsilon>0$ such that $\varepsilon^{-1}\delta_\varepsilon=q_\varepsilon\in\mathbb{N}$, $\delta_\varepsilon\to0$, and $q_\varepsilon^{-1}=\varepsilon\delta_\varepsilon^{-1}\to0$ as $\varepsilon\to0$, and we denote by $\chi_\varepsilon$ a smooth "cut-off" function such that $0\leq\chi_\varepsilon\leq1$, $\chi_\varepsilon\in C_0^\infty(\widetilde{\Omega}^\varepsilon)$, with the additional properties that $\chi_\varepsilon=1$ on $(\varepsilon^{-1}a+q_\varepsilon,\varepsilon^{-1}b-q_\varepsilon)$ and $\chi_\varepsilon'=0$ on $F_1\cap N_\varepsilon Q$. Note that such a function $\chi_\varepsilon$ varies from 0 to 1 on the interval $(\varepsilon^{-1}a,\varepsilon^{-1}a+q_\varepsilon)$, which contains $q_\varepsilon$ integer translations of the interval $(\alpha,\beta)$, and similarly varies from 1 to 0 on the interval $(\varepsilon^{-1}b-q_\varepsilon,\varepsilon^{-1}b)$. Therefore, it can be constructed by piecing together translations of smooth monotone functions $g$ on $[0,1)$ that are constant on the intervals $[0,\alpha)$, $(\beta,1]$ and satisfy the inequality $|g'(x)|\leq2(\beta-\alpha)^{-1}q_\varepsilon^{-1}$ for $x\in[\alpha,\beta]$. Clearly in this case the bound

$$\|\chi_\varepsilon'\|_{L^\infty(\widetilde{\Omega}_\varepsilon)}\leq2(\beta-\alpha)^{-1}q_\varepsilon^{-1}\tag{3.9}$$

holds. In particular, $\|\chi_\varepsilon'\|_{L^\infty(N_\varepsilon Q)}\to0$ as $\varepsilon\to0$, where we extend the function $\chi_\varepsilon$ by zero outside the interval $\widetilde{\Omega}_\varepsilon$.

For every $N \in \mathbb{N}$ we introduce the function space (cf. (2.2))

$$V_N := \{u \in H^1_\#(NQ) : u'(y) = 0 \text{ for } y \in F_1\}$$

and denote $V_N^\perp$ to be the orthogonal complement of $V_N$ in the space $H^1_\#(NQ)$ equipped with the usual inner product. For each $\varepsilon$ we set the $\chi_\varepsilon v_\varepsilon$ to take zero values on $N_\varepsilon Q \setminus \widetilde{\Omega}^\varepsilon$, and we consider the functions $U_\varepsilon := P_{V_{N_\varepsilon}}(\chi_\varepsilon v_\varepsilon)$, $V_\varepsilon := P_{V_{N_\varepsilon}^\perp}(\chi_\varepsilon v_\varepsilon)$, where $P_{V_{N_\varepsilon}}$ and $P_{V_{N_\varepsilon}^\perp}$ denote the orthogonal projections in the space $H^1_\#(N_\varepsilon Q)$ onto its subspaces $V_{N_\varepsilon}$ and $V_{N_\varepsilon}^\perp$, respectively. Note that, due to the normalization of $v_\varepsilon$ and (3.6), the bounds

$$1 \geq \|\chi_\varepsilon v_\varepsilon\|_{L^2(\widetilde{\Omega}^\varepsilon)} \geq c \tag{3.10}$$

hold for all $\varepsilon$. Further, we show that there exists a constant $C_\perp$, independent of $\varepsilon$, such that

$$\|V_\varepsilon\|_{H^1(N_\varepsilon Q)} \leq \left(c_0 q_\varepsilon^{-1} + \varepsilon C_{\mathrm{B}}\right) C_\perp. \tag{3.11}$$

The inequality (3.11) is a consequence of (3.9) and the following proposition.

PROPOSITION 3.4. *There exists a constant* $C_\perp > 0$ *such that*

$$\left\|P_{V_N^\perp} u\right\|_{H^1(NQ)} \leq C_\perp \|u'\|_{L^2(F_1 \cap NQ)}$$

*for any* $N \in \mathbb{N}$ *and any* $u \in H^1_\#(NQ)$.

Indeed, if Proposition 3.4 holds, then (3.11) follows from the a priori bounds (3.8). Note that Lemma 3.2 is one of the key ingredients in the proof of Proposition 3.4, which we give next.

*Proof.* Consider a function $u \in V_N^\perp$ for some positive integer $N$. Notice that for each $j = 0, 1, \dots, N-1$ the function

$$u_j(x) = N^{-1} \sum_{k=0}^{N-1} u(x+k) \exp(-2\pi \mathrm{i} jk/N)$$

belongs to the space $V(\theta_j)^\perp$, $\theta_j := j/N$. Indeed, for any $v \in V(\theta_j) \subset H^1_{\theta_j}(Q)$ it is clear that $v$ belongs to the space $V_N$ when extended in a quasiperiodic fashion. Further, for any $j = 0, 1, \dots, N-1$, one has

$$\int_Q u_j \overline{v} = N^{-1} \sum_{k=0}^{N-1} \int_0^1 u(x+k) \exp(-2\pi \mathrm{i} jk/N) \overline{v(x)} dx$$

$$= N^{-1} \sum_{k=0}^{N-1} \int_k^{k+1} u(x) \exp(-2\pi \mathrm{i} jk/N) \overline{v(x-k)} dx = N^{-1} \int_0^N u \overline{v},$$

and similarly

$$\int_Q u_j' \overline{v'} = N^{-1} \int_0^N u' \overline{v'}.$$

Combining the above two identities yields

$$\int_Q u_j \overline{v} + \int_Q u_j' \overline{v'} = 0 \qquad \forall v \in V(\theta_j),$$

as required.

Now using the Parseval identity and Lemma 3.2 we obtain

$$\|u\|^2_{H^1(NQ)} = N\sum_{j=0}^{N-1}\|u_j\|^2_{H^1(Q)} \le CN\sum_{j=0}^{N-1}\|u_j'\|^2_{L^2(Q_1)} = C\|u'\|^2_{L^2(F_1\cap NQ)},$$

where the positive constant $C$ does not depend on $N$. □

*Step* 2. There exists a constant $\widehat{C}$ independent of $\varepsilon$ such that

$$\|U_\varepsilon'\|_{L^2(N_\varepsilon Q)} \le \widehat{C}. \tag{3.12}$$

Indeed, by the a priori bounds (3.8), as well as (3.9) and (3.11), we find that

$$\begin{aligned}\|U_\varepsilon'\|_{L^2(N_\varepsilon Q)} &\le \|V_\varepsilon'\|_{L^2(N_\varepsilon Q)} + \|\chi_\varepsilon' v_\varepsilon + \chi_\varepsilon v_\varepsilon'\|_{L^2(N_\varepsilon Q)}\\ &\le \left(\varepsilon C_{\rm B} + c_0 q_\varepsilon^{-1}\right)C_\perp + c_0 q_\varepsilon^{-1} + (1+\varepsilon)C_{\rm B}.\end{aligned}$$

Further, the identity $\chi_\varepsilon v_\varepsilon = U_\varepsilon + V_\varepsilon$ and the bounds (3.10) imply

$$c - \left(\varepsilon C_{\rm B} + c_0 q_\varepsilon^{-1}\right)C_\perp \le \|U_\varepsilon\|_{L^2(N_\varepsilon Q)} \le 1 + \left(\varepsilon C_{\rm B} + c_0 q_\varepsilon^{-1}\right)C_\perp. \tag{3.13}$$

As the functions $U_\varepsilon$ belong to the spaces $V_{N_\varepsilon}$, by the discrete Floquet–Bloch transform the following decomposition holds: $U_\varepsilon = \sum_{j=0}^{N_\varepsilon-1} U_\varepsilon^j$ for $U_\varepsilon^j \in V(j/N_\varepsilon)$ given by

$$U_\varepsilon^j(x) = N_\varepsilon^{-1}\sum_{k=0}^{N_\varepsilon-1} U_\varepsilon(x+k)\exp(-2\pi{\rm i}jk/N_\varepsilon), \qquad x\in N_\varepsilon Q, \qquad j=0,1,\dots,N_\varepsilon-1.$$

Recalling, from section 2, that for fixed $\varepsilon$ the eigenfunctions $v^k(j/N_\varepsilon)$ form a complete sequence in the $L^2(Q)$-closure of the set $V(j/N_\varepsilon)$, we can decompose $U_\varepsilon^j$ with respect to this sequence, e.g.,

$$U_\varepsilon(x) = \sum_{j=0}^{N_\varepsilon-1}\sum_{k=1}^{\infty}\hat{U}_\varepsilon^k\big(\tfrac{j}{N_\varepsilon}\big)v^k\big(\tfrac{j}{N_\varepsilon},x\big),$$

$$U_\varepsilon'(x) = \sum_{j=0}^{N_\varepsilon-1}\sum_{k=1}^{\infty}\hat{U}_\varepsilon^k\big(\tfrac{j}{N_\varepsilon}\big)(v^k)'\big(\tfrac{j}{N_\varepsilon},x\big), \tag{3.14}$$

where $\hat{U}_\varepsilon^k(j/N_\varepsilon)\in\mathbb{C}$. The Parseval identity

$$N_\varepsilon\sum_{j=0}^{N_\varepsilon-1}\sum_{k=1}^{\infty}\Big|\hat{U}_\varepsilon^k\big(\tfrac{j}{N_\varepsilon}\big)\Big|^2 = \|U_\varepsilon\|^2_{L^2(N_\varepsilon Q)}$$

and (3.13) imply

$$\left(c - \left(\varepsilon C_{\rm B} + c_0 q_\varepsilon^{-1}\right)C_\perp\right)^2 \le N_\varepsilon\sum_{j=0}^{N_\varepsilon-1}\sum_{k=1}^{\infty}\Big|\hat{U}_\varepsilon^k\big(\tfrac{j}{N_\varepsilon}\big)\Big|^2 \le \left(1 + \left(\varepsilon C_{\rm B} + c_0 q_\varepsilon^{-1}\right)C_\perp\right)^2. \tag{3.15}$$

Denoting by $\delta(\cdot - \theta)$ the Dirac mass at $\theta$, the inequalities (3.15) can be rewritten as

$$\left(c - \left(\varepsilon C_{\mathrm{B}} + c_0 q_\varepsilon^{-1}\right) C_\perp\right)^2 \le \sum_{k=1}^{\infty} \int_0^1 \mathrm{d}\mu_\varepsilon^k \le \left(1 + \left(\varepsilon C_{\mathrm{B}} + c_0 q_\varepsilon^{-1}\right) C_\perp\right)^2, \tag{3.16}$$

$$\text{where} \quad \mathrm{d}\mu_\varepsilon^k(\theta) := N_\varepsilon \sum_{j=0}^{N_\varepsilon - 1} \left| \hat{U}_\varepsilon^k\left(\tfrac{j}{N_\varepsilon}\right)\right|^2 \delta\left(\theta - \tfrac{j}{N_\varepsilon}\right) \mathrm{d}\theta.$$

Clearly, for each $k$ the sequence $\{\mu_\varepsilon^k\}_\varepsilon$ is bounded in the space of Radon measures on $[0,1]$. Therefore, up to a subsequence, $\mu_\varepsilon^k$ weakly converges as $\varepsilon \to 0$ to some measure $\mu^k$, e.g.,

$$\int_0^1 u \ \mathrm{d}\mu_\varepsilon^k \to \int_0^1 u \ \mathrm{d}\mu^k \quad \forall u \in C[0,1]. \tag{3.17}$$

The above result follows from recalling that the space of finite Radon measures on $[0,1]$ coincides with the dual space $C[0,1]^\star$, and hence bounded sets of Radon measures are relatively compact with respect to weak star convergence in this space.

*Step* 3. Here we show that

$$c \le \sum_{k=1}^{\infty} \int_0^1 \ \mathrm{d}\mu^k \le 1, \tag{3.18}$$

which implies in particular that there exists at least one nonzero measure $\mu^{k_0}$ for some integer $k_0$. The bounds (3.18) follow from (3.16), (3.17), and the following result.

PROPOSITION 3.5. *For any $\delta > 0$ there exists a $K \in \mathbb{N}$ such that*

$$\sum_{k=K}^{\infty} \int_0^1 \mathrm{d}\mu_\varepsilon^k \le \delta \quad \forall \varepsilon.$$

*Proof.* Notice that

$$\int_{\widetilde{\Omega}_0^\varepsilon} p U_\varepsilon' \overline{U_\varepsilon'} = N_\varepsilon \sum_{k,l=1}^{\infty} \sum_{j=0}^{N_\varepsilon - 1} \int_{Q_0} p\,(x)\, \hat{U}_\varepsilon^k\left(\tfrac{j}{N_\varepsilon}\right)(v^k)'\left(\tfrac{j}{N_\varepsilon}, x\right) \overline{\hat{U}_\varepsilon^l\left(\tfrac{j}{N_\varepsilon}\right)(v^l)'\left(\tfrac{j}{N_\varepsilon}, x\right)} \mathrm{d}x$$

$$= N_\varepsilon \sum_{k=1}^{\infty} \sum_{j=0}^{N_\varepsilon - 1} \lambda_k\left(\tfrac{j}{N_\varepsilon}\right) \left| \hat{U}_\varepsilon^k\left(\tfrac{j}{N_\varepsilon}\right)\right|^2 = \sum_{k=1}^{\infty} \int_0^1 \lambda_k \mathrm{d}\mu_\varepsilon^k. \tag{3.19}$$

In the three equalities of (3.19) we use (3.14), (3.16), and the fact that $v^k(\theta)$ are orthogonal eigenfunctions of $A(\theta)$ with eigenvalues $\lambda_k(\theta)$; see section 2. Combining this observation with (3.12), we find that

$$\|p\|_{L^\infty(Q)} \widehat{C}^2 \ge \int_{\widetilde{\Omega}^\varepsilon} p |U_\varepsilon'|^2 = \sum_{k=1}^{\infty} \int_0^1 \lambda_k \mathrm{d}\mu_\varepsilon^k, \tag{3.20}$$

where we use the fact that $U_\varepsilon' = 0$ on $F_1 \cap N_\varepsilon Q$. For any integer $K \ge 2$ the inequality (3.20) immediately implies

$$\sum_{k=K}^{\infty} \int_0^1 \mathrm{d}\mu_\varepsilon^k \le \|p\|_{L^\infty(Q)} \widehat{C}^2 \min_{\theta \in [0,1)} \lambda_K(\theta)^{-1} \stackrel{K \to \infty}{\longrightarrow} 0.$$

This concludes the proof of the proposition. ∎

*Step* 4: We show next that $\lambda = \lambda_{k_0}(\theta)$ for some $k_0 \in \mathbb{N}$, $\theta \in [0,1)$. To this end, for all $\varepsilon$ and continuous $\psi$ such that $|\psi| \le 1$, we first consider a "modulation" (see [2]) $\mathcal{U}_\varepsilon \in V_{N_\varepsilon Q}$ of the function $U_\varepsilon$, given by

$$\mathcal{U}_\varepsilon(x) = \sum_{j=0}^{N_\varepsilon - 1} \psi\bigl(\tfrac{j}{N_\varepsilon}\bigr) \sum_{k=1}^{\infty} \hat{U}_\varepsilon^k\bigl(\tfrac{j}{N_\varepsilon}\bigr) v^k\bigl(\tfrac{j}{N_\varepsilon}, x\bigr).$$

Notice that due to the bound on $\psi$ and the estimates (3.12), (3.13) one has

$$\|\mathcal{U}_\varepsilon\|_{L^2(N_\varepsilon Q)} \le \|U_\varepsilon\|_{L^2(N_\varepsilon Q)} \le 1 + \bigl(\varepsilon + c_0 q_\varepsilon^{-1}\bigr)\, C_\perp C_B, \tag{3.21}$$

$$\|\mathcal{U}_\varepsilon'\|_{L^2(N_\varepsilon Q)} \le \widehat{C} \|p\|_{L^\infty(Q)} \|p^{-1}\|_{L^\infty(Q)}, \tag{3.22}$$

where the second estimate is obtained by repeating the argument of (3.19) with $U_\varepsilon$ replaced by $\mathcal{U}_\varepsilon$.

Further, we observe that $\varphi_\varepsilon = \chi_\varepsilon \mathcal{U}_\varepsilon \in \widetilde{\mathfrak{H}}$ since $H_0^1(\Omega) \subset \mathfrak{H}$, and we choose $\varphi_\varepsilon$ as a test function in the weak formulation (3.7):

$$\varepsilon^{-2} \int_{\widetilde{\Omega}_1^\varepsilon} p v_\varepsilon' \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} + \int_{\widetilde{\Omega}_0^\varepsilon} p v_\varepsilon' \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} = \lambda_\varepsilon \int_{\widetilde{\Omega}^\varepsilon} v_\varepsilon \overline{\chi_\varepsilon \mathcal{U}_\varepsilon}. \tag{3.23}$$

Notice that by construction $\chi_\varepsilon \in V_{N_\varepsilon}$; hence $\chi_\varepsilon \mathcal{U}_\varepsilon$ is piecewise constant in $\widetilde{\Omega}_1^\varepsilon$, and therefore the first term in (3.23) vanishes. We also have

$$\int_{\widetilde{\Omega}_0^\varepsilon} p v_\varepsilon' \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} = \int_{\widetilde{\Omega}_0^\varepsilon} p\, (\chi_\varepsilon v_\varepsilon)' \,\overline{\mathcal{U}_\varepsilon'} + o(1) \quad \text{ as } \varepsilon \to 0,$$

since

$$v_\varepsilon' \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} = (\chi_\varepsilon v_\varepsilon)' \,\overline{\mathcal{U}_\varepsilon'} + v_\varepsilon' \chi_\varepsilon' \overline{\mathcal{U}_\varepsilon} - v_\varepsilon \chi_\varepsilon' \overline{\mathcal{U}_\varepsilon'},$$

and by virtue of (3.21)–(3.22), the fact that $\|\chi_\varepsilon'\|_{L^\infty(N_\varepsilon Q)} \to 0$ as $\varepsilon \to 0$, and the a priori bounds (3.8), we infer that

$$\int_{\widetilde{\Omega}_0^\varepsilon} p\bigl(v_\varepsilon' \chi_\varepsilon' \overline{\mathcal{U}_\varepsilon} - v_\varepsilon \chi_\varepsilon' \overline{\mathcal{U}_\varepsilon'}\bigr)$$

$$\le \|p\|_{L^\infty(Q)} \,\|\chi_\varepsilon'\|_{L^\infty(N_\varepsilon Q)} \Bigl( \|v_\varepsilon'\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \,\|\mathcal{U}_\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} + \|\mathcal{U}_\varepsilon'\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \,\|v_\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \Bigr)$$

$$\le C \,\|\chi_\varepsilon'\|_{L^\infty(N_\varepsilon Q)} \to 0, \quad \varepsilon \to 0.$$

Next, since $\chi_\varepsilon v_\varepsilon = U_\varepsilon + V_\varepsilon$ and in view of (3.11), (3.22), we get

$$\int_{\widetilde{\Omega}_0^\varepsilon} p\, (\chi_\varepsilon v_\varepsilon)' \,\overline{\mathcal{U}_\varepsilon'} = \int_{\widetilde{\Omega}_0^\varepsilon} p U_\varepsilon' \overline{\mathcal{U}_\varepsilon'} + o(1), \qquad \varepsilon \to 0.$$

Furthermore, recalling the argument of (3.19), we have

$$\int_{\widetilde{\Omega}_0^\varepsilon} pU_\varepsilon'\overline{\mathcal{U}_\varepsilon'} = \sum_{k=1}^{\infty}\int_0^1 \lambda_k\psi \mathrm{d}\mu_\varepsilon^k. \tag{3.24}$$

Similarly we show that

$$\int_{\widetilde{\Omega}^\varepsilon} v_\varepsilon\overline{\chi_\varepsilon\mathcal{U}_\varepsilon} = \sum_{k=1}^{\infty}\int_0^1 \psi \mathrm{d}\mu_\varepsilon^k + o(1), \qquad \varepsilon\to 0. \tag{3.25}$$

Combining (3.23), (3.24), (3.25) yields

$$\sum_{k=1}^{\infty}\int_0^1 \lambda_k\psi \mathrm{d}\mu_\varepsilon^k = \lambda_\varepsilon\sum_{k=1}^{\infty}\int_0^1 \psi \mathrm{d}\mu_\varepsilon^k + o(1), \qquad \varepsilon\to 0. \tag{3.26}$$

Finally, passing to the limit $\varepsilon\to 0$ in (3.26) and using the fact that $\psi$ and $\lambda_k$, $k=1,2,\dots$, are all continuous functions of $\theta$ (see Appendix B) yields

$$\sum_{k=1}^{\infty}\int_0^1 \lambda_k\psi \mathrm{d}\mu^k = \lambda\sum_{k=1}^{\infty}\int_0^1 \psi \mathrm{d}\mu^k.$$

Using again the fact that $\lambda_k$, $k=1,2,\dots$, are continuous and recalling that $\mu^{k_0}$ is a positive measure for some $k_0$, we argue that $\lambda=\lambda_{k_0}(\theta)$ for some $\theta\in[0,1)$, $k_0$, due to the arbitrary choice of the modulating function $\psi$. ∎

**4. An example of a family with $\lim_{\varepsilon\to 0} S^\varepsilon = \sigma_{\mathrm{Bloch}}$.** Here we consider the case when the endpoints of the material domain belong to the "stiff" component. We also assume for simplicity that $(a,b)=(0,1)$ and restrict our analysis to values $\varepsilon=N^{-1}$, where $N\in\mathbb{N}$; hence $(\varepsilon^{-1}a,\varepsilon^{-1}b)=(0,N)$. Recall that $F_0$ denotes the 1-periodic extension of $Q_0=(\alpha,\beta)$ to the whole of $\mathbb{R}$ and $F_1=\mathbb{R}\backslash F_0$. For all $N\in\mathbb{N}$ we denote by $W_N$, $W_N^\perp$ the subspace

$$W_N := \left\{u\in H^1(0,N)\middle|\ u'=0 \text{ on } (0,N)\cap F_1\right\}$$

and its orthogonal complement in $H^1(0,N)$, respectively. We also denote by $P_{W_N^\perp}$ the orthogonal projection of $H^1(0,N)$ onto $W_N^\perp$.

Lemma 4.1. *There exists a constant $C>0$ independent of $N$ such that*

$$\left\|P_{W_N^\perp}u\right\|_{H^1(0,N)} \le C\left\|u'\right\|_{L^2((0,N)\cap F_1)} \qquad \forall u\in H^1(0,N). \tag{4.1}$$

*Proof.* Recall (see Proposition 3.4) that there exists $C_\perp>0$ independent of $N$ such that

$$\left\|P_{V_N^\perp}u\right\|_{H^1(0,N)} \le C_\perp\left\|u'\right\|_{L^2((0,N)\cap F_1)} \qquad \forall u\in H^1_\#(0,N),$$

where $P_{V_N}$ is the orthogonal projection of $H^1_\#(0,N)$ onto

$$V_N := \left\{u\in H^1_\#(0,N)\middle|\ u'=0 \text{ on } F_1\right\}.$$

In what follows we use this fact to prove the desired estimate (4.1). Take $\chi$ to be a smooth cut-off function such that $\chi=1$ on $(0,\alpha/2)\cup(N-(1-\beta)/2,N)$, $\chi=0$ on

$[\alpha, N-1+\beta]$, and the inequality $|\chi'| \le \max\{4/\alpha, 4/(1-\beta)\}$ holds. For fixed $u$, the function $w := u - \chi u$ is seen to belong to $H^1_\#(0,N)$ and satisfy $u - P_{V_N} w = P_{V_N^\perp} w + \chi u$. Therefore, since $V_N \subset W_N$, one has

$$\left\|P_{W_N^\perp} u\right\|_{H^1(0,N)} = \inf_{v \in W_N} \|u - v\|_{H^1(0,N)} \le \|u - P_{V_N} w\|_{H^1(0,N)}$$

$$= \left\|P_{V_N^\perp} w + \chi u\right\|_{H^1(0,N)} \le C_\perp \|w'\|_{L^2((0,N)\cap F_1)} + \|\chi u\|_{H^1(0,N)}$$

$$\le (C_\perp + 1)\bigl(\|u'\|_{L^2((0,N)\cap F_1)} + \|\chi u\|_{H^1(0,N)}\bigr).$$

It remains to prove the required bound for $\|\chi u\|_{H^1(0,N)}$. By noting the formulae

$$(\chi u)(x) = -\int_x^\alpha (\chi u)', \; x \in (0,\alpha), \text{ and } (\chi u)(x) = \int_{N-1+\beta}^x (\chi u)', \; x \in (N-1+\beta, N),$$

we find that

$$\int_0^N |\chi u|^2 \le \bigl(1 - \beta + \alpha\bigr) \int_0^N \left|(\chi u)'\right|^2$$

and

$$\int_0^N \left|(\chi u)'\right|^2 \le \max\{4/\alpha, 4/(1-\beta)\} \left(\int_0^\alpha + \int_{N-1+\beta}^N\right) |u|^2 + \left(\int_0^\alpha + \int_{N-1+\beta}^N\right) |u'|^2.$$

The result now follows from the formulae

$$u(x) = \int_0^x u' + u(0), \; x \in (0,\alpha), \text{ and } u(x) = -\int_x^N u' + u(N), \; x \in (N-1+\beta, N),$$

and the fact that the space $H^1(0,N)$ is continuously embedded in $C[0,N]$, which provides suitable bounds on $u(0)$, $u(N)$. ◻

We now consider a sequence of $\lambda_\varepsilon$ such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$, $\varepsilon^{-1} \in \mathbb{N}$, for some $\lambda \in \mathbb{R}$, and the corresponding eigenfunction sequence $v_\varepsilon \in \mathfrak{H}$ satisfies the identity

$$\int_0^N \bigl(\varepsilon^{-2}\chi_1 + \chi_0\bigr) p v_\varepsilon' \varphi' = \lambda_\varepsilon \int_0^N v_\varepsilon \varphi \quad \forall \varphi \in \mathfrak{H}, \tag{4.2}$$

for all values of $\varepsilon$ from the indicated set. Here, as before, $\mathfrak{H}$ is a subspace of $H^1(0,N)$ and $\chi_0$, $\chi_1$ are the characteristic functions of the sets $F_0$, $F_1$.

Notice first that (4.2) in combination with Lemma 4.1 implies the existence of $C > 0$ such that

$$\left\|P_{W_N^\perp} v_\varepsilon\right\|_{L^2(0,N)} \le C\varepsilon \|v_\varepsilon\|_{L^2(0,N)}.$$

Further, the function $u_\varepsilon = P_{W_N} v_\varepsilon$ is not periodic in general; however, it is piecewise constant on $(0,N) \cap F_1$. We introduce an extension $\widetilde{u}_\varepsilon$ by the formula

$$\widetilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x \in (0,N), \\ u_\varepsilon(2N - x), & x \in (N, 2N). \end{cases} \tag{4.3}$$

Clearly, the function $\widetilde{u}_\varepsilon$ is $2N$-periodic, but it is not necessarily an element of the space $V_{2N}$. However, in cases when $\widetilde{u}_\varepsilon$ does belong to $V_{2N}$, we can proceed with the Bloch measure argument in the proof of Theorem 3.1 to show that $\lambda \in \sigma_{\rm Bloch}$.

As the mapping $y \mapsto 2N - y$ takes intervals of the form $(n, n+a)$ to intervals $(2N-n-\alpha, 2N-n)$ and $(n+\beta, n+1)$ to $(2N-n-1, 2N-n-\beta)$, we find that $\widetilde{u}_\varepsilon$ belongs to the space $V_{2N}$ if

$$2N - n - (2N - n - \alpha) \geq 1 - \beta \qquad \text{and} \qquad 2N - n - \beta - (2N - n - 1) \geq \alpha,$$

e.g., $\alpha \geq 1 - \beta$ and $1 - \beta \geq \alpha$, which implies $1 - \beta = \alpha$. Hence, the following theorem holds.

THEOREM 4.2. *Assume the conditions described at the beginning of this section hold, and suppose in addition that $\alpha = 1 - \beta$. Then the limit spectrum for sequences with $\varepsilon^{-1} \in \mathbb{N}$ coincides with the Bloch spectrum $\sigma_{\rm Bloch}$.*

## 5. A modified problem with a compact perturbation, and the associated defect modes.

**5.1. Analytical set-up.** In this section we discuss a modified version of the set-up of section 1.2, as follows. Consider the operator $\widetilde{A}^\varepsilon$ defined by the sesquilinear form

(5.1)
$$(\widetilde{A}^\varepsilon u, v) = \int_a^b \Big( p_{\rm d}\chi_{\rm d}(x) + p(x/\varepsilon)\big(\varepsilon^2\chi_0(x/\varepsilon) + \chi_1(x/\varepsilon)\big)\big(1 - \chi_{\rm d}(x)\big)\Big) u'(x)\overline{v'(x)}dx,$$

$u, v \in \mathfrak{H}$. Here the space $\mathfrak{H}$ is as before (see section 1.2), and $\chi_{\rm d}$ is the indicator function of a "defect" interval $I_{\rm d}$ whose closure is assumed to be contained in $(a, b)$, and $p_{\rm d}$ is the corresponding "defect" coefficient (or "defect strength"). We denote by $\widetilde{S}^\varepsilon$ the spectrum of the operator $\widetilde{A}^\varepsilon$.

A formal two-scale asymptotic procedure carried out on the equation (cf. (1.1)) $\widetilde{A}^\varepsilon u = \lambda u$ suggests the following.

(1) The set $\lim_{\varepsilon\to 0} \widetilde{S}^\varepsilon$ is given by the union of the "Bloch spectrum" given by $\sigma_{\rm Bloch} = \bigcup_{\theta\in[0,1)} \sigma\big(A(\theta)\big)$, the "boundary spectrum" $\sigma_{\rm boundary}$ (see (3.1)), and a sequence of "defect eigenvalues" $\{p_{\rm d}\pi^2 j^2/|I_{\rm d}|^2\}_{j=1}^\infty$.

(2) Those defect eigenfunctions $u_j$, $j \in \mathbb{N}$, that correspond to the defect eigenvalues situated in the gaps of $\sigma_{\rm Bloch}$ decay exponentially away from the boundary of the defect:

$$\big|u_j(x)\big| \leq c_1 \exp\big(-c_2\varepsilon^{-1}{\rm dist}\{x, I_{\rm d}\}\big), \qquad x \in (a, b) \setminus I_{\rm d}, \qquad c_1, c_2 > 0. \tag{5.2}$$

Here $c_2$ depends on the distance of the defect eigenvalue to $\sigma_{\rm Bloch}$.

In section 5.2 we present numerical evidence that supports these claims. Note that the rate of decay in the estimate (5.2) increases as $\varepsilon \to 0$. This complements the recent results of [5], where the decay of eigenfunctions in multidimensional high-contrast problems is analyzed. It was shown in [5] that under the assumption that the matrix of the composite forms a connected set in the whole space the defect eigenfunctions corresponding to eigenvalues in the gaps of the limit spectrum satisfy the estimate $\big|u_j(x)\big| \leq c_1 \exp(-c_2{\rm dist}\{x, \partial D\}\big)$, $x \in (a, b) \setminus D$, with $c_1, c_2 > 0$, where $D$ is the defect set, and $c_2$ may depend on $j$. The higher rate of decay in (5.2) can be interpreted as a stronger localization effect in the case when the matrix of the composite is disconnected.

**5.2. Numerical results for the modified problem.** We consider a defect of length $|I_{\rm d}| = 1/2$ and strength $p_{\rm d} = 2$ in the middle of the interval $(a, b) = (0, 1)$, e.g., $I_{\rm d} = (1/4, 3/4)$. For each value of $\varepsilon$ such that $N := (4\varepsilon)^{-1}$ is a positive integer, we describe the intervals $(0, 1/4)$ and $(3/4, 1)$ on either side of the defect according to (5.1): each of them consists of $N$ cells of the same length $\varepsilon = 1/(4N)$, and in one half of each cell the coefficient in the form (the "modulus") takes the value $1/(4N)^2$, while in the other half it is equal to unity. Note that the described set-up satisfies the conditions of section 4, so in this case we expect $\lim_{\varepsilon\to 0} S^\varepsilon$ to coincide with the union of $\sigma_{\rm Bloch}$ and the sequence of defect eigenvalues.

The results of solving the above problems with finite elements are given in Tables 1 and 2. The values for the trapped mode are in good agreement with the values obtained by the asymptotic method: $\lambda_\star^{(2)} = 78.9568$, $\lambda_\star^{(3)} = 315.8273$, $\lambda_\star^{(5)} = 710.6115$, $\lambda_\star^{(6)} = 1263.3094$, where the superscript is the number of the stop band containing the related eigenvalue.

In addition, the profiles obtained for such trapped modes (see Figure 2 for the case of periodic boundary conditions) suggest that the number of half-oscillations in a trapped mode is equal to the number of the mode in the sequence, which resembles the behavior of the usual Neumann eigenfunctions on the defect. We also note that the decay of the trapped modes appears to be exponential, as can be seen in Figure 3: the larger the contrast (and hence the number of subdivisions of the string), the more localized the mode, irrespective of the boundary conditions at the endpoints of the string.

TABLE 1

*Stop bands and trapped modes for the modified problem with a defect, subject to the Dirichlet boundary conditions:* $\lambda_{\min}^{(k,128)}$ *and* $\lambda_{\max}^{(k,128)}$ *are the lower and upper bounds of the kth stop band for* $N = 128$, *and* $\lambda_\star^{(k,128)}$, $\lambda_\star^{(k,256)}$, $\lambda_\star^{(k,512)}$, $\lambda_\star^{(k,1024)}$ *are the trapped-mode eigenvalues in the kth stop band, evaluated for* $N = 128$, $N = 256$, $N = 512$, *and* $N = 1024$, *respectively.*

| Dirichlet boundary conditions | | | | | |
|---|---|---|---|---|---|
| $\lambda_{\min}^{(k,128)}$ | $\lambda_{\max}^{(k,128)}$ | $\lambda_\star^{(k,128)}$ | $\lambda_\star^{(k,256)}$ | $\lambda_\star^{(k,512)}$ | $\lambda_\star^{(k,1024)}$ |
| 11.7939 | 39.4603 | - | - | - | - |
| 65.7875 | 157.8859 | 75.7674 | 77.2502 | 78.0741 | 78.7304 |
| 187.6799 | 355.2599 | 293.9534 | 304.1141 | 309.7163 | 314.2461 |
| 386.1413 | 622.2747 | - | - | - | - |
| 662.9213 | 986.7685 | 682.6577 | 694.4984 | 702.0486 | 708.4576 |
| 1018.4394 | 1421.0468 | 1225.1298 | 1232.2190 | 1243.1182 | 1258.2799 |

TABLE 2

*Stop bands and trapped modes for the modified problem with a defect, subject to the Neumann boundary conditions:* $\lambda_{\min}^{(k,128)}$ *and* $\lambda_{\max}^{(k,128)}$ *are the lower and upper bounds of the kth stop band for* $N = 128$, *and* $\lambda_\star^{(k,128)}$, $\lambda_\star^{(k,256)}$, $\lambda_\star^{(k,512)}$, $\lambda_\star^{(k,1024)}$ *are the trapped-mode eigenvalues in the kth stop band, evaluated for* $N = 128$, $N = 256$, $N = 512$, *and* $N = 1024$, *respectively.*

| Neumann boundary conditions | | | | | |
|---|---|---|---|---|---|
| $\lambda_{\min}^{(k,128)}$ | $\lambda_{\max}^{(k,128)}$ | $\lambda_\star^{(k,128)}$ | $\lambda_\star^{(k,256)}$ | $\lambda_\star^{(k,512)}$ | $\lambda_\star^{(k,1024)}$ |
| 11.7515 | 39.4980 | - | - | - | - |
| 65.8359 | 157.8901 | 75.7676 | 77.2509 | 78.0741 | 78.7314 |
| 187.7334 | 355.2765 | 293.9539 | 304.1145 | 309.7164 | 314.3057 |
| 386.2091 | 622.2747 | - | - | - | - |
| 662.9779 | 986.7698 | 682.6578 | 694.4985 | 702.0486 | 708.6496 |
| 1018.5163 | 1421.0556 | 1225.1298 | 1232.2190 | 1243.1193 | 1258.2626 |

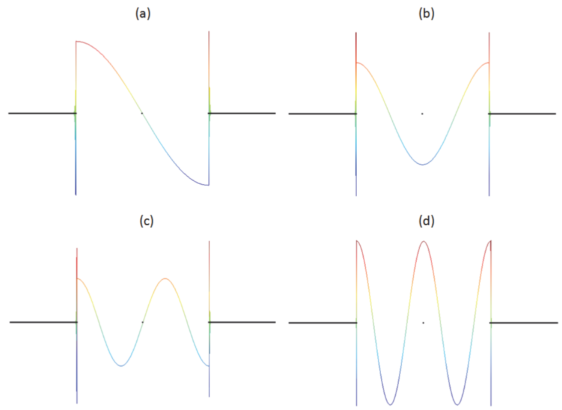


FIG. 2. *Trapped eigenmodes for the modified problem with a defect and periodic boundary conditions ($N = 1024$), corresponding to eigenfrequencies:* (a) $\lambda_\star^{(2,1024)} = 78.07$, (b) $\lambda_\star^{(3,1024)} = 314.24$, (c) $\lambda_\star^{(5,1024)} = 708.46$, (d) $\lambda_\star^{(6,1024)} = 1258.28$.

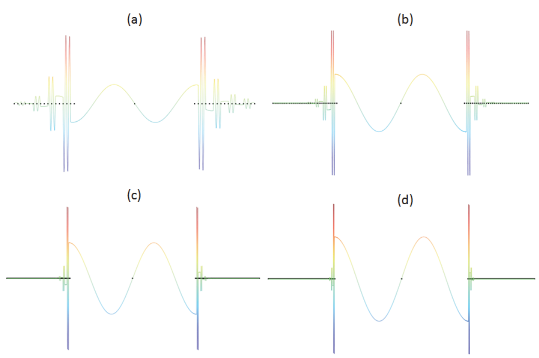


FIG. 3. *Decay of the third trapped eigenmode (located in the fifth stop band) for the modified problem with a defect and periodic boundary conditions, as a function of contrast:* (a) $\lambda_\star^{(5,32)} = 668.86$, (b) $\lambda_\star^{(5,64)} = 682.89$, (c) $\lambda_\star^{(5,128)} = 694.50$, (d) $\lambda_\star^{(5,256)} = 702.68$.

**5.3. Photonic band gaps and trapped modes in high-contrast multilayered dielectric structures.** The string problem emerges, among other contexts, in the study of wave propagation in one-dimensional photonic crystals, e.g., multilayered dielectric structures invariant along two directions. In what follows, we set these directions to be $x_1$ and $x_3$ in the usual Euclidean representation $x = (x_1, x_2, x_3)$.

We consider solutions $(\mathcal{E}, \mathcal{H})$ to the classical system of Maxwell's equations [10] that have the form

$$\mathcal{E}(x_1, x_2, x_3, t) = E(x_2)\exp\big(\mathrm{i}(\kappa x_3 - \omega t)\big), \quad \mathcal{H}(x_1, x_2, x_3, t) = H(x_2)\exp\big(\mathrm{i}(\kappa x_3 - \omega t)\big),$$

where $t$ is time, $\omega$ is the angular frequency, and $\kappa \geq 0$ is a "propagation constant." We write the Maxwell's equations for the field variable $(E, H)$:

$$\left\{\begin{array}{cl} E_3' - \mathrm{i}\kappa E_2 & = \mathrm{i}\omega\mu H_1, \\ \mathrm{i}\kappa E_1 & = \mathrm{i}\omega\mu H_2, \\ -E_1' & = \mathrm{i}\omega\mu H_3, \end{array}\right. \qquad \left\{\begin{array}{cl} -H_3' + \mathrm{i}\kappa H_2 & = \mathrm{i}\omega\varepsilon E_1, \\ -\mathrm{i}\kappa H_1 & = \mathrm{i}\omega\varepsilon E_2, \\ H_1' & = \mathrm{i}\omega\varepsilon E_3. \end{array}\right.$$

Here $\mu$ is the magnetic permeability, and $\varepsilon$ is the electric permittivity at each point of the dielectric.

We rearrange the above six equations into two groups of equations for $(E_1, H_2, H_3)$ (transverse magnetic polarization), and $(H_1, E_2, E_3)$ (transverse electric polarization). We choose $E_1$ and $H_1$ as the unknown functions within the respective groups and notice that the remaining unknowns are expressed in terms of these two scalar functions only. The equations satisfied by $E_1$, $H_1$ are

$$(E_1')' + \big(\omega^2\mu\varepsilon - \kappa^2\big)E_1 = 0, \tag{5.3}$$

$$(\varepsilon^{-1}H_1')' + \big(\omega^2\mu - \varepsilon^{-1}\kappa^2\big)H_1 = 0. \tag{5.4}$$

Note that (5.4) coincides with (1.1) when $\kappa = 0$ by setting

$$\omega^2 = \lambda, \quad \mu = 1, \quad \varepsilon^{-1}(x_2) = p(x_2/\eta)\big(\eta^2\chi_0(x_2/\eta) + \chi_1(x_2/\eta)\big), \quad x_2 \in (a,b), \quad \eta > 0,$$

where we use $\eta$ rather than $\varepsilon$ to denote the structure period in order to avoid confusion with the standard notation for electric permittivity. Our analysis in sections 2 and 3 carries over to the case $\kappa > 0$. In particular, for $p \equiv 1$ we get a $\kappa$-dependent version of the dispersion relation (2.1), as follows:

$$\frac{1}{2}(\alpha - \beta + 1)\left(\sqrt{\lambda} - \frac{\kappa^2}{\sqrt{\lambda}}\right)\sin\Big(\sqrt{\lambda}(\alpha-\beta)\Big) + \cos\Big(\sqrt{\lambda}(\alpha-\beta)\Big) = \cos(2\pi\theta).$$

Assuming infinitely conducting walls on either side of the dielectric (see, e.g., [21] for further details), we supply (5.3) and (5.4) with homogeneous Dirichlet and Neumann boundary conditions, respectively.

In the numerical solution of the above problem we employ finite elements with perfectly matched layers, e.g., anisotropic absorptive reflectionless layers (see, e.g., [21]), on the top and bottom of the computational domain. Our results are shown in Figure 4 for $\kappa = 0.1$, $N = 16$, and for the transverse electric mode with frequency $\lambda_\star = 78.34$ inside the third stop band. The latter corresponds to the first trapped mode shown in Figure 2(a), in view of the fact that for $\kappa = 0.1$ there is an additional zero-frequency stop band. The magnetic component of this mode (Figures 4(a) and (d)) clearly shares the same features as the string mode in Figure 2(a).

**Appendix A.** In this appendix we argue that for any fixed $N \in \mathbb{N}$ and $\lambda < 0$, say, $\lambda = -1$, the solutions to the problems (1.1) converge in an appropriate two-scale sense (see, e.g., [1]) to the solution of some limit problem parametrized by $N$. We first formulate the related statement for $N = 1$.

**A.1. Periodic homogenization.**

LEMMA A.1. *Set $\lambda = -1$, and let $u_\varepsilon \in \mathfrak{H}$ be the solution to the problem* (1.1). *Define a space $V_1$ by*

$$V_1 := \left\{v \in H^1_\#(Q) : v'(y) = 0 \textit{ for } y \in F_1\right\}.$$

*There exists $u(x,y) \in L^2\big((a,b); V_1\big)$ such that up to a subsequence, which we do not relabel,*

$$u_\varepsilon \overset{2}{\rightharpoonup} u(x,y), \qquad \varepsilon u_\varepsilon' \overset{2}{\rightharpoonup} \frac{\partial u}{\partial y}(x,y), \qquad \chi_1\left(\tfrac{x}{\varepsilon}\right)p\left(\tfrac{x}{\varepsilon}\right)u_\varepsilon'(x) \overset{2}{\rightharpoonup} 0,$$

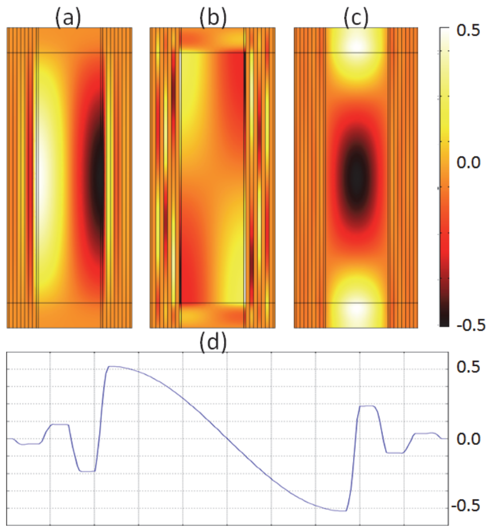


FIG. 4. *Obliquely propagating transverse electric wave in a high-contrast dielectric multilayered planar waveguide with infinitely conducting walls:* (a) *two-dimensional plot of $H_1$;* (b) *two-dimensional plot of $E_2$;* (c) *two-dimensional plot of $E_3$;* (d) *profile of $H_1$ along the horizontal centerline. Here $N = 16$, $\kappa = 0.1$, and the normalized frequency $\lambda_\star = 78.34$.*

*where "$\overset{2}{\rightharpoonup}$" is weak two-scale convergence (see* [1]*). The function $u \in L^2\big((a,b);V_1\big)$ is the unique solution to the problem*

$$\int_a^b\int_Q p(y)\chi_0(y)\frac{\partial u}{\partial y}(x,y)\overline{\frac{\partial \varphi}{\partial y}(x,y)}\;\mathrm{d}y\mathrm{d}x+\int_a^b\int_Q u(x,y)\overline{\varphi(x,y)}\;\mathrm{d}y\mathrm{d}x$$

$$=\int_a^b\int_Q f(x)\overline{\varphi(x,y)}\;\mathrm{d}y\mathrm{d}x\quad\forall\varphi\in C_0^\infty\big((a,b);V_1\big). \tag{A.1}$$

*Proof. Step* 1. Let $u_\varepsilon$ be a solution to problem (1.1), e.g.,

$$\int_a^b p\left(\tfrac{x}{\varepsilon}\right)\left(\varepsilon^2\chi_0\left(\tfrac{x}{\varepsilon}\right)+\chi_1\left(\tfrac{x}{\varepsilon}\right)\right)u_\varepsilon'(x)\overline{\varphi'(x)}\mathrm{d}x+\int_a^b u_\varepsilon(x)\overline{\varphi(x)}\mathrm{d}x=\int_a^b f(x)\overline{\varphi(x)}\mathrm{d}x \tag{A.2}$$

for all $\varphi\in\mathfrak{H}$. It is clear, by choosing $\varphi = u_\varepsilon$, that the sequences $u_\varepsilon$ and $\varepsilon u_\varepsilon'$ are bounded in $L^2(a,b)$. Therefore, by [1, Proposition 1.14] there exists $u(x,y)\in L^2\big((a,b);H^1_\#(Q)\big)$ such that up to a subsequence, $u_\varepsilon\overset{2}{\rightharpoonup}u$ and $\varepsilon u_\varepsilon'\overset{2}{\rightharpoonup}\frac{\partial u}{\partial y}$. We now show that, in fact, $u(x,y)\in L^2\big((a,b);V_1\big)$. In view of the convergence $\varepsilon u_\varepsilon'\overset{2}{\rightharpoonup}\frac{\partial u}{\partial y}$, we have

$$\lim_{\varepsilon\to0}\varepsilon\int_a^b p\left(\tfrac{x}{\varepsilon}\right)\chi_1\left(\tfrac{x}{\varepsilon}\right)u_\varepsilon'(x)\overline{\Psi\left(x,\tfrac{x}{\varepsilon}\right)}\;\mathrm{d}x=\int_a^b\int_Q p(y)\chi_1(y)\frac{\partial u}{\partial y}(x,y)\overline{\Psi(x,y)}\;\mathrm{d}y\mathrm{d}x$$
$$\forall\Psi\in L^2\big((a,b);C^\infty_\#(Q)\big).$$

Furthermore, since the sequence $\chi_1 u_\varepsilon'$ is bounded in $L^2(a,b)$, we have

$$\lim_{\varepsilon\to0}\varepsilon\int_a^b p\left(\tfrac{x}{\varepsilon}\right)\chi_1\left(\tfrac{x}{\varepsilon}\right)u_\varepsilon'\overline{\Psi\left(x,\tfrac{x}{\varepsilon}\right)}\;\mathrm{d}x=0.$$

*Step* 2. Here we show that $\chi_1\left(\frac{x}{\varepsilon}\right)p\left(\frac{x}{\varepsilon}\right)u'_\varepsilon(x)\overset{2}{\rightharpoonup}0$. Since $\chi_1\left(\frac{x}{\varepsilon}\right)u'_\varepsilon(x)$ is bounded in $L^2(a,b)$, there exists $\xi(x,y)\in L^2\big((a,b)\times Q\big)$ such that up to a subsequence, $\chi_1\left(\frac{x}{\varepsilon}\right)u'_\varepsilon(x)\overset{2}{\rightharpoonup}\xi(x,y)$. Therefore, it suffices to show that $p(y)\chi_1(y)\xi(x,y)\equiv 0$. To this end, notice that, taking $\varphi_\varepsilon(x)=\varepsilon\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)$, $\varphi\in C_0^\infty(a,b)$, $\psi(y)\in C_\#^\infty(Q)$ as test functions in (A.2), we obtain

$$\lim_{\varepsilon\to 0}\int_a^b p\left(\tfrac{x}{\varepsilon}\right)\chi_1\left(\tfrac{x}{\varepsilon}\right)u'_\varepsilon(x)\overline{\varphi'_\varepsilon}(x)\mathrm{d}x=0.$$

Notice also that since $\chi_1\left(\frac{x}{\varepsilon}\right)u'_\varepsilon(x)\overset{2}{\rightharpoonup}\xi(x,y)$, one has

$$\begin{aligned}\lim_{\varepsilon\to 0}&\int_a^b p\left(\tfrac{x}{\varepsilon}\right)\chi_1\left(\tfrac{x}{\varepsilon}\right)u'_\varepsilon(x)\overline{\varphi'_\varepsilon(x)}\mathrm{d}x\\&=\lim_{\varepsilon\to 0}\int_a^b p\left(\tfrac{x}{\varepsilon}\right)\chi_1^2\left(\tfrac{x}{\varepsilon}\right)u'_\varepsilon(x)\overline{\left(\varepsilon\varphi'(x)\psi\left(\tfrac{x}{\varepsilon}\right)+\varphi(x)\psi'\left(\tfrac{x}{\varepsilon}\right)\right)}\mathrm{d}x\\&=\int_a^b\int_Q p(y)\chi_1(y)\xi(x,y)\overline{\varphi(x)\psi'(y)}\;\mathrm{d}y\mathrm{d}x.\end{aligned}$$

It follows that

$$\int_Q p(y)\chi_1(y)\xi(x,y)\overline{\psi'(y)}\;\mathrm{d}y=0\qquad\text{for a.e. }x\in(a,b),\ \forall\ \psi\in C_\#^\infty(Q).\tag{A.3}$$

It remains to recall that for any $v\in C_0^\infty(Q_1)$ there exists $\psi\in C_\#^\infty(Q)$ such that $\psi'=v$ in $Q_1$, e.g., $\psi(y):=\int_0^y(v+\widetilde{\chi}_0)$, where $\widetilde{\chi}_0\in C_0^\infty(Q_0)$ is such that $\int_0^1(v+\widetilde{\chi}_0)=0$ .

*Step* 3. In order to show that $u$ solves (A.1), we choose $\varphi_\varepsilon(x)=\varphi\left(x,\frac{x}{\varepsilon}\right)$, where $\varphi(x,y)\in C_0^\infty\big((a,b);V_1\big)$, as a test function in (A.2) and pass to the limit $\varepsilon\to 0$, recalling that $u_\varepsilon$, $\varepsilon u'_\varepsilon$ and $\chi_1\left(\frac{x}{\varepsilon}\right)u'_\varepsilon$ two-scale converge to $u$, $\frac{\partial u}{\partial y}$, and 0, respectively. Uniqueness is shown in the standard way by observing that the problem (A.1) for $f=0$ has only the trivial solution. ∎

Clearly $C_0^\infty\big((a,b);V_1\big)$ is dense in $H:=L^2\big((a,b);V_1\big)$ with respect to the norm induced by the inner product

$$(u,v)_H:=\int_a^b\int_Q p(y)\chi_0(y)\frac{\partial u}{\partial y}(x,y)\overline{\frac{\partial\varphi}{\partial y}(x,y)}\;\mathrm{d}y\mathrm{d}x+\int_a^b\int_Q u(x,y)\overline{\varphi(x,y)}\;\mathrm{d}y\mathrm{d}x.$$

Therefore, the sesquilinear form

$$\mathcal{A}(u,\varphi):=\int_a^b\int_Q p(y)\chi_0(y)\frac{\partial u}{\partial y}(x,y)\overline{\frac{\partial\varphi}{\partial y}(x,y)}\;\mathrm{d}y\mathrm{d}x$$

defines a self-adjoint operator $A$ on $H$, and Lemma A.1 implies that the operator sequence $A^\varepsilon$ converges to $A$ in the strong two-scale resolvent sense; see [18]. This implies, in particular, that $\lim_{\varepsilon\to 0}S^\varepsilon\supset\sigma(A)$. We are now concerned with getting further information about $\sigma(A)$. To this end, notice that the $x$-dependence in the formulation (A.1) is trivial; indeed, $u(x,y)=f(x)v(y)$, where $v(y)\in V_1$ is the unique solution to the formulation

$$\int_{Q_0}pv'\overline{\varphi'}+\int_Q v\overline{\varphi}=\int_Q\overline{\varphi}\quad\forall\varphi\in V_1.\tag{A.4}$$

This observation implies, in particular, that the spectra corresponding to (A.1) and (A.4) coincide. Therefore, denoting by $A_1$ the operator defined by the sesquilinear form

$$\mathfrak{b}_1(u,v) := \int_{Q_0} pu'\overline{v'}, \quad u,v \in V_1,$$

we obtain $\lim_{\varepsilon\to 0} S^\varepsilon \supset \sigma(A) = \sigma(A_1)$.

**A.2. "*NQ*-periodic" homogenization.** In the argument above we found the two-scale limit operator $A_1$ by choosing the unit cell $Q$ to be the periodic reference cell and by passing to the two-scale limit in (1.1) as $\varepsilon \to 0$. Replacing now $Q$ with $NQ$, $N \in \mathbb{N}$, we obtain an analogue of Lemma A.1, as follows.

LEMMA A.2. *Set $\lambda = -1$, and let $u_\varepsilon$ be the solution to* (1.1). *Then, up to a subsequence, $u_\varepsilon \xrightarrow{2} u_N$, where $u_N(x,y) = f(x)v_N(y)$ and $v_N$ is the unique solution to the problem*

$$\int_{NQ} \chi_0 p v_N'\overline{\varphi'} + \int_{NQ} v_N\overline{\varphi} = \int_{NQ} \overline{\varphi} \quad \forall \varphi \in V_N,$$

*where* $V_N := \left\{v \in H^1_\#(NQ) : v'(y) = 0 \text{ for } y \in Q_1\right\}$. *Furthermore, the inclusion*

$$\lim_{\varepsilon\to 0} S^\varepsilon \supset \sigma\left(A_N\right)$$

*holds, where $A_N$ is the operator defined using the sesquilinear form*

$$\mathfrak{b}_N(u,v) := \int_{NQ} \chi_0 pu'\overline{v'}, \quad u,v \in V_N.$$

Applying Lemma A.2 for all $N \in \mathbb{N}$ yields

$$\lim_{\varepsilon\to 0} S^\varepsilon \supset \bigcup_{N\in\mathbb{N}} \sigma(A_N).$$

**A.3. Relation to the Bloch spectrum.** For each $\theta \in [0,1)$, $k = 1,2,\dots$, we denote by $\widetilde{v}^k(\theta) = \widetilde{v}^k(\theta,y)$ the extension of the eigenfunction $v^k(\theta)$ (see section 2) to the whole space $\mathbb{R}$ such that $\widetilde{v}^k(\theta,y)\exp(-2\pi \mathrm{i}\theta y)$, $y \in \mathbb{R}$, is $Q$-periodic. Notice that if $\theta = j/N$ for some integers $N, j$ such that $0 \le j \le N-1$, then for all $k = 1,2,\dots$ the functions $\widetilde{v}^k(\theta)$ are $NQ$-periodic; in particular, $\widetilde{v}^k(\theta) \in V_N$. In fact, the functions $\widetilde{v}^k(\theta)$, $k = 1,2,\dots$ are eigenfunctions of the operator $A_N$. Indeed, for all $\varphi \in V_N$ define the function $\Phi(y) := \sum_{k=0}^{N-1}\varphi(y+k)\exp(-2\pi\mathrm{i}\theta k)$, and notice that since $\Phi(y) \in V(\theta)$, one has $(A(\theta)v^k(\theta),\Phi) = \lambda_k(\theta)(v^k(\theta),\Phi)$. Therefore, writing for brevity $v$ in place of both $v^k(\theta)$ and $\widetilde{v}^k(\theta)$, we obtain

$$\int_{NQ}\chi_0 pv'\overline{\varphi} = \sum_{l=0}^{N-1}\int_Q \chi_0(y+l)p(y+l)v'(y+l)\overline{\varphi(y+l)}\mathrm{d}y = \int_Q \chi_0 pv\overline{\Phi} = \lambda_k(\theta)\int_Q v\overline{\Phi}$$

$$= \sum_{l=0}^{N-1}\lambda_k(\theta)\int_Q v(y)\exp(2\pi\mathrm{i}\theta l)\overline{\varphi(y+l)}\mathrm{d}y = \lambda_k(\theta)\int_{NQ} v\overline{\varphi}.$$

The above implies that

$$\lim_{\varepsilon\to 0} S^\varepsilon \supset \bigcup_{N\in\mathbb{N}} \sigma\left(A_N\right) \supset \bigcup_{\substack{N\in\mathbb{N}\\ 0\le j\le N-1}} \sigma\big(A(j/N)\big).$$

Finally, using the facts that the set of rational numbers $j/N$ is dense in $[0,1)$ and that the eigenvalues $\lambda=\lambda(\theta)$ are continuous with respect to $\theta$ (see Appendix B below) yields

$$\lim_{\varepsilon\to 0} S^\varepsilon \supset \bigcup_{\theta\in[0,1)} \sigma\big(A(\theta)\big).$$

**Appendix B.** Here we show that the eigenvalues of the operators $A(\theta)$ defined by (2.3) are continuous in $\theta$. The statement of continuity of $\lambda_k=\lambda_k(\theta)$, $k\in\mathbb{N}$, is not a simple consequence of the continuity of eigenvalues for the usual Floquet–Bloch decomposition. An important distinct feature of our case is the dependence on $\theta$ of the operator domain $V(\theta)$. The continuity of $\lambda_k(\theta)$ in $\theta$ is therefore shown to be a consequence of the continuity of the spaces $V(\theta)$ with respect to $\theta$; see Lemma 2.3. In fact, we show that the continuity of $V(\theta)$ leads to the operators $A(\theta)$ being continuous with respect to $\theta$ in the norm-resolvent sense, which implies as one particular consequence the continuity of $\lambda_k(\theta)$.

THEOREM B.1. *For any $\theta\in[0,1)$, let $\theta_n\in[0,1)$ be such that $\theta_n\to\theta$ as $n\to\infty$. Then the sequence $\big(A(\theta_n)+I\big)^{-1}$ converges to $\big(A(\theta)+I\big)^{-1}$ in the operator norm. In particular, the eigenvalues $\lambda_k(\theta)$, $k\in\mathbb{N}$, of $A(\theta)$ are continuous functions of $\theta\in[0,1)$, e.g., $\lim_{n\to\infty}\lambda_k(\theta_n)=\lambda_k(\theta)$.*

An essential ingredient to the proof of Theorem B.1 is the following result.

LEMMA B.2. *For any $\theta\in[0,1)$, let $\theta_n$ be such that $\theta_n\to\theta$ as $n\to\infty$ and $f_n, f\in L^2(Q)$ be such that $f_n\rightharpoonup f$ in $L^2(Q)$. Then, the solutions $u_n\in V(\theta_n)$ of the problems $\big(A(\theta_n)+I\big)u_n=f_n$ weakly converge in $H^1(Q)$ to the solution $u\in V(\theta)$ of the problem $\big(A(\theta)+I\big)u=f$.*

*Proof.* Let $u_n$ be as in the statement of the lemma; then

$$\int_Q p u_n'\overline{\varphi} + \int_Q u_n\overline{\varphi} = \int_Q f_n\overline{\varphi} \quad \forall\varphi\in V(\theta_n). \tag{B.1}$$

As the sequence $f_n$ is weakly convergent, it is bounded, and we find by choosing $\varphi=u_n$ in (B.1) that $\|u_n\|_{H^1(Q)}\le\|f_n\|_{L^2(Q)}$; e.g., the sequence $u_n$ is bounded in $H^1(Q)$. In particular, up to a subsequence, $u_n$ converges weakly in $H^1(Q)$ (hence strongly in $L^2(Q)$) to some $u\in H^1(Q)$. It it readily shown that $u\in V(\theta)$.

Furthermore, for a fixed $\varphi\in V(\theta)$ there exist, by Lemma 2.3, $\varphi_n\in V(\theta_n)$ such that $\varphi_n\to\varphi$ strongly in $H^1(Q)$. Choosing $\varphi_n$ as the test function in (B.1) and passing to the limit $n\to\infty$ shows that $u$ is a solution to

$$\int_Q p u'\overline{\varphi} + \int_Q u\overline{\varphi} = \int_Q f\overline{\varphi} \quad \forall\varphi\in V(\theta).$$

By virtue of the fact that the solution $u$ is unique, the above argument applies to any subsequence of $u_n$. Therefore, the claim holds for the whole sequence $u_n$. ∎

We shall now proceed with the proof of Theorem B.1.

*Step* 1. Let $\theta, \theta_n$ satisfy the assumptions of the theorem. First, we show that the operator sequence $R_n := \big(A(\theta_n) + I\big)^{-1}$ converges uniformly to $R := \big(A(\theta) + I\big)^{-1}$, e.g.,

$$\|R_n - R\| = \sup_{\|f\|_{L^2(Q)}=1} \|R_n f - Rf\|_{L^2(Q)} \to 0 \quad \text{as } n \to \infty.$$

For all $n$, let $f_n$ be such that

$$\sup_{\|f\|_{L^2(Q)}=1} \|R_n f - Rf\|_{L^2(Q)} \leq \|R_n f_n - Rf_n\|_{L^2(Q)} + \frac{1}{n}.$$

Since $\|f_n\|_{L^2(Q)} = 1$, the sequence $f_n$ has a subsequence that converges weakly to some $f_0 \in L^2(Q)$. Therefore, by Lemma B.2 the sequence $R_n f_n$ converges to $Rf_0$ strongly in $L^2(Q)$. Furthermore, since $R$ is compact, we infer that $Rf_n$ converges to $Rf_0$ strongly in $L^2(Q)$, and therefore

$$\sup_{\|f\|_{L^2(Q)}=1} \|R_n f - Rf\|_{L^2(Q)} \leq \|R_n f_n - Rf_0\|_{L^2(Q)} + \|Rf_n - Rf_0\|_{L^2(Q)} + \frac{1}{n}.$$

The right-hand side of the above estimate converges to zero as $n \to \infty$. The result follows as this argument holds for any subsequence of $f_n$ and therefore for the whole sequence $f_n$.

*Step* 2. We shall now show the continuity in $\theta$ of the eigenvalues $\lambda_k(\theta)$. To this end we establish that the eigenvalues $\mu_k(\theta) = \big(\lambda_k(\theta) + 1\big)^{-1}$ of the operator $\big(A(\theta) + I\big)^{-1}$ are continuous in $\theta$. To prove that $\mu_k(\theta)$ are continuous, we note that for any $f \in L^2(Q)$ one has

$$\frac{(R_n f, f)_{L^2(Q)}}{\|f\|^2_{L^2(Q)}} - \|R_n - R\| \leq \frac{(Rf, f)_{L^2(Q)}}{\|f\|^2_{L^2(Q)}} \leq \frac{(R_n f, f)_{L^2(Q)}}{\|f\|^2_{L^(Q)2}} + \|R_n - R\|,$$

and the min-max variational principle (cf. [16])

$$\mu_k(\theta) = \inf_{\substack{F \subset L^2(Q),\\ \dim F = k}} \sup_{f \in F,\ \|f\|_{L^2(Q)}=1} \frac{(Rf, f)_{L^2(Q)}}{\|f\|^2_{L^2(Q)}}$$

implies that $\big|\mu_k(\theta_n) - \mu_k(\theta)\big| \leq \|R_n - R\|$. Invoking the uniform convergence of $R_n$ to $R$ as $n \to \infty$ proves the result.

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